Problem 3I:

If $E \in \mathcal{L}$ and $m(E) > 0$, the set $E - E = \{x - y \mid x, y \in E\}$ contains an interval centered at 0.

Solution:

We use Problem 30. Hence $\exists$ I open interval such that:

$$m(E \cap I) > \frac{3}{4} m(I)$$

Let $F = E \cap I \subseteq I$. Since:

$$F - F \subseteq E - E$$

it is enough to prove the conclusion for F from the assumptions:

$$F \in \mathcal{L}, \quad F \subseteq I$$
$$m(F) > \frac{3}{4} m(I)$$

Since m is translation invariant we can assume that:

$$I = \left(-\tfrac{1}{2}m(I), \tfrac{1}{2}m(I)\right)$$

and we will prove that:

$$I \subseteq F - F.$$

This is equivalent to prove that:

$$*) \quad \forall x_0 \in I : (x_0 + F) \cap F \neq \emptyset.$$

and for this it is enough to show that:

$$**) \quad \forall x_0 \in I : m((x_0 + F) \cap F) \neq 0$$

For $x_0 \in I$ we have:

$$F = (F \cap (-x_0 + I)) \cup (F \cap (-x_0 + I)^c)$$
$$= (F \cap (-x_0 + I)) \cup (F \cap I \cap (-x_0 + I)^c)$$

a disjoint union and so:

$$m(F) = m(F \cap (-x_0 + I))$$
$$+ m(F \cap I \cap (-x_0 + I)^c)$$

by translation invariance of m:

$$= m((x_0 + F) \cap I) + m(F \cap I \cap (-x_0 + I)^c)$$

$$\leq m((x_0 + F) \cap I) + m(I \cap (-x_0 + I)^c)$$

$$\leq m((x_0 + F) \cap I) + \frac{1}{2} m(I)$$

where the last inequality follows from $x_0 \in I$.

Hence:

$$\frac{3}{4} m(I) < m(F) \leq m((x_0 + F) \cap I) + \frac{1}{2} m(I)$$

and so:

$$m((x_0 + F) \cap I) > \frac{1}{4} m(I)$$

On the other hand we have the disjoint union:

$$(x_0+F)\cap I = (x_0+F)\cap F \cup$$
$$\cup\, ((x_0+F)\cap(I\setminus F))$$

and so:

$$m((x_0+F)\cap F) =$$
$$= m((x_0+F)\cap I) - m((x_0+F)\cap(I\setminus F))$$
$$> \frac{1}{4} m(I) - m((x_0+F)\cap(I\setminus F))$$
$$\geq \frac{1}{4} m(I) - m(I\setminus F)$$
$$= \frac{1}{4} m(I) - (m(I) - m(F))$$
$$= m(F) - \frac{3}{4} m(I) > 0$$

This proves **) which implies *) and completes the solution. //