In Folland's book, we take over right after Theorem 1.16. Hence, we have the correspondences:

$$\left\{\begin{array}{l}\text{increasing} \\ \text{right-cont.} \\ \text{functions} \\ \mathbb{R} \longrightarrow \mathbb{R}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{complete} \\ \text{Borel measures} \\ \text{finite on bdd.} \\ \text{sets}\end{array}\right\}$$

$$F \longmapsto \mu_F, \ \mu_F((a,b]) = \\ = F(b) - F(a).$$

For such a measure μ we denote by $\mathcal{M}_\mu$ its domain.

Clearly:

$$\mu(K) < +\infty$$

for every compact subset K. Conversely, if μ is a measure finite on compact sets, then it is finite on bdd. sets. Hence, the above correspondence can be written:

$$\left\{\begin{array}{l}\text{increasing}\\ \text{right-cont.}\\ \text{functions}\\ \mathbb{R}\to\mathbb{R}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{complete}\\ \text{Borel measures}\\ \text{finite on}\\ \text{compact sets}\end{array}\right\}$$

For such a measure μ with domain $\mathcal{M}_\mu$:

$E \in \mathcal{M}_\mu$

$$\mu(E) = \inf\left\{ \sum_{j=1}^{+\infty} \mu((a_j, b_j]) \;\middle|\; \begin{array}{l} E \subseteq \bigcup_{j=1}^{+\infty} (a_j, b_j] \\ -\infty \leq a_j < b_j \leq +\infty \end{array} \right\}$$

Proposition:
With the previous notation: $\forall\, E \in \mathcal{M}_\mu$:

$$\mu(E) = \inf\left\{ \sum_{j=1}^{+\infty} \mu((a_j, b_j)) \;\middle|\; \begin{array}{l} E \subseteq \bigcup_{j=1}^{+\infty} (a_j, b_j) \\ -\infty \leq a_j < b_j \leq +\infty \end{array} \right\}$$

Proof:
Let us denote by $\nu(E)$ the infimum above.
If:

$$E \subseteq \bigcup_{j=1}^{+\infty} (a_j, b_j)$$

We can choose $(c_j^k)_k$ increasing and converging to b_j.
Hence:

$$(a_j, b_j) = \bigcup_{k=1}^{+\infty} (a_j, c_j^k]$$

and by continuity from below:

$$\sum_{j=1}^{+\infty} \mu((a_j, b_j)) = \sum_{j,k=1}^{+\infty} \mu((a_j, c_j^k])$$

We conclude that:

$$\mu(E) \leq \nu(E).$$

For $\varepsilon > 0$, consider a cover:

$$E \subseteq \bigcup_{j=1}^{+\infty} (a_j, b_j]$$

$\ni$

$$\sum_{j=1}^{+\infty} \mu((a_j, b_j]) \leq \mu(E) + \varepsilon$$

Let F be the increasing right-continuous function associated to μ.

$\forall j$ let δ_j be $\ni$:

$$F(b_j + \delta_j) - F(b_j) < \frac{\varepsilon}{2^j}$$

Hence:

$$E \subseteq \bigcup_{j=1}^{+\infty} (a_j, b_j + \delta_j)$$

and:

$$\nu(E) \leq \sum_{j=1}^{+\infty} \mu((a_j, b_j + \delta_j))$$

$$\leq \sum_{j=1}^{+\infty} \mu((a_j, b_j + \delta_j])$$

$$= \sum_{j=1}^{+\infty} (F(b_j + \delta_j) - F(a_j))$$

$$\leq \sum_{j=1}^{+\infty} \left(F(b_j) - F(a_j) + \frac{\varepsilon}{2^j} \right)$$

$$\leq \sum_{j=1}^{+\infty} \mu\big((a_j, b_j]\big) + \varepsilon$$

$$\leq \mu(E) + 2\varepsilon$$

$$\therefore \quad \nu(E) \leq \mu(E).$$

//

Corollary:
For μ a complete Borel measure on $\mathbb{R}$, finite on compact sets:

$\forall E \in \mathcal{M}_\mu$:

$$\mu(E) = \inf \left\{ \mu(U) \;\middle|\; \begin{array}{l} U \supseteq E \\ U \text{ open} \end{array} \right\}$$

Proof:
A set in $\mathbb{R}$ is open iff it is a countable union of disjoint open intervals. //

In what follows we consider:

$$\mathbb{R} = \bigcup_{j=1}^{+\infty} K_j$$

with K_j compact.
(e.g.:

$$\mathbb{R} = \bigcup_{j=1}^{+\infty} [-j, j]$$
)

And we can assume the sequence $\{K_j\}_{j=1}^{+\infty}$ increasing if necessary.

Proposition:
With the previous notation, let $E \in \mathcal{M}_\mu$ be given. Then $\forall \varepsilon > 0$ $\exists F \subseteq E \subseteq U$, F closed, U open $\ni$:

$$\mu(U \setminus E) < \varepsilon$$

$$\mu(E \setminus F) < \varepsilon$$

Proof:
Given E and $\varepsilon > 0$
$\exists U_j \supseteq E \cap K_j$ open $\ni$:

$$\mu(U_j) \le \mu(E \cap K_j) + \frac{\varepsilon}{2^j} < +\infty$$

Let $U = \bigcup_{j=1}^{+\infty} U_j$. It is easy to check that:

$$U \setminus E \subseteq \bigcup_{j=1}^{+\infty} (U_j \setminus (E \cap K_j))$$

Hence:

$$\mu(U \setminus E) \le \sum_{j=1}^{+\infty} \mu(U_j \setminus (E \cap K_j))$$

$$= \sum_{j=1}^{+\infty} (\mu(U_j) - \mu(E \cap K_j))$$

$$\le \varepsilon$$

Now consider E^c and let $V \supseteq E^c$ open $\ni$:

$$\mu(V \setminus E^c) < \varepsilon$$

but for $F = V^c \subseteq E$ we have:

$$E \setminus F = E \cap F^c = E \cap V$$

$$= V \cap (E^c)^c = V \setminus E^c$$

Hence:

$$\mu(E \setminus F) < \varepsilon. \quad //$$

Proposition:
With the previous notation: $\forall E \in \mathcal{M}_\mu$:

$$\mu(E) = \sup\left\{\mu(K) \,\middle|\, \begin{array}{l} K \subseteq E \\ K \text{ compact} \end{array}\right\}$$

Proof:
Clearly $\mu(E) \geq$ the sup above.
Assume that $\{K_j\}_{j=1}^{+\infty}$ is increasing.
Then, for a given E:

$$\mu(E) = \mu\left(\bigcup_{j=1}^{+\infty} E \cap K_j\right)$$

$$= \lim_{j \to +\infty} \mu(E \cap K_j)$$

For every j choose

$$F_j \subseteq E \cap K_j \subseteq K_j$$

closed such that:

$$\mu(E \cap K_j) - \mu(F_j) = \\ = \mu((E \cap K_j) \setminus F_j) < \frac{1}{j}$$

Hence F_j is compact and

$$\lim_{j \to +\infty} \mu(F_j) = \lim_{j \to +\infty} \mu(E \cap K_j) \\ = \mu(E)$$

and the claim holds. //

Now we give a characterization of $\mathcal{M}_\mu$.

Theorem:
For μ as above and $E \subseteq \mathbb{R}$, the following are equivalent:

1) $E \in \mathcal{M}_\mu$

2) $\exists\, V$ G_δ $\ni$ $E \subseteq V$ and $\mu(V \setminus E) = 0$

3) $\exists\, H$ F_σ $\ni$ $H \subseteq E$ and $\mu(E \setminus H) = 0$.

Proof:

Clearly 2) $\Rightarrow$ 1) and 3) $\Rightarrow$ 1).

<u>1) $\Rightarrow$ 2):</u>

From the results above given $E \in \mathcal{M}_\mu$ and $j \in \mathbb{N}$ $U_j \supseteq E$ open $\ni$:

$$\mu(U_j \setminus E) < \frac{1}{j}.$$

Let $V = \bigcap_{j=1}^{+\infty} U_j$. Hence:

$$E \subseteq V$$

$$V \setminus E \subseteq U_j \setminus E \quad \forall j$$

$$\therefore \mu(V \setminus E) < \frac{1}{j} \quad \forall j$$

$$\Rightarrow \mu(V \setminus E) = 0.$$

2) $\Longrightarrow$ 3):

For a given E let $V \supseteq E^c$ G_δ $\ni$:

$$\mu(V \setminus E^c) = 0$$

Let $H = V^c$ which is F_σ.
Hence:

$$H \subseteq E$$

$$E \setminus H = E \cap H^c = E \cap V$$
$$= V \cap (E^c)^c = V \setminus E$$

$\therefore$ $\mu(E \setminus H) = 0$. //

Another useful result.

Proposition:
With the previous notation let $E \in \mathcal{M}_\mu$ $\ni$ $\mu(E) < +\infty$.
Then, $\forall \varepsilon > 0$ $\exists$ A = finite union of open intervals $\ni$:

$$\mu(E \Delta A) < \varepsilon$$

Proof:
Let $\varepsilon > 0$ be given.
Then $\exists \{I_j\}_{j=1}^{+\infty}$ open intervals and K compact $\ni$:

$$K \subseteq E \subseteq \bigcup_{j=1}^{+\infty} I_j$$

$$\mu(E \setminus K) = \mu(E) - \mu(K) < \varepsilon$$

$$\sum_{j=1}^{+\infty} \mu(I_j) - \mu(E) < \varepsilon$$

In particular:

$$\sum_{j=1}^{+\infty} \mu(I_j) < +\infty.$$

Since $K \subseteq \bigcup_{j=1}^{+\infty} I_j$ we can choose a subcover.
We choose it of the form:

$$K \subseteq \bigcup_{j=1}^{n} I_j$$

and so that:

$$\sum_{j=n+1}^{+\infty} \mu(I_j) < \varepsilon$$

Let $A = \bigcup_{j=1}^{n} I_j$.

Then:

$$E \Delta A = (E \cap A^c) \cup (A \cap E^c)$$
$$\subseteq \left(E \cap \left(\bigcup_{j=n+1}^{+\infty} I_j\right)\right) \cup (A \cap K^c)$$
$$\subseteq \left(\bigcup_{j=n+1}^{+\infty} I_j\right) \cup (A \setminus K)$$

But:

$$\mu(A \setminus K) = \mu(A) - \mu(K)$$
$$\leq \sum_{j=1}^{n} \mu(I_j) - \mu(K)$$
$$\leq \sum_{j=1}^{+\infty} \mu(I_j) - \mu(E) + \varepsilon$$
$$< 2\varepsilon$$

Hence:

$$\mu(E \Delta A) \leq \mu\left(\bigcup_{j=n+1}^{+\infty} I_j\right) + \mu(A \setminus K)$$

$$\leq \sum_{j=n+1}^{+\infty} \mu(I_j) + 2\varepsilon \leq 3\varepsilon$$

//