Continuamos con el ejemplo:

$$M = \mathbb{C}^n, \quad H = \mathbb{T}^n$$

$$\omega = i \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$$

Hemos calculado:

$$X = (s_1, \ldots, s_n) \in \mathbb{R}^n$$

$$X^\#_z = (is_1 z_1, \ldots, is_n z_n) = \sum_{j=1}^{n} \left(is_j z_j \frac{\partial}{\partial z_j} - is_j \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right)$$

El paso II) es proponer funciones f ∋:

$$X_f = X^\# \qquad \forall X \in \mathbb{R}^n$$

o funciones $f_1, \ldots, f_n$ ∋:

$$X_{f_j} = X_j^\# \qquad j = 1, \ldots, n$$

para alguna base $X_1, \ldots, X_n$ de $\mathbb{R}^n$.

Esto requiere tener una fórmula para X_f. En nuestro caso la fórmula se halla como sigue.

$$X_f = \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial z_j} + b_j \frac{\partial}{\partial \bar{z}_j} \right)$$

con a_j, b_j por determinar.

Para hallar a_j, b_j usamos que:

$$df = \omega(X_f, \cdot).$$

Por un lado:

$$df = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial z_j}\, dz_j + \frac{\partial f}{\partial \bar{z}_j}\, d\bar{z}_j \right)$$

Por otro lado:

$$\omega(X_f, \cdot) = i \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j \left(\sum_{h=1}^{n} \left(a_h \frac{\partial}{\partial z_h} + b_h \frac{\partial}{\partial \bar{z}_h} \right), \cdot \right)$$

$$= i \sum_{j,h=1}^{n} a_h\, dz_j \wedge d\bar{z}_j \left(\frac{\partial}{\partial z_h}, \cdot \right)$$

$$+ i \sum_{j,h=1}^{n} b_h\, dz_j \wedge d\bar{z}_j \left(\frac{\partial}{\partial \bar{z}_h}, \cdot \right)$$

pero $dz_j \wedge d\bar{z}_j = dz_j \otimes d\bar{z}_j - d\bar{z}_j \otimes dz_j$:

$$= i \sum_{j,h=1}^{n} \left(a_h \underline{dz_j\left(\frac{\partial}{\partial z_h} \right)} d\bar{z}_j - a_h \cancelto{0}{d\bar{z}_j\left(\frac{\partial}{\partial z_h} \right)} dz_j \right)$$

$$+ i \sum_{j,h=1}^{n} \left(b_h \cancelto{0}{dz_j\left(\frac{\partial}{\partial \bar{z}_h} \right)} d\bar{z}_j - b_h \underline{d\bar{z}_j\left(\frac{\partial}{\partial \bar{z}_h} \right)} dz_j \right)$$

$$= i \sum_{j,h=1}^{n} a_h \delta_{jh}\, d\bar{z}_j - i \sum_{j,h=1}^{n} b_h \delta_{jh}\, dz_j$$

$$= i\sum_{j=1}^{n} a_j\, d\bar{z}_j - i\sum_{j=1}^{n} b_j\, dz_j$$

$$(= \omega(X_f, \cdot))$$

Comparamos con:

$$df = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial z_j}dz_j + \frac{\partial f}{\partial \bar{z}_j}d\bar{z}_j\right)$$

Por tanto, la solución es:

$$\underline{a_j} = -i\frac{\partial f}{\partial \bar{z}_j}, \quad b_j = i\frac{\partial f}{\partial z_j}$$

$$\therefore \quad X_f = i\sum_{j=1}^{n}\left(\frac{\partial f}{\partial z_j}\frac{\partial}{\partial \bar{z}_j} - \frac{\partial f}{\partial \bar{z}_j}\frac{\partial}{\partial z_j}\right).$$

Por otro lado, en nuestro caso:

$$X = (s_1, \ldots, s_n) \in \mathbb{R}^n$$

$$X^{\#}_z = i\sum_{j=1}^{n}\left(s_j z_j\frac{\partial}{\partial z_j} - s_j\bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right)$$

Para cada $X \in \mathbb{R}^n$ buscamos una función f (ó μ_X) tal que:

$$X_f = X^{\#}$$

Por el cálculo anterior f debe cumplir:

$$\frac{\partial f}{\partial z_j} = -s_j \bar{z}_j \, , \quad \frac{\partial f}{\partial \bar{z}_j} = -s_j z_j$$
$$\forall j = 1, \dots, n.$$

Una solución es:

$$f(z) = -\sum_{j=1}^{n} s_j \bar{z}_j z_j = -\sum_{j=1}^{n} s_j |z_j|^2$$

En otras palabras podemos tomar $\forall X = (s_1, \dots, s_n) \in \mathbb{R}^n$:

$$\mu_X : \mathbb{C}^n \longrightarrow \mathbb{R}$$
$$\mu_X(z) = -\sum_{j=1}^{n} s_j |z_j|^2$$

y se cumple:

$$X_{\mu_X} = X^{\#} \quad \forall X \in \mathbb{R}^n.$$

Si f, g son dos soluciones:

$$X_f = X^{\#} = X_g$$
$$\Rightarrow \quad X_f = X_g \quad \Rightarrow \quad \omega(X_f, \cdot) = \omega(X_g, \cdot)$$
$$\Rightarrow \quad df = dg \quad \Rightarrow \quad \exists \; c \in \mathbb{R} \ni f = g + c.$$

Observamos que:

$$\mu_X(t \cdot z) = \mu_X(t_1 z_1, \dots, t_n z_n)$$

$$= -\sum_{j=1}^{n} s_j |t_j z_j|^2 = \mu_X(z)$$

$\therefore$ μ_X es $\mathbb{T}^n$-invariante.

El mapeo de momento es:

$$\mu : \mathbb{C}^n \longrightarrow \mathbb{R}^n$$

$$\mu(z) = -(|z_1|^2, \ldots, |z_n|^2)$$

pues:

$$\langle (s_1, \ldots, s_n), \mu(z) \rangle = -\sum_{j=1}^{n} s_j |z_j|^2$$

μ es obviamente $\mathbb{T}^n$-invariante.