DIHEDRAL COVERINGS OF MONTESINOS KNOTS

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ABSTRACT. We determine the family of Montesinos knots k which have the 3-sphere as a dihedral quotient $S^3 \to (S^3, k)$, and we find also 'dihedral-like' coverings of certain Montesinos knots. Through understanding of the singular set of these coverings, we conclude the universality of many Montesinos knots.

1. INTRODUCTION.

Dihedral coverings. Since the time of Reidemeister (the first part of the 20th century), it is an interesting problem to understand the dihedral branched coverings of knots (§ 2), especially the dihedral coverings of the 3-sphere over itself. Fox's Quick Trip and [1] describe coverings of this type for 2-bridge knots; there are some examples of this kind of coverings for only a few more knots spread in other works on the subject.

We determine the family of Montesinos knots k, in terms of its invariants, which admit a dihedral quotient $S^3 \to (S^3, k)$ (Theorem 3.2); we get this result using knowledge on cyclic coverings of Seifert manifolds developed in [12].

It is a very difficult problem to describe in an intelligible way what is the type of the link in the preimage of a knot k under a branched covering $S^3 \to (S^3, k)$ (see, for example, the heroic struggles in [16] and [6]). With our new understanding of dihedral quotients of Montesinos knots, we are able to describe the preimage of the knot in such a covering as a union of Montesinos knots (Montesinos knots again!), and, in some cases, we are able to compute explicitly the invariants of the knots. In general we give an algorithm for such a task (§ 5).

We extend the construction of dihedral quotient to 'dihedral-like' covering (§ 6). With these new coverings, we construct, for certain Montesinos knots, k, branched coverings $S^3 \to (S^3, k)$; we are able, also in these cases, to give explicit Montesinos invariants for components of the preimage of k.

Universals. In 1982 Thurston proved the striking fact that there are universal links ([14]). A link $k \subset S^3$ is called *universal* if each closed, connected, orientable 3-manifold is a branched covering over (S^3, k). A very interesting

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problem is to describe the family of universal links in the 3-sphere. Towards this goal the following theorems are known.

Theorem 1.1 ([6]). *All hyperbolic 2-bridge links are universal.*

The 2-bridge link $\ell(b/a)$ is the Montesinos knot $m(b/a)$. A 2-bridge link $\ell(b/a)$ is hyperbolic if and only if $a \not\equiv \pm 1 \bmod b$.

The pretzel link $p(b;\alpha_1,\ldots,\alpha_t)$ is the Montesinos link $m(b/1, 1/\alpha_1,\ldots,1/\alpha_t)$. An *Uchida link* is a pretzel link $p(b;\alpha_1,\ldots,\alpha_t)$ such that at least two α_i's are even.

Theorem 1.2 ([16]). *All Uchida links are universal except for:* $p(2s, 2t)$, $p(2,-2,s)$, *with* $s,t \in Z-\{0\}$; $p(0;\pm 2,\pm 3,\mp 4)$, $p(0;\pm 3,\pm 6,\mp 2)$, $p(0;\pm 4,\pm 4,\mp 2)$, *and* $p(0;2,2,-2,-2)$.

Theorem 1.3 ([13]). *If* $|n| > 1$ *is odd, then* $p(n,n,-n)$ *is universal.*
If $n \neq -2$ *is even, then* $p(3,3,n)$ *is universal.*

Once one knows that certain knots (links) are universal, to decide if a given link k is universal, one constructs branched coverings $S^3 \to (S^3, k)$ and tries to find an already known universal link in the preimage of k. With our understanding of preimages of links in dihedral and dihedral-like coverings, we give lists of Montesinos knots which are universal (§§ 6 and 7). For this end we rely heavily on the previously stated theorems.

The paper is organized as follows. In § 2 we precisely specify the notion of dihedral quotient and recall tools to handle Seifert manifolds and Montesinos knots. In § 3 we prove the main theorem of this work (Theorem 3.2), which is an essential ingredient to get deeper in understanding dihedral coverings. § 4 is a convenient formalization of the theory of rational tangles. In § 5 we plunge into description of branch sets of dihedral quotients. Next we extend the notion of dihedral quotient to 'dihedral-like' and get some universality results in § 6. Finally in § 7 we harvest several results on universality of families of links.

2. Dihedral quotients.

A *branched covering* $\varphi : M^n \to N^n$ is a proper open map between n-manifolds such that there is a codimension 2 subcomplex $K \subset N$ with $\varphi| : M - \varphi^{-1}(K) \to N - K$ a finite covering space. For the purposes of this paper, $K \subset N$ will be a properly embedded submanifold.

The map $\varphi| : M - \varphi^{-1}(K) \to N - K$ is called *the associated covering space* of φ. The submanifold K is called the *branching* of φ. We say that 'φ is *branched along* K', 'φ is a *branched covering over* (N, K)', and that 'M is a *branched covering over* (N, K)', and write $\varphi : M \to (N, K)$.

The set $\varphi^{-1}(K)$ is called the *singular set* of φ. The *pseudo-branch* of φ is the set of points $x \in \varphi^{-1}(K)$ such that φ is a homeomorphism at x. The *branch set* of φ is the complement, in $\varphi^{-1}(K)$, of the pseudo-branch of φ.

A branched covering $\varphi : M \to (N, k)$ determines (and is determined) by a representation $\omega_\varphi : \pi_1(N - k) \to S_n$ into the symmetric group S_n. It is

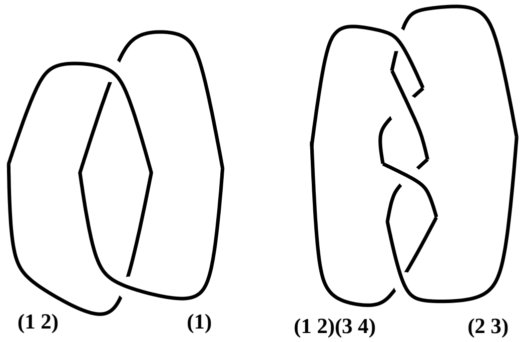


FIGURE 1. Dihedral quotients, but not dihedral coverings.

customary to call a branched covering after the nature of the image $Im(\omega_\varphi)$. If $Im(\omega_\varphi)$ is a cyclic group, then φ is called a *cyclic* covering; if $Im(\omega_\varphi)$ is a dihedral group, then φ is called a *dihedral* covering, etc.

If $B_2(k)$ is the cyclic double branched covering over (S^3, k) branched along all components of k, and $\rho : H_1(B_2(k)) \to Z_n$ is an epimorphism, the covering space $\psi : \tilde{M} \to B_2(k)$ determined by ρ has an involution $\tilde{u} : \tilde{M} \to \tilde{M}$ such that the square

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{u}} & \tilde{M} \\ \psi\downarrow & & \downarrow\psi \\ B_2(k) & \xrightarrow{u} & B_2(k) \end{array}$$

commutes, where $u : B_2(k) \to B_2(k)$ is the involution with quotient (S^3, k). Therefore one has a commutative square

$$\begin{tikzcd} & \tilde{M} \arrow[dl, "q"'] \arrow[dr, "\psi"] & \\ M \arrow[dr, "\varphi"'] & & B_2(k) \arrow[dl, "p"] \\ & (S^3, k) & \end{tikzcd}$$

where $M = \tilde{M}/\tilde{u}$. In Fox's Quick Trip, the map φ would be called a dihedral covering, for it would correspond to a dihedral representation. Actually Fox starts a discussion, and gives a procedure to obtain metacyclic representations of knot groups in [2], p 160 and ff. This procedure, translated into the 'language of commutative squares', is the one presented here in the particular case of dihedral representations. But if k is not connected, or if

n is even, it might be the case that $Im(\omega_\varphi) \leq S_n$ is not a dihedral group. See Figure 1.

We call $\varphi : M \to (S^3, k)$ a *dihedral quotient*. Notice that the branching of $q : \tilde{M} \to M$ is also the pseudo-branch of φ. For other works using the language of commutative squares for this kind of coverings, see [9] and [1].

2.1. **Standard double coverings of Seifert Manifolds.** The *Seifert manifold* M with *Seifert symbol* $M = (O, 0; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$, where $(\alpha_i, \beta_i) = 1$ and $\alpha_i \neq 0$ $(i = 1, \ldots, t)$, is constructed as follows:

Let $p_1, \ldots, p_t \in S^2$ and $S_0^2 = \overline{S^2 - \cup \mathcal{N}(p_i)}$; we obtain M as the union of $S_0^2 \times S^1$ with t solid tori $V_1, \ldots, V_t$ along the boundaries. Call $q_1, \ldots, q_t$ the boundary components of S_0^2, and h a factor $\{x\} \times S^1$ for some $x \in S_0^2$, and let $g_i : \partial V_i \to q_i \times S^1$ be a homeomorphism such that $g_i(m_i) \sim q_i^{\alpha_i} h^{\beta_i}$ for m_i a meridian of V_i, $i = 1, \ldots, t$. Then $M = (S_0^2 \times S^1) \bigcup_{\cup g_i} (\cup_i V_i)$.

The core e_i of V_i is called the *fiber of M corresponding to the ratio β_i/α_i*. In case $|\alpha_i| > 1$, e_i is called *exceptional*; otherwise e_i and the curves $h = \{x\} \times S^1$ are called *ordinary fibers of M*.

Let $M = (O, 0; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$; there is an involution $u : M \to M$, described below, associated to the Seifert symbol $(O, 0; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ (not to the manifold M), which gives rise to a 2-fold branched covering $p : M \to (S^3, k)$ branched along all components of k; the link k is called *the Montesinos knot* with symbol $k = m(\beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ (See [8]).

Let E be a great circle in S^2 which contains the points $p_1, \ldots, p_t$ in that order with respect to a given orientation of E; then reflection on S^2 along E induces a $180°$ rotation $u_0 : S_0^2 \times S^1 \to S_0^2 \times S^1$ along the 'axis' $E_0 = (E \cap S_0^2) \times \{1\} \cup (E \cap S_0^2) \times \{-1\}$. u_0 extends to each solid torus V_i in M with two arcs as fixed point set in V_i, giving the involution $u : M \to M$. In the quotient $M/u \cong S^3$ we have the picture of the branching of p in t 3-balls, $B_1 = p(V_1), \ldots, B_t = p(V_t)$, joined by a 'trivial closed braid'. See Figure 2.

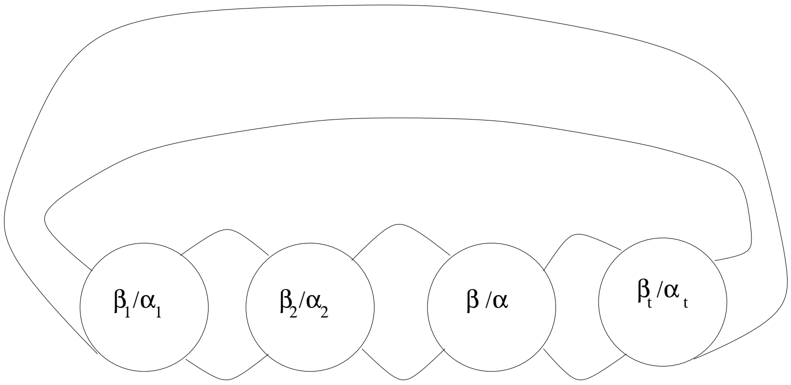


FIGURE 2

This branching, in each 3-ball, gives rise to a rational tangle, $\beta_i/\alpha_i = (B_i, a^i, m^i \cup \ell^i)$, with $a^i = p(fix(u) \cap V_i)$, $m^i = p(q_i \times \{1\}) \cup p(q_i \times \{-1\}) = m_0 \cup m_1$, and $\ell^i = p(\{x_0\} \times S^1) \cup p(\{x_1\} \times S^1) = \ell_0 \cup \ell_1$, where x_0, x_1 are the points in $E_0 \cap q_i$ in that order with respect to the orientation of E_0 (see § 4 and Figure 3).

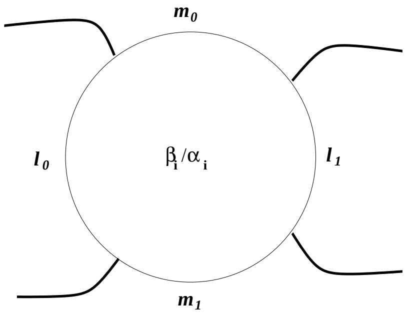


FIGURE 3

We call the covering $u : M \to M$ the *Standard involution* associated to the symbol $(O, 0; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$.

We will use, without warning, the next classification theorems.

Theorem 2.1 ([10]). *Two Seifert symbols represent homeomorphic Seifert manifolds with a fiber preserving homeomorphism if and only if one of the symbols can be transformed into the other with a finite sequence of the following moves:*

(0) Permute the ratios
(1) Add or delete $0/1$
(2) Replace the pair $\beta_i/\alpha_i, \beta_j/\alpha_j$ *with the pair* $(\beta_i + k\alpha_i)/\alpha_i, (\beta_j - k\alpha_j)/\alpha_j$.

Theorem 2.2 ([17]). *Two Montesinos symbols with at least three* α*'s, in absolute value, greater than 1 represent equivalent Montesinos knots if and only if one of the symbols can be transformed into the other with a finite sequence of the following moves:*

(0) Permute the ratios cyclically
(1) Add or delete $0/1$
(2) Replace the pair $\beta_i/\alpha_i, \beta_j/\alpha_j$ *with the pair* $(\beta_i + k\alpha_i)/\alpha_i, (\beta_j - k\alpha_j)/\alpha_j$.

In a Montesinos knot $k = m(\beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$ we allow that some of the $\alpha_i = 0$; but in this case k is a union of connected sums of 2-bridge links.

Notice that $m(\beta_1/\alpha_1, \beta_2/\alpha_2) \sim m(-(\alpha_1\beta_2 + \alpha_2\beta_1)/(\alpha_2 r_1 + \beta_2 s_1))$, the 2-bridge link $\ell(-(\alpha_1\beta_2 + \alpha_2\beta_1)/(\alpha_2 r_1 + \beta_2 s_1))$, where r_1 and s_1 are integers such that $\alpha_1 r_1 - \beta_1 s_1 = 1$.

3. DIHEDRAL COVERINGS OF MONTESINOS KNOTS.

If $M = (O, 0; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$, then the fundamental group $\pi_1(M) = \pi_1(O, 0; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t) = \langle q_1, \dots, q_t, h : [q_i, h] = 1, q_i^{\alpha_i} h^{\beta_i} = 1, q_1 \cdots q_t =$

$1\rangle$. We call the system $q_1, \ldots, q_t, h$ a *standard system of generators* for the symbol $(O, 0; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$.

We want to understand the coverings of Seifert manifolds. The next lemma, which follows from ([12], Lemma 1 and its proof), will help. Let $\varepsilon \in S_n$ denote the standard n-cycle, $\varepsilon = (1, 2, \ldots, n)$.

Lemma 3.1. *Let $\omega : \pi_1(O, 0; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t) \to S_n$ be the representation given by $\omega(h) = \varepsilon$, and $\omega(q_i) = \varepsilon^{r_i}$, $i = 1, \ldots, t$, such that $\sum r_i = 0$. Then the covering space associated to ω is*

$$\varphi : (O, 0; B_1/\alpha_1, \ldots, B_t/\alpha_t) \to (O, 0; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t),$$

where $B_i = (\beta_i + r_i\alpha_i)/n$, for $i = 1, \ldots, t$.

Also, if $\tilde{e}_i$ is the fiber corresponding to B_i/α_i, and e_i is the fiber corresponding to β_i/α_i, then $\varphi^{-1}(e_i) = \tilde{e}_i$, $i = 1, \ldots, t$. For fundamental groups it holds $\varphi(\tilde{h}) \sim h^n$, and $\varphi(\tilde{q}_i) \sim q_i h^{-r_i}$, for $\tilde{q}_1, \ldots \tilde{q}_t, \tilde{h}$ a standard system of generators of the covering. The map induced by φ on orbit surfaces is a homeomorphism.

Notice that, with the hypothesis of Lemma 3.1, it holds that $\beta_i + r_i\alpha_i \equiv 0 \mod n$ $(i = 1, \ldots, t)$, for ω is a group homomorphism.

If k is the Montesinos knot $m(\beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$, the number $\Delta(k) = \beta_1\alpha_2\cdots\alpha_t + \alpha_1\beta_2\cdots\alpha_t + \cdots + \alpha_1\alpha_2\cdots\beta_t$ is called the *determinant* of k. Let $B_2(k)$ be the double branched cover of (S^3, k); if $H_1(B_2(k))$ is infinite, then $\Delta(k) = 0$, and if $H_1(B_2(k))$ is finite, then $|H_1(B_2(k))| = |\Delta(k)|$.

Theorem 3.2. *Let k be the Montesinos knot $m(\beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$, and let n be a positive divisor of $\Delta(k)$. If $(n, \alpha_i) = 1$ for $i = 1, \ldots, t$, then there exists a dihedral quotient $\varphi : S^3 \to (S^3, k)$.*

Any n-fold dihedral quotient $\varphi : S^3 \to (S^3, k)$ has the Montesinos knot

$$m(b_1/\alpha_1, \ldots, b_t/\alpha_t)$$

as pseudo-branch, where $b_i = (\beta_i + r_i\alpha_i)/n$, $i = 1, \ldots, t$, and the integers $r_1, \ldots, r_t$ satisfy the system

$$\begin{cases} \beta_1 + r_1\alpha_1 \equiv 0 \quad \mathit{mod}\ n \\ \qquad \vdots \\ \beta_t + r_t\alpha_t \equiv 0 \quad \mathit{mod}\ n \\ \sum r_i = 0. \end{cases}$$

Proof. Assume $n|\Delta(k)$; then there is an epimorphism $\omega : H_1(B_2(k)) \to Z_n$; we regard $Z_n = \langle \varepsilon \rangle \leq S_n$, where $\varepsilon = (1, 2, \ldots, n)$. Write $\omega(q_i) = \varepsilon^{r_i}$ and $\omega(h) = \varepsilon^s$.

We have $H_1(B_2(k)) = \langle q_1, \ldots, q_t, h : q_i^{\alpha_i} h^{\beta_i} = 1, q_1 \cdots q_t = 1$, everything commutes $\rangle$. We can write $H_1(B_2(k)) = \langle x_2, \ldots, x_t, v : x_i^{\delta_i} = 1, v^\lambda x_2^{\lambda_2} \cdots x_t^{\lambda_t} = 1$, everything commutes $\rangle$, with $h = v^\ell$, $\ell = lcm\{\alpha_1, \ldots, \alpha_t\}$, $\alpha_1' = \alpha_1$, $\delta_1 = 1$, and $\delta_{i+1} = gcd\{\alpha_{i+1}, \alpha_1' \cdots \alpha_i'\}$, $\alpha_{i+1}' = \alpha_{i+1}/\delta_{i+1}$ for $i \geq 1$ ([11], Lemma 2.2).

Assume now that $(\alpha_i, n) = 1$ for $i = 1, \ldots, t$. If $\omega(x_i) = \varepsilon^{s_i}$ for some integer s_i, then $\omega(x_i^{\delta_i}) = \varepsilon^{\delta_i s_i} = (1)$; therefore $n|\delta_i s_i$. Since $\delta_i|\alpha_i$, by definition, we have that $(n, \delta_i) = 1$; then $n|s_i$ and, therefore, $\omega(x_i) = (1)$. Since ω is epimorphism, $\omega(v)$ is an n-cycle; then $\omega(h) = \omega(v)^\ell$ is an n-cycle, because $(n, \ell) = (n, lcm\{\alpha_1 \cdots \alpha_t\}) = 1$. Since $\omega(h) = \varepsilon^s$, we conclude that $(n, s) = 1$. By conjugating in $Z_n \leq S_n$, if necessary, we may assume that $\omega(h) = \varepsilon$. Since $\omega(q_1 \cdots q_t) = \varepsilon^{\Sigma r_i} = (1)$, therefore $n|\sum r_i$, or $\sum r_i = m \cdot n$ for some m; replacing r_t by $r_t - m \cdot n$, we get $\sum r_i = 0$.

By Lemma 3.1, $\psi : \tilde{M} = (O, 0; b_1/\alpha_1, \ldots, b_t/\alpha_t) \to B_2(k)$ is the n-fold cyclic covering space associated to ω, with $b_i = (\beta_i + r_i\alpha_i)/n$; also $\psi(\tilde{h}) \sim h^n$ and $\psi(\tilde{q}_i) \sim q_i h^{-r_i}$. If $\tilde{u} : \tilde{M} \to \tilde{M}$ is the standard involution for the symbol $(O, 0; b_1/\alpha_1, \ldots, b_t/\alpha_t)$, and $u : B_2(k) \to B_2(k)$ is the standard involution with $B_2(k)/u = (S^3, k)$, then the square

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{u}} & \tilde{M} \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ B_2(k) & \xrightarrow[u]{} & B_2(k) \end{array}$$

commutes, for $\psi \circ \tilde{u}(\tilde{h}) \sim \psi(\tilde{h}^{-1}) \sim h^{-n} \sim u(h^n) \sim u \circ \psi(\tilde{h})$, and $\psi \circ \tilde{u}(\tilde{q}_i) \sim \psi(\tilde{q}_i^{-1}) \sim q_i^{-1} h^{r_i} \sim u(q_i h^{-r_i}) \sim u \circ \psi(\tilde{q}_i)$. If $q : \tilde{M} \to \tilde{M}/\tilde{u}$ is the canonical projection, then q is a double branched covering over $(S^3, m(b_1/\alpha_1, \ldots, b_t/\alpha_t)) = \tilde{M}/\tilde{u}$; we have the commutative diagram

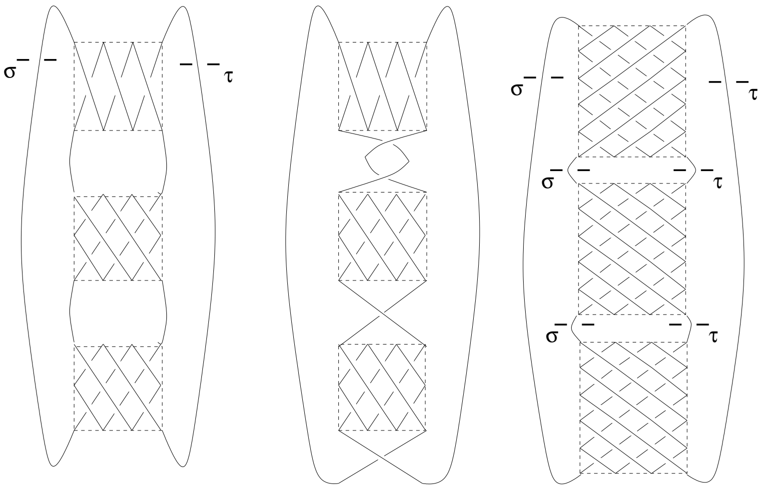
$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\psi} & B_2(k) \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ M & \xrightarrow[\varphi]{} & (S^3, k) \end{array}$$

with $\varphi : S^3 \to (S^3, k)$ a dihedral quotient, and $m(b_1/\alpha_1, \ldots, b_t/\alpha_t) \subset \varphi^{-1}(k)$ the pseudo-branch of φ. $\square$

3.1. **Remarks.** Let $k = m(\beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$, and let n be a divisor of $\Delta(k)$ such that $(n, \alpha_i) = 1$ for $i = 1, \ldots, t$. Then $k = m(nb_1/\alpha_1, \ldots, nb_t/\alpha_t)$ with $b_i = (\beta_i + r_i\alpha_i)/n$, and $\sum r_i = 0$.

3.1.1. The conclusion of Theorem 3.2 can be rephrased as: *If $n|\Delta(k)$, and $(n, \alpha_i) = 1$, then $m(b_1/\alpha_1, \ldots, b_t/\alpha_t)$ is the pseudo-branch of an n-fold dihedral quotient $S^3 \to (S^3, m(nb_1/\alpha_1, \ldots, nb_t/\alpha_t))$.*

3.1.2. Redrawing $k = m(nb_1/\alpha_1, \ldots, nb_t/\alpha_t) = m(\beta_1/\alpha_1 + r_1, \ldots, \beta_t/\alpha_t + r_t)$ one can easily visualize the representation ω in a projection of k: the meridians of each piece of one strand of the trivial 2-braid connecting the rational tangles of k (see Figure 2) go to the same permutation under ω.



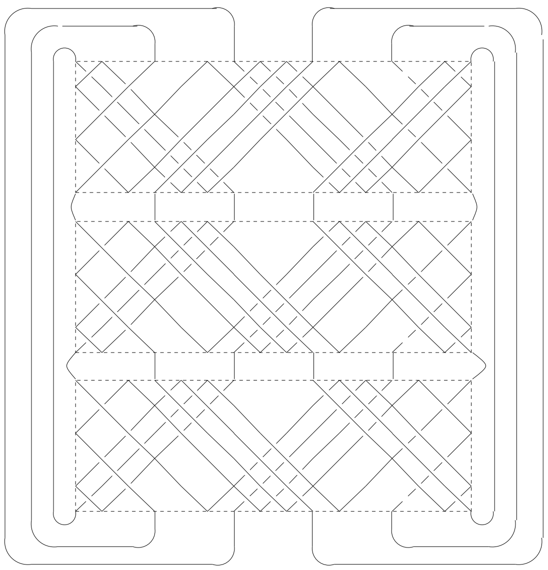


FIGURE 4

Then using similar techniques as in [6] one can draw at once the preimage of φ of Theorem 3.2 (See Figure 4 for the 5-fold dihedral covering of $m(1/3, 2/3, 2/3)$).

3.1.3. A possible partial converse of Theorem 3.2 is:

Let k be the Montesinos knot $m(\beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$. Assume n is odd and $t \geq 3$. If there exists a dihedral quotient $S^3 \to (S^3, k)$, then $(n, \alpha_i) = 1$ for $i = 1, \ldots, t$.

Notice that one cannot skip the hypothesis '$t \geq 3$ and n odd'.

For example, when $t \leq 2$, the 2-bridge knot

$$\ell = m(\beta_1/n\alpha_1, \beta_2/n\alpha_2) = m(-n(\alpha_1\beta_2 + \alpha_2\beta_1)/(n\alpha_2 r_1 + \beta_2 s_1))$$

where $n\alpha_1 r_1 - \beta_1 s_1 = 1$, has an n-fold dihedral quotient $S^3 \to (S^3, \ell)$ with pseudo-branch $m(\beta_1/\alpha_1, \beta_2/\alpha_2)$.

When n is even: if $\ell = m(\beta_1/2\alpha_1, \beta_2/2\alpha_2, \beta_3/\alpha_3, \ldots, \beta_t/\alpha_t)$, the representation $\omega : \pi_1(B_2(\ell)) \to S_2$ such that $\omega(q_1) = (1,2)$, $\omega(q_2) = (1,2)$, $\omega(q_i) = (1)$, for $i = 3, \ldots, t$, and $\omega(h) = (1)$, gives rise to a 2-fold dihedral quotient $S^3 \to (S^3, \ell)$ with pseudo-branch $m(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3, \ldots, \beta_t/\alpha_t)$ ([12], Lemma 2).

Corollary 3.3. *Let $k = m(\beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ be a universal link, and let n be a positive integer such that $(n, \alpha_i) = 1$ $(i = 1, \ldots, n)$. Then*

$$m(n\beta_1/\alpha_1, \ldots, n\beta_t/\alpha_t)$$

is a universal link.

4. Rational tangles.

A *rational tangle* is a triple $(B, a, m \cup \ell)$, where B is a 3-ball, a is a disjoint union of two properly embedded arcs in B, $a = a_0 \sqcup a_1$, and each of m and ℓ are a disjoint union of two arcs in ∂B, $m = m_0 \sqcup m_1$, $\ell = \ell_0 \sqcup \ell_1$, such that $m \cap \ell = \partial m = \partial \ell = \partial a$ and $m \cup \ell \cong S^1$, and such that the pair (B, a) is homeomorphic to $(D^2 \times I, \{x_1, x_2\} \times I)$ with $x_1, x_2 \in int(D^2)$. See Figure 5.

Two rational tangles are *equivalent* if they are homeomorphic as triples. A rational tangle $(B, a, m \cup \ell)$ contains a 'meridional disk' d_B properly embedded in B which is the image of $\delta \times I \subset D^2 \times I$ under the homeomorphism $(B, a) \cong (D^2 \times I, \{x_1, x_2\} \times I)$, where $\delta \subset D^2$ is a diameter separating x_1 and x_2. Notice that $\overline{B - \mathcal{N}(d_B)}$ is a regular neighbourhood of a, and, therefore, a_i is the arc, unique up to isotopy fixed in ∂B, which connects the ends of a_i in $B - d_B$ $(i = 0, 1)$; the isotopy class of ∂d_B in ∂B, with isotopies being the identity in ∂a, determines the isotopy class of d_B in $B - a$, and, therefore, *The curve ∂d_B determines the equivalence class of* $(B, a, m \cup \ell)$.

If $p : \tilde{B} \to B$ is the double branched cover of B branched along both a_0 and a_1, then $\tilde{m} = p^{-1}(m_0)$ and $\tilde{\ell} = p^{-1}(\ell_0)$ are a meridian-longitude pair in $\partial \tilde{B}$; if we fix an orientation on ∂B, an orientation of $\partial \tilde{B}$ is fixed; we orient $\tilde{m}$ and $\tilde{\ell}$ such that $\tilde{m} \cdot \tilde{\ell} = +1$. Now if $\tilde{d}$ is a lifting of ∂d_B in $\tilde{B}$, then $\tilde{d} \sim \tilde{\ell}^{\beta} \tilde{m}^{\alpha}$ in ∂B for some orientation of $\tilde{d}$; we associate the rational number β/α (or ∞, if $\alpha = 0$) to the tangle $(B, a, m \cup \ell)$. It is well known that the number β/α determines the equivalence class of $(B, a, m \cup \ell)$.

By pushing the arcs a into ∂B, it is possible to draw the rational tangle β/α on a square 'pillowcase' with lines of slope $\pm\beta/\alpha$ starting in the points $\{0, 1\} \times \{i/\alpha\}$ for $i = 0, 1, \ldots, \alpha$, and $\{j/\beta\} \times \{0, 1\}$ for $j = 0, 1, \ldots, \beta$ (Figure 6).

In this square pillowcase the boundary of the meridional disk $d_B = d_{\beta/\alpha}$ is drawn "in between" the arcs a_0, a_1, that is, with lines of slope $\pm\beta/\alpha$

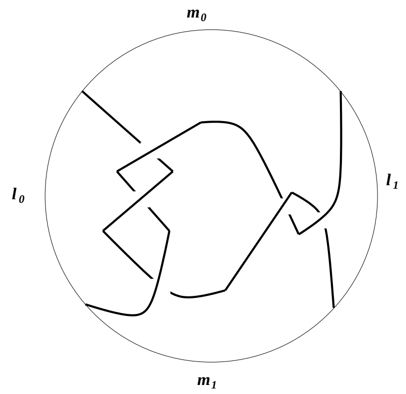


FIGURE 5

starting in the points $\{0,1\} \times \{(2i-1)/2\alpha\}$ for $i = 1, 2, \ldots, \alpha$, and $\{(2j-1)/2\beta\} \times \{0,1\}$ for $j = 1, 2, \ldots, \beta$ (Figure 7).

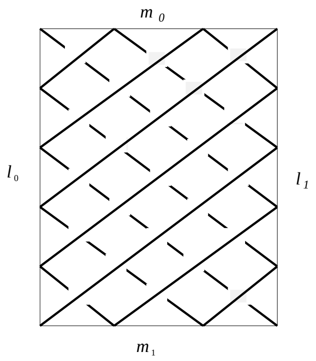


FIGURE 6. The rational tangle 3/5

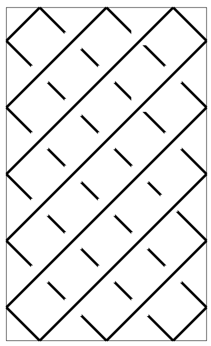


FIGURE 7. The disk 3/5

5. Branch sets in dihedral coverings.

Let k be the Montesinos knot $k = m(\beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$. Let n be a positive integer such that n divides the determinant of k, $n|\Delta(k)$, and $(n, \alpha_i) = 1$ for $i = 1, \dots, t$. Let $\varphi : S^3 \to (S^3, k)$ be the dihedral quotient guaranteed by Theorem 3.2. We will show:

The preimage $\varphi^{-1}(k)$ is a disjoint union of Montesinos knots.

We write $B_2(k) = (O, 0; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$, and $p : B_2(k) \to (S^3, k)$ the double covering branched along all components of k. Let $\psi : \tilde{M} \to B_2(k)$ be an n-fold cyclic covering space (unbranched), and let $\rho : \tilde{M} \to \tilde{M}$ be the homeomorphism of order n such that $\tilde{M}/\rho = B_2(k)$, and $\psi : \tilde{M} \to \tilde{M}/\rho = B_2(k)$ is the canonical projection.

Let $u : B_2(k) \to B_2(k)$ be the standard involution for $B_2(k)$ with respect to the symbol $(O, 0; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$, and let $\tilde{u} : \tilde{M} \to \tilde{M}$ be the standard involution for $\tilde{M}$ with respect to the symbol $(O, 0; b_1/\alpha_1, \dots, b_t/\alpha_t)$, where, if $\tilde{e}_i$ is the exceptional fiber of $\tilde{M}$ corresponding to the ratio b_i/α_i, and e_i is the exceptional fiber of $B_2(k)$ corresponding to the ratio β_i/α_i, then $\psi^{-1}(e_i) = \tilde{e}_i$; we see that $k^2 = p^{-1}(k)$ is the axis of u, that is, $k^2 = Fix(u)$; let $\tilde{k}_0$ be the axis of $\tilde{u}$, $\tilde{k}_0 = Fix(\tilde{u})$; the preimage of k^2 is $\psi^{-1}(k^2) = \tilde{k}_0 \sqcup \tilde{k}_1 \sqcup \cdots \sqcup \tilde{k}_{n-1} = \tilde{k}_0 \sqcup \rho(\tilde{k}_0) \sqcup \cdots \sqcup \rho^{n-1}(\tilde{k}_0)$. Replacing ρ by ρ^m for some m coprime with n, if necessary, we may assume that, starting with $\tilde{k}_0$, we find successively $\tilde{k}_1, \tilde{k}_2, \dots, \tilde{k}_{n-1}, \tilde{k}_0, \tilde{k}_1, \tilde{k}_2, \dots, \tilde{k}_{n-1}$ as we traverse along $\tilde{h} \subset \tilde{M}$; $\tilde{h}$ is a preimage of h, an ordinary fiber of $B_2(k)$ with $h \cap k^2 =$ two points. Notice that, then, $\tilde{k}_\ell = \tilde{u}(\tilde{k}_{n-\ell})$ for each ℓ.

We have $q : \tilde{M} \to \tilde{M}/\tilde{u} = (S^3, k_0)$ a double branched covering, and $\varphi : S^3 \to (S^3, k)$ a dihedral quotient. Then $k_0 = q(\tilde{k}_0)$ is the branching of q, and also is the pseudo-branch of φ, and the diagram of coverings

$$\begin{tikzcd} & \tilde{M} \arrow[dl, "q"'] \arrow[dr, "\psi"] & \\ M \arrow[dr, "\varphi"'] & & B_2(k) \arrow[dl, "p"] \\ & (S^3, k) & \end{tikzcd}$$

commutes.

Now we define

$$\begin{aligned} k_1 &= q(\tilde{k}_1) = q(\tilde{k}_{n-1}) \\ k_2 &= q(\tilde{k}_2) = q(\tilde{k}_{n-2}) \\ &\vdots \\ k_{(n-1)/2} &= q(\tilde{k}_{(n-1)/2}) = q(\tilde{k}_{(n+1)/2}) \end{aligned}$$

if n is odd, and

$$\begin{aligned} k_1 &= q(\tilde{k}_1) = q(\tilde{k}_{n-1}) \\ k_2 &= q(\tilde{k}_2) = q(\tilde{k}_{n-2}) \\ &\vdots \\ k_{(n-2)/2} &= q(\tilde{k}_{(n-2)/2}) = q(\tilde{k}_{(n+2)/2}) \\ k_{n/2} &= q(\tilde{k}_{n/2}) \end{aligned}$$

if n is even.

Then $\varphi^{-1}(k) = k_0 \sqcup k_1 \sqcup k_2 \sqcup \cdots$. Notice that, if n is even, $k_{n/2}$ is the trivial knot (always).

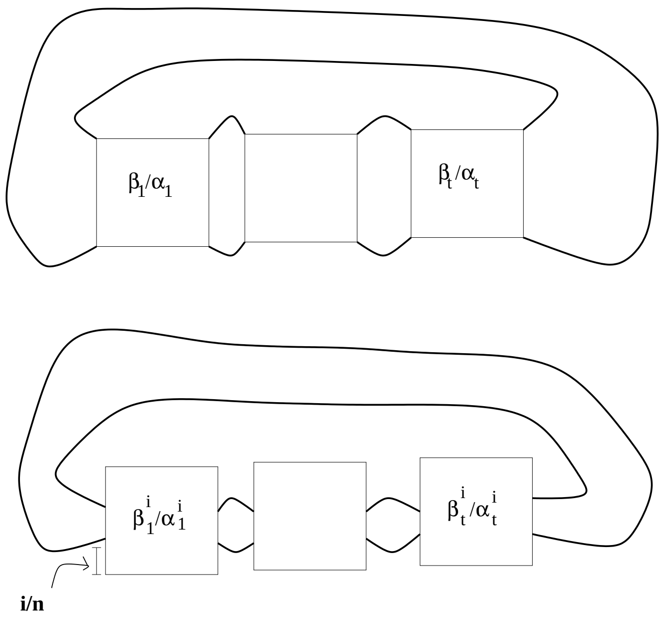


FIGURE 8

We show that each k_ℓ is a Montesinos knot (see Figure 8).

Let $i \in \{1, \ldots, t\}$; and let $j \in \{1, 2, \ldots, (n-1)/2\}$ if n is odd, or $j \in \{1, 2, \ldots, (n-2)/2\}$ if n is even.

In the solid torus $\tilde{V}_i \subset \tilde{M}$ there are two meridional disks $\tilde{D}_i, u(\tilde{D}_i)$ which separate the two arcs $\tilde{k}_j \cap \tilde{V}_i$; the image $D_i = q(\tilde{D}_i) = q(u(\tilde{D}_i))$ is a 2-disk properly embedded in the 3-ball $B_i = q(\tilde{V}_i)$ which separates the two arcs $q(\tilde{k}_j \cap \tilde{V}_i) = q(\tilde{k}_{n-j} \cap \tilde{V}_i)$ (see Figure 9). From these we get a rational tangle as follows:

Recall that, to construct $\tilde{M}$ (see § 2.), the torus $\partial\tilde{V}_i$ is glued to $\tilde{q}_i \times S^1$ in punctured $\tilde{M}$ ($\cong S^2_0 \times S^1$). Let $\tilde{h}^0 = \{x_0\} \times S^1$, and $\tilde{h}^1 = \{x_1\} \times S^1$ where x_0 and x_1 are the points in $\tilde{k}_0 \cap (\tilde{q}_i \times \{1\})$. Starting in x_a and following the

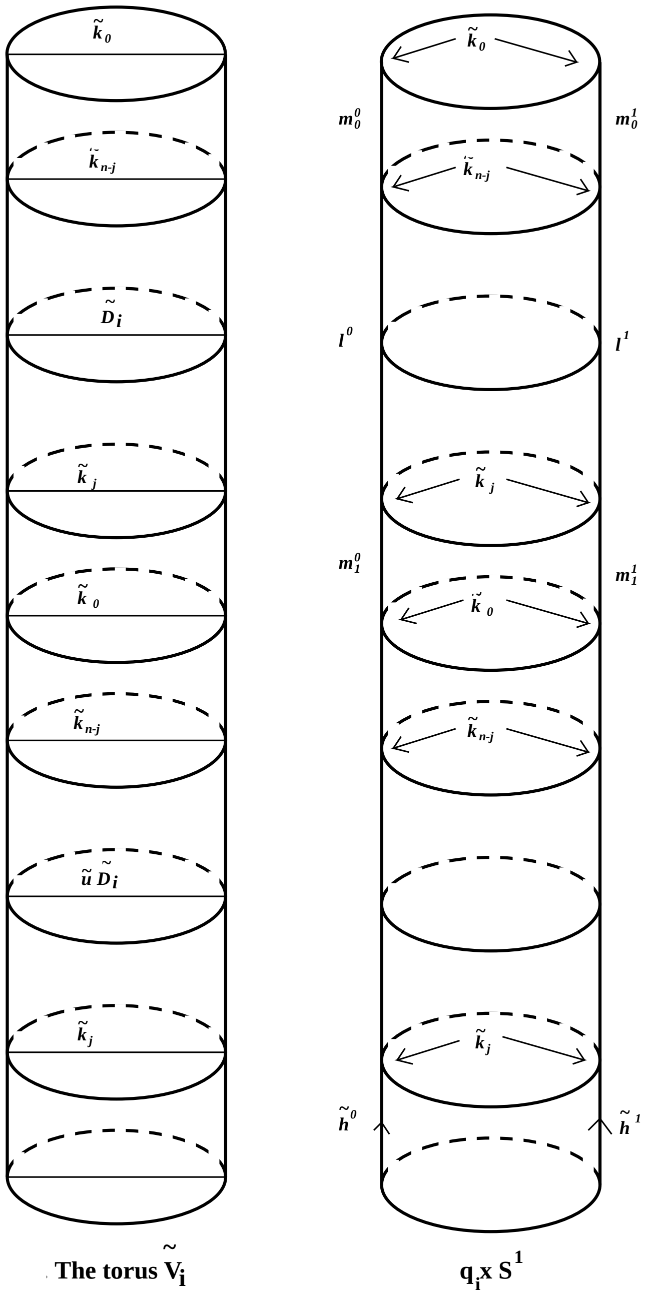


FIGURE 9. The solid torus $\tilde{V}_i$ and $q_i \times S^1$

orientation of $\tilde{h}^a$ we get three subarcs: one subarc $\tilde{m}_1^a \subset \tilde{h}^a$ connecting x_a with $\tilde{h}^a \cap \tilde{k}_j$; one subarc $\tilde{\ell}_a \subset \tilde{h}^a$ connecting $\tilde{h}^a \cap \tilde{k}_j$ with $\tilde{h}^a \cap \tilde{k}_{n-j}$; and one subarc $\tilde{m}_0^a \subset \tilde{h}^a$ connecting $\tilde{h}^a \cap \tilde{k}_j$ with $y_a \in \tilde{h}^a \cap \tilde{k}_0 - \{x_a\}$ $(a = 0, 1)$. Write $m_1 = q(\tilde{m}_1^0 \cup \tilde{q}_i \times \{1\} \cup \tilde{m}_1^1)$, $m_0 = q(\tilde{m}_0^0 \cup \tilde{q}_i \times \{y_0\} \cup \tilde{m}_0^1)$, $\ell_0 = q(\tilde{\ell}_0)$, and $\ell_1 = q(\tilde{\ell}_1)$; call $m^j = m_0 \sqcup m_1$, and $\ell^j = \ell_0 \sqcup \ell_1$; then

$$b_i^j/\alpha_i^j = (B_i, q(\tilde{k}_j \cap \tilde{V}_i), m^j \cup \ell^j)$$

is a rational tangle for some number $b_i^j/\alpha_i^j \in Q \cup \{\infty\}$, and it is defined by ∂D_i. Therefore

1. *For $j \in \{1, 2, \ldots, (n-1)/2\}$ if n is odd, or $j \in \{1, 2, \ldots, (n-2)/2\}$ if n is even, k_j is the Montesinos knot $m(b_1^j/\alpha_1^j, \ldots, b_t^j/\alpha_t^j)$.*

Now we will compute the ratios b_i^j/α_i^j.

Draw the ball B_i as a square pillowcase; then the knot k_j intersects ∂B_i in the points $\{0,1\} \times \{j/n, 1-j/n\}$.

We will compute the defining numbers $b_i^j/\alpha_i^j \in Q \cup \{\infty\}$ of a lifting d of ∂D_i in the double cover T of ∂B_i branched along $\partial B_i \cap k_j$. The contour of the square pillowcase, $m^j \cup \ell^j$, is composed by (see Figure 10)

$$m_0 = (\{0\} \times [1-j/n, 1]) \cup (I \times \{1\}) \cup (\{1\} \times [1-j/n, 1]) = v_0 \cup \hat{m}_0 \cup w_0$$

$$m_1 = (\{0\} \times [0, j/n]) \cup (I \times \{0\}) \cup (\{1\} \times [0, j/n]) = v_1 \cup \hat{m}_1 \cup w_1$$

$$\ell_0 = \{0\} \times [j/n, 1-j/n]$$

$$\ell_1 = \{1\} \times [j/n, 1-j/n].$$

Let V be the cylinder obtained by cutting ∂B_i along m^j; the boundary of V is the disjoint union of two circles $m_0^+ \cup m_0^-$ and $m_1^+ \cup m_1^-$ where $m_a^+$and m_a^- are copies of m_a in V $(a = 0, 1)$.

To construct T glue two copies, V_1, V_2, of V identifying $m_{0,1}^+ \sim m_{0,2}^-$, $m_{0,1}^- \sim m_{0,2}^+$, $m_{1,1}^+ \sim m_{1,2}^-$, and $m_{1,1}^- \sim m_{1,2}^+$, where $\partial V_a = (m_{0,a}^+ \cup m_{0,a}^-) \cup (m_{1,a}^+ \cup m_{1,a}^-)$ $(a = 1, 2)$.

To compute the defining numbers b_i^j/α_i^j of a lifting $d \subset T$ of ∂D_i with respect to m_0 and ℓ_0, we will construct a new torus $\bar{T}$ and a curve $\bar{d}$ in $\bar{T}$ which will have the defining numbers b_i^j/α_i^j with respect to some basis of $\bar{T}$.

Glue V_1 and V_2 but only along the arcs $\hat{m}_0$ and $\hat{m}_1$, that is, identify $\hat{m}_{0,1}^+ \sim \hat{m}_{0,2}^-$, etc., and obtain $\hat{T}$ which is a four-punctured torus; if in $\hat{T}$ we identify the arcs v_0, w_0, v_1, w_1 as if they were not originally cut, that is, if we identify $v_{0,1}^+ \sim v_{0,1}^-$, $w_{0,1}^+ \sim w_{0,1}^-$, $v_{0,2}^+ \sim v_{0,2}^-$, $w_{0,2}^+ \sim w_{0,2}^-$, etc., we obtain T^0 the double covering of ∂B_i branched along the corners of the pillowcase $\partial(\hat{m}_0 \cup \hat{m}_1)$. In T^0 a lifting of ∂D_i is the curve b_i/α_i, $\widetilde{\partial D_i} \sim \hat{m}_0^{\alpha_i} \hat{\ell}_0^{b_i}$, where $\hat{\ell}_0$ is the preimage in T^0 of $v_1 \cup \ell_0 \cup v_0$. We regard $\hat{T}$ as subset of T^0, and call $\hat{d} = \hat{T} \cap (\text{preimage of } \partial D_i \text{ in } T^0)$.

To obtain T from $\hat{T}$ we have still to identify, as before, $v_{0,1}^+ \sim v_{0,2}^-$, $w_{0,1}^+ \sim w_{0,2}^-$, etc. Therefore we can construct the preimage of ∂D_i in T from $\hat{d}$ as follows: each point $p^+ \in v_{0,1}^+ \cap \hat{d}$ is identified with a point $p^- \in v_{0,2}^- \cap \hat{d}$, where $p^+ \sim p^-$ in the identification $v_{0,1}^+ \sim v_{0,2}^-$; each point $p^+ \in w_{0,1}^+ \cap \hat{d}$ is identified with a point $p^- \in w_{0,2}^- \cap \hat{d}$, etc.

If we close $\hat{T}$ with four rhombi $R_1 = v_{0,1}^+ v_{0,2}^- v_{0,2}^+ v_{0,1}^-$, $R_2 = w_{0,1}^- w_{0,2}^+ w_{0,2}^- w_{0,1}^+$, $R_3 = v_{1,2}^+ v_{1,1}^- v_{1,1}^+ v_{1,2}^-$, $R_4 = w_{1,2}^+ w_{1,1}^- w_{1,1}^+ w_{1,2}^-$, in the resulting torus $\bar{T}$, we

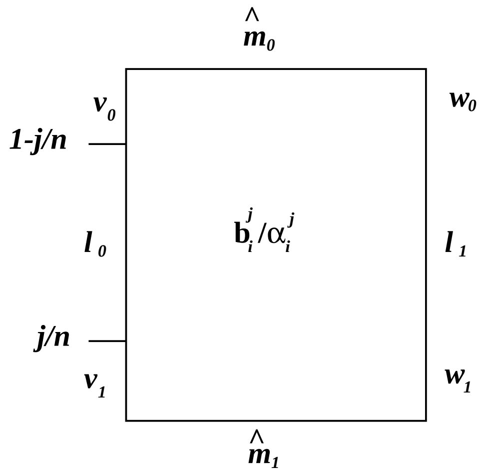


FIGURE 10. The ball B_i

close the arcs of $\hat{d}$ with vertical lines in the rhombi connecting equivalent points of the v's and w's; one of the two components, $\bar{d}_0$, of the resulting $\bar{d}$ has defining numbers b_i^j/α_i^j, $\bar{d}_0 \sim \bar{m}_0^{\pm\alpha_i^j}\bar{\ell}_0^{\pm b_i^j}$, where $\bar{m}_0$ is the union of the preimage of $\hat{m}_0$ in $\hat{T}$ with the horizontal diagonals of the rhombi R_1 and R_2 and $\bar{\ell}_0$ is the union of the preimage of ℓ_0 in $\hat{T}$ with the vertical diagonals of the rhombi R_1 and R_3. See Figure 11.

Notice that if either $b_i = 0$ or $\alpha_i = 0$, then $b_i^j/\alpha_i^j = 0$ or ∞, resp.

For practical computation of the number b_i^j/α_i^j, we may assume that $b_i \neq 0$ and $\alpha_i \neq 0$, and visualize the universal cover of $\bar{T}$ as follows:

A fundamental region for $\bar{T}$ is a square $[2u, 2u+2] \times [2v, 2v+2]$ in the plane, for u, v integers; at each vertex $(u, v) \in Z^2$, we cut the plane along the interval $J_{(u,v)} = \{u\} \times [v - j/n, v + j/n]$, and glue a thin rhombus along the resulting boundaries $(\{u\} \times [v - j/n, v])^+$, $(\{u\} \times [v, v + j/n])^-$, $(\{u\} \times [v, v + j/n])^+$, $(\{u\} \times [v - j/n, v])^-$.

A lifting of $\bar{d}_0$ is made of segments of straight lines of slope b_i/α_i and vertical segments in the glued rhombi:

Start in the point, say, $p_0 = (0, 1/2\alpha_i)$; draw the segment L_1 of the line of slope b_i/α_i starting in p_0 to the right of the interval $J_{(0,0)}$, until it hits for the first time an interval $J_{(u_1,v_1)}$ in a point $p_1 = (u_1, v_1 \pm (2j_1 - 1)/2\alpha_i)$; jump to $\bar{p}_1 = (u_1, v_1 \mp (2j_1 - 1)/2\alpha_i)$ with a vertical segment in the corresponding rhombus; now draw the segment L_2 of the line of slope b_i/α_i starting in $\bar{p}_1$ to the *left* of the interval $J_{(u_1,v_1)}$, until it hits for the first time an interval $J_{(u_2,v_2)}$ in a point $p_2 = (u_2, v_2 \pm (2j_2 - 1)/2\alpha_i)$; jump to $\bar{p}_2 = (u_2, v_2 \mp (2j_2 - 1)/2\alpha_i)$ with a vertical segment in the corresponding rhombus; now draw the segment L_3 of the line of slope b_i/α_i starting in $\bar{p}_2$ to the *right* of the interval $J_{(u_2,v_2)}$. Continuing in this fashion, eventually we draw a segment L_r which ends in an interval $J_{(u_r,v_r)}$ in a point $p_r = (u_r, v_r + 1/2\alpha_i)$ with u_r and v_r even

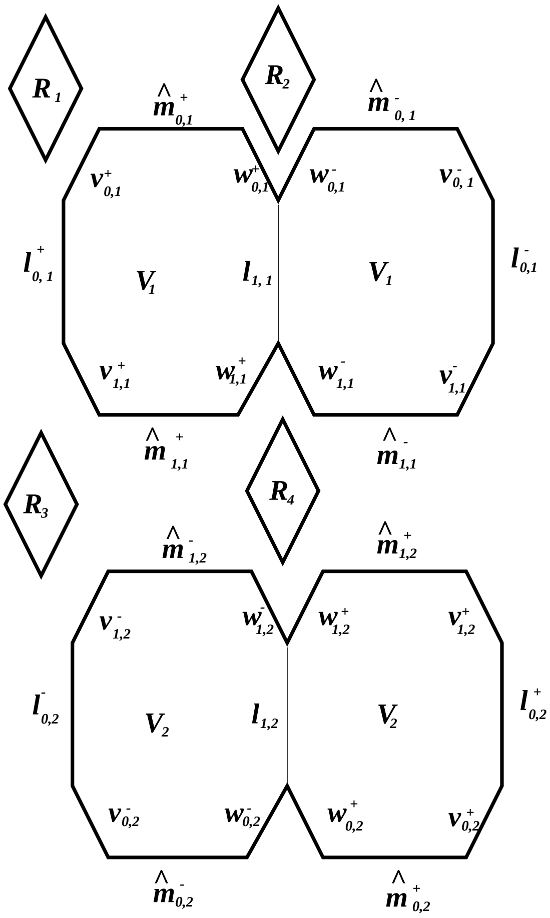


FIGURE 11

integers, and u_q and v_q are not both even for $q < r$ if $p_q = (u_q, v_q + 1/2\alpha_i)$. We see that $b_i^j/\alpha_i^j = v_r/u_r$.

Remark. With this representation in the plane it is easy to see: *(a) If $b_i\ell = -1 + \alpha_i k$, and $b_i r = 1 + \alpha_i s$ with ℓ and r minimal positive, and $j/n \in (1/2\alpha_i, 3/2\alpha_i)$, then $b_i^j/\alpha_i^j = (k-s)/(\ell - r)$.*

Similar expressions for b_i^j/α_i^j when $j/n > 3/2\alpha_i$ seem extremely complicated.

5.1. **An algorithm.** We obtain the following algorithm in the square pillowcase B_i to compute b_i^j/α_i^j.

Orient the curve $d = \partial D_i$, and let ε be the sign of b_i. Assume $j/n \in [0, 1/2)$, and write $m_0 = \{0,1\} \times [1 - j/n, 1] \cup [0,1] \times \{1\}$, $m_1 = \{0,1\} \times [0, j/n] \cup [0,1] \times \{0\}$, $\ell_0 = \{0\} \times [j/n, 1 - j/n]$

1. Mark the point $p_0 = (0, 1/2\alpha_i)$ with $+1$.

2. If the point $p_u \in d \cap \partial I^2$ is marked with $\varepsilon_u \in \{-1, +1\}$, then the straight line segment of d starting in p_u and following the given orientation of d, hits ∂I^2 in a point p_{u+1}.
If p_{u+1} has no mark, then
 If ($p_u \in m_0$ and $p_{u+1} \in m_0$) or ($p_u \in m_1$ and $p_{u+1} \in m_1$),
  then mark p_{u+1} with $\varepsilon_{u+1} = -\varepsilon_u$;
  else mark p_{u+1} with $\varepsilon_{u+1} = \varepsilon_u$.
 Go to **2.** with '$u := u + 1$'.
If p_{u+1} is already marked, then call $b =$ the sum of marks of points in m_0, and $\alpha =$ the sum of marks of points in ℓ_0. Then $b_i^j/\alpha_i^j = \varepsilon b/\alpha$.

5.2. **Rules.** Let $\beta/\alpha \in Q^* = Q \cup \{\infty\}$ with $\alpha > 0$. Let $J^0 = [0, 1/2\alpha)$, and $J^i = ((2i-1)/2\alpha, (2(i+1)-1)/2\alpha)$ for $i = 1, \ldots, [\alpha/2] - 1$, and $J^{[\alpha/2]} = ((2[\alpha/2]-1)/2\alpha, 1/2)$. Let $J = J_{\beta/\alpha} = \bigcup_{i=0}^{[\alpha/2]} J^i \subset [0, 1/2)$.

Let $\lambda \in J$. In the square pillowcase for β/α, $(B, a, m \cup \ell)$, with meridional disk $d_{\beta/\alpha}$, define

$$\begin{aligned} m_0^\lambda &= m_0 \cup \{0,1\} \times [1-\lambda, 1], \\ m_1^\lambda &= m_1 \cup \{0,1\} \times [0, \lambda], \\ \ell_0^\lambda &= \{0\} \times [\lambda, 1-\lambda], \\ \ell_1^\lambda &= \{1\} \times [\lambda, 1-\lambda], \end{aligned}$$

and $m^\lambda = m_0^\lambda \cup m_1^\lambda$, $\ell^\lambda = \ell_0^\lambda \cup \ell_1^\lambda$.

We have a function $J \to Q^*$ such that $\lambda \mapsto (\beta/\alpha)_\lambda =$ the defining rational number for $(B, a^\lambda, m^\lambda \sqcup \ell^\lambda)$ with meridional disk $d_{\beta/\alpha}$.

With this notation we get the rules

0) If $\beta < 0$, then $(\beta/\alpha)_\lambda = -(|\beta|/\alpha)_\lambda$.

a) If $\beta\ell = -1 + \alpha k$, and $\beta r = 1 + \alpha s$ with ℓ and r minimal positive, and $\lambda \in J^1$, then $(\beta/\alpha)_\lambda = (k-s)/(\ell - r)$.

b) If $\lambda, \mu \in J^i$, then $(\beta/\alpha)_\lambda = (\beta/\lambda)_\mu$.

c) Let λ be in the last interval $J^{[\alpha/2]}$. Then $(\beta/\alpha)_\lambda = b/1$ for some integer b if α is odd, and $(\beta/\alpha)_\lambda = \infty$ if α is even.

In view of rule (c), we get: *If $n > 2\alpha_i$, and $(n, \alpha_i) = 1$ ($i = 1, \ldots, t$), then $\varphi^{-1}(k)$ contains a rational link, where $k = m(n\beta_1/\alpha_1, \ldots, n\beta_t/\alpha_t)$, and $\varphi : S^3 \to (S^3, k)$ is an n-fold dihedral quotient,* for, by the assumption $n > 2\alpha_i$, there is an integer $j \in \{1, \ldots, (n-1)/2\}$ such that $j/n \in J^{[\alpha_i/2]}$, the last interval for the tangle β_i/α_i, ($i = 1, \ldots, t$).

d) If $\beta/\alpha = \pm 1/\alpha$ and $\lambda \in J^u$, then $(\beta/\alpha)_\lambda = \pm 1/(\alpha - 2u)$.

More generally, recalling the construction of the tangle with number β/α in [8], we get

e) Assume $\alpha > 2k|\beta|$ for some k positive. If $\lambda \in J^{k|\beta|}$, then $(\beta/\alpha)_\lambda = \beta/(\alpha - 2k|\beta|)$.

In the special case $\alpha \equiv \pm 1 \bmod \beta$, examining carefully the signs in the square pillowcase given by Algorithm 5.1, one can see:

f) Assume $\alpha = \beta\ell \pm 1$, and $\beta > 0$; let r be such that $0 \leq r \leq [\ell/2]$.

If $t \in \{0, 1, \ldots, [\beta/2]\}$ and $\lambda \in J^{r\beta+t}$, then

$$\left(\frac{\beta}{\alpha}\right)_\lambda = \frac{\beta - 2t}{(\beta - 2t)(\ell - 2r) \pm 1}.$$

If $t \in \{[\beta/2]+1, [\beta/2]+2, \ldots, \beta - 1\}$ and $\lambda \in J^{r\beta+t}$, then

$$\left(\frac{\beta}{\alpha}\right)_\lambda = \frac{2t - \beta}{(2t - \beta)(\ell - 2r - 2) \pm 1}.$$

In particular we compute that the first $\beta + 1$ terms (when $r = 0$) of the sequence $\left\{\left(\frac{\beta}{\alpha}\right)_\lambda\right\}_{t=1}^{\beta}$ are:

$$\frac{\beta}{\alpha} = \frac{\beta}{\beta\ell \pm 1}, \frac{\beta - 2}{(\beta - 2)\ell \pm 1}, \frac{\beta - 4}{(\beta - 4)\ell \pm 1}, \ldots$$

$$\ldots, \begin{cases} \frac{4}{4\ell \pm 1}, \frac{2}{2\ell \pm 1}, 0, \frac{2}{2(\ell - 2) \pm 1}, \frac{4}{4(\ell - 2) \pm 1}, \ldots, \frac{\beta}{\alpha - 2\beta} & \text{if } \beta \text{ even} \\ \frac{3}{3\ell \pm 1}, \frac{1}{\ell \pm 1}, \frac{1}{(\ell - 2) \pm 1}, \frac{3}{3(\ell - 2) \pm 1}, \ldots, \frac{\beta}{\alpha - 2\beta} & \text{if } \beta \text{ odd} \end{cases}$$

Proposition 5.1. *Let q be an odd integer, $q \notin \{-11, -7, -5, -3, -1, 1, 3, 5\}$, and let k be the pretzel knot $k = p(2, q, q) = m(1/2, \pm 1/|q|, \pm 1/|q|)$. Then there exists a $|q + 4|$-fold dihedral covering $\varphi : S^3 \to (S^3, k)$ such that*

1) If $|q| \equiv 1 \bmod 4$, then either Montesinos knot $m(1/2, -1/5, -1/5) \subset \varphi^{-1}(k)$ or $m(1/2, -2/9, -2/9) \subset \varphi^{-1}(k)$.

2) If $|q| \equiv -1 \bmod 4$, then either Montesinos knot $m(-1/2, 3/5, 3/5) \subset \varphi^{-1}(k)$ or $m(-1/2, 2/3, 2/3) \subset \varphi^{-1}(k)$.

Proof. Since q is odd, $(q + 4, q) = (q, 2) = 1$; since $\Delta(k) = q^2 + 2q + 2q = q(q + 4)$, by Theorem 3.2, there exists $\varphi : S^3 \to (S^3, k)$ a $|q + 4|$-dihedral covering.

"(1)" Assume $|q| = 4\beta + 1$; then $k = m(\frac{|q+4|}{2}, \frac{-|q+4|\beta}{|q|}, \frac{-|q+4|\beta}{|q|})$, and, therefore, the pseudo-branch of φ is $k_0 = m(\frac{1}{2}, \frac{-\beta}{|q|}, \frac{-\beta}{|q|})$.

If β is odd, write $i = (\beta - 1)/2$; then $i/|q + 4| \in J^i$ for the tangle $-\beta/|q|$, that is, $(2i - 1)/2|q| < i/|q + 4| < (2i + 1)/2|q|$; and $i/|q + 4| \in J^0$ for the tangle $1/2$, that is, $i/|q + 4| < 1/4$.

By Rule (f), $\left(\frac{-\beta}{|q|}\right)_{i/|q+4|} = -1/5$, and $\left(\frac{1}{2}\right)_{i/|q+4|} = 1/2$. We conclude $k_i = m(1/2, -1/5, -1/5) \subset \varphi^{-1}(k)$.

If β is even, write $i = \beta/2 - 1$; then $i/|q + 4| \in J^i$ for the tangle $-\beta/|q|$; and $i/|q + 4| \in J^0$ for the tangle $1/2$. By Rule (f), $\left(\frac{-\beta}{|q|}\right)_{i/|q+4|} = -2/9$, and $\left(\frac{1}{2}\right)_{i/|q+4|} = 1/2$. We conclude $k_i = m(1/2, -2/9, -2/9) \subset \varphi^{-1}(k)$.

"(2)" Assume $|q| = 4\beta - 1$; then $k = m(\frac{-|q+4|}{2}, \frac{|q+4|\beta}{|q|}, \frac{|q+4|\beta}{|q|})$, and, therefore, the pseudo-branch of φ is $k_0 = m(-\frac{1}{2}, \frac{\beta}{|q|}, \frac{\beta}{|q|})$.

If β is odd and $q \geq 7$, write $i = (\beta+5)/2$. Then $i/(q+4) \in J^{(\beta+3)/2}$ for the tangle $\beta/|q|$; and $i/(q+4) \in J^0$ for the tangle $-1/2$. By Rule (f), $\left(\frac{\beta}{|q|}\right)_{i/(q+4)} = 3/5$, and $\left(\frac{-1}{2}\right)_{i/(q+4)} = -1/2$. We conclude $k_i = m(-1/2, 3/5, 3/5) \subset \varphi^{-1}(k)$.

If β is odd and $q < -11$, then $|q+4| = -(q+4) > 7$; write $i = (\beta+1)/2$. Then $-i/(q+4) \in J^{(\beta+3)/2}$ for the tangle $\beta/|q|$; and $-i/(q+4) \in J^0$ for the tangle $-1/2$. Again $k_i = m(-1/2, 3/5, 3/5) \subset \varphi^{-1}(k)$.

If β is even and $q \geq 7$, write $i = \beta/2 + 2$. Then $i/(q+4) \in J^{\beta/2+1}$ for the tangle $\beta/|q|$; and $i/(q+4) \in J^0$ for the tangle $-1/2$. By Rule (f), $\left(\frac{\beta}{|q|}\right)_{i/(q+4)} = 2/3$, and $\left(\frac{-1}{2}\right)_{i/(q+4)} = -1/2$. We conclude $k_i = m(-1/2, 2/3, 2/3) \subset \varphi^{-1}(k)$.

If β is even and $q < -11$, then $|q+4| = -(q+4) > 7$; write $i = \beta/2$. Then $\left(\frac{\beta}{|q|}\right)_{-i/(q+4)} = 2/3$; and $\left(\frac{-1}{2}\right)_{-i/(q+4)} = -1/2$. Again $k_i = m(-1/2, 2/3, 2/3) \subset \varphi^{-1}(k)$. $\square$

We finish this section rewriting a version of Theorem 3.2.

Corollary 5.2. *Let $(n, \alpha_i) = 1$, and $k = m(n\beta_1/\alpha_1, \ldots, n\beta_t/\alpha_t)$. If $\varphi : S^3 \to (S^3, k)$ is an n-fold dihedral quotient, and $k_j \subset \varphi^{-1}(k)$ is as before with $j < n/2$, then*

$$k_j = m((\beta_1/\alpha_1)_{j/n}, \ldots, (\beta_t/\alpha_t)_{j/n}).$$

6. Dihedral-like coverings of Montesinos knots.

Let $k \subset S^3$ be a link. Let $\varphi : M \to (S^3, k)$ be an m-fold branched covering, and let $\omega = \omega_\varphi : \pi_1(S^3 - k) \to S_m$ be the representation determined by φ.

Assume that $order(\omega(\mu)) = p$ for each meridian μ of k. Write $\langle \mu^p \rangle_\pi$ for the normal closure of $\langle \mu^p ; \mu \text{ a meridian of } k \rangle$ in $\pi_1(S^3 - k)$; then ω factors through the quotient $\pi_1(S^3 - k)/\langle \mu^p \rangle_\pi$

$$\begin{tikzcd} \pi_1(S^3 - k) \arrow[rr, "\omega"] \arrow[dr] & & S_m \\ & \pi_1(S^3 - k)/\langle \mu^n \rangle_\pi \arrow[ur, "\bar{\omega}"'] & \end{tikzcd}$$

It is known (see [4], Ch. 2, § 4, with the slight necessary modifications to the argument) that this quotient is a semi-direct product

$$\pi_1(S^3 - k)/\langle \mu^p \rangle_\pi \cong Z_p \ltimes \pi_1(B_p(k)),$$

where $B_p(k)$ denotes the p-fold cyclic covering of S^3 branched along all components of k. The generator of Z_p is the class of a meridian, $\bar{\mu}$, and acts on $\pi_1(B_p(k))$ as the isomorphism induced by the homeomorphism of order p, $\rho : B_p(k) \to B_p(k)$, such that $B_p(k)/\rho = (S^3, k)$.

Let $\psi : \tilde{M} \to B_p(k)$ be the m-fold covering space determined by $\bar{\omega}| : \pi_1(B_p(k)) \to S_m$, which might be not transitive, and therefore, $\tilde{M}$ might be not connected.

Now assume $p = 2$. Then $\bar{\mu}$ (which is the isomorphism induced by the involution ρ) leaves invariant $\psi_{\#}\pi_1(\tilde{M}) \leq \pi_1(B_2(k))$, and, therefore, ρ lifts to an involution $\tilde{\rho} : \tilde{M} \to \tilde{M}$, with quotient a 2-fold cyclic branched covering $q : \tilde{M} \to M = \tilde{M}/\tilde{\rho}$. The quotient-induced m-fold branched covering is the original $\varphi : M \to (S^3, k)$, and we have a commutative diagram

$$\begin{tikzcd} & \tilde{M} \arrow[dl, "q"'] \arrow[dr, "\psi"] & \\ M \arrow[dr, "\varphi"'] & & B_n(k) \arrow[dl, "p"] \\ & (S^3, k) & \end{tikzcd}$$

The branching of q is the pseudo-branch of φ (if any).

Conversely, if we start with a representation

$$\omega : \pi_1(S^3 - k)/\langle \mu^2 \rangle_\pi = Z_2 \ltimes \pi_1(B_2(k)) \to S_m,$$

we get a commutative diagram of coverings

$$\begin{tikzcd} & \tilde{M} \arrow[dl, "q"'] \arrow[dr, "\psi"] & \\ M \arrow[dr, "\varphi"'] & & B_2(k) \arrow[dl, "p"] \\ & (S^3, k) & \end{tikzcd}$$

with q a 2-fold covering branched along the pseudo-branch of φ.

1. Let $k = m(\beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$. Then $\pi_1(S^3 - k)/\langle \mu^2 \rangle_\pi$ equals the semidirect product $Z_2 \ltimes \pi_1(O, 0; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$; the quotient, obtained by 'killing' an ordinary fiber of $B_2(k)$, gives the epimorphism

$$\pi_1(S^3 - k)/\langle \mu^2 \rangle_\pi \to Z_2 \ltimes \Delta(\alpha_1, \ldots, \alpha_t).$$

In terms of generators and relations, $Z_2 \ltimes \Delta(\alpha_1, \ldots, \alpha_t) = \langle \mu, q_1, \ldots, q_t : \mu^2 = 1, q_1^{\alpha_1} = 1, \ldots, q_t^{\alpha_t} = 1, q_1 \cdots q_t = 1, q_i^{\mu} = q_i^{-1} \rangle = \langle \mu, q_1, \ldots, q_{t-1} : \mu^2 = 1, q_1^{\alpha_1} = 1, \ldots, q_{t-1}^{\alpha_{t-1}} = 1, (q_1 \cdots q_{t-1})^{\alpha_t} = 1, q_i^{\mu} = q_i^{-1} \rangle$. To find a representation of $Z_2 \ltimes \Delta(\alpha_1, \ldots, \alpha_t)$ into S_n, is equivalent to find permutations $\tau, \sigma_1, \ldots, \sigma_{t-1} \in S_n$ such that $\tau^2 = (1)$, $\sigma_i^{\alpha_i} = (1)$, $\sigma_i^{\tau} = \sigma_i^{-1}$ ($i = 1, \ldots, t-1$), and $(\sigma_1 \cdots \sigma_{t-1})^{\alpha_t} = (1)$. This task simplifies when $t = 3$, for it is possible to easily draw the product of two permutations ([7]), which amounts to drawing the induced branched covering on the orbit surfaces of $\psi : \tilde{M} \to B_2(k)$.

In the computations of the following representations, the GAP programming language ([3]) was very useful. We will use, for computations of Seifert symbols, Lemma 2 of [12].

2. Consider $k = m(\beta_1/2, \beta_2/3, \beta_3/3)$. We find $\sigma_1 = (1)$, $\sigma_2 = (1,2,3)$, $\tau = (2,3)$ (the other two meridians of k go to $(2,3)$ and $(1,2)$); we compute $\tilde{M} = (O, 0; \beta_1/2, \beta_1/2, \beta_1/2, \beta_2/1, \beta_3/1)$ ([12], Lemma 2), and the involution of $B_2(k)$ lifts to the involution $\tilde{u}$ in $\tilde{M}$ with axis crossing the fibers corresponding to one of the $\beta_1/2$ ratios, and $\beta_2/1$, $\beta_3/1$, and interchanging the other two fibers corresponding to $\beta_1/2$, as seen in Figure 12.

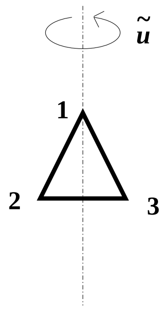


FIGURE 12. $\sigma_1 = (1)$, and $\sigma_2 = (1,2,3)$

We have $\varphi_1 : M = \tilde{M}/\tilde{u} \to (S^3, k)$ a 3-fold (simple) branched covering with pseudo-branch the branching of the quotient $\tilde{M} \to M$.

Let $\tilde{M}_0$ be the result of drilling out the fibers $\beta_1/2$ in $\tilde{M}$ interchanged by $\tilde{u}$; then $\tilde{M}_0/\tilde{u}$ is the complement of a trivial knot T in S^3 and the branching looks as in Figure 13: the link $m(\beta_1/2, \beta_2/1, \beta_3/1) \subset \tilde{M}_0/\tilde{u}$. To construct $\tilde{M}/\tilde{u}$, we have to fill $\tilde{M}_0/\tilde{u}$ with a solid torus whose meridian goes around $m^2\ell^{\beta_1}$ for m, ℓ a meridian-longitude pair of T.

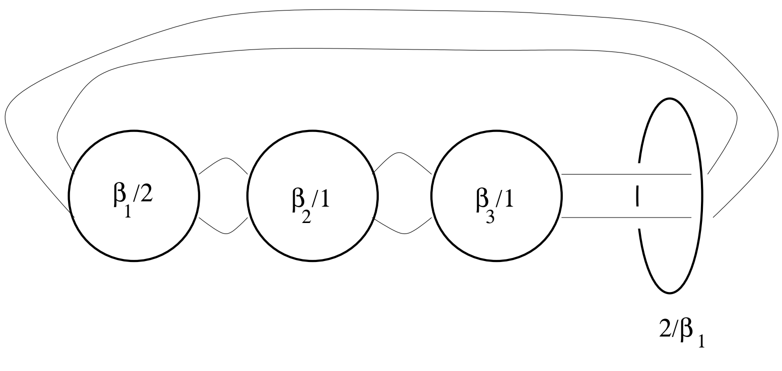


FIGURE 13

Therefore $M = \tilde{M}/\tilde{u}$ is the lens space $L(2, \beta_1)$. If $\varphi_2 : S^3 \to M$ is the universal covering space, then $\varphi_2^{-1}(\tilde{M}_0/\tilde{u})$ is the complement of a trivial knot where we still have to perform surgery $1/\beta_1$; then the preimage in S^3 of the pseudo-branch of φ_1 is the Montesinos knot

$$\tilde{k} = m\left(\frac{\beta_1}{2}, \frac{\beta_2}{1}, \frac{\beta_3}{1}, \frac{\beta_1}{2}, \frac{\beta_2}{1}, \frac{\beta_3}{1}, \frac{2\beta_1}{1}\right) = m\left(-\frac{12\beta_1 + 8\beta_2 + 8\beta_3}{6\beta_1 + 4\beta_2 + 4\beta_3 - 1}\right).$$

This link is a torus link if and only if $6\beta_1 + 4\beta_2 + 4\beta_3 - 1 \equiv \pm 1 \mod 12\beta_1 + 8\beta_2 + 8\beta_3$, that is, if and only if $\Delta(k)/3 = 3\beta_1 + 2\beta_2 + 2\beta_3 = \pm 1$. Then $\tilde{k}$ is a universal link if and only if $\Delta(k) \neq \pm 3$ (Theorem 1.1). Notice that the equality $\Delta(k) = \pm 3$ implies that k is the pretzel knot $p(\pm 2, \mp 3, \mp 3)$, which is a torus knot. We have $\varphi = \varphi_1 \circ \varphi_2 : S^3 \to (S^3, k)$ a branched covering with $\tilde{k} \subset \varphi(k)$; we conclude that:

The Montesinos knot $k = m(\beta_1/2, \beta_2/3, \beta_3/3)$ is a universal link if and only if $\Delta(k) \neq \pm 3$.

3. Consider $k = m(\beta_1/2, \beta_2/3, \beta_3/5)$. We find $\sigma_1 = (2,3)(4,5)$, $\sigma_2 = (1,2,4)$, $\tau = (1,2)(3,5)$ (the other two meridians of k go to $(2,5)(3,4)$ and $(2,4)(3,5)$); we compute $\tilde{M} = (O, 0; \beta_1/2, \beta_1/1, \beta_1/1, \beta_2/3, \beta_2/3, \beta_2/1, \beta_3/1)$ ([12], Lemma 2), and the involution of $B_2(k)$ lifts to the involution $\tilde{u}$ in $\tilde{M}$ with axis crossing the fibers corresponding to $\beta_1/2$, $\beta_2/1$, $\beta_3/1$, and interchanging by pairs the fibers corresponding to $\beta_1/1$ and $\beta_2/3$, as seen in Figure 14.

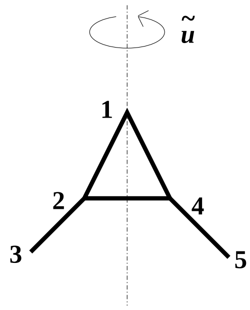


FIGURE 14. $\sigma_1 = (2,3)(4,5)$, and $\sigma_2 = (1,2,4)$

We have $\varphi : M = \tilde{M}/\tilde{u} \to (S^3, k)$ a 5-fold branched covering with pseudo-branch the branching of the quotient $\tilde{M} \to M$. To draw this pseudo branch we still have to perform surgeries $1/\beta_1$, and $3/\beta_2$ as before (Figure 15); passing to the universal cover of $M = L(3, 3\beta_1 + \beta_2)$ we obtain

$$\begin{aligned}\tilde{k} &= m\left(\frac{\beta_1}{2}, \frac{\beta_2}{1}, \frac{\beta_3}{1}, \frac{\beta_1}{2}, \frac{\beta_2}{1}, \frac{\beta_3}{1}, \frac{\beta_1}{2}, \frac{\beta_2}{1}, \frac{\beta_3}{1}, \frac{2(3\beta_1 + \beta_2)}{1}\right) \\ &= m\left(\frac{\beta_1}{2}, \frac{\beta_1}{2}, \frac{\beta_1}{2}, \frac{6\beta_1 + 5\beta_2 + 3\beta_3}{1}\right).\end{aligned}$$

If $\beta_1 = \pm 1$, then $\tilde{k}$ is an Uchida pretzel chain,

$$\tilde{k} = p(\pm 6 + 5\beta_2 + 3\beta_3; \pm 2, \pm 2, \pm 2) = \begin{cases} (3, 6 + 5\beta_2 + 3\beta_3) & \text{if } \beta_1 = 1 \\ (3, -9 + 5\beta_2 + 3\beta_3) & \text{if } \beta_1 = -1 \end{cases}$$

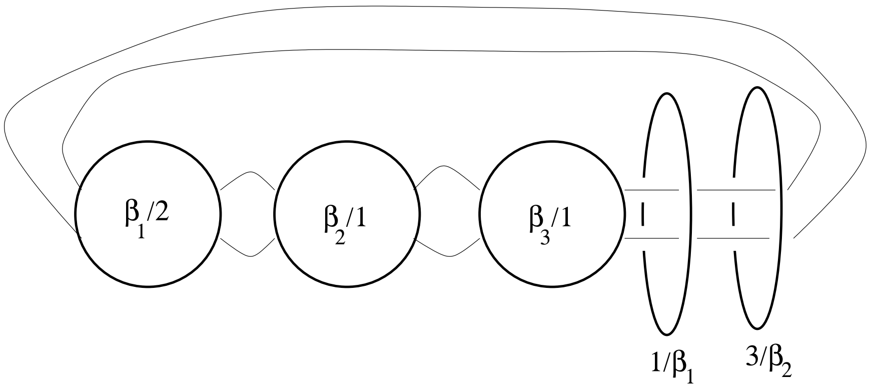


FIGURE 15

which is universal if $6 + 5\beta_2 + 3\beta_3 \neq -2, -1$ for $\beta_1 = 1$, and is universal if $-9 + 5\beta_2 + 3\beta_3 \neq -2, -1$ for $\beta_1 = -1$ ([15]). If $\beta_1 = 1$, then the equality $6 + 5\beta_2 + 3\beta_3 = -2, -1$ is equivalent to $\Delta(k) = \pm 1$; also if $\beta_1 = -1$, the equality $-9 + 5\beta_2 + 3\beta_3 = -2, -1$ is equivalent to $\Delta(k) = \pm 1$. Note that $\Delta(k) = \pm 1$ implies that k is the pretzel knot $= p(\pm 2, \mp 3, \mp 5)$, which is a torus knot. We conclude that:

The Montesinos knot $k = m(\pm 1/2, \beta_2/3, \beta_3/5)$ *is a universal link if and only if* $\Delta(k) \neq \pm 1$.

Note that any $m(\gamma_1/2, \gamma_2/3, \gamma_3/5)$ is of the form $m(\pm 1/2, \beta_2/3, \beta_3/5)$.

4. Consider $k = m(\beta_1/2, \beta_2/3, \beta_3/7)$. We find $\sigma_1 = (2,3)(4,6)(5,7)(8,9)$, $\sigma_2 = (1,2,4)(3,5,7)(6,9,8)$, $\tau = (2,3)(4,6)(5,7)(8,9)$ (the other two meridians of k go to $(2,6)(3,4)(5,9)(7,8)$ and $(2,4)(3,6)(5,8)(7,9)$); we compute

$$\tilde{M} = (O, 0; \beta_1/2, \beta_1/1, \beta_1/1, \beta_1/1, \beta_1/1, \beta_2/1, \beta_2/1, \beta_2/1, \beta_3/1, \beta_3/7, \beta_3/7)$$

([12], Lemma 2), and the involution of $B_2(k)$ lifts to the involution $\tilde{u}$ in $\tilde{M}$ with axis crossing the fibers corresponding to $\beta_1/2$, $\beta_2/1$, $\beta_3/1$, and interchanging by pairs the fibers corresponding to $\beta_1/1$, $\beta_2/1$, and $\beta_3/7$ as seen in Figure 16.

We have $\varphi : M = \tilde{M}/\tilde{u} \to (S^3, k)$ a 9-fold branched covering with pseudo-branch the branching of the quotient $\tilde{M} \to M$. To draw this pseudo-branch we still have to perform surgeries $1/\beta_1$, $1/\beta_1$, $1/\beta_2$, and $7/\beta_3$ as before (Figure 17); passing to the universal cover of $M = L(7, 14\beta_1 + 7\beta_2 + \beta_3)$ we obtain

$$\tilde{k} = m\Bigg(\overbrace{\frac{\beta_1}{2}, \ldots, \frac{\beta_1}{2}}^{7 \text{ times}}, \overbrace{\frac{\beta_2}{1}, \ldots, \frac{\beta_2}{1}}^{7 \text{ times}}, \overbrace{\frac{\beta_3}{1}, \ldots, \frac{\beta_3}{1}}^{7 \text{ times}}, \frac{2(14\beta_1 + 7\beta_2 + \beta_3)}{1}\Bigg)$$

$$= m\Bigg(\overbrace{\frac{\beta_1}{2}, \ldots, \frac{\beta_1}{2}}^{7 \text{ times}}, \frac{28\beta_1 + 21\beta_2 + 9\beta_3}{1}\Bigg).$$

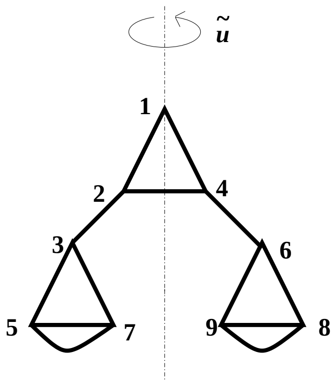


FIGURE 16. $\sigma_1 = (2,3)(4,6)(5,7)(8,9)$, and $\sigma_2 = (1,2,4)(3,5,7)(6,9,8)$

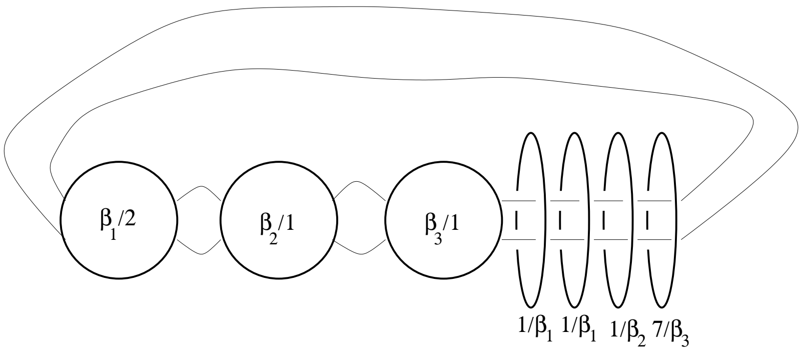


FIGURE 17

If $\beta_1 = \pm 1$, then $\tilde{k}$ is a universal Uchida pretzel link,

$$\tilde{k} = p(\pm 28 + 21\beta_2 + 9\beta_3; \overbrace{\pm 2, \ldots, \pm 2}^{7 \text{ times}}).$$

We conclude that:

The Montesinos knot $k = m(\pm 1/2, \beta_2/3, \beta_3/7)$ *is a universal link.*

Note that any $m(\gamma_1/2, \gamma_2/3, \gamma_3/7)$ is of the form $m(\pm 1/2, \beta_2/3, \beta_3/7)$, and that, in particular, the Fintushel-Stern knot, $k = p(-2, 3, 7)$, is universal; this famous knot has $\Delta(k) = 1$ and, therefore, has no dihedral quotients.

5. Using the same representations, and following the ideas below in paragraphs **2.**, **3.**, and **4.**, we obtain also:

a) If $|x| > 1$, *then the pretzel link* $k = p(e; 2x, 3y, 3z)$ *is universal,* for, in this case, following the constructions as in **2.**, $\tilde{k}$ is the universal Uchida pretzel link

$$\tilde{k} = p(\pm(2 + (4x+1)e); \overbrace{2x, \ldots, 2x}^{2|x| \text{ times}}, \overbrace{y, \ldots, y}^{2|x| \text{ times}}, \overbrace{z, \ldots, z}^{2|x| \text{ times}}).$$

a.1) Similarly, if $|y|>1$ or $|z|>1$, then the pretzel link $k=p(2,3y,3z)$ is universal.

a.2) If $|y|>1$ or $|z|>1$, and $\beta_2\equiv\pm1$ mod y and $\beta_3\equiv\pm1$ mod z, then the Montesinos knot $k=m(1/2,\beta_2/3y,\beta_3/3z)$ is universal, for, as before, it holds that $p(-2(-1+k_2+k_3);2,2,\pm y,\pm y,\pm z,\pm z)$, which is a universal Uchida link, is in the preimage of k in a branched covering, where $\beta_2=k_2y\pm1$ and $\beta_3=k_3z\pm1$.

b) If $|y|>1$, or $|z|>1$, then the pretzel link $k=p(\pm2,\pm3y,\pm5z)$ is universal, for, in this case, following the constructions as in **3.**, $\tilde{k}$ is the universal Uchida pretzel link

$$\tilde{k}=p(2(\pm3y\pm1);\ \overbrace{\pm2,\dots,\pm2}^{3|y|\text{ times}},\overbrace{\pm y,\dots,\pm y}^{3|y|\text{ times}},\overbrace{\pm z,\dots,\pm z}^{3|y|\text{ times}}).$$

c) If $z>0$, then the pretzel link $k=p(\pm2,\pm3,\pm7z)$ is universal, for, in this case, following the constructions as in **4.**, $\tilde{k}$ is the universal Uchida pretzel link

$$\tilde{k}=p(2(\pm14z\pm7z\pm1);\ \overbrace{\pm2,\dots,\pm2}^{7z\text{ times}},\overbrace{\pm3,\dots,\pm3}^{7z\text{ times}},\overbrace{\pm z,\dots,\pm z}^{7z\text{ times}}).$$

6. Consider $k=p(\pm2,5y,5z)$ $(y,z\neq0)$. We find permutations $\sigma_1=(2,3)(5,7)(6,9)(10,11)$, $\sigma_2=(1,2,4,8,5)(3,6,7,10,11)$, and $\tau=(2,5)(3,7)(4,8)(10,11)$ (the other two meridians of k go to $(1,2)(3,10)(4,5)(6,7)$ and $(2,7)(3,5)(4,8)(6,9)$); we compute

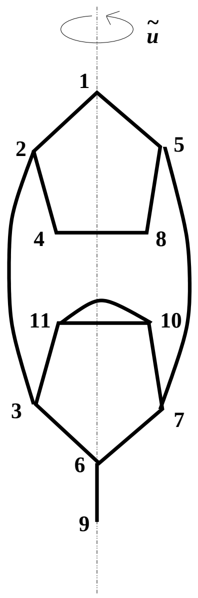


FIGURE 18. $\sigma_1=(2,3)(5,7)(6,9)(10,11)$, and $\sigma_2=(1,2,4,8,5)(3,6,7,10,11)$

$\tilde{M}=(O,0;\pm1/2,\pm1,\pm1,\pm1,\pm1,\pm1/2,\pm1/2,1/5y,1/y,1/y,1/5z,1/z,1/z),$

([12], Lemma 2), and the involution $\tilde{u}$ in $\tilde{M}$ (see Figure 18) gives as quotient $M = L(2,1)$. In the universal cover of M we find, as preimage of k,

$$\tilde{k} = p(\pm 10; \pm 2, \pm 2, 5y, 5y, y, y, y, y, 5z, 5z, z, z, z, z)$$

which is a universal Uchida link. We conclude

The pretzel knot $p(\pm 2, 5y, 5z)$ is universal ($y, z \neq 0$).

7. We think that it is not possible to obtain the 3-sphere as a branched covering over $(S^3, m(\beta_1/2, \beta_2/p, \beta_3/q))$ with this kind of dihedral-like coverings, except for the pairs (p,q) as in paragraphs **2.**–**6.**.

7. Universal Montesinos knots.

We collect the following results.

Theorem 7.1. *Let q be an odd integer, $q \neq -1, -3, -7, -11$. Then the pretzel knot $k = p(2, q, q)$ is universal.*

Proof. If $q = 1$, then $k = m(5/2)$ is universal (Theorem 1.1). If $q = 3$, k is universal by § 6.6. If $q = \pm 5$, k is universal by § 6.6. Assume $|q| > 5$, $q \neq -7, -11$. By Proposition 1 in § 5.1, there is a dihedral quotient $\varphi : S^3 \to (S^3, k)$ such that $\varphi^{-1}(k)$ contains one of the knots $p_1 = m(1/2, -1/5, -1/5)$, $p_2 = m(1/2, -2/9, -2/9)$, $p_3 = m(-1/2, 3/5, 3/5)$, or $p_4 = m(-1/2, 2/3, 2/3)$.

Now p_1 is universal by § 6.6; p_2 is universal by § 6.5 (a.2); p_4 is universal by § 6.2; $p_3 = m(7/2, -7/5, -7/5)$ has a 7-fold dihedral quotient with pseudo-branch p_1 which is universal, so, p_3 is universal. We conclude that k is universal. □

We remark that $p(2, -1, -1) =$ trefoil knot, and $p(2, -3, -3) = \tau_{3,4}$ are not universal.

Question. Are the knots $p(2, -7, -7)$ and $p(2, -11, -11)$ universal?

Theorem 7.2. *Let n be a positive integer such that $n > \alpha(\alpha - 2)/2$ and $(n, \alpha) = (n, \alpha_i) = 1$, where $\alpha_1, \dots, \alpha_t$ are odd positive integers with $\alpha \geq \alpha_i + 2$ for each i. Let $\varepsilon_i \in \{-1, +1\}$ ($i = 1, \dots, t$) such that $\sum \varepsilon_i \neq -2, -1, 0, 1$. Then*

$$k = m(n\varepsilon_1/\alpha_1, \dots, n\varepsilon_t/\alpha_t, n/\alpha)$$

is a universal link.

Proof. The hypothesis $n > \alpha(\alpha - 2)/2$ implies that $(\alpha - 2)/2\alpha - (\alpha_i - 2)/2\alpha_i \geq (\alpha - 2)/2\alpha - (\alpha - 4)/2(\alpha - 2) > 1/n$; therefore there is some $j \in \{1, \dots, (n-1)/2\}$ such that j/n is in the last interval $J^{[\alpha_i/2]}$ for the tangle ε_i/α_i, and in the next-to-last interval $J^{[\alpha_i/2]-1}$ for the tangle $1/\alpha$ (see § 5.2.). Then, in an n-fold dihedral quotient $\varphi : S^3 \to (S^3, k)$, we have $k_j = m(\varepsilon_1/1, \dots, \varepsilon_t/1, 1/3) = m(\sum \varepsilon_i + 1/3) \subset \varphi^{-1}(k)$ (§ 5.2.(c)) if α odd, and $k_j = m(\varepsilon_1/1, \dots, \varepsilon_t/1, 1/2) = m(\sum \varepsilon_i + 1/2)$ if α even. Since $m(\sum \varepsilon_i + 1/3)$ and $m(\sum \varepsilon_i + 1/2)$ are non-torus 2-bridge links, by the choice of the ε_i, we see that $m(\sum \varepsilon_i + 1/3)$ and $m(\sum \varepsilon_i + 1/2)$ are universal (Theorem 1.1); therefore k is universal. □

From Theorem 1.2 in the Introduction and Corollary 3.3 it follows

Theorem 7.3. *Let $b, \alpha_1, \dots, \alpha_t$ be integers such that $p(b, \alpha_1, \dots, \alpha_t)$ is a universal Uchida link. Let n be an integer such that $(n, \alpha_i) = 1$ for $i = 1, \dots, t$. Then*

$$m(nb/1, n/\alpha_1, \dots, n/\alpha_t)$$

is a universal link.

From Theorem 1.3 in the Introduction and Corollary 3.3 it follows

Theorem 7.4. *If $|p| > 1$ is odd and $(n, p) = 1$, then $m(n/p, n/p, -n/p)$ is a universal link.*

If $p \neq -2$ is even and $(n, p) = 1$, then $m(n/3, n/3, n/p)$ is a universal link.

7.1. **Montesinos knots up to ten crossings and Hatcher-Oertel knots.**

7.1.1. *Montesinos knots.* We analyze the Montesinos knots up to 10 crossings. We do not consider here 2-bridge knots, but only 'real' Montesinos knots.

1. *Non-universals.* The knots $8_{19} = p(-2, 3, 3) = \tau_{3,4}$, and $10_{124} = p(-2, 3, 5) = \tau_{3,5}$ are torus knots and not universal.

2. *Universals.* By § 6.2, the knots 8_5, 8_{10}, 8_{15}, 8_{20}, 8_{21}, 9_{16}, 9_{24}, 9_{28}, 10_{76}, 10_{77}, and 10_{78} are universal.

By § 6.3, the knots 9_{22}, 9_{25}, 9_{30}, 9_{36}, 9_{42}, 9_{43}, 9_{44}, 9_{45}, 10_{46}, 10_{47}, 10_{48}, 10_{49}, 10_{70}, 10_{71}, 10_{72}, 10_{73}, 10_{125}, 10_{126}, and 10_{127} are universal.

By § 6.4, the knots 10_{50}, 10_{51}, 10_{52}, 10_{53}, 10_{54}, 10_{55}, 10_{56}, 10_{57}, 10_{128}, 10_{129}, 10_{130}, 10_{131}, 10_{132}, 10_{133}, 10_{134}, and 10_{135} are universal (also all of them have a dihedral quotient with the Fintushel-Stern knot as pseudo-branch).

By § 6.5 (a), or Theorem 7.3, the knots 10_{61}, 10_{62}, 10_{63}, 10_{64}, 10_{65}, 10_{66}, 10_{139}, 10_{140}, 10_{141}, 10_{142}, 10_{143}, and 10_{144} are universal.

The knots 10_{58}, 10_{60}, 10_{136}, and 10_{138} have the pretzel knot $p(-2, 5, 5)$ as pseudo-branch in a dihedral covering of 13, 17, 3 and 7 sheets, resp. The knot 10_{59} has the pretzel knot $p(-2, -5, 5)$ as pseudo-branch in a 3-fold dihedral covering. Then, using § 6.6, we see that 10_{58}, 10_{59}, 10_{60}, 10_{136}, and 10_{138} are universal knots.

The knots $9_{37} = m(1/3, 2/3, 2/3) = m(-5/3, 5/3, 5/3)$, $9_{46} = m(-1/3, 1/3, 1/3)$, and $10_{74} = m(1/3, 1/3, 5/3) = m(-7/3, 7/3, 7/3)$ are universal by Theorem 7.4.

We found sixty six universal knots.

3. *Undecided.* By now, we cannot decide about universality of ten knots: 9_{35}, 9_{48}, 10_{67}, 10_{68}, 10_{69}, 10_{75}, 10_{137}, 10_{145}, 10_{146}, and 10_{147}

4. *No dihedral quotients.* The following fourteen knots do not have the 3-sphere as a dihedral quotient (yet some of them have S^3 as a dihedral-like covering, as noted before): 8_{10}, 8_{19}, 8_{20}, 9_{35}, 9_{46}, 9_{48}, 10_{75}, 10_{124}, 10_{137}, 10_{139}, 10_{140}, 10_{143}, 10_{145}, and 10_{147}.

7.1.2. *Hatcher-Oertel knots.* The following knots are mentioned in [5].

1. $k = m(2/5, 3/7, -1/3, -5/8)$, $\Delta(k) = 109$. In the 109-fold dihedral quotient, $k_{33} = m(2/1, 1/1, 1/3, 1/4, -1/10) = m(-31/22)$ which is universal and, therefore, k is universal.

2. $k = m(-15/32, 3/11, 7/41)$, $\Delta(k) = 365$. In the 365-fold dihedral quotient, $k_{33} = m(5/14, 2/5, -1/1) = m(-17/4)$ which is universal and, therefore, k is universal.

4. $k = m(11/53, 17/43, -13/21)$, $\Delta(k) = 773$. In the 773-fold dihedral quotient, $k_{19} = m(-24/49, 13/33) = m(-155/3)$ which is universal and, therefore, k is universal.

5. $k = m(1/3, 3/5, -3/4, -2/7, 3/11, -5/13)$, $\Delta(k) = 12869$. In the 12869-fold dihedral quotient, $k_{2758} = m(1/1, -1/2, 1/1, 1/1) = m(7/2)$ which is universal and, therefore, k is universal.

6. $k = m(1/3, 1/3, -1/3, -2/5, 1/5, -3/4, 2/3)$, $\Delta(k) = 405$. Therefore k has not the 3-sphere as a dihedral quotient.

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