CIRCULAR HANDLE DECOMPOSITIONS OF FREE GENUS ONE KNOTS

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Abstract. We determine the structure of the circular handle decompositions of the family of free genus one knots. Namely, if k is a free genus one knot, then the handle number $h(k) = 0$, 1 or 2, and, if k is not fibered (that is, if $h(k) > 0$), then k is almost fibered. For this, we develop *practical* techniques to construct circular handle decompositions of knots with free Seifert surfaces in the 3–sphere (and compute handle numbers of many knots), and, also, we characterize the free genus one knots with more than one Seifert surface. These results are obtained through analysis of spines of surfaces on handlebodies. Also we show that there are infinite families of free genus one knots with either $h(k) = 1$ or $h(k) = 2$.

1. Introduction

In the study of the topology of a given 3–manifold, M, it has been useful to consider regular real-valued Morse functions $f : M \to \mathbb{R}$ where M has some smooth structure. A regular real-valued Morse function on M corresponds to a handle decomposition of M of the form $M = b_0 \cup B_1 \cup P_1 \cup \cdots \cup B_r \cup P_r \cup b_3$ where b_0 is a collection of 0-handles, B_j is a collection of 1–handles, P_j is a collection of 2–handles, and b_3 is a collection of 3–handles, in such a way that the i–handles of the decomposition are neighbourhoods of the critical points of index i of the Morse function ($j = 1, \ldots, r$, and $i = 0, 1, 2, 3$). In a celebrated paper ([14]), M. Scharlemann and A. Thompson introduced the concept of *thin position* for 3–manifolds; their idea is to build the manifold as described above (that is, step by step: adding to the set b_0 the set B_1, and then adding P_1, and then adding B_2, and so on) with a sequence of selected sets of 1–handles and sets of 2–handles chosen to keep the boundaries of the intermediate steps as simple as possible.

Now if a 3–manifold M satisfies $H^1(M; \mathbb{Q}) \neq 0$, then there are essential (non-nulhomotopic) regular Morse functions $f : M \to S^1$, and one can always find this kind of functions having only critical points of index 1 and 2 (see Section 2.2). Such a function corresponds to a *circular handle decomposition* $M = F \times [0, 1] \cup B_1 \cup P_1 \cup \cdots \cup B_r \cup P_r$ where F is a properly embedded surface in M, B_j is a collection of 1–handles, and P_j is a collection of 2–handles (the handles are glued along, say, $F \times \{1\}$), and, as above, the set of i–handles of the decomposition correspond to the critical points of index i of the Morse function. With this kind of circular handle decompositions we may also require that the intermediate steps be as simple

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as possible: that requirement acquires the notion of thin position for circular handle decompositions. The existence of these decompositions gives rise to numerical topological invariants such as the (circular) handle number, $h(M) = \sum_{1=1}^{r} \#(B_i)$ where the sum $\sum \#(B_i)$ is minimal among all circular handle decompositions; also, when the decomposition is in thin position, we obtain the circular width, $cw(M)$ (See Section 2.4).

Outstanding examples of manifolds that admit circular handle decompositions, are the exteriors of links in S^3. In this case the interesting intermediate surfaces in the decomposition are Seifert surfaces for the given link (these intermediate surfaces have no closed components, and, if the decomposition is in thin position, they are a sequence of Seifert surfaces which are alternately incompressible and weakly incompressible. See [9], Theorem 3.2, where there is a statement for knots, but its proof works verbatim for links).

Then it is interesting to find explicit constructions of circular handle decompositions of the exterior of a given link which are minimal (that is, that realize the handle number), or that are in thin position. In [2], although in other context, explicit minimal circular handle decompositions of the exterior of the 250 knots in Rolfsen's table are given (of these knots, 117 are fibered, and 132 have handle number one. The Perko knot, $10_{161} = 10_{162}$ is fibered). As far as we know, there are no other previously published explicit constructions of circular handle decompositions of exteriors of links in the 3–sphere.

In this paper we are interested mainly in the circular handle structures of the family of free genus one knots:

In the first part of this work (Section 3) we develop techniques to construct explicit circular decompositions of link exteriors for links that admit a free Seifert surface; these decompositions are interesting, of course, when the free Seifert surface used in the construction is of minimal genus for the link. The information needed to construct these decompositions for the exterior of a given link is encoded in some spine of a free Seifert surface of the link. In this sense, the techniques developed in Section 3 (and through all this paper) could be regarded as elements for a possible theory of spines of surfaces on handlebodies that might be worthy of consideration. As applications we construct minimal circular decompositions for all rational knots and links and, also, for a family of pretzel knots, namely, pretzel knots of the form $P(\pm 3, q, r)$ with $|q|, |r|$ odd integers ≥ 3. These circular decompositions for both families of links are all minimal and have handle number one; they are also in thin position, giving also the circular width of each link considered. This last family gives examples of non-fibered knots whose handle number is strictly less than their tunnel number (Remark 3.11). Also, it is shown that free genus one knots have handle number ≤ 2 (Corollary 3.6).

Secondly (Section 4), we construct circular handle decompositions for the exteriors of all pretzel knots of the form $P(p, q, r)$ with $|p|, |q|, |r|$ odd integers ≥ 5, and we show that these decompositions are minimal with handle number two (Theorem 4.1), and are also in thin position, giving the circular width equal to 6 for each of these knots. These examples answer a question posed in [11] (Remark 4.5).

Next, in Section 5, we give a characterization of the free genus one knots that admit at least two different (non-parallel) Seifert surfaces of genus one. This characterization is given in terms of the existence of a special spine for the given genus one free Seifert surface of the knot (see Theorem 5.2).

If the exterior of a link ℓ in S^3 admits a circular decomposition of the form $E(\ell) = F \times [0,1] \cup B_1 \cup P_1$, and this decomposition is in thin position, we say that ℓ is an *almost fibered link*.

Using the characterization given in Section 5 we show, in the final part of this work, that all (non-fibered) free genus one knots are almost fibered (Theorem 6.7).

It follows from the proof of Theorem 6.7, that the free genus one knots with handle number two have a unique minimal Seifert surface (that is, free genus one knots with at least two genus one Seifert surfaces have handle number one). It is an interesting open problem to determine the family of free genus one knots with handle number two.

2. Preliminaries

Unless explicitly stated, we will use the word 'knot' for a knot or a link in S^3. That is, we will emphasize connectedness if needed. Otherwise, we will admit non-connected knots.

Let X be a manifold and let $Y \subset X$ be a sub-complex. We write $E(Y) = \overline{X - \mathcal{N}(Y)}$ for the *exterior* of Y in X where $\mathcal{N}(Y)$ is a regular neighbourhood of Y in X.

Let X be a manifold and let $Y \subset X$ be a properly embedded submanifold. Y is called *∂–parallel* in X, or *parallel into ∂X*, if there is an embedding $e : (Y, \partial Y) \times I \to (X, \partial X)$, such that $e_0 : Y \to Y$ is the identity, and $e_1(Y) \subset \partial X$. If Y is ∂–parallel in X with embedding $e : (Y, \partial Y) \times I \to (X, \partial X)$, then the submanifold $e(Y \times I)$ is called a *∂–parallelism for Y*. Notice that if Y is disconnected with components $Y_1, \dots, Y_n$, and Y is ∂–parallel in X with a ∂–parallelism W, then W is a disjoint union of ∂–parallelisms $W_1, \dots, W_n$ for $Y_1, \dots, Y_n$, respectively.

2.1. Seifert Surfaces.

Let $k \subset S^3$ be a knot, and let F be a Seifert surface for k; that is, F is an orientable surface and $\partial F = k$. Then, by drilling out a small neighbourhood, $\mathcal{N}(k)$, of k, the surface $\widehat{F} = F \cap E(k)$ is a properly embedded surface in $E(k)$, the exterior of k in S^3, and one may assume that $\partial \widehat{F}$ is parallel to k in $\mathcal{N}(k)$. Usually, we identify F with $\widehat{F}$; but, more appropriately, we start with $F \subset E(k)$ a Seifert surface for k. Seifert surfaces may be disconnected, but they are not allowed to contain closed components. The *genus* $g(k)$ of a knot k is the minimal genus among all Seifert surfaces for k.

A surface $F \subset S^3$ is called *free* if $E(F)$ is a handlebody. The *free genus* of a knot k, $g_f(k)$, is the minimal genus among all free Seifert surfaces for k.

In this work we will be interested mainly in free genus one knots.

2.2. Handle decompositions of rel ∂ Cobordisms.

Let W be a cobordism rel ∂ between surfaces with no closed components, $\partial_+ W$ and $\partial_- W$. A *moderate handle decomposition of* W is a decomposition of the form $W \cong \partial_+ W \times I \cup (1\text{–handles}) \cup$

(2–handles). Given W, a cobordism rel ∂ between surfaces with no closed components, ∂_+W and ∂_-W, it is easy to find a moderate decomposition as above by considering a triangulation of the exterior $E(\partial_+W) = \overline{W - \mathcal{N}(\partial_+W)}$.

Given a cobordism W and a moderate handle decomposition for W, one can find a regular Morse function $f : W \to I$ which realizes the handle decomposition of W. That is, f only has critical points of index 1 and 2, and neighbourhoods of the critical points of f correspond to the 1 and 2–handles of W, and the preimage of each regular level of f is a properly embedded surface in W. We will call such a Morse function a *moderate Morse function.*

2.3. **Circular decompositions.** Let k be a knot in S^3. Since $H_1(E(k))$ is a free Abelian group of positive rank, we can always find an essential (non-nulhomotopic) moderate Morse function $f : E(k) \to S^1$. Any such Morse function, as in Subsection 2.2, induces a decomposition

$$E(k) = (F \times I) \cup B \cup P$$

where $F \subset E(k)$ is a Seifert surface for k, B is a set of n 1–handles glued along, say, $F \times \{1\}$, and P is a set of the same number, n, of 2–handles glued along the same side.

We call such a decomposition a *circular handle decomposition of* $E(k)$ *based on* F, and write $h(F) = n$, the *handle number of* F, where n is the minimal number of 1–handles among all circular handle decompositions of $E(k)$ based on F. The *circular handle number of* k, or simply the *handle number of* k, $h(k)$, is the minimal $h(F)$ among all Seifert surfaces $F \subset E(k)$. Notice that $h(k) = 0$ if and only if k is a fibered knot.

By rearranging the critical points of a moderate Morse function $f : E(k) \to S^1$, we can thin a circular handle decomposition of $E(k)$:

$$E(k) = (F \times I) \cup B_1 \cup P_1 \cup B_2 \cup P_2 \cup \cdots \cup B_\ell \cup P_\ell$$

where B_i is a set of 1–handles glued along $F \times \{1\}$, and P_i is a set of 2–handles, $i = 1, \ldots, \ell$ (of course, it is not always possible to thin a given circular handle decomposition).

For $i = 1, \ldots, \ell$, the set $W_i = (F \times [\frac{1}{2}, 1]) \cup B_1 \cup P_1 \cup \cdots \cup B_i$ gives a moderate handle decomposition for the rel ∂ cobordism W_i with $\partial_+W_i = F \times \{\frac{1}{2}\}$. Write $S_i = \partial_-W_i$. Now we define

$$c(S_i) = \sum_{j=1}^{n_i} (1 - \chi(G_{i,j}))$$

where χ stands for Euler characteristic, and $G_{i,1}, \ldots, G_{i,n_i}$ are the components of S_i (Notice that there are no closed components of S_i for, F has no closed components and the handle decomposition is moderate). Order the surfaces $S_{\sigma(1)}, S_{\sigma(2)}, \ldots, S_{\sigma(\ell)}$ in such a way that $c(S_{\sigma(i)}) \geq c(S_{\sigma(i+1)})$ for $i = 1, \ldots, \ell - 1$, where σ is a permutation in the symbols $1, \ldots, \ell$. Then the *circular width* of this decomposition is the tuple $(c(S_{\sigma(1)}), c(S_{\sigma(2)}), \ldots, c(S_{\sigma(\ell)}))$. The *circular width of* k, $cw(k)$, is the minimal circular width, with respect to lexicographic order, among all thinned circular decompositions of $E(k)$ based on all possible Seifert surfaces for k.

Let $k \subset S^3$ be a knot such that its circular width has the form $cw(k) = (n)$. Then we write $cw(k) = n$, or $cw(k) \in \mathbb{Z}$. If k is a non-fibered knot and $cw(k) \in \mathbb{Z}$, then k is said to be an *almost fibered* knot.

Remark 2.1. **Equivalence of knots**. Let $k, \ell \subset S^3$ be two knots. If the pairs (S^3, k) and (S^3, ℓ) are homeomorphic, then their exteriors also are homeomorphic, $E(k) \cong E(\ell)$; and therefore, the exteriors of k and ℓ have homeomorphic handle decompositions. We regard two knots as being *equivalent* if their corresponding pairs are homeomorphic.

Remark 2.2. **Construction of circular decompositions**. To describe (or, rather, to actually *construct*) a decomposition

$$E(k) = (F \times I) \cup B \cup P$$

where B is a set of 1–handles, and P is a set of 2–handles, it is convenient to write

$$E(k) = (F \times [\tfrac{1}{2}, 1]) \cup B \cup P \cup (F \times [0, \tfrac{1}{2}]).$$

Then to obtain (describe) this circular decomposition we can either

(1) Start with a regular neighbourhood $\mathcal{N}(F)$ of F in $E(k)$. Then add a number of 1–handles to $\mathcal{N}(F)$ (the elements of B) on one side, say $F \times \{1\}$, and then add the same number of 2–handles (the elements of P) on the same side.
The complement of the union above is a regular neighbourhood of $F \times \{0\}$ in $E(k)$. Or

(2) Start with $E(F)$, the exterior of F in $E(k)$. Then drill a number of 2–handles (the elements of B) out of $E(F)$. Now drill the same number of 1–handles (the elements of P) out of $E(F)$.
Here one should be careful that the drilled out 2–handles intersect $\partial E(F)$ on the same side, say $F \times \{1\}$, and that the following drilled out 1–handles intersect the remaining boundary of $E(F)$ on the same side.
The result of this drilling is a regular neighbourhood of $F \times \{0\}$ in $E(k)$.

Of course, in (1) above, '$\mathcal{N}(F)$' stands for $F \times [\frac{1}{2}, 1]$, and in (2), '$E(F)$' stands for the exterior $\overline{E(k) - F \times [\frac{1}{2}, 1]}$. To describe a thinned circular decomposition, one proceeds similarly, but now there will be several steps. Note that, in this kind of decomposition, a thinned decomposition, the number of 1–handles and the number of 2–handles at each step are not necessarily the same.

We emphasize that the main use of the program outlined in (1) is to describe an explicit circular handle decomposition of some given example.

Remark 2.3. **Decompositions of non almost fibered knots**. Now start with a circular decomposition

$$E(k) = (F \times [\tfrac{1}{2}, 1]) \cup B_1 \cup P_1 \cup B_2 \cup P_2 \cup \cdots \cup B_\ell \cup P_\ell \cup (F \times [0, \tfrac{1}{2}])$$

which realizes $cw(k)$, the circular width of k. For $i = 1, \ldots, \ell$, the set $V_i = (F \times [\frac{1}{2}, 1]) \cup B_1 \cup P_1 \cup \cdots \cup B_i \cup P_i$ gives a moderate handle decomposition for the rel ∂ cobordism V_i with $\partial_+ V_i = F \times \{\frac{1}{2}\}$. Write $T_i = \partial_- V_i$. Then the ℓ disjoint surfaces $T_1, T_2, \ldots, T_\ell = F$ are incompressible in $E(k)$ and are non-parallel by pairs (see [9], Theorem 3.2. As noted in the Introduction, the theorem also holds for non-connected knots). That is, *If k is non fibered and not an almost fibered knot, then k has at least two non-parallel incompressible Seifert surfaces.*

Remark 2.4. **Decompositions of pairs**. Let $k \subset S^3$ be a knot with Seifert surface $F \subset E(k)$. There is a copy of F, $F_0 \subset \partial E(F)$, such that $E(F)$ is a cobordism rel ∂ between $F_0 = \partial_+ E(F)$ and $\partial_- E(F)$. We commit an abuse of notation by identifying F with F_0. To find a circular decomposition of $E(k)$ based on F is the same as finding a moderate handle decomposition of the rel ∂ cobordism $E(F)$. A *handle decomposition of the pair* $(E(F), F)$ is, by definition, a handle decomposition of the rel ∂ cobordism $E(F)$.

Now let $\ell \subset S^3$ be another knot with Seifert surface $G \subset E(\ell)$. If there is a homeomorphism of pairs $(E(F), F) \cong (E(G), G)$, then the handle decompositions of the pairs $(E(F), F)$ and $(E(G), G)$ (as well as those of $E(F)$ and $E(G)$ as rel ∂ cobordisms) are in 1-1 correspondence via the given homeomorphism. That is:

To find circular decompositions of $E(k)$ based on F, we need only to construct moderate handle decompositions of the homeomorphism class of the pair $(E(F), F)$. In particular, it is not necessary to regard $E(F)$ as embedded in S^3.

This remark is very helpful in the search of circular decompositions.

2.4. **Spines.** Let X be either a handlebody or a surface with boundary. A *spine* of X is a graph $\Gamma \subset X$ such that X is a regular neighbourhood of Γ. In this work we mainly consider spines of the form $\Gamma \cong \bigvee_{i=1}^n S^1$, a wedge of circles. We write $\Gamma = a_1 \vee \cdots \vee a_n$ to emphasize the circles involved, and we assume that the curves a_i carry a given orientation. Notice that it is allowed for Γ to be a single simple closed curve.

Let $k \subset S^3$ be a knot, and let $F \subset E(k)$ be a Seifert surface for k. A regular neighbourhood $\mathcal{N}(F)$ of F in $E(k)$ admits a product structure $\mathcal{N}(F) = F \times I$ where $\partial F \times I = \mathcal{N}(k) \cap \mathcal{N}(F)$. A spine $\Gamma \subset F \times \{0\}$, $\Gamma \cong \bigvee_{i=1}^n S^1$, is also a spine for $\mathcal{N}(F)$, and the graph Γ induces a product structure $\mathcal{N}(F) = G \times I$, where, say, $G \times \{0\}$ is a regular neighbourhood of Γ in $\partial\mathcal{N}(F)$ (here, of course, G is isotopic to F in $\partial\mathcal{N}(F)$). A spine $\Gamma \subset F \times \{0\}$ is also a graph $\Gamma \subset \partial E(F)$. A spine for F, $\Gamma \subset F \times \{0\}$ (or $\Gamma \subset F \times \{1\}$), is called a *spine for F on $\partial\mathcal{N}(F)$*. Also, we say that Γ is a *spine for F on $\partial E(F)$*.

If Γ is a spine for F on $\partial E(F)$, and G is a regular neighbourhood of Γ in $\partial E(F)$, then a *handle decomposition for the pair* $(E(F), \Gamma)$ is, by definition, a handle decomposition for the pair $(E(F), G)$.

Let $\Gamma = a_1 \vee \cdots \vee a_n$ be a spine for F on $\partial E(F)$, and let $t(a_i)$ be a Dehn twist on F along the curve a_i. If $\widetilde{\Gamma}$ is the graph obtained from Γ by replacing the curve a_j by the curve $t(a_i)(a_j)$, then $\widetilde{\Gamma}$ is also a spine for F. The graph $\widetilde{\Gamma}$ is called *the spine for F obtained from Γ by sliding a_j along $a_i^{\pm 1}$* $(i, j \in \{1, \ldots, g\})$.

Remark 2.5. Notice that if $\widetilde{\Gamma}$ is another spine for F on $\partial E(F)$, and $\widetilde{G}$ is a regular neighbourhood of $\widetilde{\Gamma}$ in $\partial E(F)$, then the pairs $(E(F), \Gamma)$ and $(E(F), \widetilde{\Gamma})$ usually are not homeomorphic, but the pairs $(E(F), F)$ and $(E(F), \widetilde{G})$ are homeomorphic. Thus:

To find circular decompositions of $E(k)$ based on F, we need only to construct moderate handle decompositions of the homeomorphism class of a pair $(E(F), \Gamma)$ for some spine Γ for F on $\partial E(F)$.

Remark 2.6. Let $F \subset S^3$ be a connected orientable surface with boundary $k = \partial F$. If a spine Γ for F on $\partial\mathcal{N}(F)$ is also a spine for $E(F)$, then k is a fibered knot with fiber F. Indeed, $E(F)$ is a handlebody (for it is an irreducible 3–manifold with connected boundary, and with free fundamental group), and both $\mathcal{N}(F)$ and $E(F)$ admit a product structure of the form $G \times I$, where G is a regular neighbourhood of Γ in $\partial\mathcal{N}(F) = \partial E(F)$.

2.5. **Whitehead diagrams.** Let H be a genus g handlebody, and let $x_1, \dots, x_g$ be a system of meridional disks for H. The exterior $E(x_1 \cup \dots \cup x_g)$ is a 3–ball with $2g$ fat vertices $x_1, \bar{x}_1, \dots, x_g, \bar{x}_g$ on its boundary, where $x_i = x_i \times \{0\}$ and $\bar{x}_i = x_i \times \{1\}$ are the copies of x_i in the product structure $\mathcal{N}(x_i) = x_i \times I \subset H$, $i = 1, \dots, g$.

There is a 1-1 correspondence between isotopy classes of systems of meridional disks $\{x_1, \dots, x_g\}$ for H, and homotopy classes of spines of the form $a_1 \vee \cdots, \vee a_g \subset H$ such that $\#(a_i \cap x_i) = 1$, and $a_i \cap x_j = \emptyset$ for $i \neq j$, $i = 1, \dots, j$. It is convenient to commit an abuse of notation, and write both $\{x_1, \dots, x_g\}$ for a meridional system of disks for H, and $\{x_1, \dots, x_g\}$ for the corresponding basis of $\pi_1(H)$ represented by the curves $a_1, \dots, a_g$ in the 1-1 correspondence above. Throughout this paper we adhere to this abuse of notation.

A graph $\Gamma = a_1 \vee \cdots \vee a_n \subset \partial H$ intersects $E(x_1 \cup \cdots \cup x_g)$ in a set of subarcs of the curves a_i; some of these arcs intersect in the base point of Γ. These arcs together with $x_1, \bar{x}_1, \dots, x_g, \bar{x}_g$ form a graph G with $2g$ fat vertices immersed on $\partial E(x_1 \cup \cdots \cup x_g)$. The base point of Γ appears in the drawing on $\partial E(x_1 \cup \cdots \cup x_g)$ as the intersection of some edges of G, but the base point of Γ is not considered a vertex of G. We require that the graph G has no loops, that is, that there are no edges with ends in the same fat vertex of G. In our examples, we will be able to realize this assumption —no loops in G— through the use of some isotopies of H. For each i we number the ends of the arcs in x_i and $\bar{x}_i$ in such a way that the gluing homeomorphisms, which recover H from $E(x_1 \cup \cdots \cup x_g)$, identify equally numbered points. The immersion of the graph G in $\partial E(x_1 \cup \cdots \cup x_g)$, together with these numberings, is called *the Whitehead diagram of the pair* (H, Γ) associated to the system of meridional disks $x_1, \dots, x_g \subset H$ (see Figure 1). The graph G is called the *Whitehead graph* of the corresponding Whitehead diagram.

Let X be a graph, and let e, f be two edges of X; we say that e and f are *parallel* if they connect the same pair of vertices of X. The *simple graph associated to* X is the graph obtained from X by replacing each parallelism class of edges of X by a single edge, and deleting each loop in X (if any).

If X is a connected graph, a vertex v of X is called a *cut vertex* of X if $X - \{v\}$ is not connected. Notice that a loop-less graph X contains a cut vertex if and only if the simple graph associated to X contains a cut vertex.

Let $\mathcal{F}$ be a free group with basis Y, and let A be a set of cyclically reduced words on $Y \cup Y^{-1}$, regarded as elements of $\mathcal{F}$. The *genuine Whitehead graph of* A is the graph, Γ, with vertex set $Y \cup Y^{-1}$, and if $\alpha \in A$, when cyclically α contains the word of length two $v_1 v_2$, then there is an edge in Γ from v_1 to v_2^{-1} for, $v_1, v_2 \in Y \cup Y^{-1}$. If α is of length 1, $\alpha = v$, then there is an edge from v to v^{-1}. If A is a set of elements of $\mathcal{F}$, we can replace A with a set A' of cyclically reduced words representing the conjugacy classes of the elements of A, and then the *genuine Whitehead graph* of A is, by definition, the genuine Whitehead graph

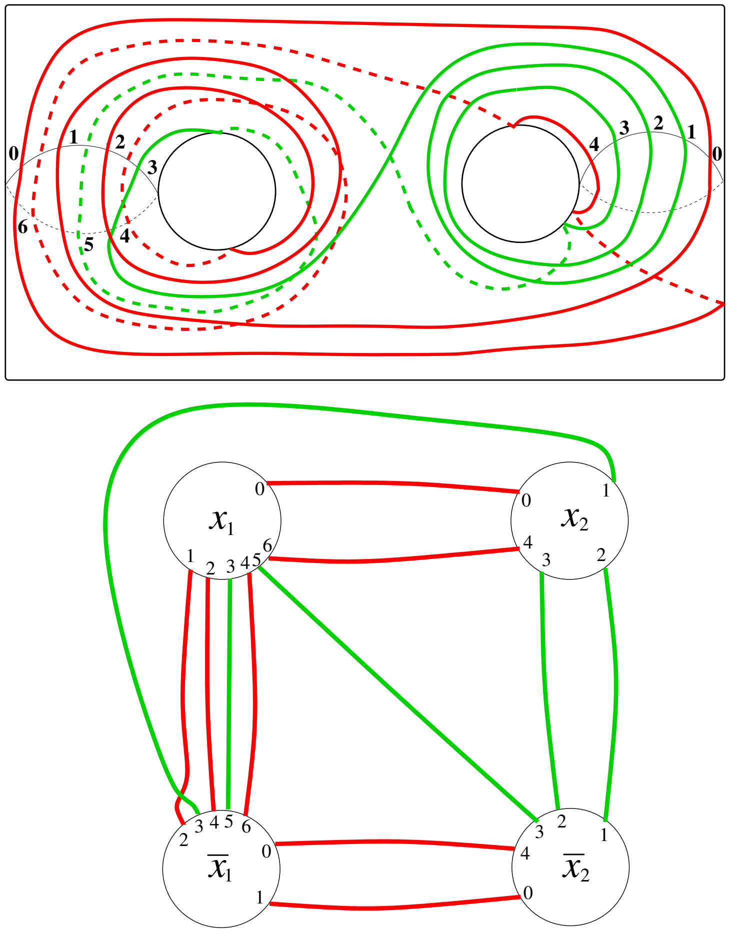


FIGURE 1. A Whitehead diagram associated to the exterior of the pretzel knot $p(5, 5, 5)$.

of A'. The genuine Whitehead graph of a set of elements of $\mathcal{F}$ is regarded as being embedded in 3–space and also contains no loops.

Let $\mathcal{F}$ be a free group and let A be a set of elements of $\mathcal{F}$. Then A is called *separable* if there exists a non-trivial splitting $\mathcal{F} \cong \mathcal{F}_1 * \mathcal{F}_2$ such that each $\alpha \in A$ represents, up to conjugacy, an element of $\mathcal{F}_j$ for some j.

Theorem 2.7 (Theorem 2.4 of [15])**.** *Let A be a set of elements of a free group $\mathcal{F}$ with genuine Whitehead graph Γ. If Γ is connected and if A is separable in $\mathcal{F}$, then there is a cut vertex in Γ.*

The following result follows from Theorem 2.7 and is included here for future reference.

Corollary 2.8. *Let $\Gamma = a_1 \vee \cdots \vee a_n$ be a wedge of n simple closed curves embedded in the boundary of a handlebody H. Assume that for some Whitehead diagram of the pair (H, Γ), the Whitehead graph of this diagram is connected and has no cut vertex. Then Γ intersects every essential disk of H.*

Proof. Let G be the Whitehead graph of the pair (H, Γ) with respect to some system of meridional disks $\{x_1, \ldots, x_g\}$, such that G has no cut vertex and is connected. In particular G has no loops. If we regard G as a graph G' embedded in 3–space so that the base point of Γ vanishes, then G' is the genuine Whitehead graph of the set of elements of $\pi_1(H)$ represented by $\{a_1, \ldots, a_n\}$ with respect to the basis $\{x_1, \ldots, x_g\}$. Since G is connected and has no cut vertex, it follows that G' is also connected and has no cut vertex (recall that the base point of Γ is not part of G; then G and G' are isomorphic graphs). If there is an essential disk in H disjoint with Γ, then the set of elements of $\pi_1(H)$ represented by $\{a_1, \ldots, a_n\}$ clearly is separable, and by Theorem 2.7, G' has a cut vertex or is disconnected. Since G' is connected and has no cut vertex, it follows that Γ intersects every essential disk of H.

□

2.6. **Handle slides.** Handle slides in a handlebody are conveniently visualized when 'translated' into a Whitehead diagram. Figure 2 shows the effect of sliding the handle corresponding to the disk x_2 along the handle corresponding to x_1. But, of course, in the final step, the meridional disks $x_1, \bar{x}_1, x_2, \bar{x}_2$ in the drawing are no longer the same disks, but are their images after the handle slide in the handlebody (The effect of such a handle slide in the fundamental group of the handlebody is a *Whitehead automorphism*. See [15]).

2.6.1. *∂–parallel arcs in handlebodies.* Let $k \subset S^3$ be a knot, and let $F \subset E(k)$ be a free Seifert surface for k. Also let Γ be a spine for F on $\partial E(F)$. In Remark 2.2 (2) a program is outlined to construct a circular decomposition for $E(k)$. It starts by drilling some 2–handles out of $E(F)$ disjoint with F. A 2–handle $P \subset E(F)$ is a product $P = D^2 \times I$ such that $(D^2 \times I) \cap \partial E(F) = D^2 \times \{0, 1\}$, and it is determined by its 'co-core' $\gamma = \{0\} \times I$. This co-core, γ, can be visualized in $E(F)$ as a properly embedded arc with ends disjoint with Γ.

Given two properly embedded arcs γ and γ' in $E(F)$ disjoint with Γ, if the triples $(E(F), \Gamma, \gamma)$ and $(E(F), \Gamma, \gamma')$ are homeomorphic, then the pairs $(E(\gamma), \Gamma)$ and $(E(\gamma'), \Gamma)$ are homeomorphic, and, therefore, have homeomorphic handle decompositions. In this sense, we say that γ and γ' *induce homeomorphic handle decompositions* of $(E(F), \Gamma)$. Also we say, as an abuse of language, that γ and γ' are *equivalent 2–handles.*

Let k be a knot and let $F \subset E(k)$ be a free Seifert surface for k. To exhibit a one-handled decomposition of $E(k)$ based on F, we may follow the program outlined in Remark 2.2 (2). We consider a properly embedded arc $\gamma \subset E(F)$ disjoint with $F \times \{0\}$. If the arc γ corresponds to the single 2–handle to be drilled out of $E(F)$, then γ is called *the arc of the handle decomposition.* In this case we know that γ is parallel into $\partial E(F)$ (see Corollary 4.3 below).

Consider a system of meridional disks $x_1, \ldots, x_g \subset E(F)$. If the arc γ is ∂–parallel into $\partial E(F)$, let z be a ∂–parallelism disk for γ. After an isotopy of $E(F)$ which keeps Γ fixed point-wise, we may assume that z is disjoint with the disks $x_1, \ldots, x_g$. Then γ can be visualized in the Whitehead diagram of $(E(F), \Gamma)$, with respect to $x_1, \ldots, x_g \subset E(F)$, as a properly embedded arc in $E(x_1 \cup \cdots \cup x_g)$ disjoint with G, where G is the corresponding Whitehead graph. After drilling out the 2–handle P, which is a regular neighbourhood of γ, we are 'adding a new

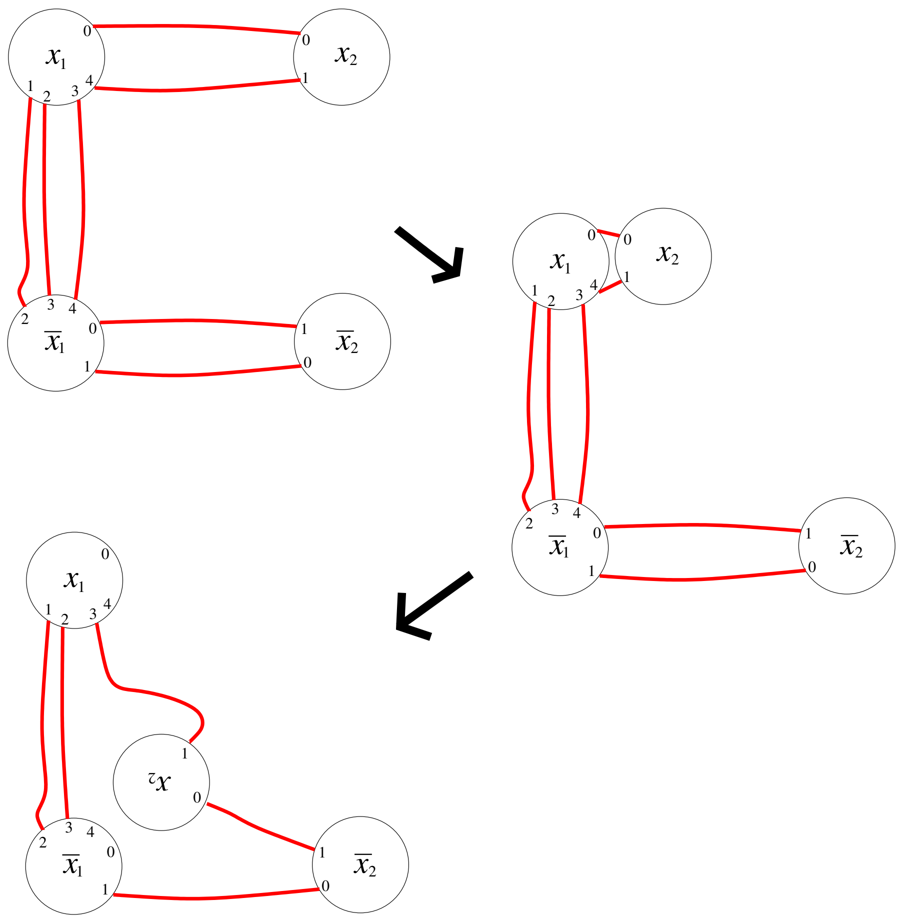


Figure 2. A handle slide.

handle' to $E(F)$; that is, the exterior $E(\gamma) \subset E(F)$ is homeomorphic to $E(F)$ plus one 1–handle. We obtain a Whitehead diagram for $(E(\gamma), \Gamma)$ with respect to $x_1, \ldots, x_g, z$, adding two fat vertices z and $\bar{z}$ as in Figure 3.

This new diagram may contain a cut vertex v. When there is a cut vertex v in G, this vertex decomposes the graph G into two non-trivial graphs X_1 and X_2. One of these graphs, say X_1, does not contain $\bar{v}$. Then we can slide the part corresponding to graph X_1 along the handle defined by disk v. If, after sliding, there appear cut vertices, we continue sliding along some cut vertex on and on. See Figures 4 and 5. Since each such handle slide lowers the 'complexity' of the graph, that is, the sum of all valences of the fat vertices of the corresponding Whitehead graph, eventually we end up with, either:

(1) *A disconnected diagram.* See the last drawing of Figure 5. Then there are obvious essential disks in $E(\gamma)$ disjoint with Γ (more precisely, disjoint with the *image* of Γ on the diagram after the slides); the boundary of these essential disks are curves that separate the components of the current Whitehead graph. If a neighbourhood of one of these disks is a 1–handle B

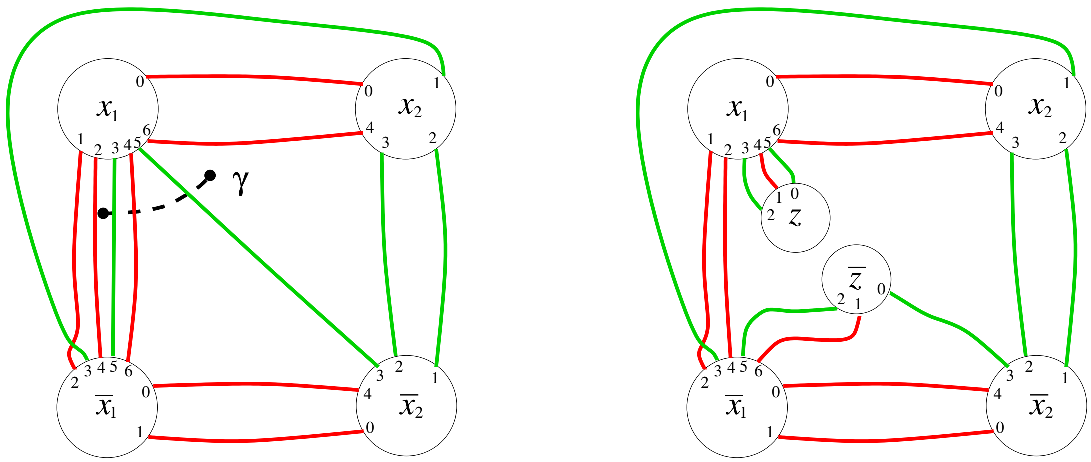


FIGURE 3. Drilling out a 2–handle.

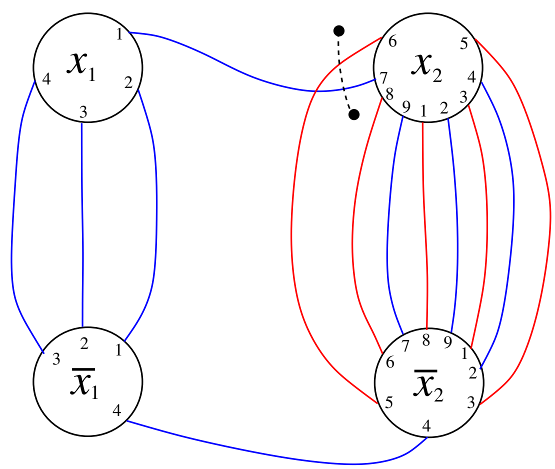


FIGURE 4

inside $E(\gamma)$, after drilling out B, either $E(\gamma \cup B)$ is a regular neighbourhood of $F = F \times \{0\}$, as the last drawing in Figure 5 where the disk labeled x_1 corresponds to B, and we have found a circular one-handled decomposition of $E(k)$ based on F according to the program outlined in Remark 2.2 (2). Or we have to drill out yet another 2–handle from $E(F)$ (to construct possibly a decomposition with more than one 1–handle), or we have to restart the program choosing a different first 2–handle to drill out.

Or we are left with:

(2) *A connected diagram with no cut vertices, and we cannot go on with this plan.* By Corollary 2.8, the chosen arc is not part of a one-handled circular decomposition. To continue the program in Remark 2.2 (2), we have to drill out yet another 2–handle from $E(F)$ (to construct possibly a decomposition

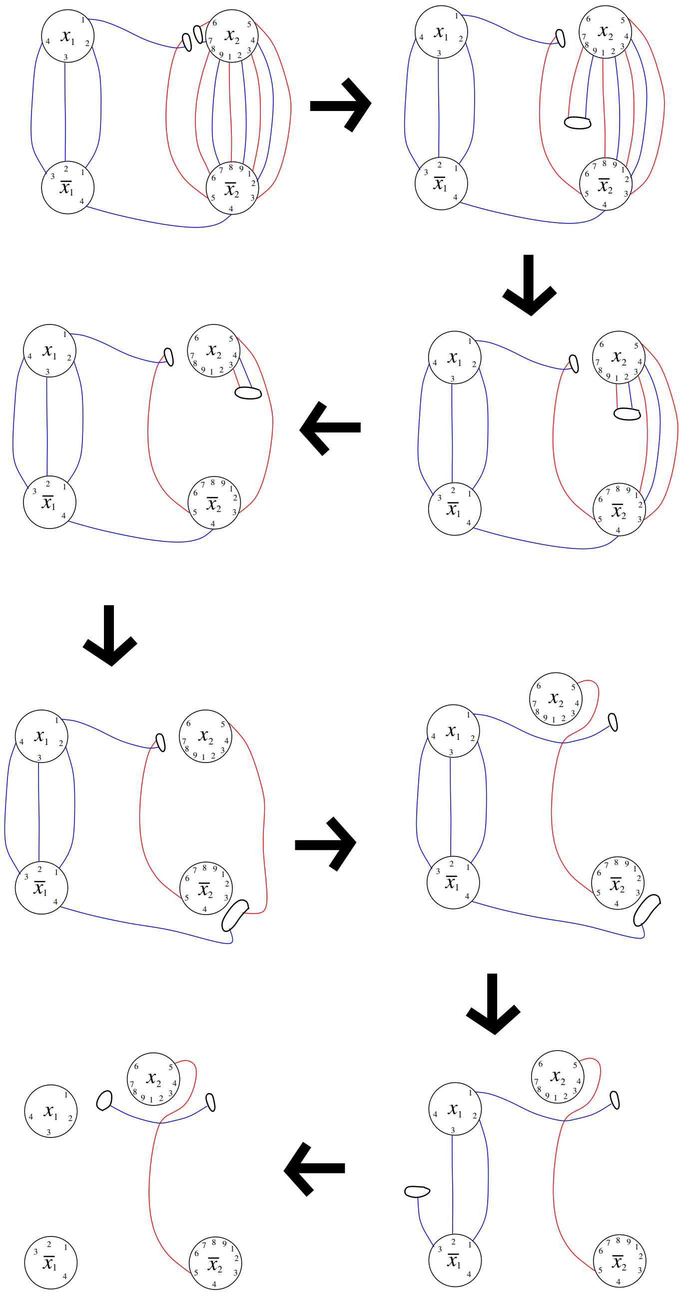


FIGURE 5

with more than one 1–handle), or we have to restart the program choosing a different first 2–handle to drill out.

Now let γ and γ' be two ∂–parallel properly embedded arcs in $E(F)$ disjoint with Γ, with ∂–parallelism disks z and z', respectively; let $\{x_1, \ldots, x_g\}$ be a meridional system of disks for $E(F)$, and let G be the corresponding Whitehead graph with respect to this system of disks. Then, by an isotopy of $E(F)$, we may assume that z and z' are contained in $E(x_1 \cup \cdots \cup x_g)$ and (the images of) γ and γ' are disjoint with G.

Assume that for two faces of G, that is, two connected components $A, B \subset \partial E(x_1 \cup \cdots \cup x_g) - G$, the face A contains an endpoint of γ and one of γ', and the face B contains the other two endpoints of γ and γ'. Then there is an isotopy of $E(x_1 \cup \cdots \cup x_g)$ that fixes point-wise G and sends γ onto γ'. Such an isotopy exists for, being γ and γ' ∂–parallel, they are unknotted properly embedded arcs in the 3–ball $E(x_1 \cup \cdots \cup x_g)$, and the isotopy can be chosen to fix G, for the endpoints of the arcs are, by pairs, in components of $\partial E(x_1 \cup \cdots \cup x_g) - G$. Then we see that a class of 'equivalent' 2–handles in the Whitehead diagram of $(E(F), \Gamma)$ with respect to $x_1, \ldots, x_g$ is determined by pairs of faces of G in $\partial E(x_1 \cup \cdots \cup x_g)$ (and conversely). That is, for ∂–parallel properly embedded arcs $\gamma, \gamma' \subset E(x_1 \cup \cdots \cup x_g)$, the triples $(E(x_1 \cup \cdots \cup x_g), G, \gamma)$ and $(E(x_1 \cup \cdots \cup x_g), G, \gamma')$ are homeomorphic if and only if γ and γ' connect the same pair of faces of G.

This is a very useful fact. To search for a one-handled decomposition, one must only test a finite number of ∂–parallel arcs in some Whitehead diagram, and analyze as above: there are as many ∂–parallel arcs to check as pairs of faces of the corresponding Whitehead graph.

We end this section with some definitions. Assume the arc γ is boundary parallel into $\partial E(F)$. Let z be a ∂–parallelism disk for γ such that $\partial z = \gamma \cup \gamma_z^B$, where γ_z^B is an arc in $\partial E(F)$. Then, after a small isotopy of z, if necessary, γ_z^B intersects the edges of Γ transversely in a finite number of points. If $e_1, \ldots, e_n$ are the edges of Γ that intersect γ_z^B and each e_i intersects only once with γ_z^B, we say that γ *encircles the edges* $e_1, \ldots, e_n$. If γ encircles the edges $e_1, \ldots, e_n$, and all e_i are incident in the vertex ξ of Γ, we say that the arc γ *is around the vertex* ξ. Notice that if $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}$ are all the edges incident in the vertex ξ of Γ, and γ is around vertex ξ encircling the edges $e_1, \ldots, e_n$, then γ also encircles the edges $e_{n+1}, \ldots, e_{n+m}$. The *length* of γ in Γ is the minimal number of intersection points of γ_z^B and Γ among all ∂–parallelism disks z for γ.

3. Primitive elements in spines

Let $\mathcal{F}$ be a free group. An element $x \in \mathcal{F}$ is called *primitive* if x is part of some basis of $\mathcal{F}$. A set of elements $x_1, x_2, \ldots, x_k \in \mathcal{F}$ are called *associated primitive elements* if they are contained in some basis of $\mathcal{F}$.

Let H be a genus g handlebody. A simple closed curve $\alpha \subset H$ represents a primitive element in $\pi_1(H)$ if and only if there is an essential properly embedded disk $D \subset H$ such that $\alpha \cap D$ consists of a single point. A set of simple closed curves $\alpha_1, \ldots, \alpha_k \subset H$ represent a set of associated primitive elements in $\pi_1(H)$ if and only if there is a system of meridional disks $D_1, D_2, \ldots, D_g \subset H$ such that, up to renumbering, $\alpha_i \cap D_i$ consists of a single point, and $\alpha_i \cap D_j = \emptyset$ for $i \neq j$, $i = 1 \ldots, k$, and $j = 1, \ldots, g$.

Theorem 3.1. *Let $k \subset S^3$ be a knot, and let $F \subset E(k)$ be a free Seifert surface for k. Assume $E(F)$ is a handlebody of genus g.*

If there exists a graph $\Gamma = a_1 \vee \cdots \vee a_g$ such that Γ is a spine for F on $\partial E(F)$, and the ℓ curves $a_1, \ldots, a_\ell$ represent associated primitive elements of $\pi_1(E(F))$, then the handle number $h(F) \leq g - \ell$.

Proof. We follow the plan in Remark 2.2 (2): we will exhibit a system of properly embedded arcs (the arcs β_j^I, below) which are the co-cores of $g-\ell$ 2–handles to be drilled out of $E(F)$, and a system of $g-\ell$ 2–disks ($D_{\ell+1}, \ldots, D_g$, below) which define the co-cores of $g-\ell$ 1–handles to be drilled out of $E(F \cup \bigcup_j \beta_j^I)$

Let $D_1, D_2, \ldots, D_g \subset E(F)$ be a system of meridional disks for $E(F)$ such that $|a_i \cap D_i| = 1$, and $a_i \cap D_j = \emptyset$ for $i \neq j$, $i = 1, \ldots, \ell$, and $j = 1, \ldots, g$. This system of meridional disks exists, since $a_1, \ldots, a_\ell$ represent associated primitive elements of $\pi_1(E(F))$.

Let $P \subset E(F)$ be a regular neighbourhood of the base point $x_0 \in \partial E(F)$ (x_0 is also the base point of the graph Γ). We visualize P as a $2g$–gonal prism. See Figure 6. For $i = 1, \ldots, g$, let T_i be a regular neighbourhood of a_i in $E(F)$ such that $T_i \cap T_j = P$ if $i \neq j$. Write $\widehat{T}_i = \overline{T_i - P}$; then $\widehat{T}_i$ is a 3–ball. The intersection, $\widehat{T}_i \cap P = d_i^+ \cup d_i^-$, is the disjoint union of two 2–disks d_i^+ and d_i^- (see Figure 6). Also write $\partial d_i^+ = \beta_i^B \cup \beta_i^I$ where β_i^B is an arc in $\partial E(F)$, and β_i^I is a properly embedded arc in $E(F)$. Finally, write $A_i = \overline{\partial T_i - (d_i^+ \cup d_i^- \cup \partial E(F))}$ which is a 2–disk.

The arcs $\beta_{\ell+1}^I, \ldots, \beta_g^I$ are the co-cores of 2–handles in $E(F)$ to be drilled out, according to the plan in Remark 2.2 (2):

Notice that the exterior of each β_i^I, $E(\beta_i^I) = \overline{E(F) - \mathcal{N}(\beta_i^I)} \cong \overline{E(F) - \mathcal{N}(A_i)}$, and this homeomorphism is the identity map outside a small neighbourhood of A_i.

Consider $V = \overline{E(F) - (\widehat{T}_{\ell+1} \cup \widehat{T}_{\ell+2} \cup \cdots \cup \widehat{T}_g)}$. Then V is a genus g handlebody and $E(F)$ is a regular neighbourhood of V. We see that

$$\overline{E(F) - \cup_{\ell+1}^g \mathcal{N}(\beta_i^I)} \cong \overline{E(F) - \cup_{\ell+1}^g \mathcal{N}(A_i)} \cong V \cup (g-\ell \text{ 1–handles })$$

where the $g-\ell$ 1–handles are the $g-\ell$ balls $\widehat{T}_i$ attached along the disks d_i^+, d_i^-, $i = \ell+1, \ldots, g$.

By the choice of the disks $\{D_i\}$, we see that $\overline{V - \cup_{\ell+1}^g \mathcal{N}(D_i \cap V)}$ is a regular neighbourhood of $a_1 \vee \cdots \vee a_\ell$. Then $\overline{E(F) - (\cup_{\ell+1}^g \mathcal{N}(\beta_i^I) + \cup_{\ell+1}^g \mathcal{N}(D_i \cap V))}$ is a regular neighbourhood of Γ. In other words, $\mathcal{N}(F) \cup \{\mathcal{N}(\beta_i^I) | i = \ell+1, \ldots, g\} \cup \{\mathcal{N}(D_i \cap V) | i = \ell+1, \ldots, g\}$ determines a circular handle decomposition of $E(k)$ based on F, as in Remark 2.2 (2). Therefore, $h(F) \leq g - \ell$.

□

Remark 3.2. To describe the circular decomposition constructed in the proof of Theorem 3.1, which may not be easy to visualize, we may use the program in Remark 2.2 (1) as follows:

- Find a set of compression disks in $E(F)$ for $a_1, \ldots, a_\ell$, that is, the disks $D_{\ell+1}, \ldots, D_g$ in the proof of Theorem 3.1.
- Put on $\partial \mathcal{N}(F)$ a neighbourhood of the arc β_i^I circling the curve a_i, for $i = \ell+1, \ldots, g$. See Figure 7.

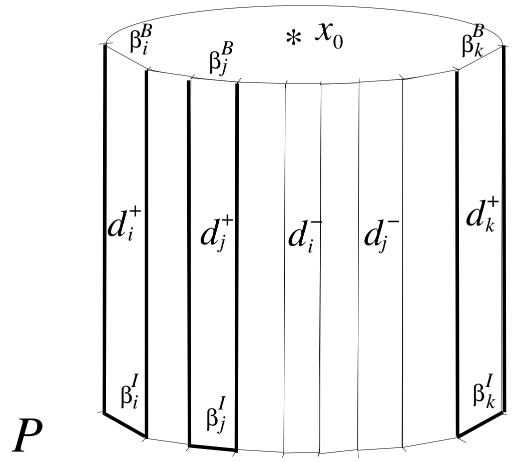


FIGURE 6. The neighbourhood of x_0.

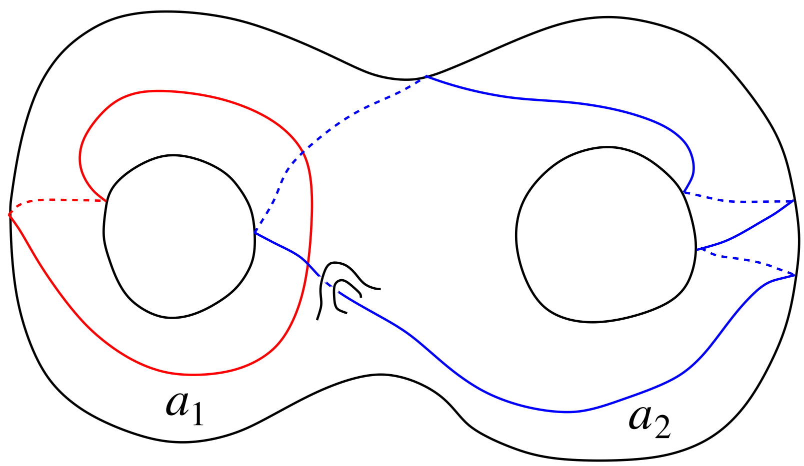


FIGURE 7. Put 1–handles.

- Enlarge the neighbourhoods of the arcs on $\partial\mathcal{N}(F)$ like 'tunnels'. See Figures 8, and 9.
- Glue what is left of the compression disks to $\mathcal{N}(F)$. See Figure 10.

By the proof of Theorem 3.1, the exterior of $\mathcal{N}(F)\cup$(1–handles)$\cup$(compression disks) is a neighbourhood of a parallel copy of F.

The case "$\ell = g$":

Corollary 3.3. *Let $k \subset S^3$ be a knot, and let F be a free Seifert surface for k. Assume $E(F)$ is a handlebody of genus g.*

If there exists a graph $\Gamma = a_1 \vee a_2 \vee \cdots \vee a_g$ such that Γ is a spine for F on $\partial E(F)$, and the curves $a_1, \ldots, a_g$ form a basis of $\pi_1(E(F))$, then k is a fibered knot with fiber F.

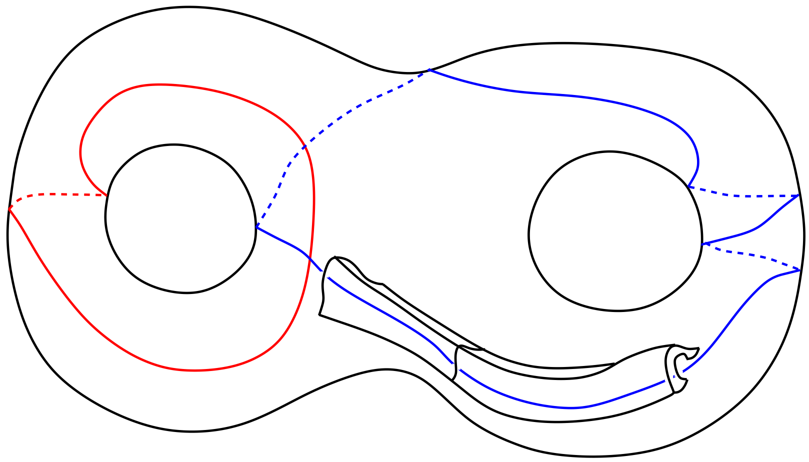


FIGURE 8. Enlarge 1–handles.

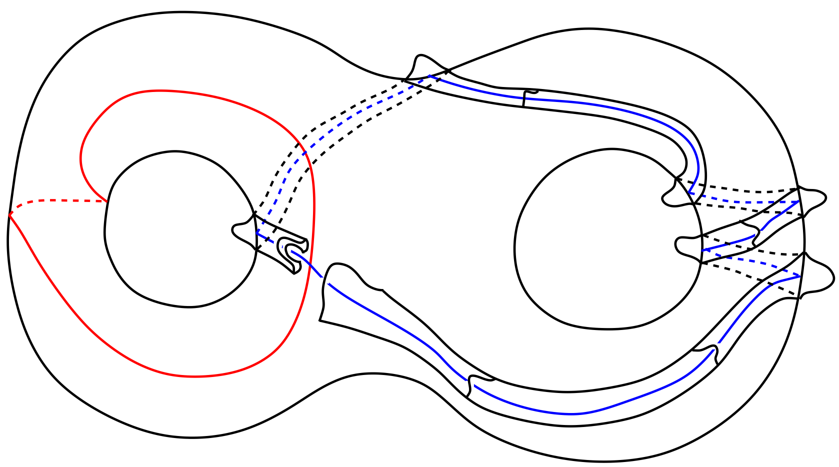


FIGURE 9. Full 1–handle enlargement.

Proof. In this case $h(F) = 0$, therefore, $E(F)$ admits a product structure $E(F) = F \times I$ induced by Γ, and k is fibered with fiber F. $\square$

The case "$\ell = 0$":

Corollary 3.4. *Let $k \subset S^3$ be a knot, and let $F \subset E(k)$ be a free Seifert surface for k. Assume $E(F)$ is a handlebody of genus g.*

Then $h(k) \leq g$.

Proof. By Theorem 3.1, considering $\ell = 0$, we have $h(F) \leq g$. Therefore, $h(k) \leq g$. $\square$

Remark 3.5. Corollary 3.4 asserts that for a connected knot k, $h(k) \leq 2g_f(k)$. See [6] for another proof of this fact (a fact called the 'Free Genus Estimate' in [6]).

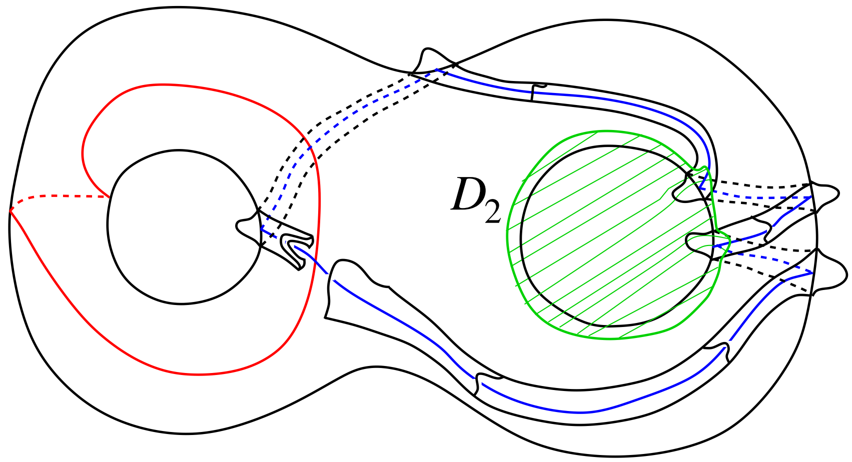


FIGURE 10. Put compression disks.

Corollary 3.6. *If k is a connected free genus one knot, then $h(k) = 0, 1,$ or 2.*

□

Remark 3.7. At this point, it follows from Corollary 3.6, that if k is a connected free genus one knot, and k is not fibered (that is, $k \neq 3_1, 4_1$), then $cw(k) = 4$, $cw(k) = 6$, $cw(k) = (4, 4)$, or $cw(k) = (4, 4, 4)$ (here we use Remark 2.3 and Lemma 4.2 of [16]).

As it was mentioned in the Introduction, free genus one knots are almost fibered. By Theorem 6.7 below, it follows that $cw(k) \in \{4, 6\}$.

Example 3.8. Rational knots. *If $k \subset S^3$ is a non-fibered rational knot, then $h(k) = 1$. Also $cw(k) = 4g(k)$ if k is connected, and $cw(k) = 4g(k) + 1$ otherwise.*

Let $k \subset S^3$ be a rational knot. Then k is encoded with a continued fraction of the form $[2b_1, 2b_2, \ldots, 2b_g]$ where g is even or odd if k is connected or not, respectively. Here $b_1, \ldots, b_g$ are non-zero integers . Now k has a minimal genus Seifert surface F as in Figure 11 (see [1], Answer 1.19). This surface is free. Note that $g(F) = g/2$ if k is connected, and $g(F) = (g-1)/2$ otherwise.

In a neighbourhood V of this surface we can find a spine $\Gamma \subset F \times \{0\} \subset \partial V$ with $\Gamma = a_1 \vee a_2 \vee \cdots \vee a_g$, as in Figure 12. For the obvious meridional disks, $x_1, x_2, \ldots, x_g$, of the handlebody $E(F)$, corresponding to a basis $\{x_1, x_2, \ldots, x_g\}$ of $\pi_1(E(F))$, the curves $a_1, a_2, \ldots, a_g$ represent the elements $x_1^{b_1}$, $x_2^{b_2}x_1$, $x_3^{b_3}x_2 \ldots$, $x_{g-1}^{b_{g-1}}x_{g-2}$, $x_g^{b_g}x_{g-1}$ of $\pi_1(E(F))$, respectively.

If each $|b_i| = 1$, then $a_1, a_2, \ldots, a_g$ represent a basis of $\pi_1(E(F))$, and, by Corollary 3.3, k is fibered with fiber F.

If some $|b_i| \geq 2$, then $\{x_g, x_2^{b_2}x_1, x_3^{b_3}x_2 \ldots, x_{g-1}^{b_{g-1}}x_{g-2}, x_g^{b_g}x_{g-1}\}$ is a basis for $\pi_1(E(F))$; it follows that the curves $a_2, a_3, \ldots, a_g \subset \Gamma$ represent associated primitive elements of $\pi_1(E(F))$, and, by Theorem 3.1, $h(k) \leq h(F) = 1$. By the second part of the statement of Answer 1.19 of [1], k is not fibered. Therefore, $0 < h(k) = h(F) = 1$, and $cw(k) = 2g$ if k is connected, and $cw(k) = 2g + 1$ otherwise.

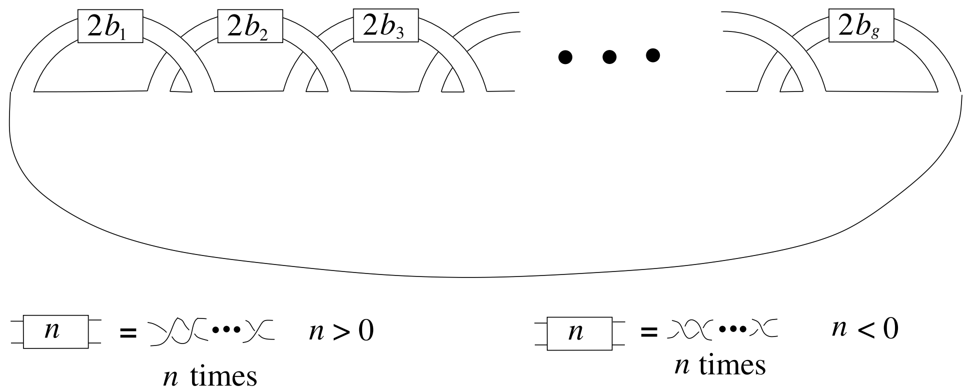


FIGURE 11. A minimal Seifert surface for the knot $k = [2b_1, 2b_2, \ldots, 2b_n]$.

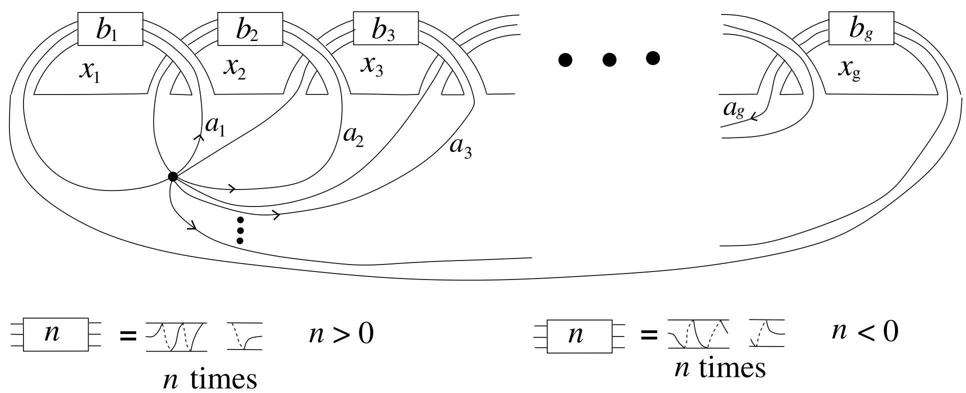


FIGURE 12. A spine for $k = [2b_1, 2b_2, \ldots, 2b_g]$ in $\partial\mathcal{N}(F)$.

Remark 3.9. In Theorem 3.21 of [3] it is claimed that the result in Example 3.8, the one-handledness of rational knots, is known, but unpublished.

Example 3.10. Pretzel knots. *The pretzel knot* $k = P(\pm 3, q, r)$ *with* $|q|, |r|$ *odd integers* ≥ 3*, has* $h(k) = 1$ *and, therefore,* $cw(k) = 4$.

Let k be the pretzel knot $P(p, q, r)$ with p, q, r odd integers. Then k is a connected knot, and the 'black surface' F of a standard projection of k is a free genus one Seifert surface for k. See Figure 13. If $|p|, |q|, |r| \geq 3$, it is known that (1) k has a unique incompressible Seifert surface (see [4]), namely, the free black surface F of genus one; (2) k has tunnel number two (see [7]); (3) $h(k) \leq 2$ (see Corollary 3.6); (4) since $t(k) \neq 1$, k is not a rational knot; (5) also k is not fibered (that is, $k \neq 3_1, 4_4$).

For any permutation s, t, u of p, q, r, the pair (S^3, k) is homeomorphic to a pair (S^3, ℓ) where ℓ is a pretzel knot $P(s, t, u)$. Also, by a reflection, $P(p, q, r)$ is equivalent to $P(-p, -q, -r)$. Then, by Remark 2.1, we may assume that it holds either, *Case 1*: "$p, q, r > 0$", or *Case 2*: "$p < 0$ and $q, r > 0$".

There is a spine shown in Figure 13 for the surface $F \times \{0\} \subset \partial\mathcal{N}(F)$. This spine is a θ–graph. To obtain a wedge of circles as a spine $\Gamma = a_1 \vee a_2 \subset F \times \{0\} \subset \partial\mathcal{N}(F)$,

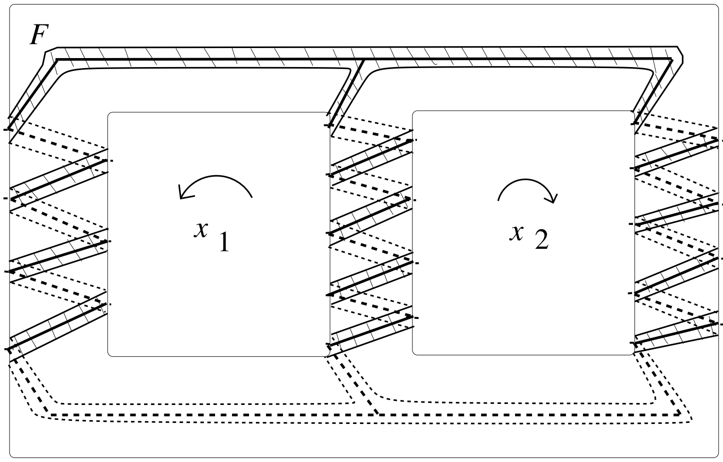


FIGURE 13. Black surface for $P(7, 9, 9)$.

we slide the middle edge of the θ–graph to the left. In *Case 1*, "$p, q, r > 0$", we obtain the upper part of Figure 14; and, in *Case 2*, "$p < 0$ and $q, r > 0$", after using an isotopy to avoid unnecessary intersections of the curve a_2 with the disk x_1, we obtain the lower part of Figure 14. We see that, writing $\pi_1(E(F)) \cong \langle x_1, x_2 : -\rangle$:

Case 1, $(p, q, r > 0)$, the curves a_1 and a_2 represent the elements $x_2^{(r+1)/2} x_1^{-(p-1)/2}$ and $x_1^{(p+1)/2}(x_2x_1)^{(q-1)/2}$, respectively, in $\pi_1(E(F))$, or,

Case 2, $(p < 0$ and $q, r > 0)$, the curves a_1 and a_2 represent the elements $x_2^{(r+1)/2} x_1^{(|p|+1)/2}$ and $x_1^{-(|p|-3)/2}(x_2x_1)^{(q-3)/2}x_2$, respectively, in $\pi_1(E(F))$.

Assume the number $3 \in \{|p|, q, r\}$.

In *Case 1*, "$p, q, r > 0$", using a homeomorphism of S^3, we may assume $p = 3$. In this case the curve $a_1 \simeq x_2^{(r+1)/2} x_1^{-1}$ represents a primitive element of $\pi_1(E(F))$ for, the set $\{x_2^{(r+1)/2} x_1^{-1}, x_2\}$ is a basis of $\pi_1(E(F))$. Therefore, by Theorem 3.1, $h(k) = h(F) = 1$, and $cw(k) = 4$.

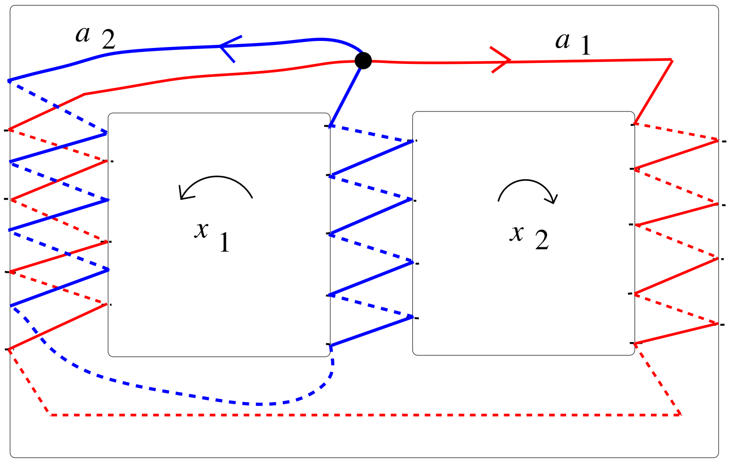
In *Case 2*, "$p < 0$ and $q, r > 0$", if $p = -3$, then the curve $a_2 \simeq (x_2x_1)^{(q-3)/2}x_2$ represents a primitive element of $\pi_1(E(F))$ for, the set $\{(x_2x_1)^{(q-3)/2}x_2, x_2x_1\}$ is a basis of $\pi_1(E(F))$. If $q = 3$ or $r = 3$, we may assume that $q = 3$, and then the curve $a_2 \simeq x_1^{(|p|-3)/2}x_2$ represents a primitive element of $\pi_1(E(F))$ for, the set $\{x_1^{-(|p|-3)/2}x_2, x_1\}$ is a basis of $\pi_1(E(F))$.

In both cases, $p = -3$, or q or $r = 3$, we conclude by Theorem 3.1, $h(k) = h(F) = 1$, and $cw(k) = 4$.

Remark 3.11. If $|q|, |r|$ are odd integers ≥ 3, then $k = P(\pm 3, q, r)$ has tunnel number two. Then the family of pretzel knots $\{P(\pm 3, q, r) : |q|, |r| \text{ odd integers } \geq 3\}$ is a family of examples of non-fibered knots k for which the strict inequality $h(k) < t(k)$ holds (compare with [10], where it is proved that $h(k) \leq t(k)$).

4. PRETZEL KNOTS: THE CASE $|p|, |q|, |r| \geq 5$

In this section we show:



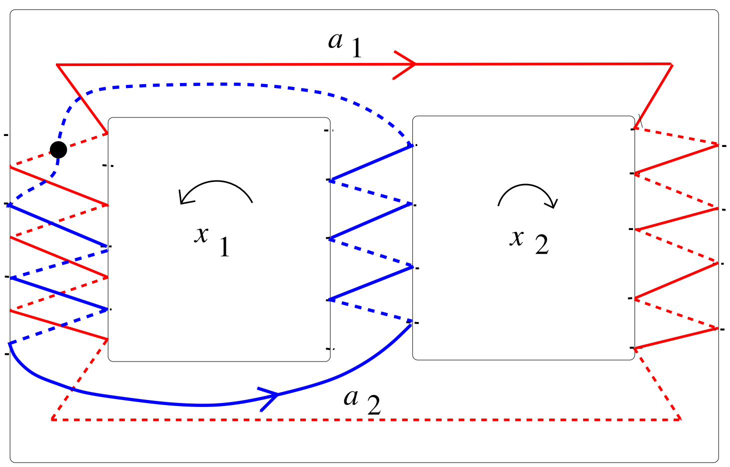


FIGURE 14. Spines for $P(p,q,r)$.

Theorem 4.1. *The free genus one Seifert surface for a pretzel knot $P(p,q,r)$ with $|p|,|q|,|r| \geq 5$ has handle number two.*

As noted in Example 3.10, when dealing with the pretzel knot $k = P(p,q,r)$ we may assume: *Case 1:* "$p,q,r > 0$", or *Case 2:* "$p < 0$ and $q,r > 0$".

4.1. **Handle decompositions of $E(P(p,q,r))$.**

Lemma 4.2. *Let V be a handlebody and let $\alpha \subset V$ be a properly embedded arc. If the exterior $E(\alpha) \subset V$ is a handlebody, then α is parallel into ∂V.*

Proof. By hypothesis, $\pi_1(E(\alpha))$ is a finitely generated free group. If $\mathcal{N}(\alpha) = D^2 \times I$ is a regular neighbourhood of α in V, let $\mu = \partial D^2 \times \{1/2\}$ be a meridian of $\mathcal{N}(\alpha)$. If $N\langle\mu\rangle$ denotes the normal closure of the element represented by μ in $\pi_1(E(\alpha))$, then $\pi_1(E(\alpha))/N\langle\mu\rangle$ is isomorphic to the fundamental group of the space obtained from $E(\alpha)$ by adding a 2–handle along μ. Then $\pi_1(E(\alpha))/N\langle\mu\rangle \cong \pi_1(V)$ is a free group. It follows that μ represents a primitive element in $\pi_1(E(\alpha))$ (see [17], Theorem 4). Thus, there is an essential disk $\delta \subset E(\alpha)$ such that the number of points $\#(\delta \cap \mu) = 1$. After an isotopy, we may assume that $\partial\delta \cap \partial N(\alpha) = \gamma$ is an arc, and $\partial\delta = \beta \cup \gamma$ where β is an arc contained in ∂V.

There is a product 2–disk $Z =$ (radius of D^2) $\times I$ between γ and α, with $Z \subset \mathcal{N}(\alpha)$ for some product structure $D^2 \times I$ of $\mathcal{N}(\alpha)$. Then δ can be extended to a disk $\delta' = Z \cup \delta$ whose boundary is a union of arcs $\alpha \cup \beta'$ with $\beta' \subset \partial V$ (and $\beta \subset \beta'$). Therefore, α is parallel into ∂V. □

Corollary 4.3. *Let F be a free Seifert surface for a knot k. Suppose F has handle number one and let α be the core of the 1–handle of a one-handled circular decomposition of $E(k)$ based on F. Then α is parallel into $\partial E(F)$.*

Proof. As in Remark 2.2 (2), the one-handled decomposition of the pair $(E(F), F)$ is constructed by, first, drilling a 2–handle out of $E(F)$ disjoint with, say, $F \times \{1\}$. This 2–handle has as co-core the arc α of the statement (cf. Remark 2.6.1). After drilling out α, we, secondly, drill one 1–handle B out of the exterior $E(\alpha) \subset E(F)$ with B disjoint with $F \times \{1\}$. The result of this drilling is a regular neighbourhood of the surface $F \times \{0\}$ in $E(k)$ which is a handlebody. Therefore, the exterior $E(\alpha)$ in $E(F)$ is the union of the neighbourhood of $F \times \{0\}$ and the 1–handle B; that is, $E(\alpha)$ is a handlebody. By Lemma 4.2 we conclude that α is parallel into $\partial E(F)$. □

Proof of Theorem 4.1. Let F be the free genus one Seifert surface for $k = P(p, q, r)$ with $|p|, |q|, |r|$ odd integers ≥ 5.

For the sake of contradiction, we assume that F has handle number one. By Corollary 4.3, the core γ of the 1–handle of the circular decomposition of $E(k)$ based on F is parallel into $\partial E(F)$. By assumption, there is also a 2–handle $B \cong I \times D^2$ that completes the decomposition, such that the exterior $E(\gamma \cup B) \subset E(\gamma)$ is a regular neighbourhood of F in $E(k)$, and ∂B is disjoint with F. In particular the core, $\{1/2\} \times D^2$, of B is an essential disk in $E(\gamma)$ disjoint with F. We will show that any essential disk in $E(\gamma)$ intersects F, obtaining the desired contradiction.

Case 1: "$p, q, r > 0$". Let $\Gamma = a_1 \vee a_2$ be the spine for F given in Example 3.10. By Remark 2.5, we only need to analyze the handle decompositions of $(E(F), \Gamma)$. There is an obvious system of meridional disks $x_1, x_2 \subset E(F)$ as depicted in the upper part of Figure 14. The Whitehead diagram for $(E(F), \Gamma)$ with respect to x_1, x_2 looks like Figure 15.

In the corresponding Whitehead graph G we see:

- Four fat vertices corresponding to the meridional disks x_1 and x_2.
- There are $(q-1)/2$ horizontal edges connecting $\bar{x}_1$ and x_2, and $(q-1)/2$ horizontal edges connecting x_1 and $\bar{x}_2$; all these horizontal arcs belong to the curve a_2.
- There are $(r-1)/2$ vertical edges connecting x_2 and $\bar{x}_2$; one diagonal edge connecting x_1 and x_2, and one diagonal edge connecting $\bar{x}_1$ and $\bar{x}_2$; all these vertical and diagonal edges belong to the curve a_1.
- Finally, connecting x_1 with $\bar{x}_1$, we find, going from right to left in Figure 15, first an arc belonging to a_2, and then we find $(p-3)/2$ pairs of arcs belonging consecutively to a_1 and a_2; and a last arc belonging to a_2 which crosses with the diagonal arc from x_1 to x_2 on the base point of Γ.

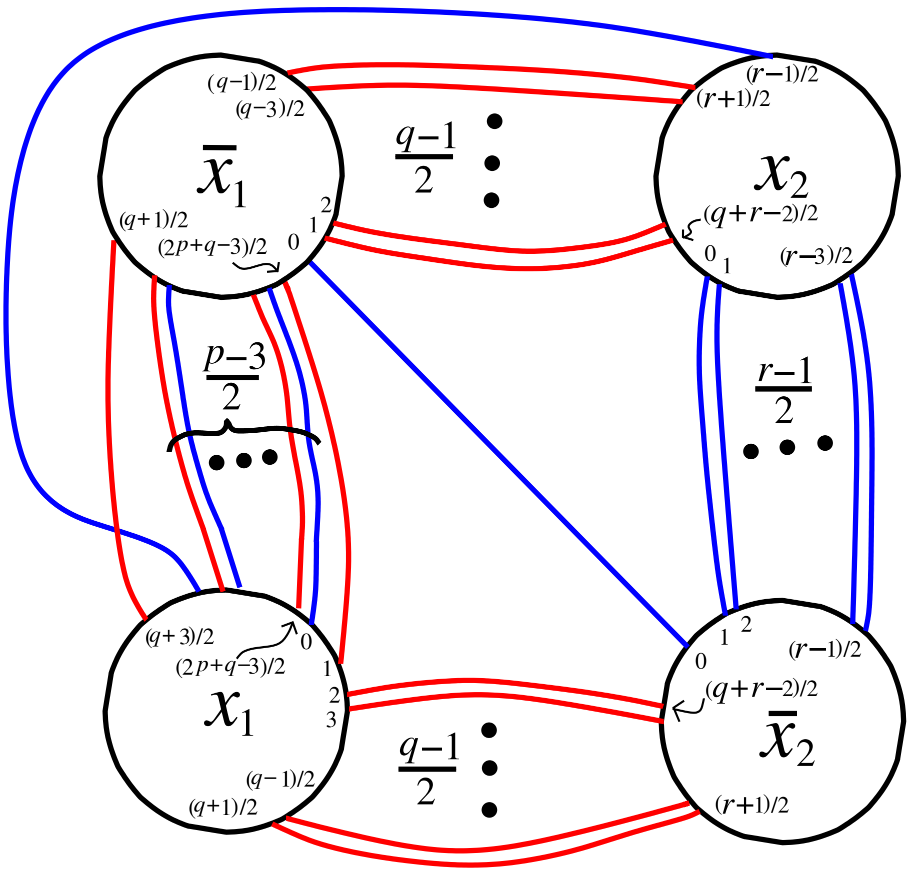


Figure 15

Claim 0: Let z be a ∂–parallelism disk for the arc γ in $E(F)$. Then the disk z contains at least one point of a_1 and one point of a_2.

Proof. Let G_i be the Whitehead graph of the pair $(E(F), a_i)$ with respect to x_1, x_2 $(i = 1, 2)$. See Figure 16. After sliding the handle defined by the disk x_2 along the handle defined by $\bar{x}_1$ on the right side of Figure 16, the image of the graph G_2 looks like Figure 17. Since these graphs are connected and contain no cut vertex, it follows from Corollary 2.9 that any essential disk in $E(F)$ intersects a_i $(i = 1, 2)$. Now, the exterior $E(\gamma)$ can be regarded as a copy of $E(F)$ plus one 1–handle defined by the disk z. Assume $z \cap a_2 \neq \emptyset$. If $z \cap a_1 = \emptyset$, then a_1 is contained in the copy of $E(F) \subset E(\gamma)$. By hypothesis there is an essential disk $\Delta \subset E(\gamma)$ such that $\Delta \cap (a_1 \cup a_2) = \emptyset$. Now, $\Delta \cap z \neq \emptyset$, otherwise Δ is a subset of the copy of $E(F) \subset E(\gamma)$ missing the extra 1–handle, and $\Delta \cap a_1 = \emptyset$, contradicting that any essential disk in $E(F)$ intersects a_1. Through isotopies, we may assume that $\Delta \cap z$ is a set of disjoint arcs. Then the intersection of Δ with the copy of $E(F) \subset E(\gamma)$, that is, the set $\Delta \cap \overline{(E(\gamma) - \mathcal{N}(z))}$, is a set of disjoint properly embedded disks $\Delta_1, \ldots, \Delta_n \subset E(F)$. Since Δ is not parallel to z in $E(\gamma)$, at least one Δ_i is essential in $E(F)$, otherwise Δ would be parallel into $\partial E(\gamma)$. We obtain again an essential disk in $E(F)$ disjoint with a_1, which is a contradiction as above, and, therefore, $z \cap a_1 \neq \emptyset$. □

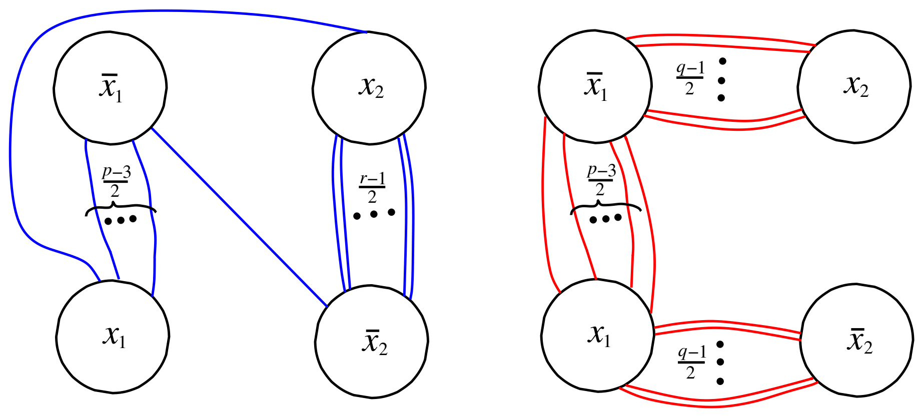


FIGURE 16. The graphs of curves a_1 and a_2

The arc γ, being ∂–parallel in $E(F)$ by Corollary 4.3, can be isotoped into this Whitehead diagram as a properly embedded arc with ends disjoint with G (that is, after an isotopy of $E(F)$, we may assume that γ is disjoint with the system of disks x_1 and x_2). Recall that we are assuming that γ is the core of a 1–handle of a one-handled circular decomposition of $E(k)$ based on F. Therefore, after drilling out γ, there is an essential disk in $E(\gamma)$ disjoint with Γ; that is, after drilling out γ, and obtaining a new Whitehead diagram with six fat vertices with Whitehead graph G', there is a sequence of handle slides of $E(\gamma)$ that disconnect the graph G', giving an essential disk in $E(F)$ disjoint with Γ (see Section 2.6).

Let G_i be the Whitehead graph of the pair $(E(F), a_i)$ with respect to x_1, x_2. See Figure 16. After drilling out the arc γ from the diagram of G_i, we obtain a new Whitehead diagram for $(E(\gamma), a_i)$ with six fat vertices, corresponding to x_1, x_2, and z, and with Whitehead graph G'_i. Performing the handle slides of $E(\gamma)$ as above, the image of the graph G'_i will be also disconnected, giving an essential disk in $E(\gamma)$ disjoint with a_i $(i = 1, 2)$.

Notice that if we drill out an arc of length one in G_i and perform handle slides, the image of G_i is disconnected (it contains four isolated fat vertices), $i = 1, 2$. We deal with this kind of arcs after Claims 1 and 2.

Claim 1: Let α be a properly embedded arc in $(E(F), a_2)$, disjoint with a_2, such that α is parallel into $\partial E(F)$, and α has length at least two in G_2. Then any essential disk in $E(\alpha)$ intersects a_2.

Proof. The arc α *minimally* encircles a number of edges of the graph G_2. For example, the arc that encircles the two diagonal edges in Figure 17 actually has length 0.

Now, after sliding the handle defined by the disk x_2 along the handle defined by $\bar{x}_1$ on the right side of Figure 16, the image of the graph G_2 looks like Figure 17. The fat vertices of this graph are also obtained from the images of the disks x_1 and x_2 after the slide. We still call this new graph and new disks G_2, and x_1, x_2,

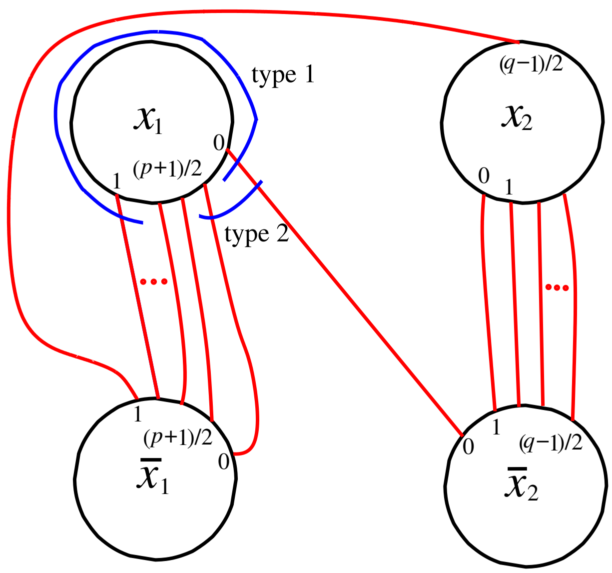


FIGURE 17

respectively. This graph has $(q-3)/2$ vertical edges connecting x_2 with $\bar{x}_2$, one diagonal edge connecting x_2 with $\bar{x}_1$, one diagonal edge connecting x_1 with $\bar{x}_2$, and there are $(p-1)/2$ vertical arcs connecting x_1 with $\bar{x}_1$.

Let z be a minimal ∂–parallelism disk for α in $E(F)$, and let G be the Whitehead graph of $(E(\alpha), a_2)$ with respect to x_1, x_2, and z, which is obtained from G_2, by cutting along z and adding two fat vertices z and $\bar{z}$.

Case "Length of $\alpha = 2$*"*: Since $p \geq 5$, there are at least two vertical edges connecting x_1 and $\bar{x}_1$. Then there are two types of arcs of length two for the edges of G_2 around x_1 as in Figure 17, for, any arc encircling two consecutive edges of G_2 connecting x_1 and $\bar{x}_1$ can be slid in $E(F)$ into an arc of type 1 or type 2. See Figure 18 where the arcs that can be slid in $E(F)$ into an arc of type 2 are shown.

After drilling out the arc α, if α is of type 1, or of type 2, the new Whitehead graph contains a cut vertex (see Figure 19).

After sliding handles, as in Section 2.6.1, we end up with a graph G_2' with its simple associated graph a cycle of six vertices and six edges; that is, this simple graph contains no cut vertex. Therefore, G_2' contains no cut vertex, and by Corollary 2.8, a_2 intersects every essential disk of $E(\alpha)$.

If $q \geq 7$, there are at least two vertical edges connecting x_2 and $\bar{x}_2$. Then, by symmetry, the analysis of arcs of length two around x_2 and $\bar{x}_2$ is the same as for arcs of length two around x_1 and $\bar{x}_1$.

If $q = 5$, there is a single vertical edge connecting x_2 and $\bar{x}_2$, and, then, there are no arcs of length two around x_2 or $\bar{x}_2$.

For arcs not around a vertex of G_2, there are two more types of arcs of length two as in Figure 20, but, after drilling out the arc α of type 3 or 4, the new Whitehead graph contains no cut vertex, and then, by Corollary 2.8, a_2 intersects every essential disk of $E(\alpha)$.

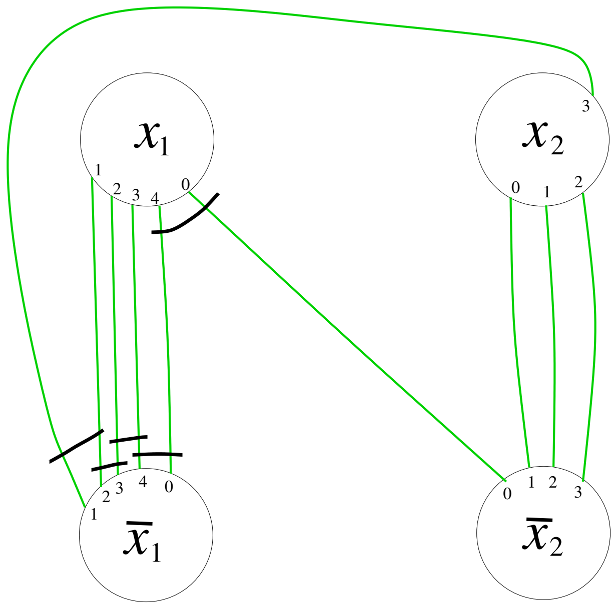


FIGURE 18

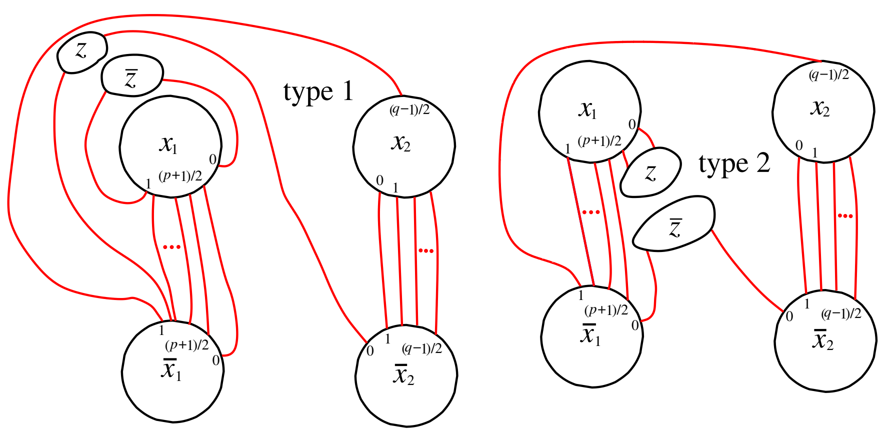


FIGURE 19

Case "Length of $\alpha \geq 3$*"*: If α is an arc around x_i, we may assume that the length of α in G_2 is between 3 and $degree(x_i)/2$ (see last paragraph of Section 2.6.1), and α contains a sub-arc of type 1 or 2. After drilling out the arc α and sliding, if there appear cut vertices, we end up with a graph with its simple associated graph a cycle with six vertices and six edges. Therefore, a_2 again intersects every essential disk of $E(\alpha)$.

If α is of length at least 3, and α contains a sub-arc of type 3 or 4, then, after drilling out the arc α, the new Whitehead graph contains no cut vertex, and, by Corollary 2.8, we conclude that a_2 intersects every essential disk of $E(\alpha)$.

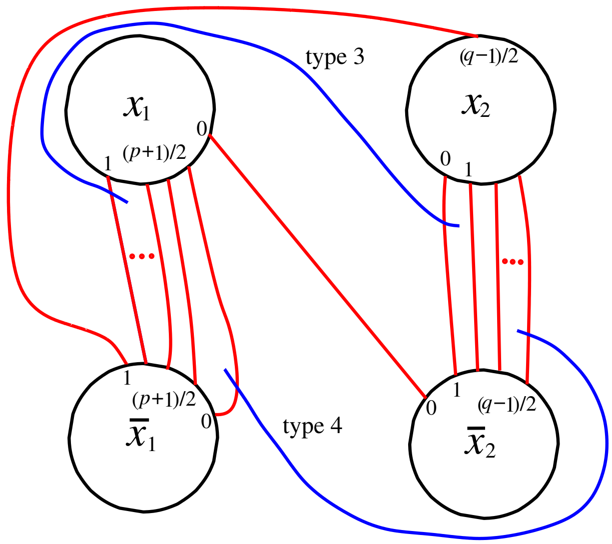


FIGURE 20

By the final remarks of Section 2.6.1, the arcs of type 1-4 exhaust all arcs to be considered as arcs of a one-handled decomposition for G_2.

□

Claim 2: Let α be a properly embedded arc in $(E(F), a_1)$, disjoint with a_1, such that α is parallel into $\partial E(F)$, and α has length at least two in G_1. Then any essential disk in $E(\alpha)$ intersects a_1.

Proof. The Whitehead graph G_1 of $(E(F), a_1)$ has a shape as in Figure 17, but with $(r-1)/2$ vertical edges connecting x_2 with $\bar{x}_2$, one diagonal edge connecting x_2 with $\bar{x}_1$, one diagonal edge connecting x_1 with $\bar{x}_2$, and there are $(p-3)/2$ vertical arcs connecting x_1 with $\bar{x}_1$.

A similar (symmetric) analysis as in Claim 1, gives that a_1 intersects every essential disk of $E(\alpha)$.

□

We are assuming that, after drilling out the arc γ, there is a set of handle slides of $E(\gamma)$ that disconnect the graph G', giving an essential disk in $E(F)$ disjoint with Γ.

By Claims 1 and 2, γ is of length one in G_1, and of length one in G_2. If γ is around one fat vertex ξ of G, it might happen that γ encircles exactly one edge of G_1, and all but one edge of G_2, or vice versa. In this case, γ is around either x_2 or $\bar{x}_2$. There are four arcs around x_2, and four arcs around $\bar{x}_2$ of this kind. The four arcs with this property around $\bar{x}_2$ can be slid in $E(F)$ and become equivalent to the four arcs around x_2 in Figure 21; see Section 2.6.1. After drilling out γ, there is a cut vertex in the new Whitehead graph, and a single handle slide produces a graph G' with no cut vertices. By Corollary 2.8, there are no essential disks disjoint with G in $E(\gamma)$. Another possibility is that γ encircles all but one edge of G_1 and

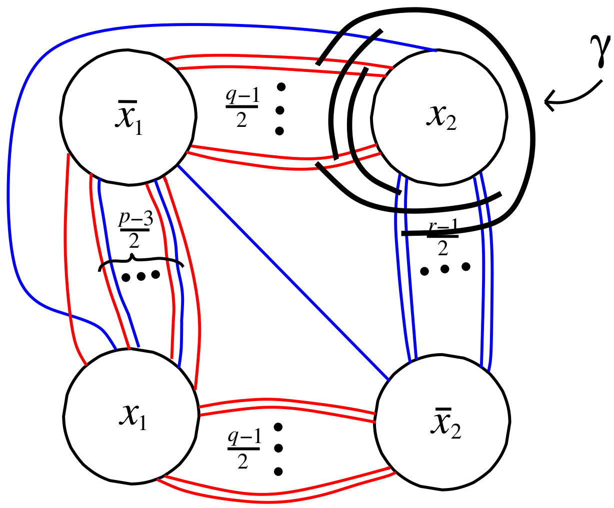


FIGURE 21

all but one edge of G_2, but in this case, γ also encircles exactly one edge of G_1, and exactly one edge of G_2.

There are four types of arcs of length two encircling exactly one edge of G_1 and exactly one edge of G_2 (see Figure 22). Again, any arc encircling two edges of G, one of G_1 and one of G_2 can be slid in $E(F)$ into an arc of type 1, type 2, type 3, or type 4; see Section 2.6.1.

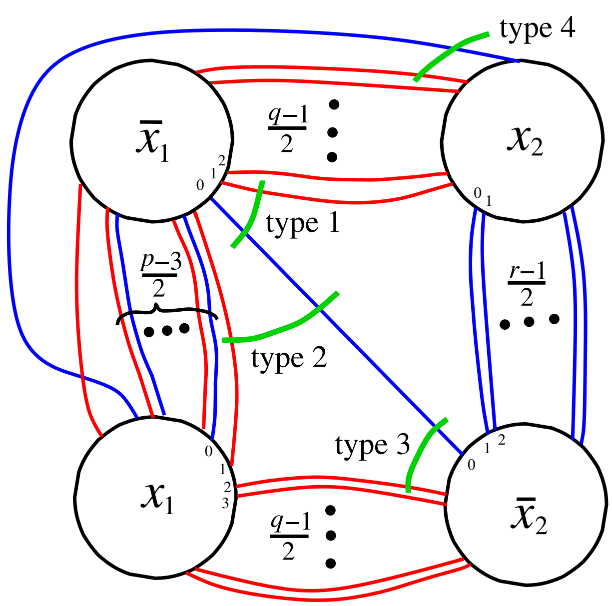


FIGURE 22

After drilling out the arc γ, if γ is of type 1, type 2, type 3, or type 4, the new Whitehead graph contains a cut vertex. After sliding, we end up with a graph G' with its simple associated graph as one of the drawings in Figure 23. Since these graphs contain no cut vertex, by Corollary 2.8, we conclude that any essential disk

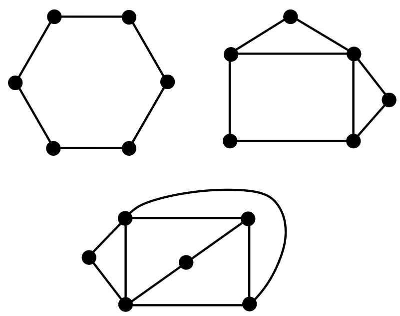


FIGURE 23

in $E(\gamma)$ intersects G, and, therefore, intersects $\Gamma \subset F$. This contradiction shows that $h(F) \neq 1$. Since $k = P(p, q, r)$ is not fibered, and $h(F) \leq 2$, by Corollary 3.6, it follows that $h(F) = 2$, when $p, q, r \geq 5$.

This finishes Case 1.

Case 2: "$p < 0$, and $q, r > 0$". As in Example 3.10, we construct a spine $\Gamma = a_1 \vee a_2$ for F starting with the spine shown in Figure 13, but now we slide the middle edge of the θ–graph rightwards. The spine Γ looks like Figure 24, and the Whitehead diagram for $(E(F), \Gamma)$ with respect to the system of disks x_1, x_2 is as in Figure 25. By Remark 2.5, we only need to analyze the handle decompositions of $(E(F), \Gamma)$.

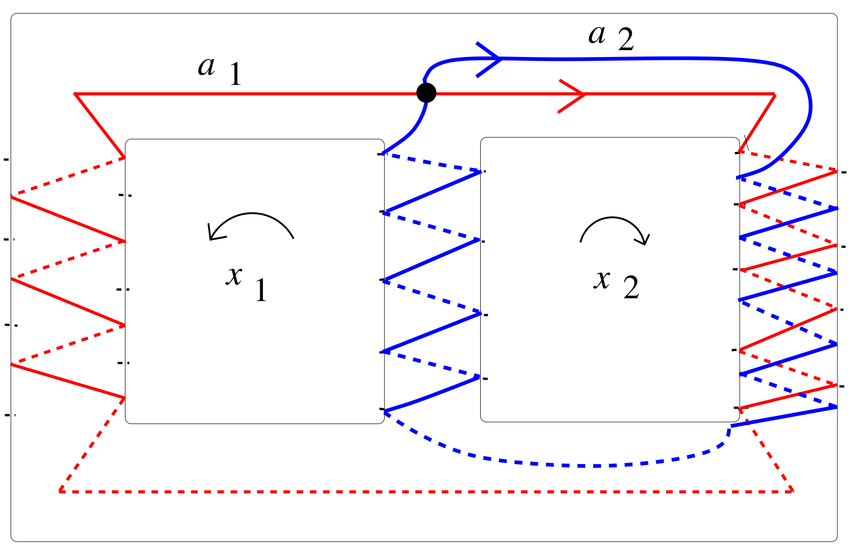


FIGURE 24

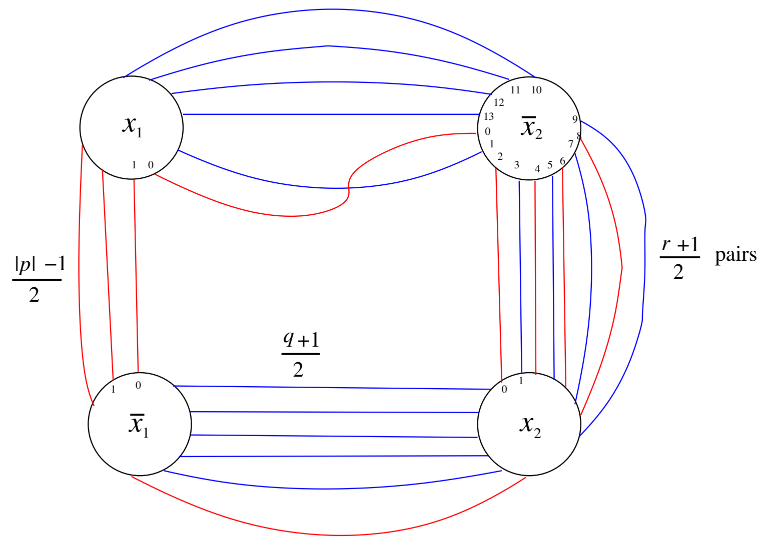


FIGURE 25

The Whitehead graphs G_1 and G_2 of the pairs $(E(F), a_1)$ and $(E(F), a_2)$, respectively, are shown in Figure 26. Although these diagrams are similar to the diagrams in Figure 16 of Case 1, the configuration of the diagram for a_1 here is not the same as the configuration of the positive case (Case 1); that is, the corresponding Whitehead diagrams are not isomorphic.

However, the analysis of the different properly embedded arcs in the Whitehead diagrams of $(E(F), a_1)$, $(E(F), a_2)$, and $(E(F), \Gamma)$, giving rise to a possible one-handled decomposition, is completely similar as in Case 1.

The Whitehead diagram for $(E(F), a_2)$ is isomorphic to the corresponding Whitehead diagram of Case 1. Then

Claim 1: Let α be a properly embedded arc in $(E(F), a_2)$, disjoint with a_2, such that α is parallel into $\partial E(F)$, and α has length at least two in G_2. Then any essential disk in $E(\alpha)$ intersects a_2.

□

Claim 2: Let α be a properly embedded arc in $(E(F), a_1)$, disjoint with a_1, such that α is parallel into $\partial E(F)$, and α has length at least two in G_1. Then any essential disk in $E(\alpha)$ intersects a_1.

Proof. We first analyze arcs of length 2 in G_1. The arcs around vertices x_1 and $\bar{x}_1$ are shown in Figure 27. There are only two types after sliding the arcs in $E(F)$. After drilling out the arc α, if α is of type 1, or of type 2, the new Whitehead graph contains a cut vertex, but after sliding handles, as in Section 2.6.1, we end up with a graph G_1' with its simple associated graph a cycle of six vertices and six edges; that is, this simple graph contains no cut vertex. Therefore, G_1' contains no cut vertex, and by Corollary 2.8, a_2 intersects every essential disk of $E(\alpha)$.

For arcs of length 2 around the vertices x_2 and $\bar{x}_2$, the analysis is identical to Case 1.

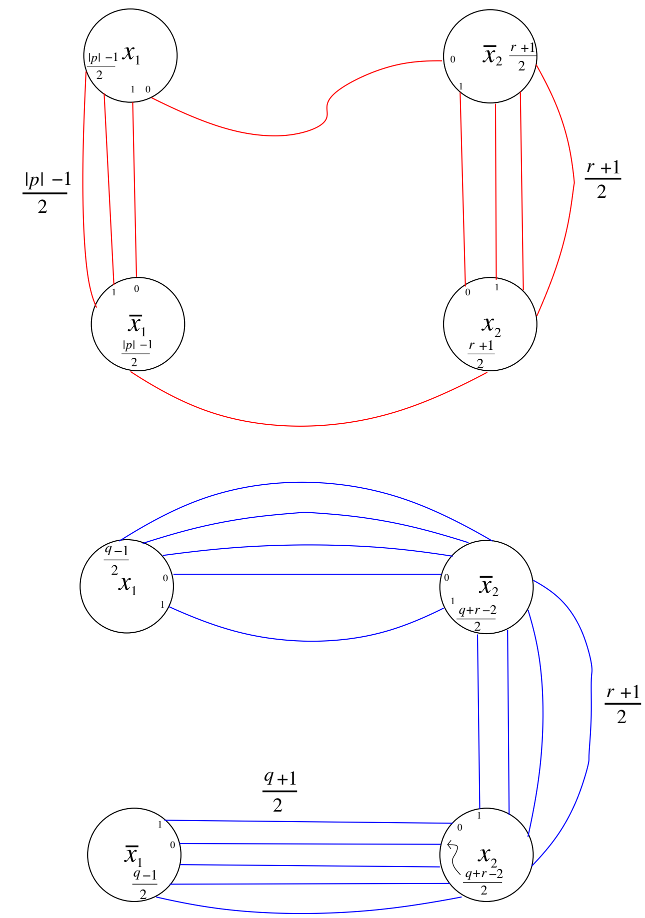


Figure 26

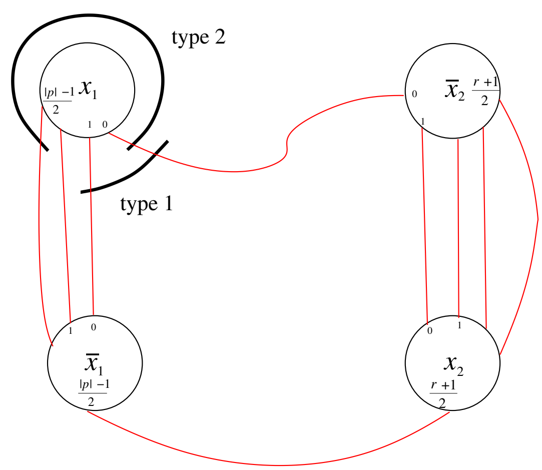


Figure 27

For arcs not around a vertex of G_1, there are two more types of arcs of length two as in Figure 28, but, after drilling out the arc α of type 3 or 4, the new

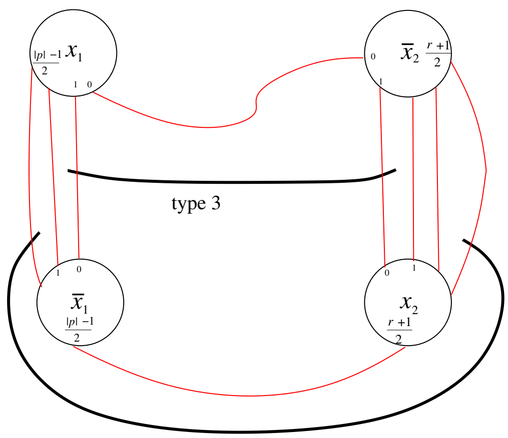


FIGURE 28

Whitehead graph contains no cut vertex, and then, by Corollary 2.8, a_2 intersects every essential disk of $E(\alpha)$.

For arcs of length at least three, we follow the same argument as in Case 1, and conclude that a_2 intersects every essential disk of $E(\alpha)$.

□

Recall that we are assuming that γ is the core of a 1–handle of a one-handled circular decomposition of $E(k)$ based on F. In view of Claims 1 and 2, as in Case 1, we see that the arc γ encircles exactly one edge of G_1, and exactly one edge of G_2.

There are four types of arcs of length two encircling exactly one edge of G_1 and exactly one edge of G_2 (see Figure 29). For, any arc encircling two edges of G, one of G_1 and one of G_2 can be slid in $E(F)$ into an arc of type 1, type 2, type 3, or type 4 (Section 2.6.1).

After drilling out the arc γ, if γ is of type 1, type 2, type 3, or type 4, the new Whitehead graph contains a cut vertex. After sliding, we end up with a graph G' with its simple associated graph as one of the drawings in Figure 30. Since these graphs contain no cut vertex, by Corollary 2.8, we conclude that any essential disk in $E(\gamma)$ intersects G, and, therefore, intersects $\Gamma \subset F$. Thus, $h(F) \neq 1$. Since $k = P(p,q,r)$ is not fibered, and $h(F) \leq 2$, by Corollary 3.6, it follows that $h(F) = 2$, when $p \leq -5$ and $q, r \geq 5$.

This finishes Case 2, and also the proof of Theorem 4.1.

□

Corollary 4.4. *Let k be the pretzel knot $P(p,q,r)$ with $|p|, |q|, |r| \geq 5$. Then $cw(k) = 6$.*

Proof. Since k has a unique incompressible Seifert surface, by Remark 2.3 it follows that $cw(k) \in \mathbb{Z}$. By Theorem 4.1, $cw(k) = 6$. □

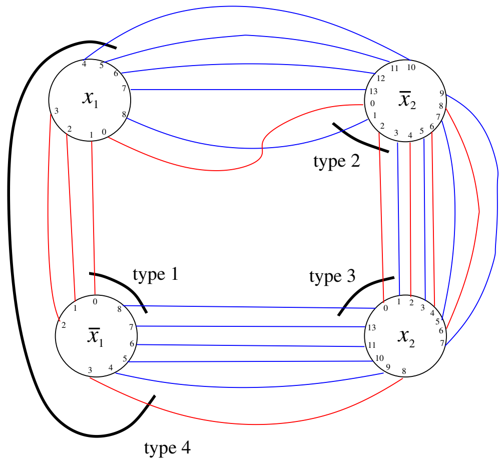


FIGURE 29

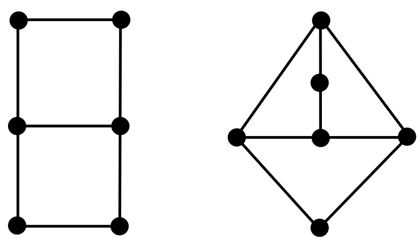


FIGURE 30

Remark 4.5. Theorem 4.1 gives a family of knots of genus one and handle number two. This answers in the affirmative a question in [6]: Does there exist a knot k with $h(k) > g(k)$?

5. Genus one essential surfaces and powers of primitive elements

In this section we show that if k is a free genus one knot with at least two non-isotopic Seifert surfaces, then the free Seifert surface of k admits a special type of spine. This result is essential to prove the main theorem of Section 6 (Theorem 6.7).

Lemma 5.1. *Let H be a handlebody of genus $g \geq 2$, and let $\alpha \subset \partial H$ be a simple closed curve. Assume that there is a primitive element $p \in \pi_1(H)$ such that α represents an element conjugate with p^n for some $n \in \mathbb{Z}$, $n \neq 0$. Then there is an essential 2–disk $D \subset H$ such that $D \cap \alpha = \emptyset$.*

Proof. Consider a basis $\{p, q_2, \ldots, q_g\}$ for $\pi_1(H)$. Then $\pi_1(H) = \langle p \rangle * \langle q_2, \ldots, q_g \rangle$ is a non-trivial splitting, and α is conjugate with $p^n \in \langle p \rangle$. Then $\{\alpha\}$ is separable in $\pi_1(H)$, and the disk D is obtained by Theorem 3.2 of [15]. □

Let $\Gamma \cong a_1 \vee a_2$ be a graph in the boundary of a genus two handlebody H. We say that a_2 *spoils disks for* a_1 if for any essential disk $D \subset H$ such that $D \cap a_1 = \emptyset$, the number of points $\#(D \cap a_2) \geq 2$.

Theorem 5.2. *Let $k \subset S^3$ be a non-trivial connected knot, and let $F \subset E(k)$ be a free genus one Seifert surface for k.*

There is another genus one Seifert surface for k which is not equivalent to F if and only if there exists a spine $\Gamma = a_1 \vee a_2$ for F in $\partial\mathcal{N}(F)$ such that a_1 represents an element conjugate to g^n with $n \geq 2$ for some primitive element $g \in \pi_1(E(F))$, and a_2 spoils disks for a_1.

Proof. Let $\Gamma = a_1 \vee a_2$ be a spine for F such that a_1 represents an element conjugate to g^n with $n \geq 2$ for some primitive element $g \in \pi_1(E(F))$, and a_2 spoils disks for a_1

Let $D \subset E(F)$ be an essential properly embedded disk such that $a_1 \cap D = \emptyset$, which is given by Lemma 5.1. We may assume that $H_1 = \overline{E(F) - \mathcal{N}(D)}$ is a solid torus. Let A_1 be a regular neighbourhood of a_1 in $\partial E(F)$; then $A_1 \subset \partial H_1$. Write $B_1 = \overline{\partial H_1 - A_1}$. Since $|n| \geq 2$, the annuli A_1 and B_1 are non-parallel in H_1. We push $Int(B_1)$ into H_1 to obtain B_1', a properly embedded annulus in H_1.

Let $\mathcal{N}(a_2) \subset \partial E(F)$ be a regular neighbourhood of a_2 such that $A_1 \cap \mathcal{N}(a_2)$ is a rectangle; then $B_2 = \overline{\mathcal{N}(a_2) - A_1}$ is a 'band' (that is, a 2–disk) such that $B_2 \cap A_1 = B_2 \cap B_1'$ is a pair of arcs in $\partial B_1'$. Then $G = B_2 \cup B_1'$ is a genus one Seifert surface for k (we push $Int(G)$ slightly into $E(F)$ to get a properly embedded surface in $E(F)$).

Now, $\widehat{G} = G \cap H_1$ is the union of the annulus B_1' with the disk components of $\widehat{B}_2 = B_2 \cap H_1$. Notice that $\partial \widehat{B}_2 \subset B_1 \subset \partial H_1$.

By hypothesis $\#(a_2 \cap D) \geq 2$; thus, $\widehat{G}$ is disconnected, and the components of $\widehat{G}$ are $B_1' \cup$ (two 2–disks of $\widehat{B}_2$), and at least one sub-disk $z \subset \widehat{B}_2$ with $\partial z \subset Int(B_1)$.

Since $|n| \geq 2$, we cannot push B_1' onto A_1 in H_1. Then a ∂–parallelism for $\widehat{G}$ in H_1 contains a ∂–parallelism W for B_1' onto B_1, but then W contains the 2–disk $z \subset \widehat{G}$. Therefore, $\widehat{G}$ is not parallel into ∂H_1. We conclude that G is not boundary parallel in $E(F)$ for, a ∂–parallelism for G induces a ∂–parallelism for $\widehat{G}$. It follows that G and F are not equivalent. This finishes sufficiency.

Now, if there is another genus one Seifert surface for k which is not equivalent to F, we can find still another non-equivalent genus one Seifert surface $G \subset E(k)$ for k such that G and F have disjoint interiors; see [13]. We write $k = G \cap \partial E(F)$.

The surface G splits $E(F)$ into two handlebodies, $H_0 \cup H_1 = \overline{E(F) - \mathcal{N}(G)}$, of genus two for H_0 and H_1 are irreducible and, since G is π_1–injective into H_0 and H_1, it follows that H_0 and H_1 are π_1–injective into $E(F)$; therefore, H_0 and H_1 have free fundamental groups. We assume $\partial H_i = G \cup (F \times \{i\})$ plus a neighbourhood of k, $i = 0, 1$. By considering a system of disks for the handlebody $E(F)$, we see that there is a disk $D \subset E(F)$ that ∂–compresses G in $E(F)$, and D is contained in, say H_0, and is properly embedded in H_0.

Then k is a $((1,0),(n,m))$–curve in ∂H_0 (Lemma 4.3 of [16]) with $|k \cap D| = 2$. See Figure 31.

Cutting H_0 along D we obtain a solid torus $V \subset H_0$ such that $\widehat{G} = \overline{G \cap V}$ is an (n,m)–torus annulus in ∂V; and the complementary annulus $\widehat{F} = \overline{\partial V - \widehat{G}}$ contains, and is isotopic, to $(F \times \{0\}) \cap V$ in ∂V with an isotopy fixed outside a regular neighbourhood of D.

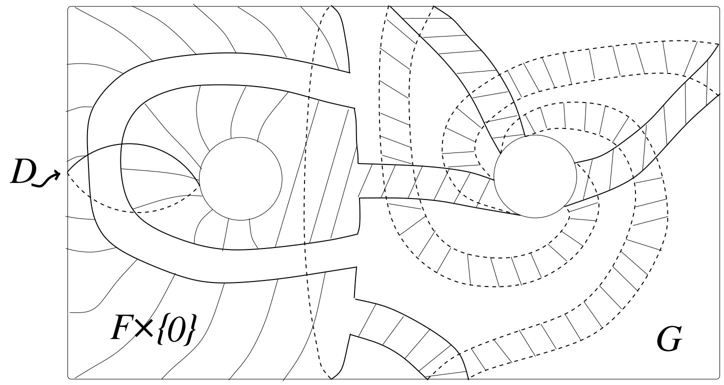


FIGURE 31. Surfaces G and $F \times \{0\}$ in H_0.

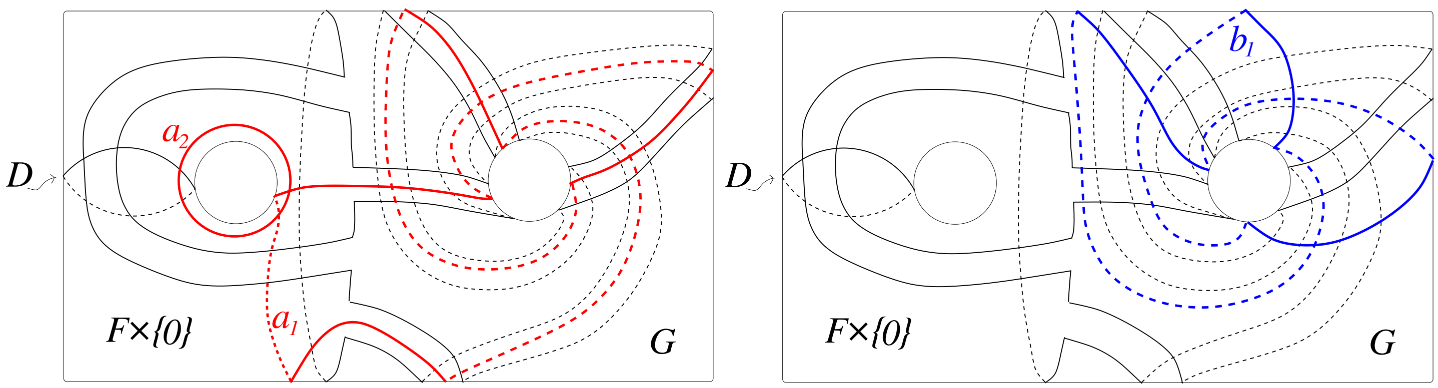


FIGURE 32. $\Gamma = a_1 \vee a_2$ and b_1.

Let $a_1 \subset F \times \{0\}$ be the core of the annulus $\widehat{F}$, and let $b_1 \subset \widehat{G}$ be the core of the annulus $\widehat{G}$.

Let $C' \subset \partial H_0$ be a 2–disk that contains the pair of disks $\partial H_0 \cap \mathcal{N}(D)$, and let $C \subset H_0$ be a properly embedded disk with $\partial C = \partial C'$. Now let $Z \subset H_0$ be a meridional disk such that $Z \cap C = \emptyset$. Then $\widetilde{F} = (F \times \{0\}) \cap (\overline{H_0 - \mathcal{N}(Z)})$ contains a (1,0)–annulus A in the solid torus $\overline{H_0 - \mathcal{N}(Z)}$. Let $a_2 \subset Int(F \times \{0\})$ be the core of A, where we can arrange that $a_1 \cap a_2$ is just one point. Then $\Gamma = a_1 \vee a_2$ is a spine for F. See Figure 32.

The curve a_2 spoils disks for a_1 in $E(F)$ for, otherwise, there is an essential disk $D \subset E(F)$ such that $D \cap a_1 = \emptyset$, and the number of points $\#(D \cap a_2) < 2$. If $D \cap a_2 = \emptyset$, since Γ is a spine for F, the surface F is contained in the solid torus $E(D) \subset E(F)$; it follows that F is compressible in $E(D)$, and, thus, F is compressible in $E(F)$. But, since k is non-trivial, and $g(F) = 1$, F is incompressible in $E(k)$. Then $D \cap a_2$ is just one point, and $D \cap \partial F$ is a set of two points. We may assume that D intersects $k = \partial G$ in exactly two points. Since G is incompressible, we may arrange that $D \cap G$ is just one arc. Now, this arc is essential in G for, otherwise, we can slide G along D, and obtain G' homotopic to G in $E(F)$ such that G' is contained in the solid torus $E(D)$; then G' is not π_1–injective, and, since G and G' are homotopic embeddings, thus, G is not π_1–injective; but that makes G compressible. Then $\widehat{G} = G \cap E(D)$ is an annulus, therefore, $\widehat{G}$ is parallel into $\partial E(D)$. Using the disk D we can extend this parallelism to a parallelism of G into $\partial E(F)$, contradicting that G is essential in $(E(F), k)$.

Now, $a_1 \subset F \times \{0\}$ represents, up to conjugacy, the same element as $b_1 \subset G$ in $\pi_1(H_0)$ for, they are disjoint curves on a torus, and therefore, parallel.

Observe that, since G is not parallel to $F \times \{0\}$, we have $|n| \geq 2$. In particular $\widehat{G}$ and $\widehat{F}$ are not parallel in V.

We now explore H_1.

Recall D is a ∂–compression disk for G in $E(F)$; in particular $D \cap \partial E(F)$ is an arc. It follows that, to recover $E(F)$ from $\overline{E(F) - \mathcal{N}(D)}$, we attach to $\overline{E(F) - \mathcal{N}(D)}$ the 3–ball $\mathcal{N}(D)$ along a disk. Then $\overline{E(F) - \mathcal{N}(D)}$ is a genus 2 handlebody. In fact $E(F)$ is a regular neighbourhood of $\overline{E(F) - \mathcal{N}(D)}$. In particular, the inclusion induces an isomorphism $\pi_1(\overline{E(F) - \mathcal{N}(D)}) \to \pi_1(E(F))$.

Since $\overline{E(F) - \mathcal{N}(D)} = H_1 \cup_{\widehat{G}} V$, then $H_1 \cup_{\widehat{G}} V$ is a genus two handlebody. Therefore, the core b_1 of $\widehat{G}$ represents a primitive element $\beta_1 \in \pi_1(H_1)$ for, if $\pi_1(V) = \langle v; - \rangle$, then b_1 represents v^n, which is not primitive in V. The element β_1 is part of a basis, say, $\pi_1(H_1) = \langle w, \beta_1 : - \rangle$. By Seifert-van Kampen, $\pi_1(E(F)) \cong \pi_1(H_1 \cup_{\widehat{G}} V) = \langle w, \beta_1, v : \beta_1 = v^n \rangle \cong \langle w, v : - \rangle$. That is, v is primitive in $\pi_1(E(F))$, and b_1 represents v^n.

□

6. Free genus one knots are almost fibered

In this section we show that all free genus one knots are almost fibered. We start with an outline of the plan of the proof:

Start with a non-fibered free genus one knot k with a genus one free Seifert surface $F \subset E(k)$. If k has a unique Seifert surface, then k is almost fibered (Remark 2.3). If k has another non-isotopic Seifert surface, as in Remark 3.7, k has a genus one Seifert surface not isotopic to F. By Theorem 5.2, there is a spine $\Gamma = a_1 \vee a_2$ for F in $\partial \mathcal{N}(F)$ such that a_1 represents an element conjugate to g^p with $p \geq 2$ for some primitive element $g \in \pi_1(E(F))$, and a_2 spoils disks for a_1. By Lemma 5.1, we can find an essential disk $\Delta \subset E(F)$ with $\Delta \cap a_1 = \emptyset$, and the exterior $E(\Delta) = \overline{E(F) - \mathcal{N}(\Delta)}$ is the disjoint union of two solid tori, V_0, V_1 with, say, $a_1 \subset \partial V_0$. We regard $\Delta \subset \partial V_0$. Then $\Gamma \cap V_0$ consists of the curve a_1, which is a (p, q)–curve in V_0, and an arc with endpoints on $\partial \Delta$ intersecting a_1 in exactly one point, and a set of parallel arcs with endpoints on $\partial \Delta$ which are disjoint with a_1. See Figure 36.

In Section 6.1 we show how to find a properly embedded arc in V_0 disjoint with Γ which, in Section 6.2, is shown to be the core of the 1–handle of a one-handled circular decomposition for $E(k)$ based on F. In this analysis, the disk Δ is regarded as 'unreachable', and should be thought as very near the point at infinity. That is, all homeomorphisms in this subsection will fix point-wise the disk Δ.

6.1. **Handles for torus manifolds.** Let p and q be a pair of coprime integers. Consider the points $\{s_\ell\}_{\ell=1}^p \subset S^1$ with $s_\ell = e^{2\pi i\ell/p}$; also let $\widetilde{V}$ be the cylinder $D^2 \times I$, and write $s_\ell^I = s_\ell \times I \subset \widetilde{V}$. The rotation ρ_q of angle $2\pi q/p$ on D^2 gives a quotient $P : (\widetilde{V}, \cup_{\ell=1}^p s_\ell^I) \to (V, \alpha)$, where V is the solid torus obtained from $\widetilde{V}$ by identifying $(z, 0)$ with $(\rho_q(z), 1)$ for each $z \in D^2$, and α is the simple closed curve on ∂V obtained as the image of the union $\cup_{\ell=1}^p s_\ell^I$ in this quotient. The rotation ρ_q acts on $\{s_\ell\}_{\ell=1}^p$ as the cyclic permutation of order p such that $\rho_q(s_i) = s_{i+q}$ where subindices are taken $\mod p$. We consider also a fixed point $\infty \in \alpha$, the 'point at infinity'. The homeomorphism type of the pair (V, α) is called *the* (p, q)*–torus sutured manifold*, or simply *the* (p, q)*–manifold*. Throughout this section we assume $0 < q < p$. Notice that the (p, q)–torus sutured manifold (V, α) is not a sutured manifold, but α is a spine of a small regular neighbourhood $\mathcal{N}(\alpha) \subset \partial V$, and the pair $(V, \mathcal{N}(\alpha))$ is a true sutured manifold with suture α.

In the following, we perform several operations on the (p, q)–manifold (drilling of arcs, homeomorphisms, etc.), and it will be done in such a way that the point at infinity of the manifold will remain fixed.

Let $x \subset V$ be the meridional disk $P(D^2 \times \{0\})$. From the pair $(\widetilde{V}, \cup_{\ell=1}^p s_\ell^I)$ we give a Whitehead diagram for the (p, q)–manifold (V, α) associated to x as follows:

We regard $\tilde{V} = D^2 \times I$ as the exterior $E(x) \subset V$, and write x and $\bar{x}$ for $D^2 \times \{0\}$ and $D^2 \times \{1\}$, respectively. The arcs $s_1^I, \ldots, s_p^I$ are the edges of G, the corresponding Whitehead graph with fat vertices x and $\bar{x}$. To obtain a Whitehead diagram, we have to number the endpoints of $s_1^I, \ldots, s_p^I$. In a plane projection of the graph G, we assume that the unbounded face of G contains the edges s_q^I and s_{q+1}^I. See Figure 33. The point at infinity is either the middle point of s_q^I, or the middle point of s_{q+1}^I. If $\infty \in s_q^I$, then we rename $v_j = (s_j, 0)$ and $\bar{v}_j = (\rho_q(s_j), 1) = (s_{j+q}, 1)$; if $\infty \in s_{q+1}^I$, we rename $v_j = (s_{j+q}, 0)$ and $\bar{v}_j = (\rho_q(s_{j+q}), 1) = (s_{j+2q}, 1)$ where subindices are taken $\mod p$. In any case, we number the point v_i with the number i, and the point $\bar{v}_j$ with the number j $(i, j = 1, \ldots, p)$. Also, we write α_i for the edge of G such that $v_i \in \alpha_i$. This diagram and the corresponding Whitehead graph are called *the* (p, q)*–diagram* and *the* (p, q)*–graph*, respectively. Notice that the edge α_1 connecting x with $\bar{x}$ starting at the point numbered $1 \in x$ ends at the point numbered $p - q + 1 \in \bar{x}$.

Remark 6.1. Consider a Whitehead diagram of a pair (V, α) associated to x where V is a solid torus, α is a simple closed curve on ∂V, and x is a meridional disk of V. If in the fat vertices of the Whitehead diagram of (V, α), the points corresponding to ends of edges are numbered with elements of the set $\{1, \ldots, p\}$ consecutively in the positive (negative) direction on x (on $\bar{x}$), in a compatible way with the gluing homeomorphism to recover the V, then if the edge connecting x with $\bar{x}$ starting at the point numbered $1 \in x$ ends at the point numbered $t \in \bar{x}$, then $t = p - q + 1$; that is, the Whitehead diagram corresponds to the (p, q)–torus sutured manifold with $q = p - t + 1$.

Let (V, α) be the (p, q)–torus sutured manifold, and let G be the Whitehead graph of (V, α) with respect to a meridional disk $x \subset V$. Let γ be a properly embedded arc in V, such that γ is around the vertex x in the Whitehead diagram of (V, α) with respect to x, and γ encircles the edges $\alpha_1, \ldots, \alpha_q$. Also, assume that γ lies 'above' the point $\infty \in \alpha$, that is, γ is between ∞ and x. See Figure 33.

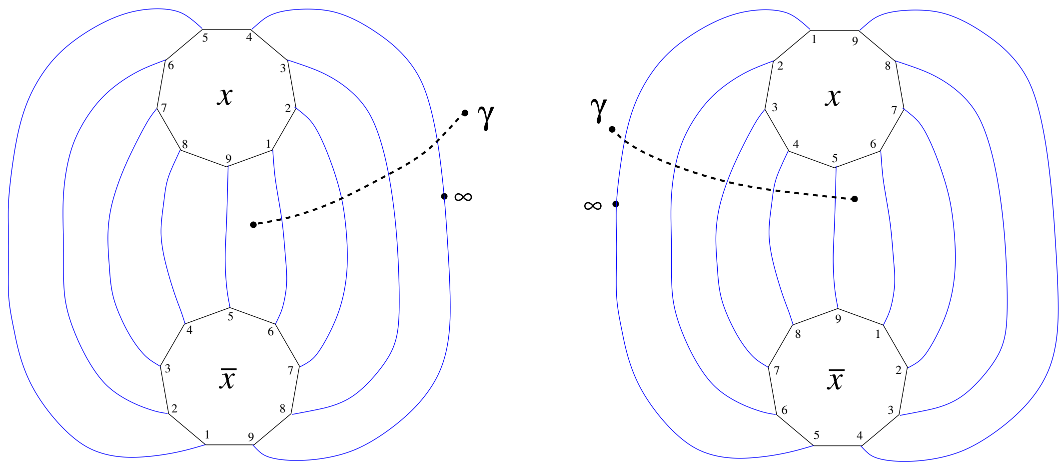


FIGURE 33. Whitehead diagrams for the (9,4)–manifold and the (9,5)–manifold

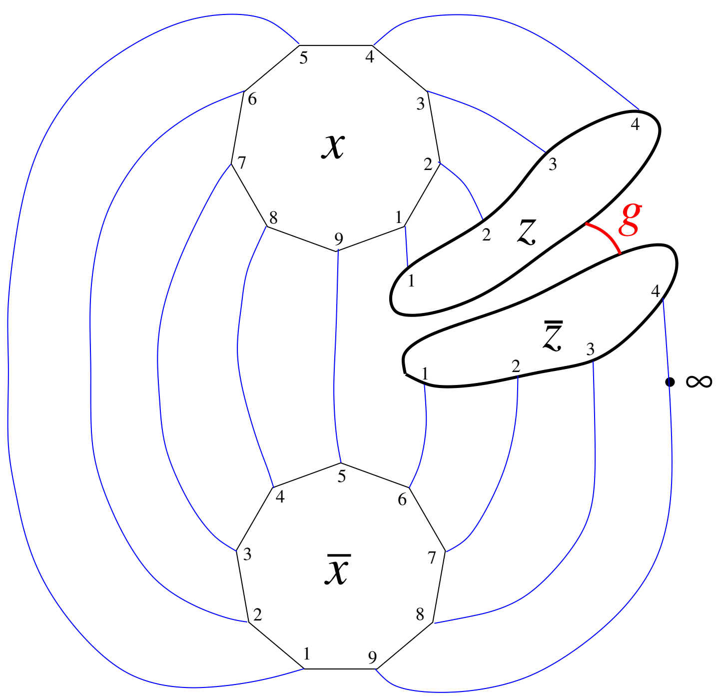


FIGURE 34

The arc γ is called *the canonical 2–handle of length* q for the (p, q)–manifold. Note that the arc γ is the co-core of a 2–handle in V.

If we drill out the canonical 2–handle of length q, we obtain a Whitehead diagram with respect to the system of disks $x, z \subset E(\gamma) \subset V$ where z is the obvious ∂–parallelism disk for γ. See Figure 34. We refer to this Whitehead diagram as the

Whitehead diagram obtained by drilling out the canonical 2–handle of length q of the (p, q)–manifold. Notice that the arc g in Figure 34 is a 'longitude' for the handle defined by z. That is, if we glue back the disks z and $\bar{z}$ and kill the longitude g with a 2–handle, we recover the Whitehead diagram of the (p, q)–manifold. In practice, we just join the ends of the edges in z with the ends of the edges in $\bar{z}$ with parallel arcs on the diagram, and delete the disks z and $\bar{z}$ from the picture, and we get the Whitehead diagram of the (p, q)–manifold back.

Let G be the graph of the Whitehead diagram obtained by drilling out the canonical 2–handle of length q of the (p, q)–manifold. Then G is a graph with four fat vertices $x, \bar{x}, z$, and $\bar{z}$; there are q edges connecting z and x; there are q edges connecting $\bar{z}$ and $\bar{x}$; and there are $p-q$ edges connecting x with $\bar{x}$. Compare with Figure 34. Note that x is a cut vertex of G (and z and $\bar{z}$ are *not* cut vertices); then we can slide the handle corresponding to z along the handle defined by x

After sliding, if the new disk x is still a cut vertex, we can again slide the new disk z along the new disk x, and so on. Let G' be the image of the graph G after κ handle slides of z along x. The graph G' is called *the κ–slid graph obtained from the (p, q)–graph G.*

Lemma 6.2. *Let p, q be a pair of coprime integers, $0 < q < p$, and assume that*

$$p = \kappa_1 q + r_1, \ \text{with } 0 \leq r_1 < q, \ \text{and } \kappa_1 \geq 1$$

Let G be the graph of the Whitehead diagram obtained by drilling out the canonical 2–handle of length q of the (p, q)–manifold, and let G' be the κ_1–slid graph obtained from the (p, q)–graph G. Then G' is the graph of the Whitehead diagram obtained by drilling out the canonical 2–handle of length r_1 of the (q, r_1)–manifold. The point at infinity is a fixed point of these handle slides.

Proof. In the Whitehead graph G, the ends of the edges connecting the disk z with the disk x, are numbered $1, 2, \ldots, q$ in the disk x; these ends are the points $v_1, v_2, \ldots, v_q$ in ∂x. Then, after sliding z along x, the new disk z carries the edges with ends that were numbered $1, 2, \ldots, q$ in $\bar{x}$. Thus, now the ends of the edges connecting z and x, after the slide, have ends which are the image of the rotation ρ_q of angle $2\pi q/p$ of the points $v_1, v_2, \ldots, v_q$; that is, the ends are the points $v_{q+1}, v_{q+2}, \ldots, v_{2q}$ which are numbered $q+1, q+2, \ldots, 2q$ in x.

We see that after sliding $\kappa_1 - 1$ times z along x, the ends of the edges connecting z and x are numbered $(\kappa_1 - 1)q + 1, (\kappa_1 - 1)q + 2, \ldots, \kappa_1 q$ in x. Then after sliding κ_1 times z along x, the points still connected by edges in x are numbered $\kappa_1 q + 1, \kappa_1 q + 2, \ldots, p$. Now, by hypothesis $p = \kappa_1 q + r_1$, then $\kappa_1 q + 1 = p - r_1 + 1$, which means that there are r_1 points left in x. That is, see Figure 35, we have a graph, the image of G after the slides, with fat vertices $x, \bar{x}, z, \bar{z}$; there are r_1 edges connecting x with z; there are r_1 edges connecting $\bar{x}$ with with $\bar{z}$; and there are $q - r_1$ edges connecting z with $\bar{z}$. Now, the edge with one end in z numbered with 1 has the other end numbered with $p - r_1 + 1 \in x$; and the edge with one end in $\bar{x}$ numbered with $p - r_1 + 1$ has the other end in $\bar{z}$ numbered with $q - r_1 + 1$.

Therefore, the new diagram is the Whitehead diagram obtained by drilling out the canonical 2–handle of length r_1 of the (q, r_1)–manifold. Since the disks $\bar{x}$, and $\bar{z}$ were never touched, the point at infinity is a fixed point of the handle slides.

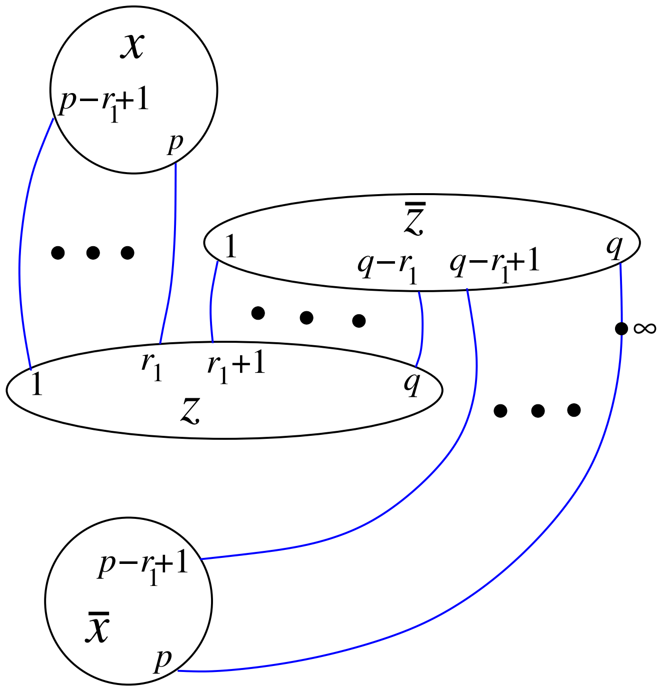


FIGURE 35. After sliding z along x

Notice that if $q = 1$, then $\kappa_1 = p$, and $r_1 = 0$, and everything is easier: The image graph G above, in this case, replacing the values of q and r_1, has four fat vertices $x, \bar{x}, z, \bar{z}$; there are 0 edges connecting x with z; there are 0 edges connecting $\bar{x}$ with with $\bar{z}$; and there is 1 edge connecting z with $\bar{z}$. That is, after canceling the handle defined by x, we obtain the (1,0)–manifold. $\square$

Corollary 6.3. *Let r_1, r_2 be a pair of coprime integers, $0 < r_2 < r_1$. Assume*

$$\begin{array}{lll} r_1 = & \kappa_1 r_2 + r_3, & 0 < r_3 < r_2 \\ r_2 = & \kappa_2 r_3 + r_4, & 0 < r_4 < r_3 \\ & \vdots & \vdots \\ r_{n-1} = & \kappa_{n-1} r_n + 1, & 0 < 1 < r_n \\ r_n = & \kappa_n, & \end{array}$$

with $\kappa_i \geq 1$, $i = 1, \ldots, n$.

Let G be the graph of the Whitehead diagram obtained by drilling out the canonical 2–handle of length r_2 of the (r_1, r_2)–manifold. Let G_1 be the κ_1–slid graph obtained from the (r_1, r_2)–graph G. For $i = 1, \ldots, n-1$, let G_{i+1} be the κ_{i+1}–slid graph obtained from the (r_i, r_{i+1})–graph G_i.

Then G_n is the graph of the Whitehead diagram obtained by drilling out the canonical 2–handle of length 0 of the $(1, 0)$–manifold (V, α).

The point at infinity is a fixed point of these handle slides.

$\square$

Remark 6.4. The graph G_i in the statement of Corollary 6.3 is the graph of the Whitehead diagram obtained by drilling out the canonical 2–handle of length r_{i+2} of the (r_{i+1}, r_{i+2})–manifold. Then G_i is a graph with four fat vertices $\xi, \bar{\xi}, \zeta$, and $\bar{\zeta}$. The symbols ξ and ζ stand for the symbols x and z in some order (that is, the sets $\{\xi, \zeta\}$ and $\{x, z\}$ are equal, but just as unordered sets). There are r_{i+2} edges connecting ζ and ξ; there are r_{i+2} edges connecting $\bar{\zeta}$ and $\bar{\xi}$; and there are $r_{i+1} - r_{i+2}$ edges connecting ξ with $\bar{\xi}$.

Remark 6.5. Let p, q be a pair of coprime integers, and assume that $p/q = [\kappa_1, \dots, \kappa_n]$, as a continued fraction, with $\kappa_i \geq 1$ for each i.

(1) Write $p_i/q_i = [\kappa_1, \dots, \kappa_i]$ with p_i, q_i coprimes. Write $p_0 = 1, p_{-1} = 0$, and $q_0 = 0, q_{-1} = 1$. It is well known that $p_i = \kappa_i p_{i-1} + p_{i-2}$, and $q_i = \kappa_i q_{i-1} + q_{i-2}$; also $p_i q_{i-1} - p_{i-1} q_i = (-1)^i$ for $i \geq 1$. (Article 337 and 338 of [5]). Since $\kappa_i \geq 1$, one easily shows $p_i > q_i > 0$ for $i \geq 1$. In particular, $p > q > 0$. Note also that $p_{i+1} > p_i$.
(2) Let r, s be the two coprime integers p_{n-1}, q_{n-1}, respectively, and let (V, α) be the (p, q)–manifold. Then the (r, s)–torus curve can be drawn on ∂V as a simple closed curve, β, which intersects α exactly at the point at infinity for $ps - qr = \pm 1$. Note that if n is even, then the point at infinity is at the right in the Whitehead diagram, and if n is odd, it is at the left, as in Figure 33. The curve β can be visualized on the Whitehead diagram of the (p, q)–manifold as a set of new edges connecting the fat vertices, and disjoint with the Whitehead graph, and a single new edge intersecting the Whitehead graph at the point at infinity. Conversely, the curve α can be visualized in a similar way on the Whitehead diagram of the (r, s)–manifold.
    Notice that between two edges of α, there is at most one edge of β for, $p > r$.

Theorem 6.6. *Assume $p/q = [\kappa_1, \dots, \kappa_n]$ with p, q coprime, and $\kappa_i \geq 1$ for each i. Let r, s be the pair of coprime integers such that $r/s = [\kappa_1, \dots, \kappa_{n-1}]$. Let (V, α) be the (p, q)–manifold, and let $\beta \subset \partial V$ be the (r, s)–torus curve such that α intersects β exactly at the point at infinity.*

If $\gamma \subset V$ is the canonical 2–handle of length q of the (p, q)–manifold, then the exterior $E(\gamma)$ is a regular neighbourhood of $\alpha \cup \beta$.

Proof. Let G be the graph of the Whitehead diagram obtained by drilling out the canonical 2–handle of length q of the (p, q)–manifold, but including the arcs of the curve β. Call *α–edges* the edges of G corresponding to the (p, q)–torus curve α, and *β–edges* the edges of G corresponding to the (r, s)–torus curve β.

Writing $r_1 = p$, and $r_2 = q$, the statement $p/q = [\kappa_1, \dots, \kappa_n]$ with $\kappa_i \geq 1$ means: there are integers $r_3, \dots, r_n$ such that

$$\begin{array}{lll}
r_1 = & \kappa_1 r_2 + r_3, & 0 < r_3 < r_2 \\
r_2 = & \kappa_2 r_3 + r_4, & 0 < r_4 < r_3 \\
 & \vdots & \vdots \\
r_{n-1} = & \kappa_{n-1} r_n + 1, & 0 < 1 < r_n \\
r_n = & \kappa_n. &
\end{array}$$

See Remark 6.5, (1). Writing $\rho_1 = r$ and $\rho_2 = s$, the statement $r/s = [\kappa_1, \ldots, \kappa_{n-1}]$ means: there are integers $\rho_3, \ldots, \rho_{n-1}$ such that

$$\begin{array}{lll} \rho_1 = & \kappa_1\rho_2 + \rho_3, & 0 < \rho_3 < \rho_2 \\ \rho_2 = & \kappa_2\rho_3 + \rho_4, & 0 < \rho_4 < \rho_3 \\ & \vdots & \vdots \\ \rho_{n-2} = & \kappa_{n-2}\rho_{n-1} + 1, & 0 < 1 < \rho_{n-1} \\ \rho_{n-1} = & \kappa_{n-1}. & \end{array}$$

Notice that the canonical 2–handle of length q for the (p,q)–manifold is the canonical 2–handle of length q for the α–edges of G, but it is also the canonical 2–handle of length s for the β–edges of G. Then the graph G_{n-1} of Corollary 6.3 (Remark 6.4) contains four fat vertices $\xi, \bar{\xi}, \zeta$, and $\bar{\zeta}$. Note $r_{n+1} = 1$; then there is a single α–edge connecting ζ and ξ; there is a single α–edge connecting $\bar{\zeta}$ and $\bar{\xi}$; and there are $r_n - 1$ α-edges connecting ξ with $\bar{\xi}$. Note that $\rho_n = 1$ and $\rho_{n+1} = 0$; then there is a single β–edge connecting ξ with $\bar{\xi}$ intersecting the α–edge connecting $\bar{\zeta}$ and $\bar{\xi}$ at the point at infinity; and there are no more β–edges. The graph G_n is obtained by sliding ζ through ξ the number $\kappa_n = r_n$ of times. Then G_n has a single α–edge connecting ξ with $\bar{\xi}$ and a single β–edge connecting ζ with $\bar{\zeta}$ intersecting at the point at infinity. The corollary follows.

Notice that when $q = 1$, $n = 1$, the graph $G_{n-1} = G$.

□

6.2. **One-handledness of knots.**

Theorem 6.7. *If k is a non-fibered free genus one knot in S^3, then k is almost fibered.*

Proof. Let $k \subset S^3$ be a knot, and let $F \subset E(k)$ be a genus one free Seifert surface for k. Assume k is not almost fibered. Then, by Remark 2.3 and Corollary 3.6, k has another genus one Seifert surface disjoint and not equivalent to F. By Corollary 5.2 there is a spine $\Gamma = a_1 \vee a_2$ for F in $\partial\mathcal{N}(F)$ such that a_1 represents an element conjugate to g^p with $p \geq 2$, for some primitive element $g \in \pi_1(E(F))$, and a_2 spoils the disks of a_1. We shall show that the existence of such graph Γ implies $h(F) = 1$, and, since F is of minimal genus, therefore, $cw(k) = 4$. This contradiction gives the theorem.

By Corollary 5.1, there is an essential 2–disk $\Delta \subset E(F)$ such that $\Delta \cap a_1 = \emptyset$. We may assume that the exterior $E(\Delta) \subset E(F)$ is not connected, and is the union of two solid tori H_0 and H_1 with $a_1 \subset H_0$. There is a copy of Δ in ∂H_0; then $a_1 \subset \partial H_0 - \Delta$. Write $T = \overline{\partial H_0 - \Delta}$; T is a once punctured torus. A properly embedded arc $\alpha \subset T$ is called a rel Δ curve in ∂H_0, and is visualized as the arc α union a properly embedded arc in Δ with the same ends as α. Or rather, we may regard Δ as a point at infinity of the torus $T/\partial\Delta$.

We have that a_1 is a (p,q)–torus curve in H_1 for some q (this implies that we have fixed a longitude-meridian pair in ∂H_0; by changing the longitude-meridian pair, we may assume that $0 < q < p$). The intersection $a_2 \cap H_0 = a_2 \cap \partial H_0$ is a set of disjoint arcs $c \cup b_1 \cup \cdots \cup b_m \subset \partial H_0$ with ends in $\partial\Delta$ and such that $b_i \cap a_1 = \emptyset$ for each i, and the set $c \cap a_1$ is a single point, the base point of Γ.

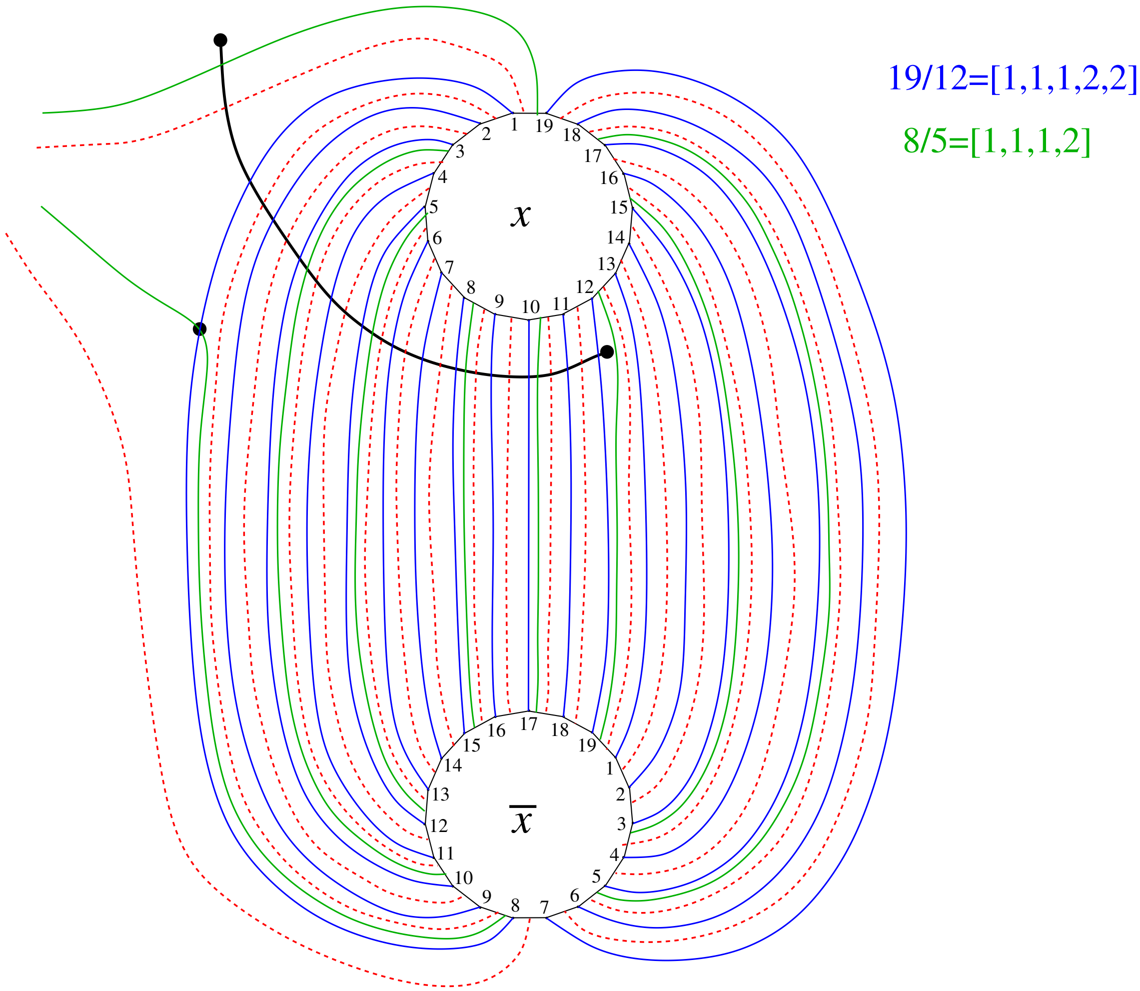


FIGURE 36. The (19,12) and (8,5)–torus curves

Regarding c as a rel Δ curve, c is an (r, s)–torus rel Δ curve in H_0 with $ps-qr = \pm 1$. Since $ps - qr = \pm 1$, any other pair (r', s') such that $ps' - qr' = \pm 1$ is of the form $(r', s') = (r + \ell p, s + \ell q)$ for some integer ℓ. Then by sliding a_2 along $a_1^{\pm 1}$ several times, we obtain a new spine for F. By Remark 2.5, we may assume that the arc c is an (r, s)–torus rel Δ curve in H_0 where, if $p/q = [\kappa_1, \ldots, \kappa_n]$ as a continued fraction with terms $\kappa_i \geq 1$, then $r/s = [\kappa_1, \ldots, \kappa_{n-1}]$.

Since $b_1, \ldots, b_m \subset \partial H_0 - (Int(\Delta) \cup a_1 \cup c) \cong D^2$, then each of $b_1, \ldots, b_m$ are rel Δ curves parallel to a_1.

Now, consider the graph G of the Whitehead diagram of the (p, q)–manifold (H_0, a_1), and include in G the edges induced by the rel ∂ curves $c, b_1, \ldots, b_m$. By deforming the diagram, we may assume that Δ is contained in a small neighbourhood of the point at infinity which is the base point of Γ, the point of intersection of c and a_1. Let γ be the canonical 2–handle of length q for (H_0, a_1). In the Whitehead diagram, we place γ in such a way that it starts by encircling the arc c coming

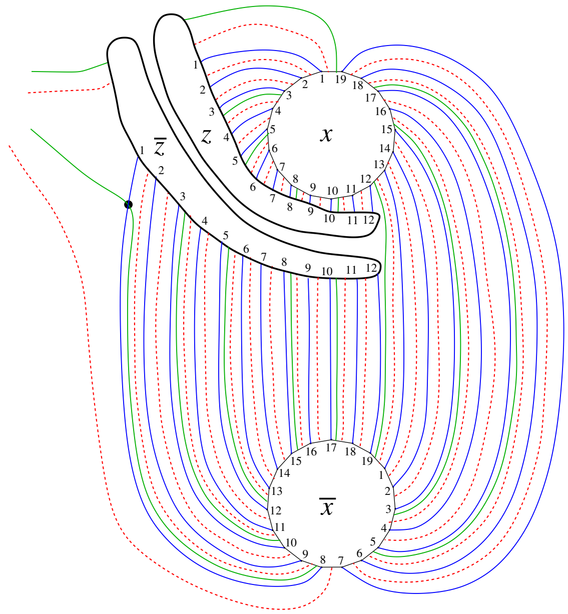


FIGURE 37. Slide z along x

from infinity, and then encircles the q edges belonging to a_1 and whatever is in the middle, and nothing more (that is, after encircling the last edge belonging to a_1, the arc γ does not encircle any arc belonging to c or $b_1, \ldots, b_m$). See Figure 36 where the dotted line is a set of parallel arcs. We drill out γ and, by Theorem 6.6, if we slide handles in the Whitehead diagram obtained by drilling γ out of H_0, we obtain a sequence of diagrams as in Figures 37-42. All handle slides fix point-wise the small neighbourhood of the point at infinity, and, thus, also the disk Δ.

The resulting Whitehead graph on ∂H_0 consists of four fat vertices $\xi, \bar{\xi}, \zeta, \bar{\zeta}$; there is a single a_1–edge connecting ξ and $\bar{\xi}$, and a single c–edge connecting ζ with $\bar{\zeta}$ intersecting in the base point of Γ (In Figure 36, $\xi = z$ and $\zeta = x$). Notice that the c–arc is actually two arcs, one connecting ζ with $\partial\Delta$, and the other connecting $\partial\Delta$ with $\bar{\zeta}$. Without lost of generality, this last arc contains the base-point of Γ.

Let v be a meridional disk for H_1 disjoint with Δ. Then ξ, ζ and v is a system of meridional disks for the handlebody $E(\gamma)$. Write $\pi_1(E(\gamma)) = \langle \xi, \zeta, v : - \rangle$. Then a_1 represents the element ξ, and a_2 represents an element $\bar{\zeta} \cdot W(\xi, v)$ where $W(\xi, v)$ is a word in the letters ξ and v. Since $\{\xi, \bar{\zeta} \cdot W(\xi, v), \zeta\}$ is a basis for $\pi_1(E(\gamma))$, it follows that a_1 and a_2 represent associated primitive elements. Then we can find a system of disks D_1, D_2, D_3 for $E(\gamma)$ such that $a_i \cap D_i$ is exactly one point, and $a_i \cap D_j = \emptyset$ for $i \neq j$, $i = 1, 2$, and $j = 1, 2, 3$. Therefore, $\overline{E(\gamma) - \mathcal{N}(D_3)}$ is a regular neighbourhood of $\Gamma = a_1 \vee a_2$. We conclude that D_3 is the co-core of a 1–handle that, together with γ, gives a one-handled circular decomposition for $E(k)$ as in Remark 2.2 (2). Since k is not fibered, it follows that $h(k) = 1$, and that k is almost fibered. This contradiction finishes the proof of the theorem.

□

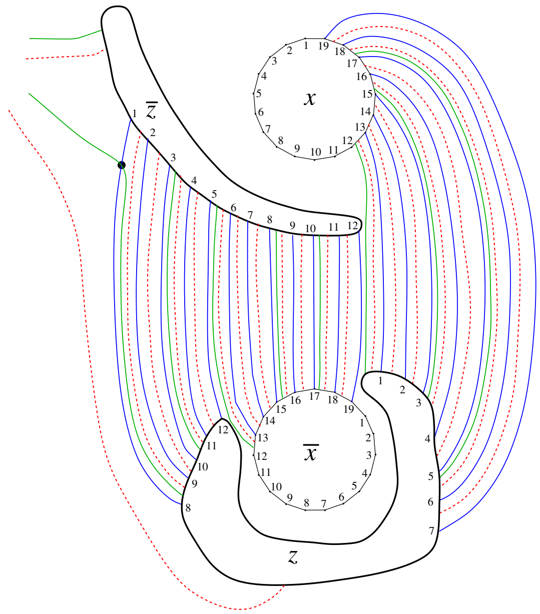


FIGURE 38. Slide x along z

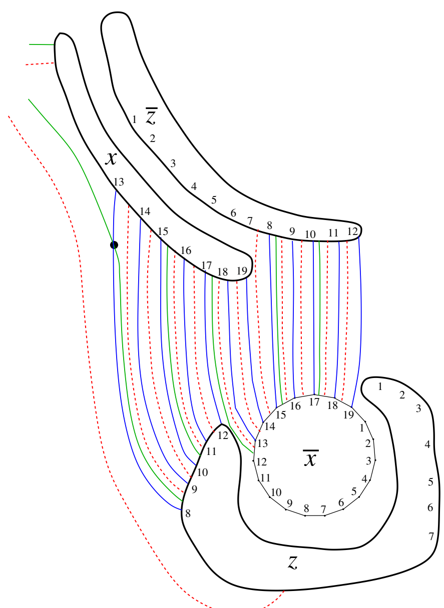


FIGURE 39. Slide $\bar{z}$ along $\bar{x}$

Remark 6.8. By [10], a tunnel number one knot admits a one-handled circular decomposition based on some not specified surface. In [12] genus one knots with tunnel number one were classified, and it turns out that these knots are free genus one knots. Let k be a non fibered genus one knot with tunnel number one. In

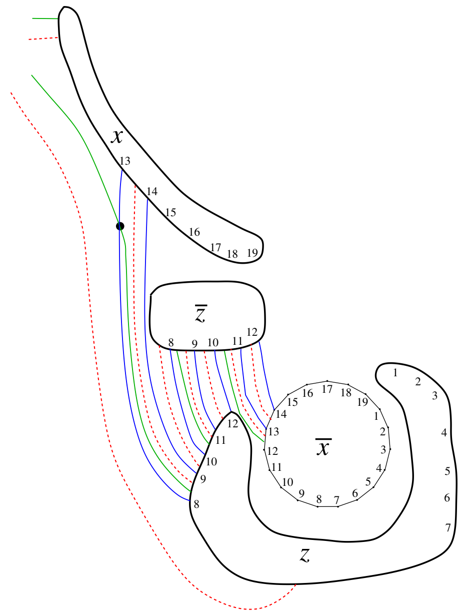


FIGURE 40. Slide twice $\bar{x}$ along $\bar{z}$

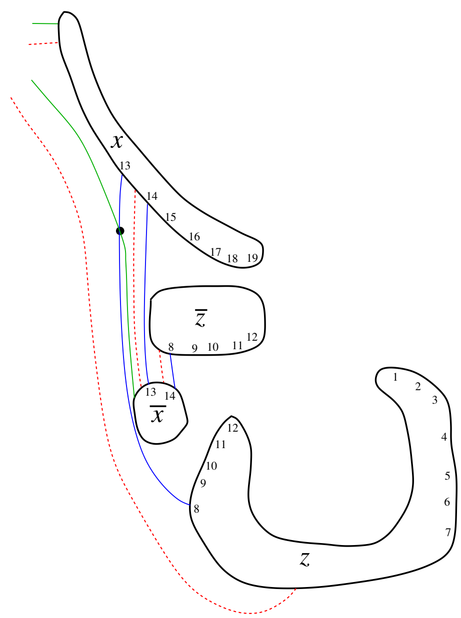


FIGURE 41. Slide twice $\bar{z}$ along $\bar{x}$

Example 3.8, we considered the case that k is simple, and in the proof of Theorem 6.7, we considered the case that k is not simple. It follows that for these knots, their circular width is realized with a one-handled circular decomposition based on a minimal (genus one) free Seifert surface.

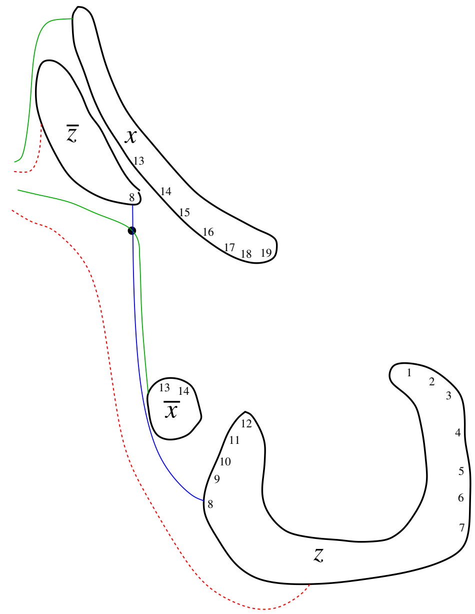


Figure 42. A long slide of x deletes curve

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