A NOTE ON RANKS OF GROUPS AND HEEGAARD GENERA OF COVERING SPACES OF 3-MANIFOLDS

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Abstract. We show that there are closed 3-manifolds M with infinite fundamental group which have a finite covering space $\tilde{M} \to M$ with Heegaard genus $h(\tilde{M}) < h(M)$. These also give rise to examples of groups G with infinite order and a finite index subgroup $H \leq G$ such that $rank(H) < rank(G)$.

1. Introduction

Let G be an infinite group, and let H be a finite index subgroup of G. Is it possible for the rank of H to satisfy $rank(H) < rank(G)$? Thinking in terms of group presentations for G and H, where the rank of some free groups are involved, an affirmative answer for the question above would seem unlikely.

Assume the infinite group G is the fundamental group of a closed 3-manifold M; then the finite index subgroup $H \leq G$ corresponds to a finite-fold covering space $\varphi : \tilde{M} \to M$ with the index of H in G equals the number of sheets of φ. A similar question as the one above (but not quite equivalent; see [1]) would be, is it possible for the Heegaard genus of $\tilde{M}$ to satisfy $h(\tilde{M}) < h(M)$? Thinking of specific examples, or reviewing some 'asymptotic' evidence as in [3] and [2], one again would think that an affirmative answer is unlikely.

We answer both questions in the affirmative (see Corollaries 2.4 and 2.5).

2. Coverings of Seifert manifolds

Let M be the orientable Seifert manifold with orientable orbit surface of genus g and Seifert symbol $(Oo, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$, where $\alpha_1, \beta_1, \ldots, \alpha_t, \beta_t$ are integers with $\alpha_i \geq 1$ and $(\alpha_i, \beta_i) = 1$ for $i = 1, \ldots, t$.

Then the fundamental group $\pi_1(M) = \langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_t, h : q_1^{\alpha_1} h^{\beta_1} = 1, \ldots, q_t^{\alpha_t} h^{\beta_t} = 1, q_1 \cdots q_t = [a_1, b_1] \cdots [a_g, b_g]$, everything commutes with $h\rangle$ where $a_1, b_1, \ldots, a_g, b_g$ represent a basis for the fundamental group of the orbit surface of M (a good and fast introduction to the Theory of Seifert Manifolds is [4]).

For a natural number n we denote by S_n the symmetric group on n symbols, and we write $\varepsilon_n = (1, 2, \ldots, n) \in S_n$ for the standard n-cycle.

Lemma 2.1. *Let M be the Seifert manifold with symbol $(Oo, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ and $g \geq 0$. Let $r_1, \ldots, r_t$ be integers such that $\alpha_i r_i + \beta_i \equiv 0 \mod n$ for $i = 1, \ldots, t$, and assume $\sum r_i = 0$. Then there is an n-fold cyclic covering space*

$$(Oo, g; B_1/\alpha_1, \ldots, B_t/\alpha_t) \to M$$

where the integer $B_i = (\alpha_i r_i + \beta_i)/n$ for $i = 1, \ldots, t$.

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Proof. Let M_0 be the result of drilling out from M the fibered solid tori defined by the ratios $\beta_1/\alpha_1, \ldots, \beta_t/\alpha_t$. Define the representation $w : \pi_1(M_0) \to S_n$ such that $w(h) = \varepsilon_n$, $w(q_i) = \varepsilon_n^{r_i}$, and $w(a_j) = w(b_j) = 1$ $(i = 1, \ldots, t; j = 1, \ldots, g)$. From Lemma 1 of [4] we obtain an n-fold branched covering $\varphi : (Oo, g; B_1/\alpha_1, \ldots, B_t/\alpha_t) \to M$ with numbers B_i as in the statement of this lemma. Since the assignments $w(h) = \varepsilon_n$, $w(q_i) = \varepsilon_n^{r_i}$, and $w(a_j) = w(b_j) = 1$ are compatible with the defining relations of $\pi_1(M)$, then w in fact defines a homomorphism $w : \pi_1(M) \to S_n$; we conclude that φ is a true (unbranched) covering space. □

Corollary 2.2. *For any integers $g \geq 0$, $\alpha \geq 1$ and β such that $(\alpha, \beta) = 1$, and $|\beta| \geq 2$, there is a $|\beta|$-fold covering space $(Oo, g; \pm 1/\alpha) \to (Oo, g; \beta/\alpha)$.*

Proof. If we set $r_1 = 0$, then, using Lemma 2.1, we obtain $B_1 = \beta/|\beta| = \pm 1$, and a $|\beta|$-fold covering space $(Oo, g; \pm 1/\alpha) \to (Oo, g; \beta/\alpha)$. □

Lemma 2.3. *Let M be the Seifert manifold with symbol $(Oo, g; \beta/\alpha)$ and $g \geq 0$. Then the Heegaard genus of M and the rank of the fundamental group of M are*

$$h(M) = rank(\pi_1(M)) = \begin{cases} 2g & \textit{if } \beta = \pm 1 \\ 2g + 1 & \textit{otherwise.} \end{cases}$$

Proof. Following the proof of Theorem 1.1 of [1], one can get easily a Heegaard decomposition for M of genus $2g$ if $\beta = \pm 1$, and a Heegaard decomposition for M of genus $2g + 1$ if $\beta \neq \pm 1$. Therefore

$$h(M) \leq \begin{cases} 2g & \text{if } \beta = \pm 1 \\ 2g + 1 & \text{otherwise.} \end{cases}$$

Recall one always has $rank(\pi_1(M)) \leq h(M)$. Now one computes
$H_1(M) = \langle a_1, b_1, \ldots, a_g, b_g, q, h : q^\alpha h^\beta = 1, [q, h] = 1, [a_j, h] = [b_j, h] = 1,$
$q = [a_1, b_1] \cdots [a_g, b_g] \rangle_{Ab} = \langle a_1, b_1, \ldots, a_g, b_g, q, h : q^\alpha h^\beta = 1, q = 1 \rangle_{Ab} =$
$= Z^{2g} \oplus \langle h : h^\beta = 1 \rangle = \begin{cases} Z^{2g} & \text{if } \beta = \pm 1 \\ Z^{2g} \oplus Z_{|\beta|} & \text{otherwise,} \end{cases}$ where the subindex '*Ab*' indicates the image of the Abelianization homomorphism. In particular

$$h(M) \geq rank(\pi_1(M)) \geq \begin{cases} 2g & \text{if } \beta = \pm 1 \\ 2g + 1 & \text{otherwise.} \end{cases}$$

□

Corollary 2.4. *Let α, β be a pair of relatively prime integers with $\alpha \geq 1$ and $|\beta| \geq 2$, and let M be the Seifert manifold with symbol $(Oo, g; \beta/\alpha)$.*

If $g > 0$ then $\pi_1(M)$ is of infinite order and M has a finite covering space $\tilde{M} \to M$ such that the Heegaard genus $h(\tilde{M}) < h(M)$.

Proof. Follows from Corollary 2.2 and Lemma 2.3 □

Corollary 2.5. *There is an infinite family of groups G such that*

(1) *G has infinite order,*
(2) *G has a subgroup of finite index H, and*
(3) *$rank(H) < rank(G)$.*

Proof. For each integer $g > 0$, and each pair of integers $\alpha \geq 1$ and β such that $(\alpha, \beta) = 1$, and $|\beta| \geq 2$, we set $G_{g,\alpha,\beta} = \pi_1(Oo, g; \beta/\alpha)$, and $H_{g,\alpha,\beta} = \pi_1(Oo, g; \pm 1/\alpha)$. The result follows from Corollary 2.2. □

Remark 2.6. In the case of M an orientable Seifert manifold with non-orientable orbit surface it is possible to prove the following

Proposition 2.7 ([5]). *Let α, β be a pair of relatively prime integers with $\alpha \geq 1$ and $|\beta| \geq 2$; let $g < 0$, and let M be the Seifert manifold with symbol $(Oo, g; \beta/\alpha)$.*

If $g < -1$, then $\pi_1(M)$ is of infinite order and M has a finite covering space $\tilde{M} = (Oo, g; \pm 1/\alpha) \to M$ such that the Heegaard genus $h(\tilde{M}) < h(M)$, and $rank(\pi_1(\tilde{M})) < rank(\pi_1(M))$.

Question 2.8. *Is it possible to find a closed hyperbolic 3-manifold M and a finite covering space $\tilde{M} \to M$ with Heegaard genus $h(\tilde{M}) < h(M)$?*

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