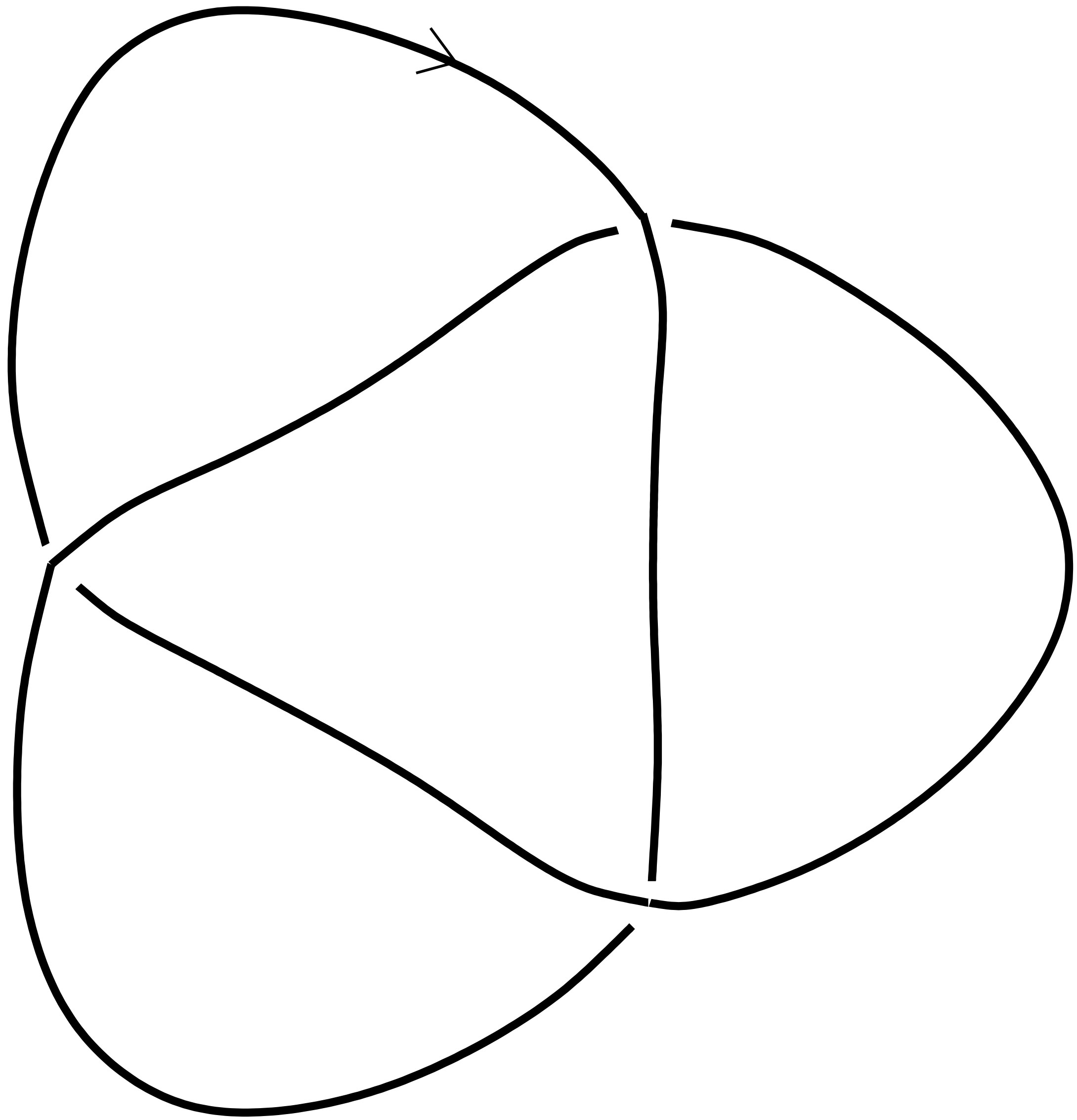
writhe

Tomamos un diagrama D de un nudo k y marcamos una "orientación" de k en el diagrama.

(o sea, una dirección en la que se recorre la curva k)

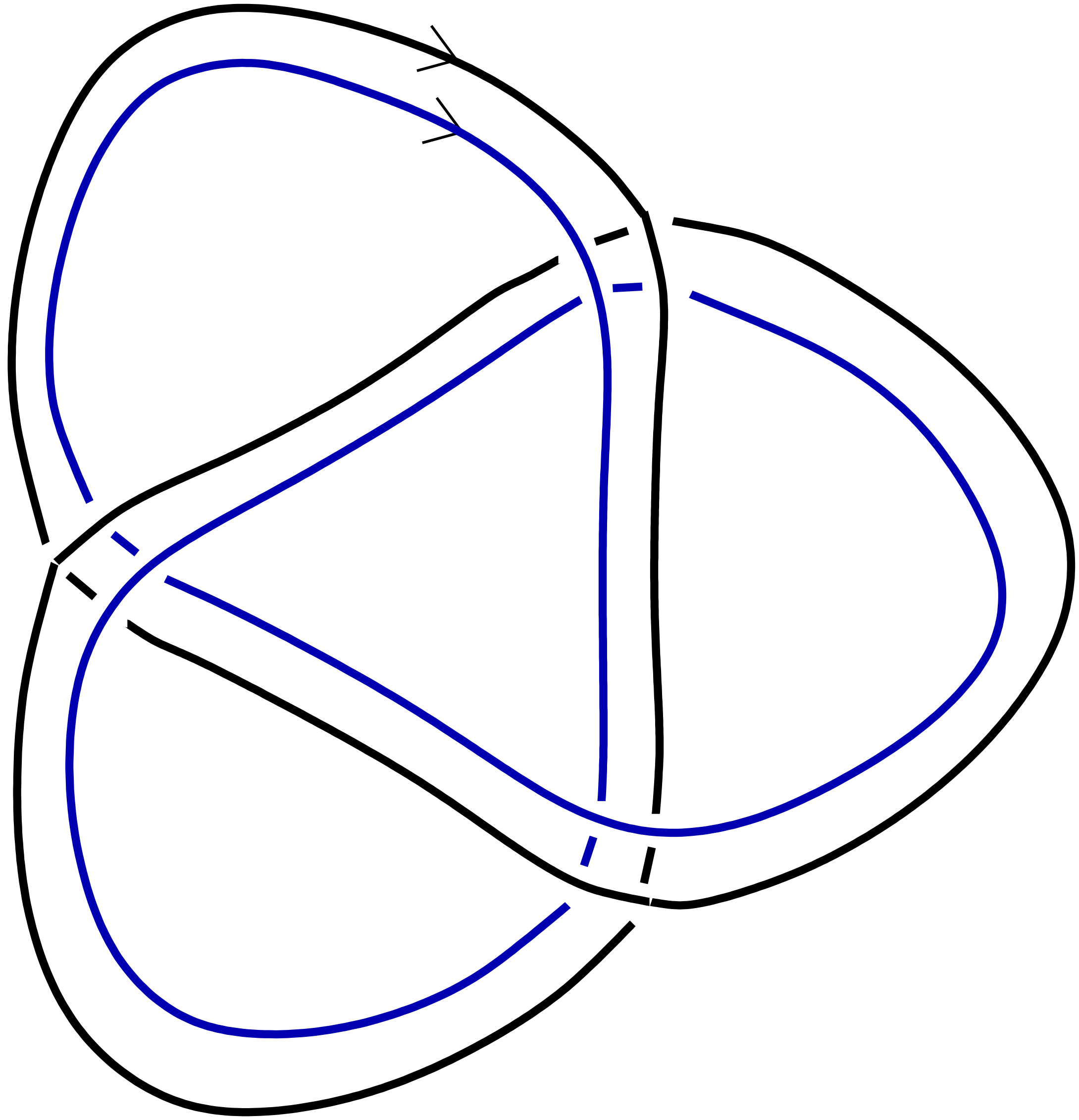
writhe



Dibujamos sobre el diagram D una curva “paralela” al nudo que sea “idéntica”.

Obtenemos así un nuevo diagrama D' que contiene dos curvas paralelas.

writhe

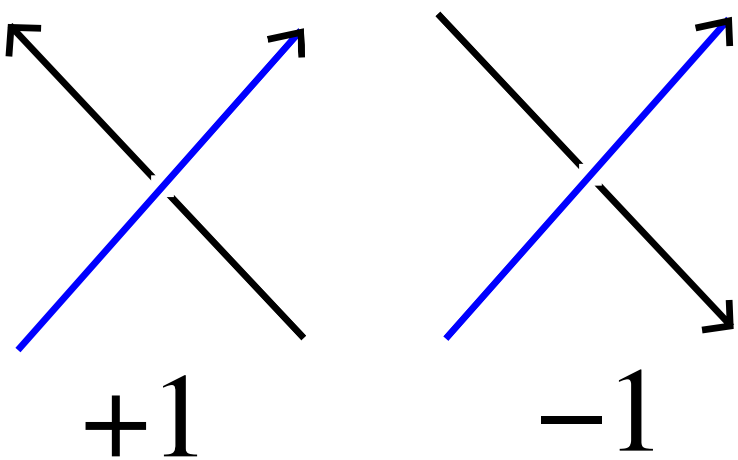


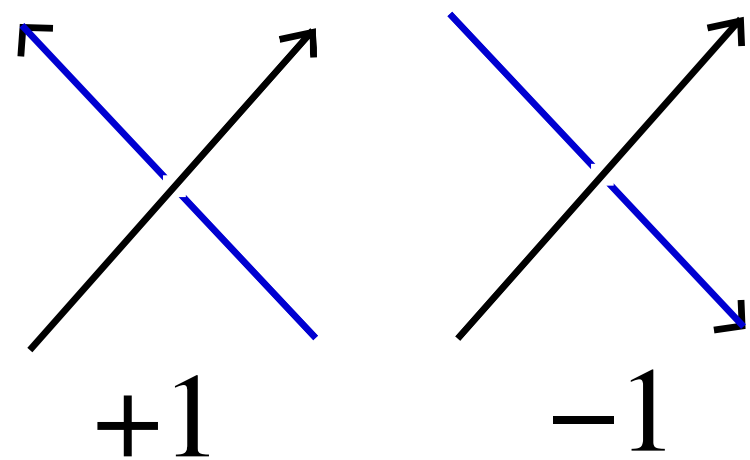
Este nuevo diagrama tiene el doble de cruces que el original.

writhe

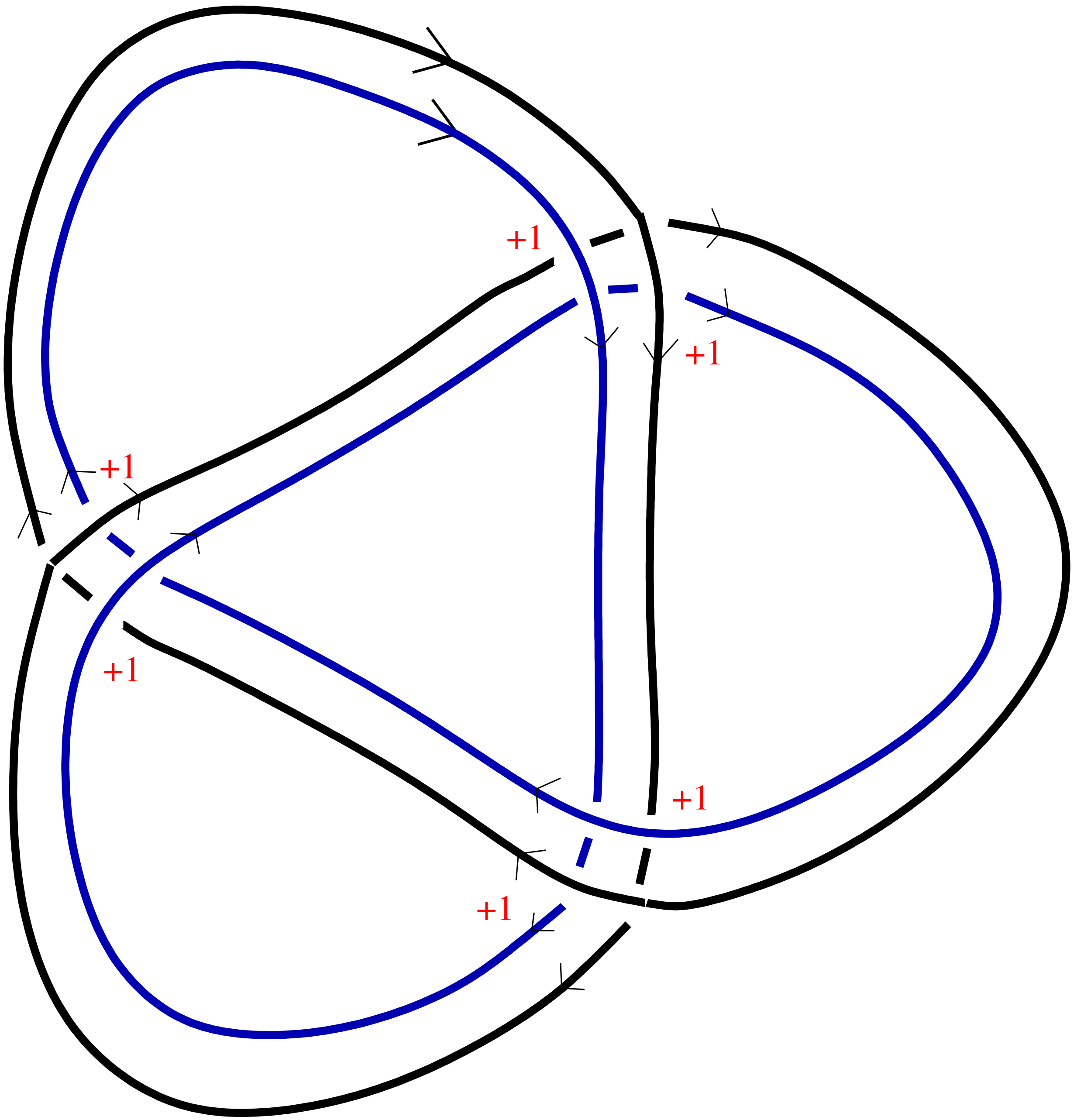
Vamos a considerar sólo los cruces entre el nudo original y su copia.

Por cada cruce “positivo” de una componente con la otra contamos 1; y por cada cruce “negativo” contamos -1:





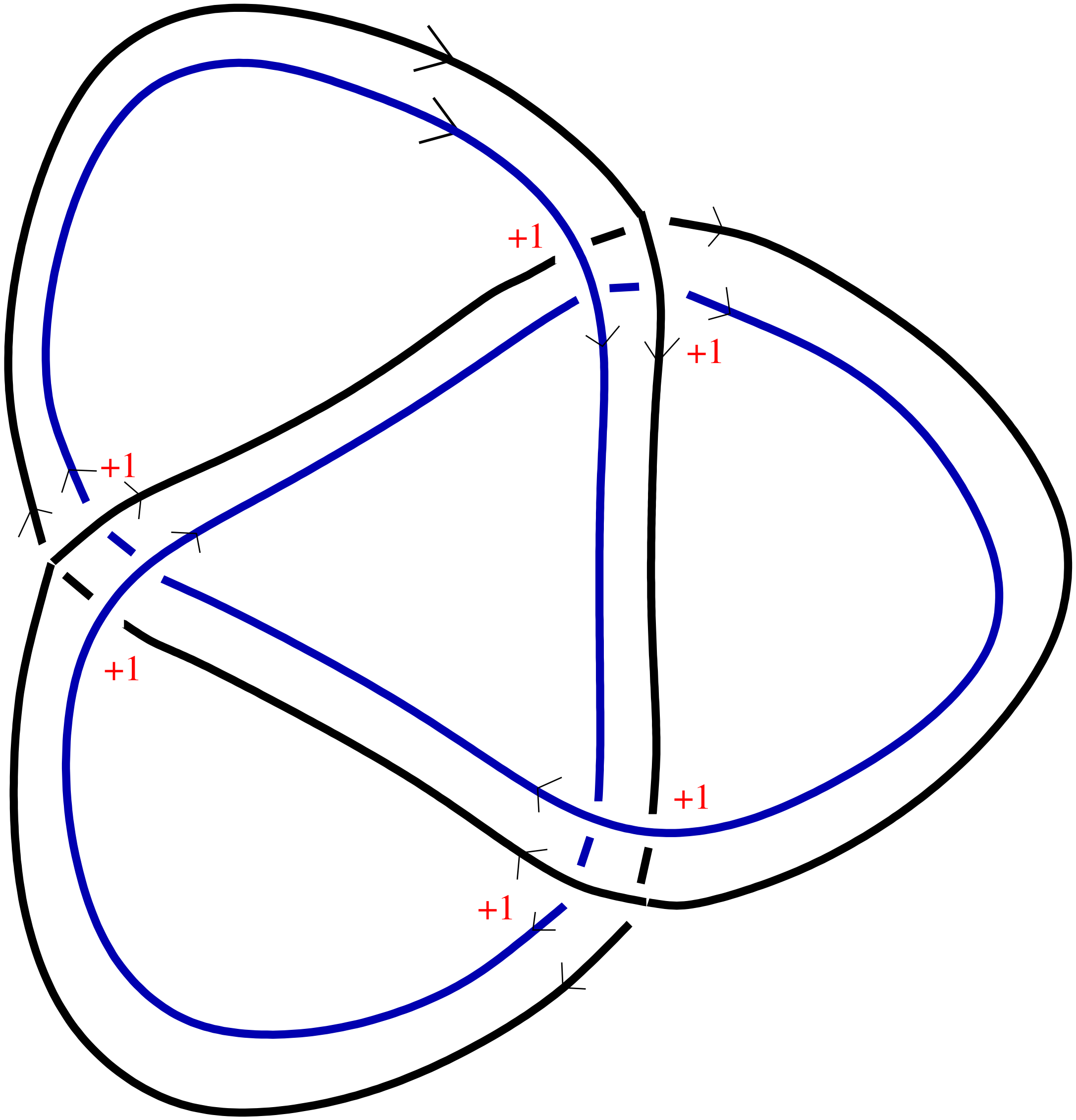
writhe



writhe

A la mitad del resultado de sumar todos los unos y menos unos lo llamamos el número de retorcimiento de D (el writhe de D).

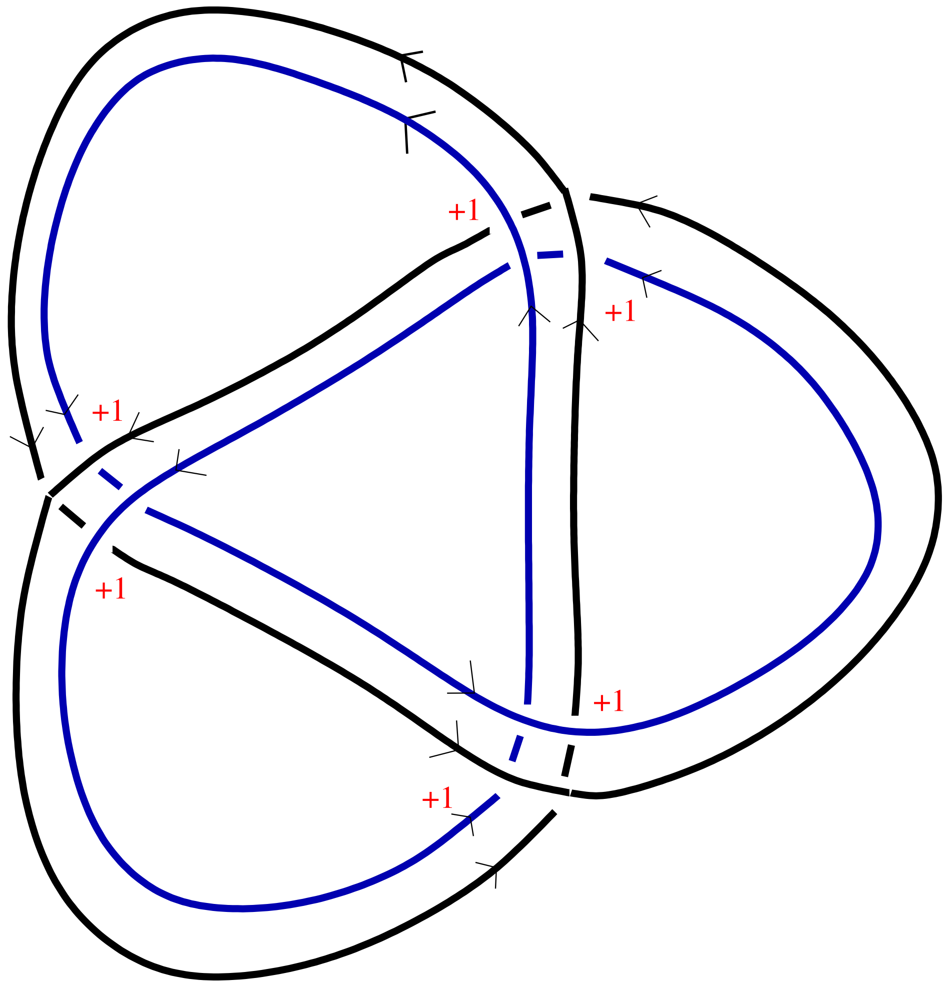
writhe



En este ejemplo tenemos writhe$= 6/2 = 3$.

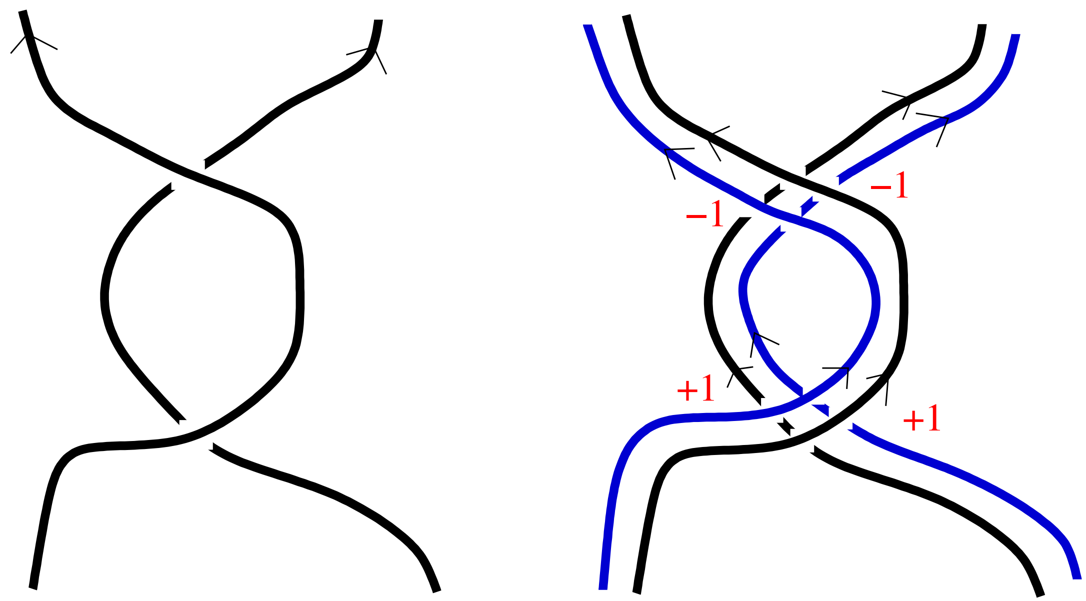
writhe

(Si hubiéramos orientado el nudo al revés, obtendríamos el mismo resultado)



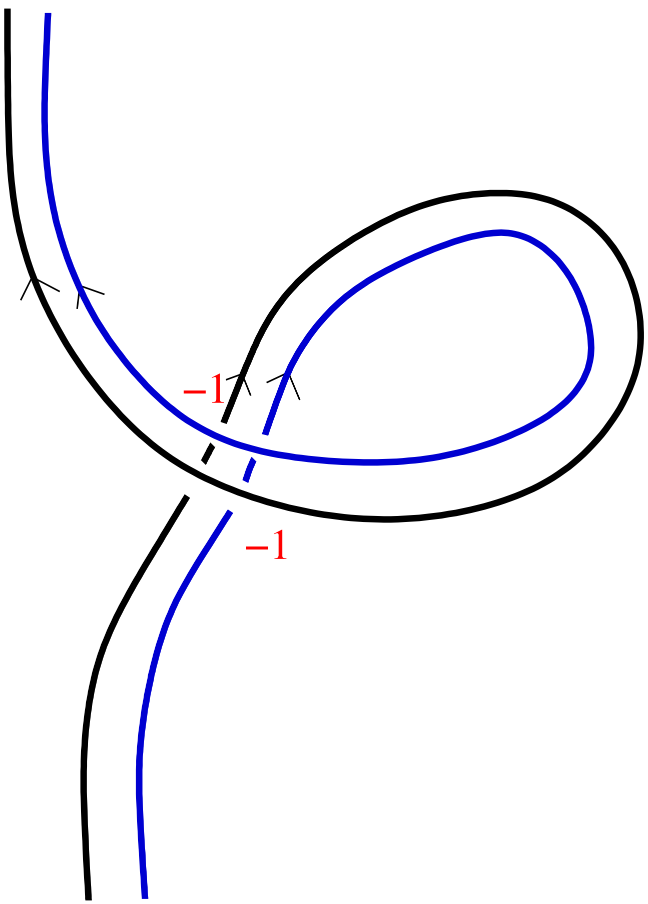
No es difícil convencerse de que el writhe de un diagrama no cambia con las movidas de Reidemeister II y III.

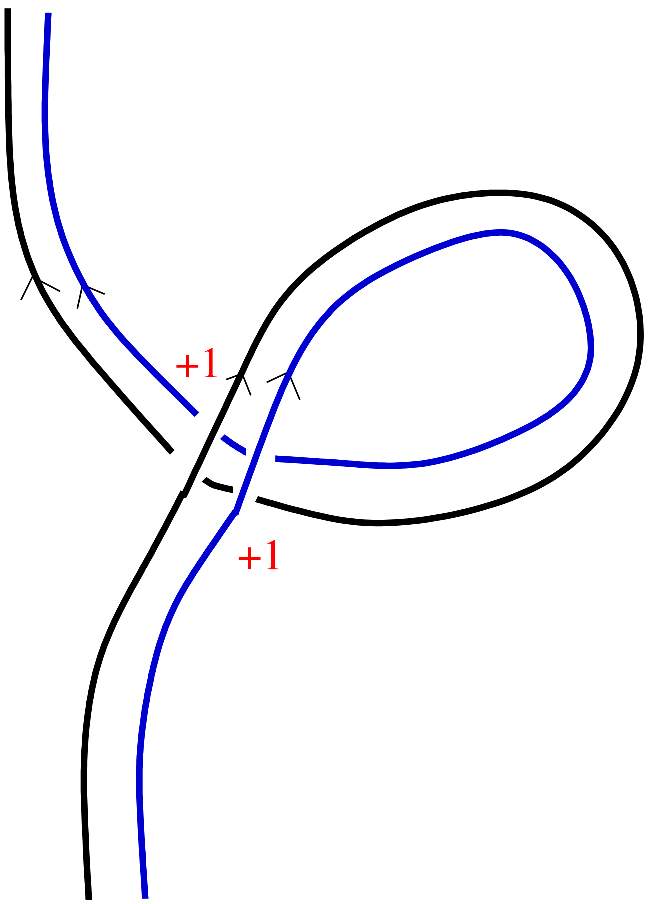
Por ejemplo:



writhe

Pero la movida de Reidemeister I cambia el writhe en ± 1.



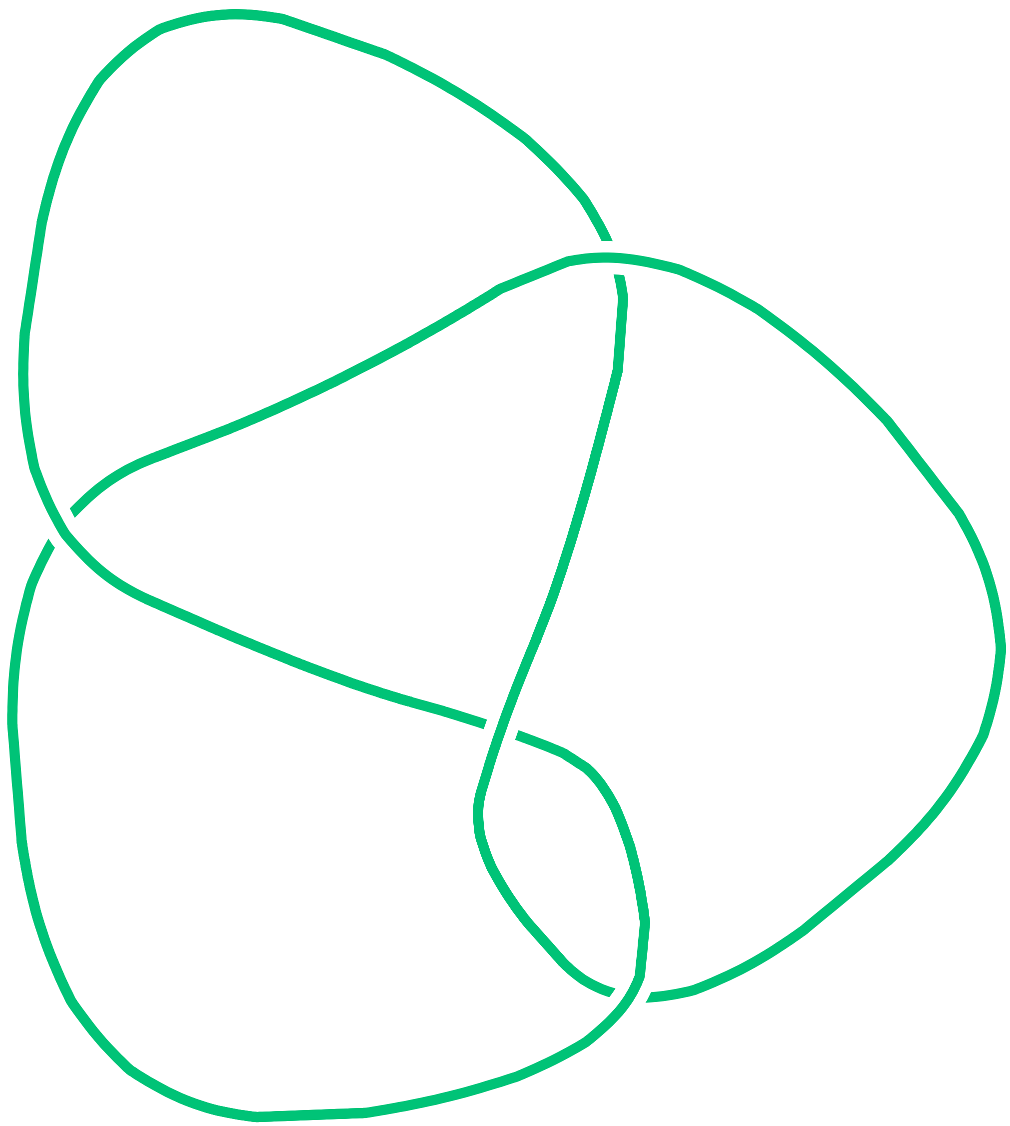


writhe

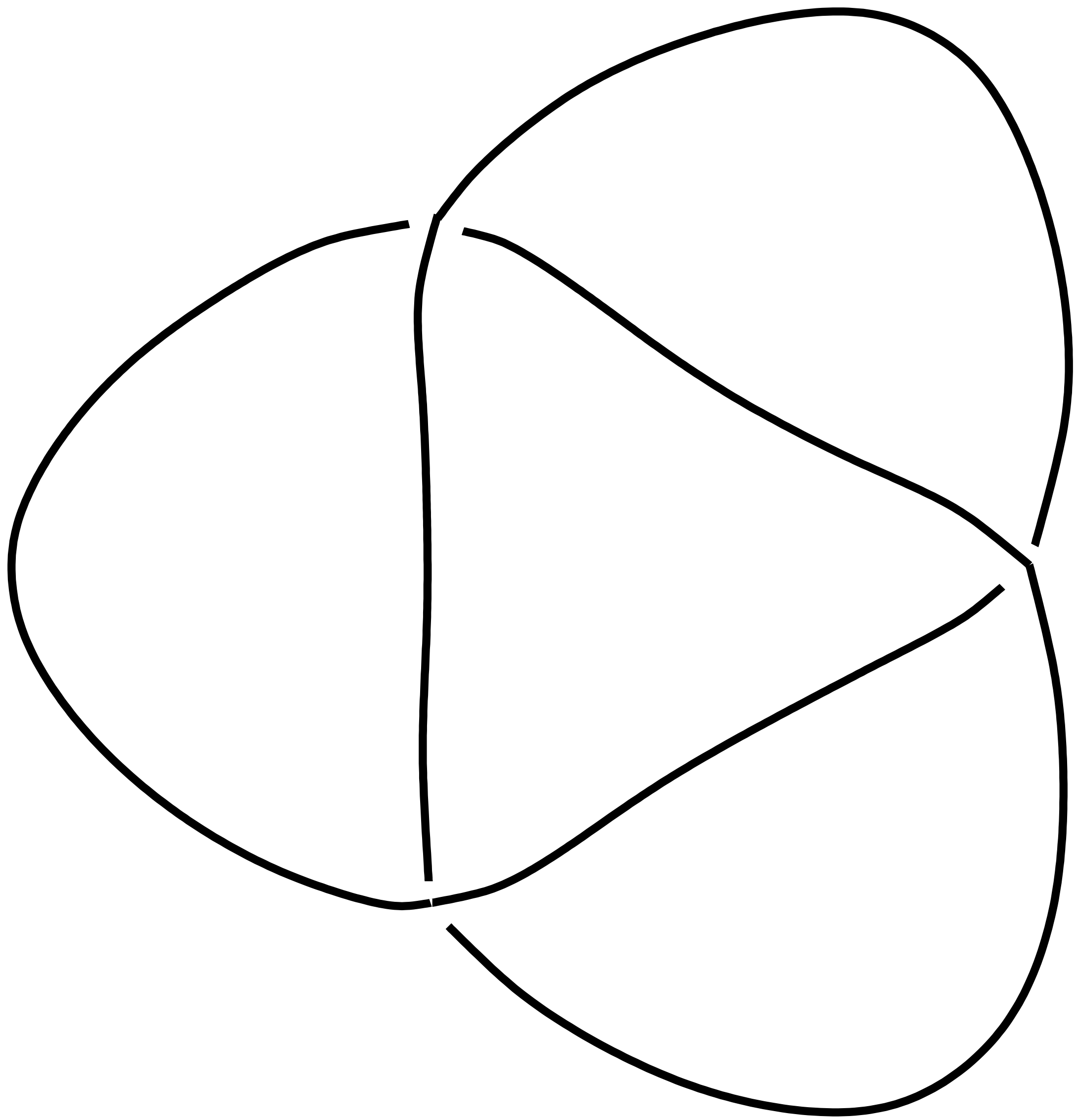
No obtuvimos un invariante.

Pero este writhe va a ser muy útil.

Calcula el writhe del ocho:



Calcula el writhe de la imagen en un espejo del trébol:



Kauffman

Kauffman

A cada diagrama D de un nudo k vamos a asociarle una expresión algebraica (un polinomio).

Estos polinomios van a tener (de momento) tres variables x, y y z.

$$k \mapsto \text{ polinomio}$$

Para un diagrama D de un nudo k, vamos a escribir su polinomio como

$$[D] = [D](x, y, z).$$

Para calcular el polinomio paréntesis, primero declaramos que

1) $[\mathrm{circu}] = 1$

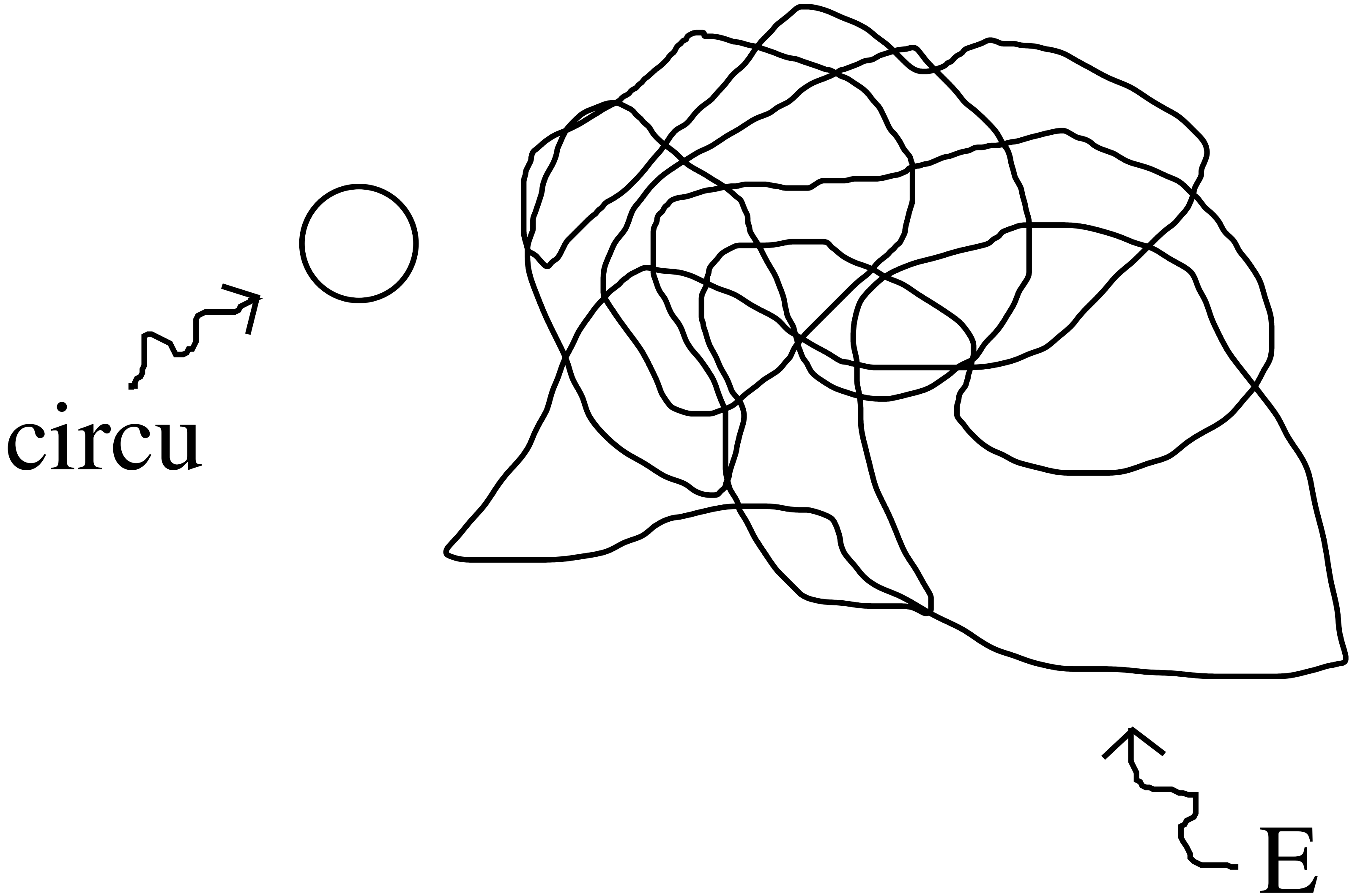
donde circu es el diagrama del no-nudo sin cruces.

(este "1" es el polinomio constante 1)

Si tenemos un diagrama D con una circunferencia (un no-nudo) que está alejada del resto del diagrama, escribimos

$$D = \mathrm{circu} \cup E$$

donde E es lo que queda del diagrama D después de borrar la circunferencia.

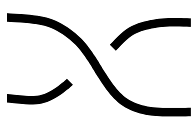
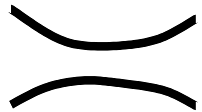
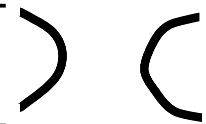


El diagrama $D = \mathrm{circu} \cup E$

Declaramos ahora que

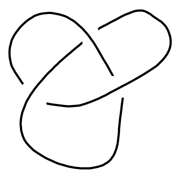
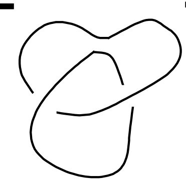
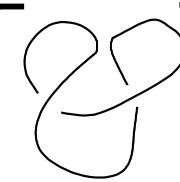
2) $[\mathrm{circu} \cup E] = z \cdot [E]$.

Finalmente declaramos que

3) [] $= x$ [] $+ y$ []

O sea, si en un cruce de un diagrama hacemos los cambios indicados, los polinomios paréntesis de los diagramas involucrados están relacionados como dice (3).

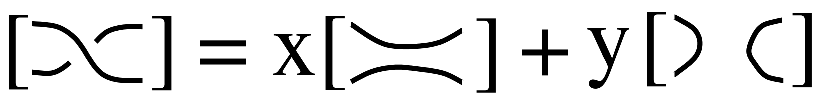
Ejemplo

[] = x[]+y[]

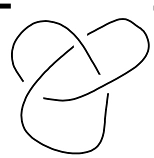
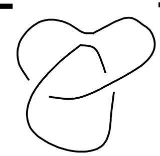
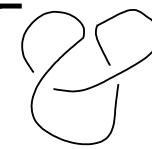
Axiomas de Kauffman

1) $[\text{circu}] = 1$

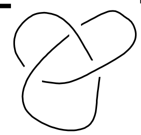
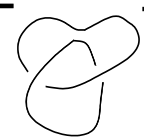
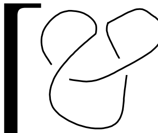
2) $[\text{circu} \cup E] = z \cdot [E]$.

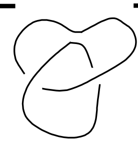
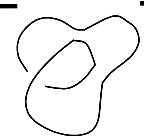
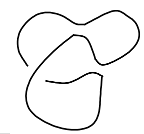
3) 

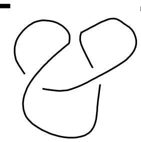
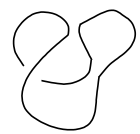
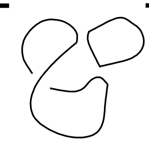
Ejemplo

[] = x [] + y []

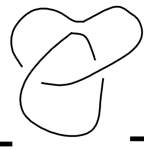
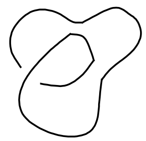
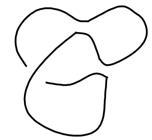
Ejemplo

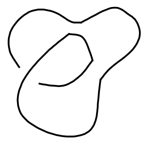
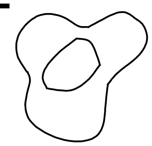
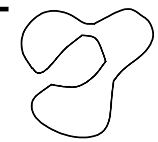
[] = x [] + y []

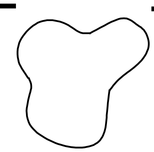
[] = x [] + y []

[] = x [] + y []

Ejemplo

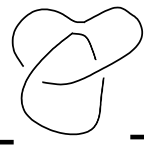
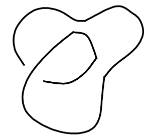
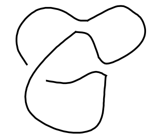
[] = x [] + y []

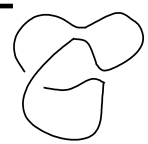
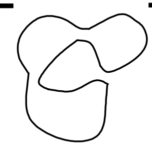
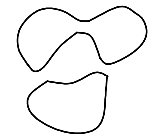
[] = x [] + y []

= xz [] + y

$$= xz + y$$

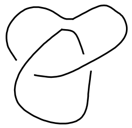
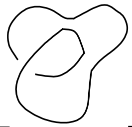
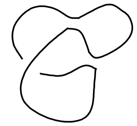
Ejemplo

[] = x [] + y []

[] = x [] + y []

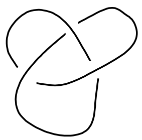
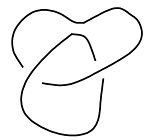
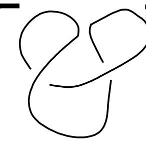
$$= x + yz$$

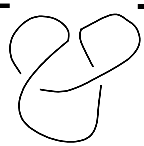
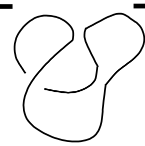
Ejemplo

[] $=x$ [] $+y$ []

$$=x(xz+y)+y(x+yz)$$

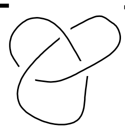
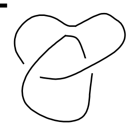
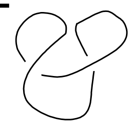
Ejemplo

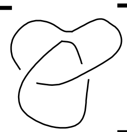
[] = x[]+y[]

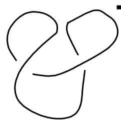
[]=x[]+y[]

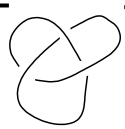
$$=x(x+yz)+yz(x+yz)$$

Ejemplo

[] $= x$ [] $+ y$ []

[] $= x(xz+y)+y(x+yz)$

[] $= x(x+yz)+yz(x+yz)$

[] $= x(x(xz+y)+y(x+yz))$

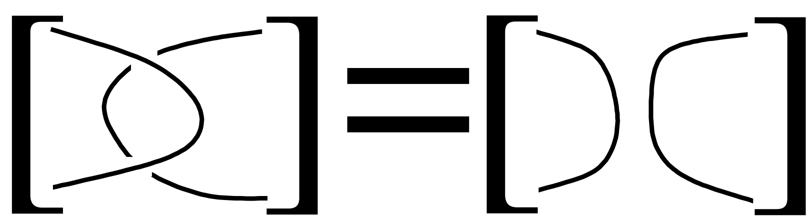
$$+\, y(x(x+yz)+yz(x+yz))$$

$$= x^3z + 3x^2y + 3xy^2z + y^3z^2$$

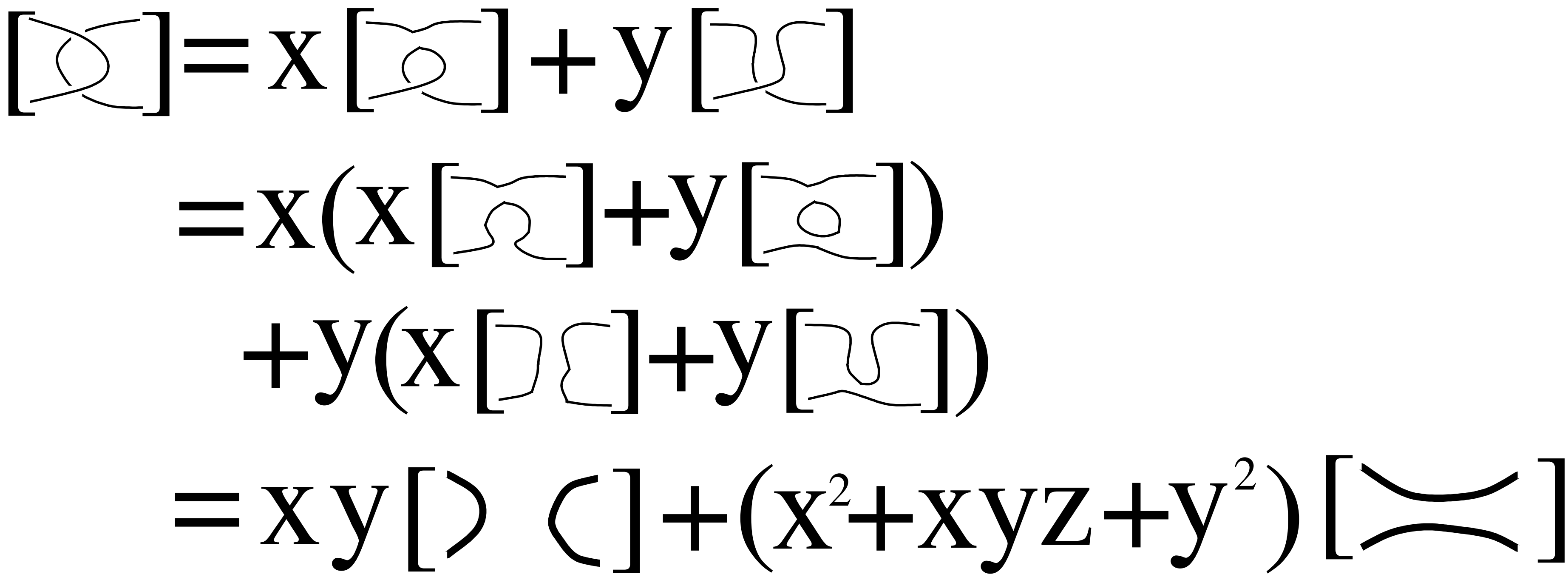
Bueno.

Nos gustaría mucho que este polinomio paréntesis se comportara bien con las movidas de Reidemeister.

Por ejemplo, quisiéramos que



Veamos:



Nos estorban los coeficientes xy y $x^2 + xyz + y^2$.

Pues, los quitamos.

Vamos a declarar que $xy = 1$ y $x^2 + xyz + y^2 = 0$.

Esto lo hacemos para que nos salgan las cuentas.

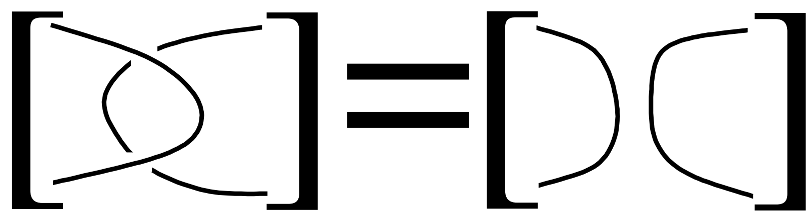
Obtenemos

$$y = \frac{1}{x} = x^{-1}$$

y

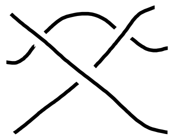
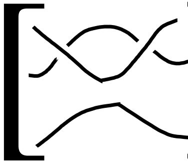
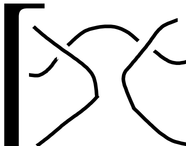
$$z = -x^2 - x^{-2}.$$

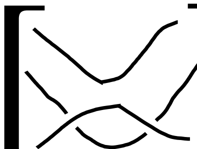
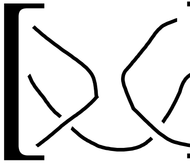
Con estos nuevos valores para y y z, el polinomio paréntesis se vuelve un polinomio en una sola variable x y (lo más importante) ahora se cumple que

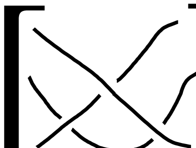


Además

$$(y = x^{-1})$$

[] $= x$ [] $+ y$ []

$= x$ [] $+ y$ []

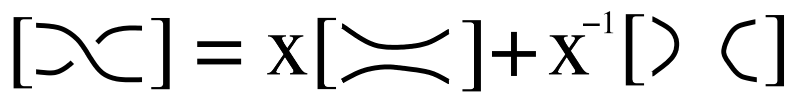
$=$ []

¡La movida III sale gratis!

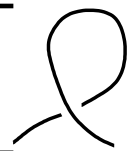
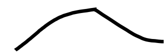
Con el cambio $y = x^{-1}$ y $z = -x^2 - x^{-2}$, los axiomas de Kauffman se ven como

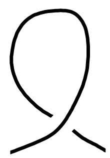
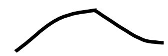
1) $[\text{circu}] = 1$

2) $[\text{circu} \cup E] = (-x^2 - x^{-2})[E]$.

3) 

Pero, si hacemos los cálculos para la movida I:

$\Big[$  $\Big] = -X^{3} \Big[$  $\Big]$

$\Big[$  $\Big] = -X^{-3} \Big[$  $\Big]$

De nuevo no nos queda un invariante...

Pero a Kauffman se le ocurrió hacer lo siguiente:

Tomamos un diagrama D de un nudo k y escribimos el siguiente polinomio

$$f_k(x) = (-x)^{-3(\mathrm{writhe}(D))}[D]$$

Y ¡resulta!

(writhe(D) es el retorcimiento de D)

Queda ver (como ejercicio fácil) que el polinomio f de un diagrama no cambia con las movidas de Reidemeister.

O sea, tenemos

Teorema. Si los nudos k y ℓ son equivalentes, entonces sus polinomios (calculados en cualesquiera diagramas) resultan

$$f_k(x) = f_\ell(x).$$

Si recordamos

$$[\text{trébol}] = x^3z + 3x^2y + 3xy^2z + y^3z^2$$

Hacemos los cambios $y = x^{-1}$ y $z = -x^2 - x^{-2}$

$$\begin{aligned}[\text{trébol}] &= x^3(-x^2 - x^{-2}) + 3x^2(x^{-1}) + 3x(x^{-1})^2(-x^2 - x^{-2}) \\ &\quad + (x^{-1})^3(-x^2 - x^{-2})^2 \\ \\ &= -x^5 - x^{-3} + x^{-7}\end{aligned}$$

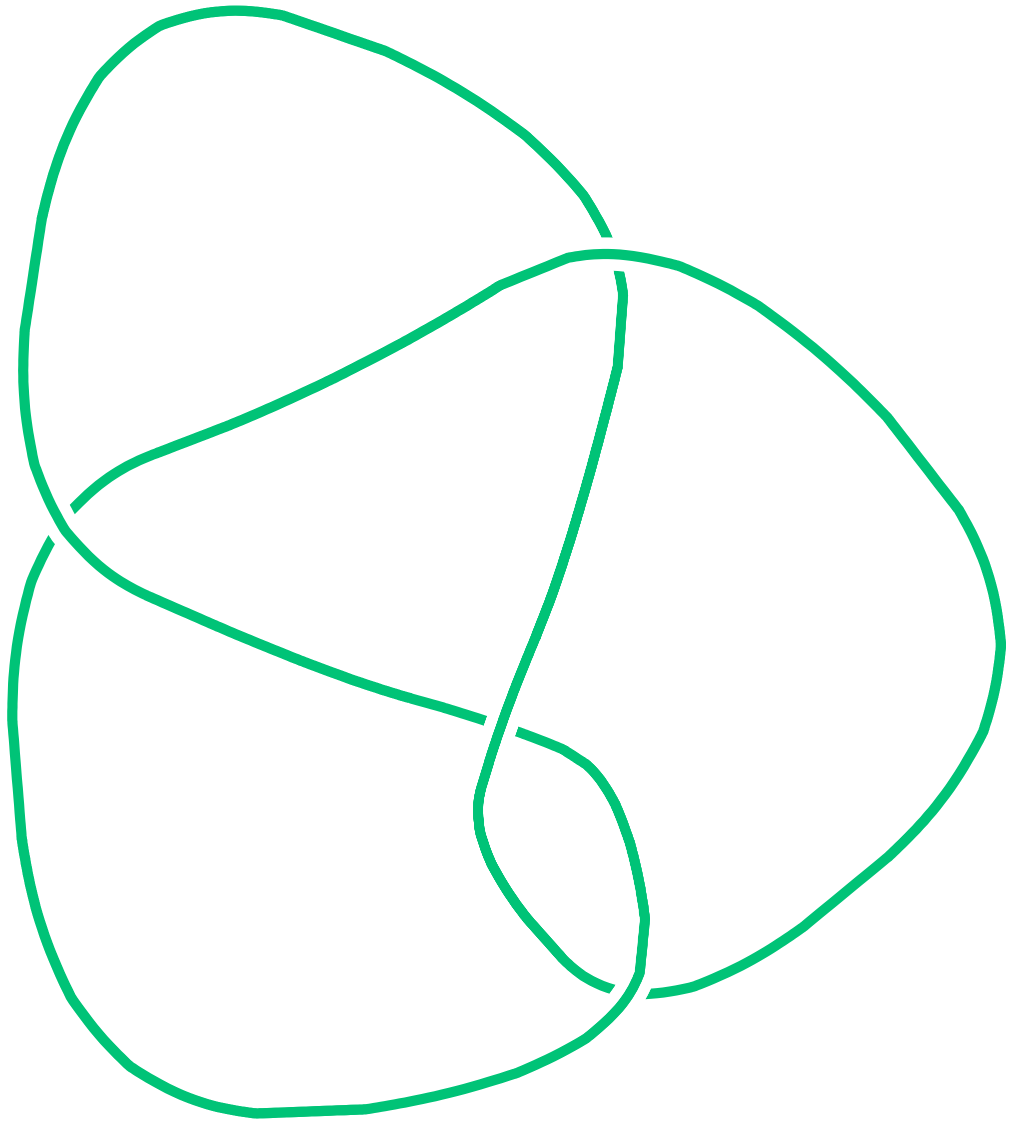
Ya sabemos que el writhe de este dibujo es 3.

Entonces el polinomio

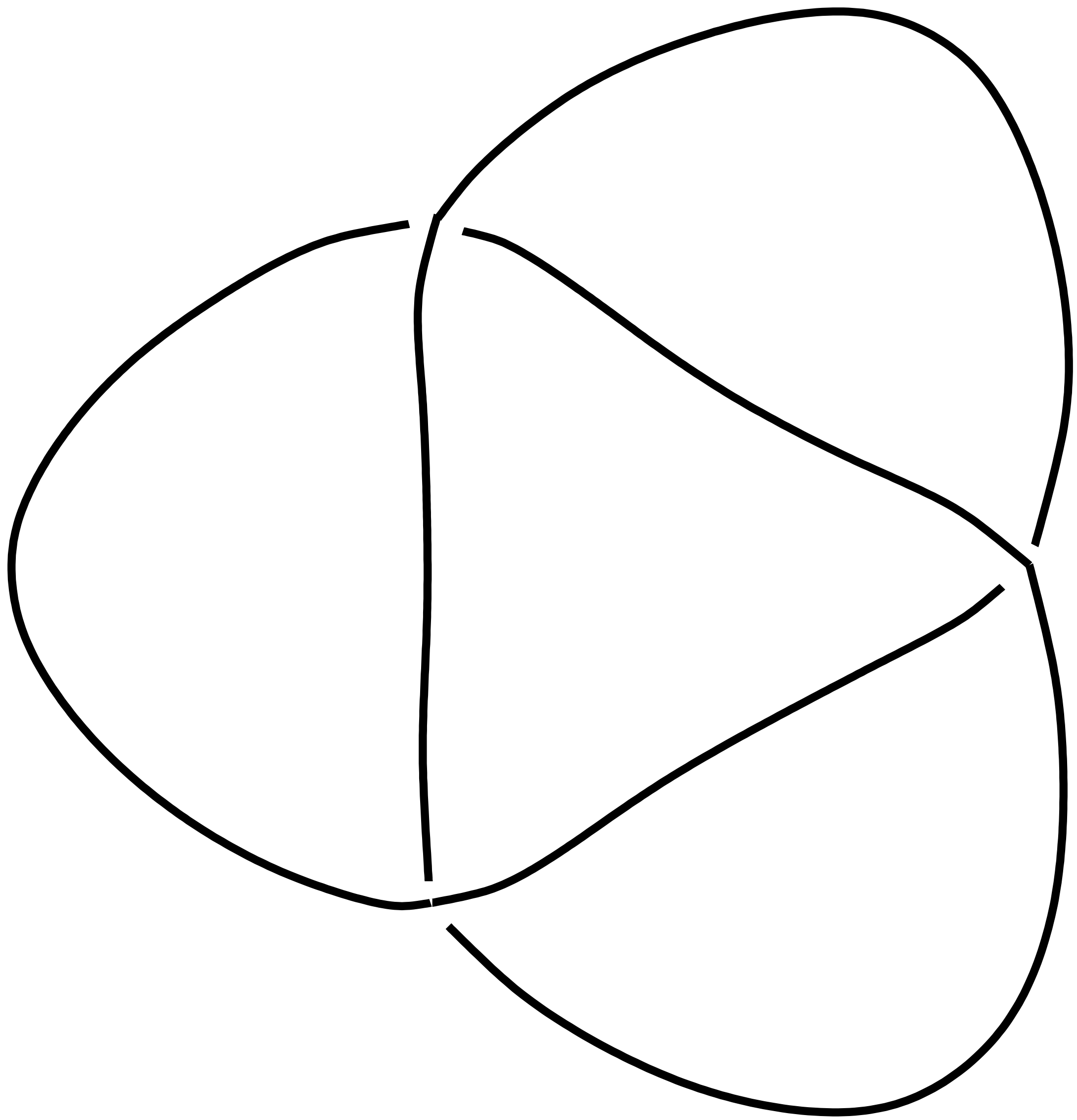
$$f_{\text{trébol}}(x) = (-x)^{-3\text{writhe}(\text{trébol})}[\text{trébol}] = -x^{-9}[\text{trébol}]$$

$$= x^{-4} + x^{-12} - x^{-16}$$

Calcula el polinomio f del ocho:

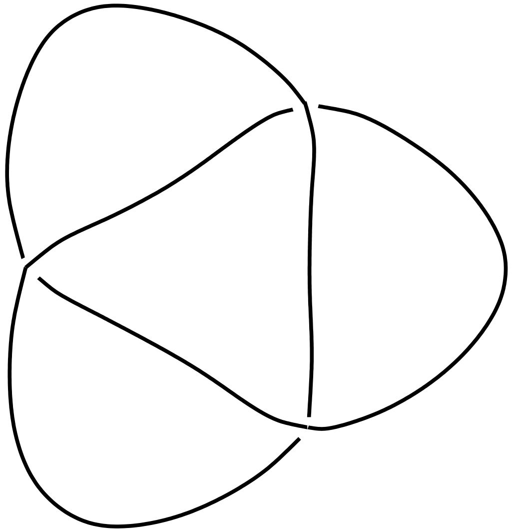


Calcula el polinomio f de la imagen en un espejo del trébol:

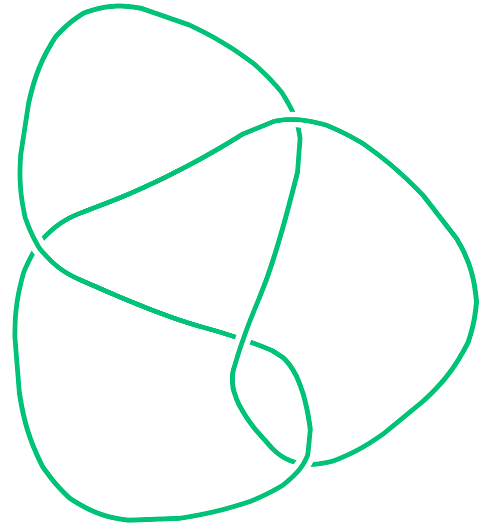


Nota: Se puede probar lo siguiente:

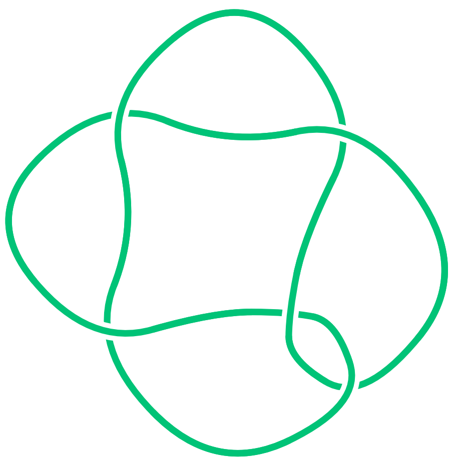
Teorema. Tomamos un nudo k y su reflejado en un espejo k^*. Entonces $[k](x) = [k^*](x^{-1})$ y $f_k(x) = f_{k^*}(x^{-1})$.



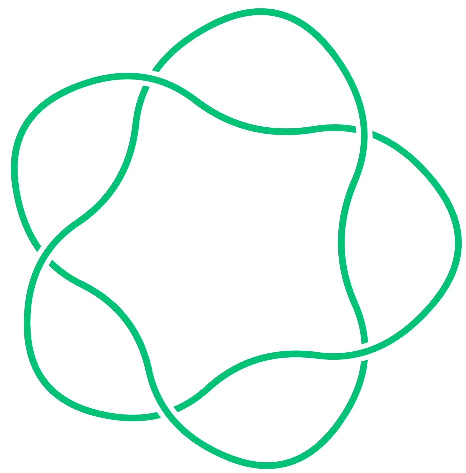
$x^{-4} + x^{-12} - x^{-16}$



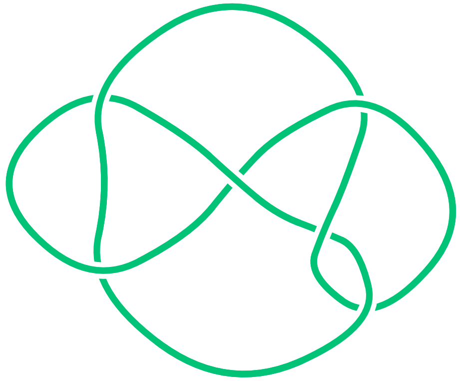
$x^{8} - x^{4} + 1 - x^{-4} + x^{-8}$



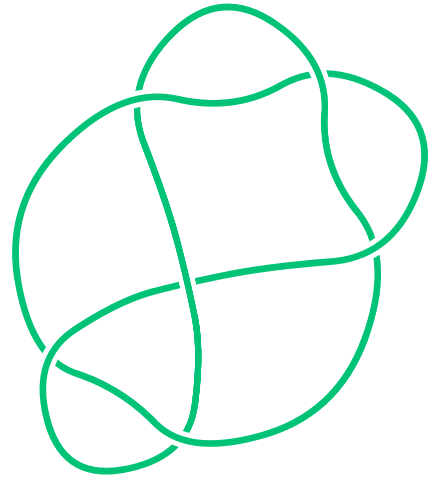
$x^{-4} - x^{-8} + 2x^{-12} - x^{-16} + x^{-24} - x^{-28}$



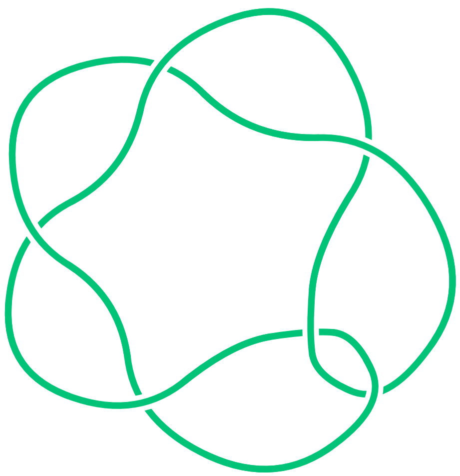
$-x^{-8} + x^{-16} - x^{-20} + x^{-24} - x^{-28}$



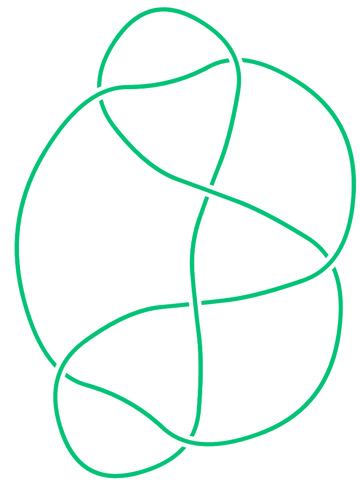
$-x^{12} + 2x^{8} - 2x^{4} + 3 - 2x^{-4} + 2x^{-8} - x^{-12}$



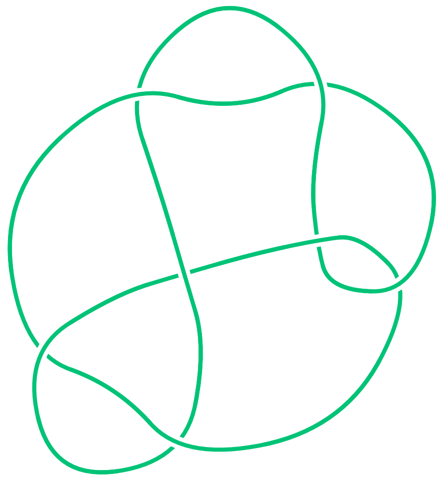
$x^{4} - 1 + 2x^{-4} - 2x^{-8} + 2x^{-12} - 2x^{-16} + x^{-20}$



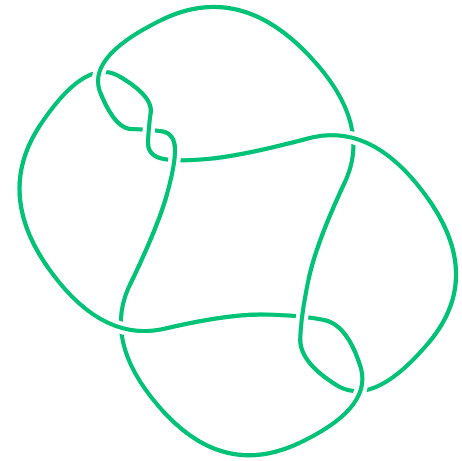
$x^{8} - x^{4} + 2 - 2x + x^{-4} - x^{-12} + x^{-16}$



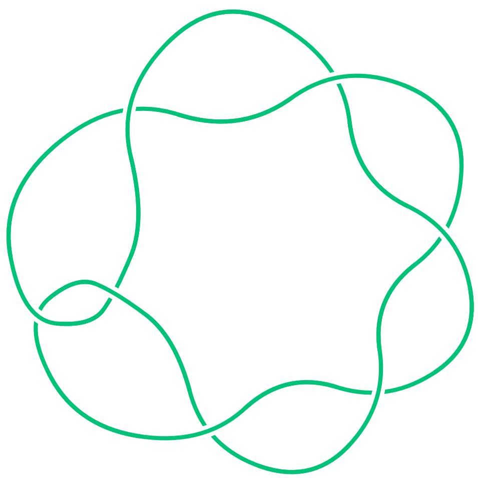
$x^{16} - 2x^{12} + 3x^{8} - 4x^{4} + 4 - 3x^{-4} + 3x^{-8} - x^{-12}$



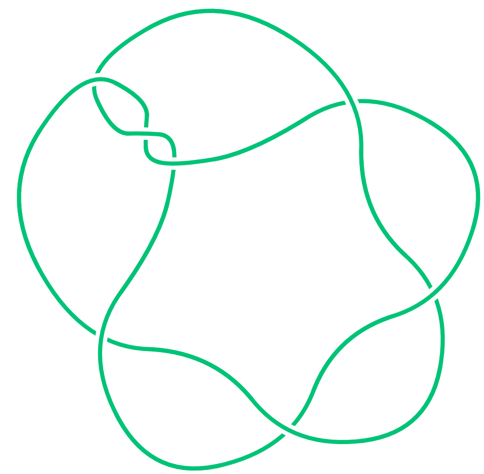
$x^{4} - 2 + 3x^{-4} - 3x^{-8} + 4x^{-12} - 3x^{-16} + 2x^{-20} - x^{-24}$



$x^{2} - x^{3} + 3x^{4} - 3x^{5} + 3x^{6} - 3x^{7} + 2x^{8} - x$



$x^{-4} - x{-}8 + 2x^{-12} - 2x^{-16} + 2x^{-20} - x^{-24} + x^{-28} - x^{-32}$



$x^{-8} - x^{-12} + 2x^{-16} - 2x^{-20} + 3x^{-24} - 2x^{-28} + x$

Notación

3	1	1	4	1	0	1	−1				
4	1	−2	2	1	−1	1	−1	1			
5	1	1	6	1	−1	2	−1	1	−1		
5	2	2	7	1	0	1	−1	1	−1		
6	1	−3	3	−1	2	−2	3	−2	2	−1	
6	2	−1	5	1	−1	2	−2	2	−2	1	
6	3	−2	4	1	−1	2	−2	1	−1	1	
7	1	−4	3	1	−2	3	−4	4	−3	3	−1
7	2	−1	6	1	−2	3	−3	4	−3	2	−1
7	3	2	9	1	−1	3	−3	3	−3	2	−1
7	4	1	8	1	−1	2	−2	2	−1	1	−1
7	5	2	9	1	−1	2	−2	3	−2	1	−1
7	6	1	8	1	−2	3	−2	3	−2	1	−1
7	7	3	10	1	0	1	−1	1	−1	1	−1