The trefoil knot is as universal as it can be

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ABSTRACT. Let $\tau_{p,q} \subset S^3$ denote the p,q-torus knot. It is known that if $\varphi : M \to S^3$ is a branched covering branched along $\tau_{p,q}$, then M is an orientable Seifert manifold with orientable base and non-zero Euler number. We prove a converse of this fact: If M is an orientable Seifert manifold with orientable base and non-zero Euler number, M is a branched covering of S^3 with branching in the knot $\tau_{p,q}$ for any p,q, $(p,q)=1$, and $2 \leq p < q$.

As an application of our techniques we compute, in terms of Seifert invariants, the cyclic branched coverings of S^3 branched along $\tau_{p,q}$.

1. *Introduction.* Since the fundamental paper of Seifert [11] the study of branched coverings of fibered spaces (see Section 2) with branching along fibers has been posed as an interesting question. In particular, the study of branched coverings of the fibrations of the 3-sphere has occupied an ample space in the literature.

Gordon and Heil proved ([7]) that any covering of the 3-sphere branched along the p,q-torus knot is an orientable Seifert manifold with orientable base. Moreover, since the Euler number (see Section 2) of the fibering of the 3-sphere with torus knots is never zero, one has that the Euler number of a branched covering of the 3-sphere with branching in a torus knot is neither zero ([9]).

In this work we prove a converse of the previous result, namely, each orientable Seifert manifold with orientable base and non-zero Euler number is a covering of the 3-sphere branched in the p,q-torus knot for any p,q, $(p,q)=1$, and $2 \leq p < q$ (Section 5).

In fact Gordon and Heil proved that a branched covering of *any* Seifert manifold with branching along fibers is also a Seifert manifold. To compute branched coverings of an arbitrary fibered space one can try similar methods and apply the techniques developed in the following sections.

One of the favourite themes in the study of branched coverings of fibered spaces is the cyclic coverings of the torus knots. As an application of the developed tools in

Section 3, we give, in terms of Seifert invariants, a list of all the cyclic coverings of the p, q-torus knot for any p, q (Section 4).

In [11], Seifert obtains homology spheres as cyclic coverings of the 3-sphere branched along the p, q-torus knot. He finds that $(n, pq) = 1$ is sufficient for the n-cyclic covering of the p, q-torus knot to be a homology sphere. Subsequently it was verified ([1], [6], [5], [8], [2], [3]) that the condition $(n, pq) = 1$ is also necessary. By just searching the statement of Theorem 1 in Section 4, we recover this nice result.

This paper is organized as follows. Section 2 contains some definitions and useful remarks. In Section 3 we develope the main tools for our computations; this section is the technical support for the rest of the paper. In Section 4 we give a list of the cyclic coverings of S^3 branched along the p, q-torus knot, $\tau_{p,q}$. Finally Section 5 contains the statement and proof of the Main Theorem (Theorem 2).

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2. *Preliminaries.* A *branched covering* between two n-manifolds M and N is an open, proper map $\varphi : M \to N$ which is finite-to-one. The usual way to check that an open map $\varphi : M \to N$ is a branched cover is to find a subcomplex $K \subset N$ of codimension two such that the restriction $\varphi| : M - \varphi^{-1}(K) \to N - K$ is a finite covering space. The subcomplex K, usually a submanifold, is called the *branching* of φ, and $\varphi^{-1}(K)$ is called the *singular set* of φ. The covering space $\varphi| : M - \varphi^{-1}(K) \to N - K$ is called the *associated covering space* of φ.

It is known that the associated covering determines the branched covering: Any finite covering space $\psi : M \to N$ can be completed, by a well known compactification process, into a branched covering $\bar{\psi} : \bar{M} \to \bar{N}$ (See [4]).

A covering space of n sheets $\psi : X \to Y$ defines a representation $\omega_\psi : \pi_1(Y) \to S_n$ into the symmetric group of n symbols, S_n, as follows: Number the preimage of the base point $\psi^{-1}(*) = \{1, 2, \ldots, n\}$, and for a class $[\alpha] \in \pi_1(Y)$, consider the liftings $\alpha_1, \alpha_2, \ldots, \alpha_n : I \to X$, where α_i starts in the point $i \in \psi^{-1}(*)$; then define $\omega([\alpha])(i) = \alpha_i(1)$ $(i = 1, 2, \ldots, n)$.

Now a homomorphism $\omega : \pi_1(Y) \to S_n$ determines a covering space (connected if and only if ω is transitive) of n sheets $\psi_\omega : X \to Y$: The covering space corresponding to the subgroup $\omega^{-1}(St(1)) \leq \pi_1(Y)$. A representation of ψ, ω_ψ, is conjugate to the homomorphism ω (it only depends on the numbering of $\psi^{-1}(*)$).

Along this work we will describe a branched covering of a manifold M by giving a codimension two submanifold $K \subset M$ and a representation $\omega : \pi_1(M - K) \to S_n$.

Seifert manifolds. Let $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_t, \beta_t$ be integers such that $\alpha_i > 0$ and $(\alpha_i, \beta_i) = 1$ for $i = 1, 2, \ldots, t$. The *Seifert manifold* M associated to the *Seifert symbol* $(O, g; \beta_1/\alpha_1, \beta_2/\alpha_2, \ldots, \beta_t/\alpha_t)$ is assembled as follows:

Let F be an orientable closed surface of genus g, and let $D_1, D_2, \ldots, D_t \subset F$ be t disjoint 2-disks. Write $F_0 = \overline{F - \cup D_i}$, and $M_0 = F_0 \times S^1$. If $\partial F_0 = q_1 \sqcup q_2 \sqcup \cdots \sqcup q_t$ and $h = \{x\} \times S^1$ for some $x \in F_0$, we let $m_i \subset q_i \times S^1$ be a simple closed curve such that $m_i \sim q_i^{\alpha_i} h^{\beta_i}$. Let $V_1, V_2, \ldots, V_t$ be solid tori with meridians $\mu_1, \mu_2, \ldots, \mu_t$, resp., and let $\eta_i : \partial V_i \to q_i \times S^1$ be a homeomorphism such that $\eta_i(\mu_i) = m_i$ for $i = 1, 2, \ldots, t$. Then the Seifert manifold M associated to the symbol $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ is

$$M = M_0 \bigcup_{\cup \eta_i} (\cup V_i).$$

The circles $\{x\} \times S^1$ for $x \in F_0$ are called the *ordinary fibers* of M and the core e_i of V_i is called an *exceptional fiber of order* α_i in case $\alpha_i > 1$; otherwise e_i is also an ordinary fiber ($i = 1, 2, \ldots, t$). The surface F is called the *orbit surface* or the *base* of M. Note that collapsing each fiber of M into a point gives us an identification $p : M \to F$.

The choices in the construction of M are taken into account by the

THEOREM ([9]). *Two Seifert symbols represent homeomorphic Seifert manifolds by a fiber preserving homeomorphism if and only if one of the symbols can be changed into the other by a finite sequence of the following moves:*

0. Permute the ratios.

1. Add or delete $\dfrac{0}{1}$.

2. Replace the pair $\dfrac{\beta_i}{\alpha_i}, \dfrac{\beta_j}{\alpha_j}$ *by* $\dfrac{\beta_i + k\alpha_i}{\alpha_i}, \dfrac{\beta_j - k\alpha_j}{\alpha_j}$. □

Notice that the manifold M associated to $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ can be recovered unambiguously from $M_0 = F_0 \times S^1$ and the curves $m_i = q_i^{\alpha_i} h^{\beta_i}$, $i = 1, 2, \ldots, t$. We call the pair $(F_0 \times S^1, \{m_i\})$ a *kit* for $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$.

A branched covering of a solid torus V branched along the core e of V is completely determined, up to covering equivalence, by a covering space of ∂V. Therefore to describe a branched covering of $(O,g;\beta_1/\alpha_1,\ldots,\beta_t/\alpha_t)$ branched along fibers it suffices to construct a covering space of a kit for $(O,g;\beta_1/\alpha_1,\ldots,\beta_t/\alpha_t,0/1,\ldots,0/1)$. In this way we obtain a branched covering branched (at most) along the exceptional fibers and the 'added' $\frac{0}{1}$-fibers.

From the symbol $(O,g;\beta_1/\alpha_1,\ldots,\beta_t/\alpha_t)$ one can write down a presentation of the fundamental group: Let $a_1,\ldots,a_{2g}$ be a basis of the orbit surface; then

$\pi_1(O,g;\beta_1/\alpha_1,\ldots,\beta_t/\alpha_t) \cong\langle a_1,\ldots,a_{2g},q_1,q_2,\ldots,q_t,h : q_1q_2\cdots q_t = \prod[a_{2i-1},a_{2i}]$, $[h,a_i]=1$, $[h,q_j]=1$, $q_j^{\alpha_j}h^{\beta_j}=1\rangle$.

If $(F_0\times S^1,\{m_i\})$ is a kit for $(O,g;\beta_1/\alpha_1,\ldots,\beta_t/\alpha_t)$ then

$\pi_1(F_0\times S^1) \cong \langle a_1,\ldots,a_{2g},q_1,q_2,\ldots,q_t,h : [h,a_i]=1,[h,q_j]=1,q_1q_2\cdots q_t = \prod[a_{2i-1},a_{2i}]\rangle$.

Then a set of permutations $\omega(a_1),\ldots,\omega(a_{2g}),\omega(q_1),\ldots,\omega(q_t),\omega(h)\in S_n$ extends into a representation $\omega:\pi_1(F_0\times S^1)\to S_n$ if and only if $\omega(q_1)\omega(q_2)\cdots\omega(q_t) = \prod[\omega(a_{2i-1}),\omega(a_{2i})]$, $[\omega(h),\omega(a_i)]=(1)$, $[\omega(h),\omega(q_j)]=(1)$ (the symbol (1) is the identity permutation).

All the representations that we will consider in this work will have $\omega(a_i)=(1)$ for $i=1,2,\ldots,2g$.

LEMMA 0 ([11]). *Let F be a compact orientable surface with $\partial F\neq\emptyset$. If $\varphi:\tilde{M}\to F\times S^1$ is a finite covering space, then there exists $\tilde{F}\subset\tilde{M}$ an orientable surface such that $\tilde{M}=\tilde{F}\times S^1$ and φ preserves the 'fibers' $\{\tilde{x}\}\times S^1$ for each $\tilde{x}\in\tilde{F}$.* □

Remark. In the covering space $\varphi:\tilde{M}\to F\times S^1$ of Lemma 0 a component C of the preimage $\varphi^{-1}(F)$ can be used as the surface $\tilde{F}$ if and only if the 'fiber' $\varphi^{-1}(\{a\}\times S^1)=\{x_1\}\times S^1\sqcup\{x_2\}\times S^1\sqcup\cdots\sqcup\{x_k\}\times S^1$ intersects C in exactly k points (if and only if k is the number of sheets of φ). Thus if $\varphi^{-1}(F)$ is connected, then $\tilde{F}=\varphi^{-1}(F)$ if and only if the fiber h of $F\times S^1$ goes to the identity under the representation $\omega:\pi_1(F\times S^1)\to S_n$ associated to φ.

If $M=(O,g;\beta_1/\alpha_1,\ldots,\beta_t/\alpha_t)$, then $e(M)=-\sum\frac{\beta_i}{\alpha_i}$ is called de *Euler number of M*.

THEOREM ([9]). *Let M and M' be Seifert manifolds with base spaces F and F', and let $f:M\to M'$ be an orientation preserving fiber preserving map. Let the degree of the induced map on a typical fiber be n and the degree of the induced map*

$\bar{f} : F \to F'$ *be* m. *Then* $e(M) = (m/n)e(M')$. $\square$

In particular if $\varphi : M \to M'$ is a branched covering and $e(M') \neq 0$, then $e(M) \neq 0$.

Along this work we will denote by $\varepsilon = \varepsilon_n \in S_n$ to the standard n-cycle $\varepsilon = (1, 2, \ldots, n)$. Also (p, q) will denote the greatest common divisor of the integers p and q.

3. *Basic lemmas.* Let $(F \times S^1, \{m_i\})$ be a kit for $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$; let $\omega : \pi_1(F \times S^1) \to S_n$ be a representation, and let $\varphi : \tilde{M} \to F \times S^1$ be the covering space associated to ω. If $\{\tilde{m}_j\}$ is a set of components of $\varphi^{-1}(\cup m_i)$, one for each component of $\varphi^{-1}(\cup(q_i \times S^1))$, then $(\tilde{M}, \{\tilde{m}_j\})$ is a kit for some Seifert symbol $(O, \tilde{g}; B_1/A_1, \ldots, B_u/A_u)$; for, by means of a Fox compactification, if we extend φ into a branched covering $\bar{\varphi} : \tilde{M} \bigcup_{\cup \tilde{\eta}_j} (\cup_{j=1}^{u} \tilde{V}_j) \to (F \times S^1) \bigcup_{\cup \eta_i} (\cup_{i=1}^{t} V_i)$, then, for each j, $\bar{\varphi} : \tilde{V}_j \to V_{i_j}$ is a continuous map and, therefore, sends a meridian of $\tilde{V}_j$ into a meridian of V_{i_j} (more precisely, $\bar{\varphi}^{-1}(m_{i_j}) = \varphi^{-1}(m_{i_j})$ is a set of meridians of $\tilde{V}_j$); and, by Lemma 0, we know that $\tilde{M}$ is a product $\tilde{M} = \tilde{F} \times S^1$.

Thus, given a representation $\omega : \pi_1(F \times S^1) \to S_n$, the main task for us will be to find out which are the numbers $\tilde{g}, u, A_1, B_1, \ldots, A_u, B_u$ by 'understanding' the preimages of the meridians $\varphi^{-1}(m_i)$. In this section we will compute these numbers for some special representations $\omega : \pi_1(F \times S^1) \to S_n$, namely, representations ω such that $\omega(h)$ is the identity or $\omega(h)$ is an n-cycle.

TORUS LEMMA. *Let* T *be a torus and let* $a, b \subset T$ *be a basis for* $\pi_1(T)$. *Let* $\omega : \pi_1(T) \to S_n$ *be the representation such that* $\omega(a) = \varepsilon^s$ *and* $\omega(b) = \varepsilon^r$. *If* $\varphi : \tilde{T} \to T$ *is the covering space associated to* ω *and* $(n, s) = 1$, *then there exists* $\tilde{b} \subset \tilde{T}$ *a simple closed curve such that* $\tilde{a} = \varphi^{-1}(a)$ *and* $\tilde{b}$ *is a basis for* $\pi_1(\tilde{T})$, *and* $\varphi(\tilde{a}) = a^n$, *and* $\varphi(\tilde{b}) = ba^{-s^* r}$ *where* $ss^* \equiv 1 \pmod{n}$.

Proof. Cut T along a and b to obtain a fundamental square V with boundary b^+, a^+, b^-, a^- so that we recover T from V by identifying b^+ with b^- and a^+ with a^-. See Figure 1.

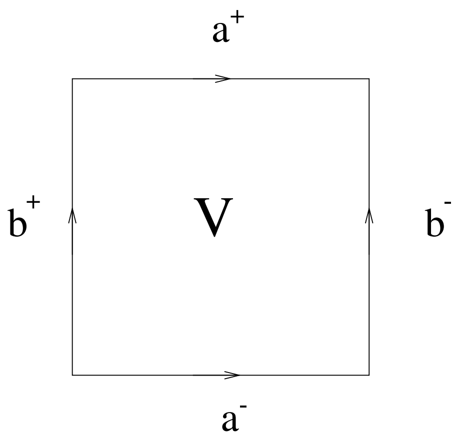

Figure 1.

Let $V(1), V(2), \ldots, V(n)$ be n copies of V such that the boundary of $V(i)$ is $b^+(i), a^+(i), b^-(i), a^-(i)$. Then $\tilde{T}$ is obtained from $\{V(i)\}$ by identifying $b^+(i)$ with $b^-(\varepsilon^s(i))$ and $a^+(j)$ with $a^-(\varepsilon^r(j))$, $i = 1, 2, \ldots, n$. See Figure 2.

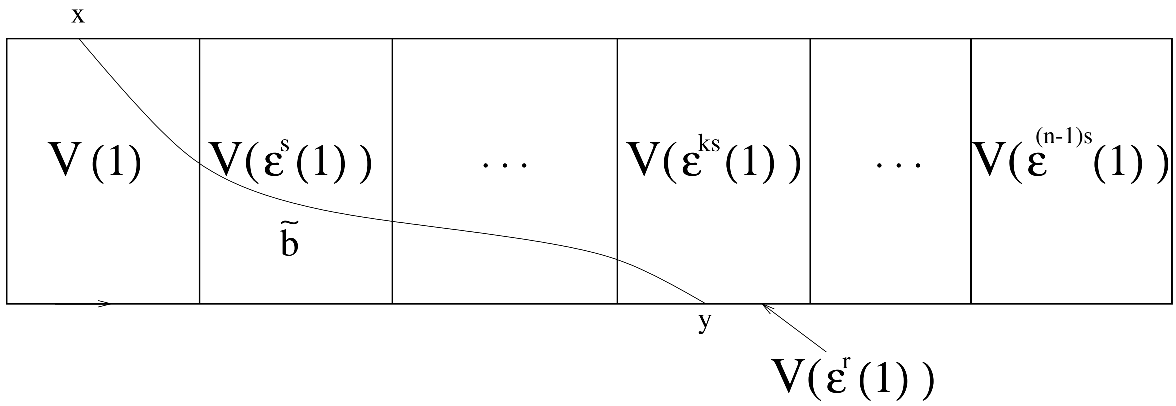

Figure 2.

We let $\tilde{a} \subset \tilde{T}$ be the preimage of a, $\varphi^{-1}(a) = a^+(1) \cup a^+(\varepsilon^s(1)) \cup a^+(\varepsilon^{2s}(1)) \cup \cdots \cup a^+(\varepsilon^{(n-1)s}(1))$. Clearly $\varphi(\tilde{a}) = a^n$ and, therefore, $\tilde{a}$ is an essential simple closed curve on $\tilde{T}$. If $x \in a^+(1)$, then x is glued with some $y \in a^-(\varepsilon^{ks}(1))$ for some k. We let $\tilde{b}$ be a simple 'shortest' path from y to x as in Figure 2. Then $\tilde{b}$ is a simple closed curve on $\tilde{T}$ which intersects $\tilde{a}$ in the single point $x = y \in \tilde{T}$; therefore $\tilde{a}$ and $\tilde{b}$ is a basis for $\pi_1(\tilde{T})$. Now, by construction of $\tilde{T}$, $y \in a^-(\varepsilon^r(1))$; we have that $\varepsilon^{ks} = \varepsilon^r$ and, therefore, $ks \equiv r \pmod n$, or $k \equiv s^* r \pmod n$, where $ss^* \equiv 1 \pmod n$. Since $\tilde{b}$

intersects k vertical lines in Figure 2 and $\tilde{a}$ in one point, we have that $\varphi(\tilde{b}) = ba^{-k}$ (the sign is a minus because we oriented $\tilde{b}$ from y to x); by replacing, if necessary, $\tilde{b}$ with $\tilde{b}\tilde{a}^t$ for some t, we may assume that $\tilde{b}$ intersects exactly s^*r vertical lines in Figure 2 obtaining the required $\varphi(\tilde{b}) = ba^{-s^*r}$. □

LEMMA 1. *Let $(F \times S^1, \{m_i\})$ be a kit for $(O, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$. Let $\omega : \pi_1(F \times S^1) \to S_n$ be the representation such that $\omega(a_j) = (1)$ for $j = 1, \dots, 2g$; $\omega(q_i) = \varepsilon^{r_i}$ for $i = 1, \dots, t$ and $\sum r_i = 0$; and $\omega(h) = \varepsilon^s$ with $(n, s) = 1$. If $\varphi : \tilde{M} \to F \times S^1$ is the covering associated to ω and $\tilde{m}_i$ is a component of $\varphi^{-1}(m_i)$ $(i = 1, \dots, t)$, then $(\tilde{M}, \{\tilde{m}_i\})$ is a kit for $(O, g; B_1/A_1, \dots, B_t/A_t)$, where*

$$A_i = \frac{n\alpha_i}{d_i}, \qquad B_i = \frac{\beta_i + s^* r_i \alpha_i}{d_i}$$

and $d_i = (n, \beta_i s + r_i \alpha_i) = (n, \beta_i + s^* r_i \alpha_i)$, $i = 1, \dots, t$, and $ss^* \equiv 1 \pmod{n}$.

Proof. Let $\tilde{F} \subset \tilde{M}$ be the surface such that $\tilde{M} \cong \tilde{F} \times S^1$ as in Lemma 0. Since $\varphi| : \varphi^{-1}(F) \to F$ is a (possibly disconnected) covering space of n sheets, we have $\chi(\varphi^{-1}(F)) = n\chi(F)$. Write $\tilde{h} = \varphi^{-1}(h)$; since $\tilde{h}$ intersects $\varphi^{-1}(F)$ in n points (for $\omega(h) = \varepsilon^s$ and $(n, s) = 1$), then the projection $p : \tilde{M} \to \tilde{F}$ restricted to $\varphi^{-1}(F)$, $p| : \varphi^{-1}(F) \to \tilde{F}$, is a map of degree n; since there are no exceptional fibers in $\tilde{M}$, we have that $p| : \varphi^{-1}(F) \to \tilde{F}$ is a (possibly disconnected) covering space; therefore $\chi(\varphi^{-1}(F)) = n\chi(\tilde{F})$, and $\chi(F) = \chi(\tilde{F})$. The number of boundaries of $\tilde{F}$ equals the number of boundaries of $\tilde{M}$; since the representation ω restricted to each $q_i \times S^1$ is transitive ($\omega(h) = \varepsilon^s$ and $(n, s) = 1$), then $\tilde{M}$ has t boundaries; therefore F and $\tilde{F}$ have the same number of boundaries and, since both are orientable, we conclude $F \cong \tilde{F}$.

By the Torus Lemma, there is a curve $\tilde{q}_i \subset \varphi^{-1}(q_i \times S^1)$ such that $\tilde{q}_i$ and $\tilde{h}$ form a basis for $\pi_1(\varphi^{-1}(q_i \times S^1))$, $\varphi(\tilde{h}) \simeq h^n$, and $\varphi(\tilde{q}_i) \simeq q_i h^{-s^* r_i}$ ($ss^* \equiv 1 \pmod{n}$); there is a basis $\tilde{a}_1, \dots, \tilde{a}_{2g} \subset \tilde{F}$ such that $\varphi(\tilde{a}_i) \simeq a_i$ (for the induced map $\bar{\varphi} : \tilde{F} \to F$ is a homeomorphism). We compute

$$\varphi\left(\tilde{q}_1\tilde{q}_2\cdots\tilde{q}_t\left(\prod[\tilde{a}_{2i-1}, \tilde{a}_{2i}]\right)^{-1}\right) \simeq q_1 h^{-s^* r_1} q_2 h^{-s^* r_2} \cdots q_t h^{-s^* r_t}\left(\prod[a_{2i-1}, a_{2i}]\right)^{-1} \simeq h^{-s^* \Sigma r_i} q_1 q_2 \cdots q_t \left(\prod[a_{2i-1}, a_{2i}]\right)^{-1} \simeq 1$$

(recall $\sum r_i = 0$). Since φ is a covering space, the morphism induced by φ in fundamental groups is 1-1; therefore $\tilde{q}_1\tilde{q}_2\cdots\tilde{q}_t \simeq \prod[\tilde{a}_{2i-1}, \tilde{a}_{2i}]$ in $\tilde{M}$; it follows that $\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_t$ span an orbit surface for $\tilde{M}$ that, without loss of generality, we may assume is $\tilde{F}$.

Now since $\omega(m_i) = \omega(q_i^{\alpha_i} h^{\beta_i}) = \varepsilon^{\alpha_i r_i + s\beta_i}$ has order n/d_i (recall $d_i = (n, \alpha_i r_i +$

$s\beta_i)$), $\omega(m_i)$ has exactly d_i orbits; and, then, $\varphi^{-1}(m_i)$ has d_i components. We see that $\varphi(\tilde{m}_i) = m_i^{n/d_i} = q_i^{\alpha_i n/d_i} h^{\beta_i n/d_i}$, where $\tilde{m}_i$ is a component of $\varphi^{-1}(m_i)$. If $\tilde{m}_i = \tilde{q}_i^{A_i}\tilde{h}^{B_i}$, then $\varphi(\tilde{m}_i) = (q_i h^{-s^* r_i})^{A_i}(h^n)^{B_i} = q_i^{A_i} h^{nB_i - s^* r_i A_i}$; comparing exponents we see that $A_i = \dfrac{\alpha_i n}{d_i}$, and $B_i = \dfrac{\beta_i + s^* r_i \alpha_i}{d_i}$ as claimed. □.

COROLLARY. *If $\sum r_i = 0$ and $(n,s) = 1$, then $(O, g; B_1/A_1, \ldots, B_t/A_t)$ is an n-fold branched covering of $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$, where*

$$A_i = \frac{n\alpha_i}{d_i}, \qquad B_i = \frac{\beta_i + s^* r_i \alpha_i}{d_i}$$

and $d_i = (n, \beta_i s + r_i \alpha_i) = (n, \beta_i + s^* r_i \alpha_i)$, $i = 1, \ldots, t$. □

LEMMA 2. *Let $(F \times S^1, \{m_i\})$ be a kit for $(O, g;\ \beta_1/1,\ \ldots,\ \beta_u/1,\ \beta_{u+1}/\alpha_{u+1},\ \ldots,\ \beta_{u+t}/\alpha_{u+t})$ with $\alpha_i > 1$ for $i = u+1, \ldots, u+t$. Let $\omega : \pi_1(F \times S^1) \to S_n$ be a transitive representation such that $\omega(a_i) = (1)$ for $i = 1, \ldots, 2g$; $\omega(q_i) = \varepsilon^{s_i}$ with $(n, s_i) = 1$ for $i = 1, \ldots, u$; $\omega(q_i)$ is a product of disjoint d_i-cycles and has k_i fixed points for $i = u+1, \ldots, u+t$; and $\omega(h) = (1)$. If $\varphi : \tilde{M} \to F \times S^1$ is the covering associated to ω and $\{\tilde{m}_j\}$ is a set of componets of $\varphi^{-1}(\bigcup_{i=1}^{u+t} m_i)$, one for each component of $\varphi^{-1}(\bigcup_{i=1}^{u+t}(q_i \times S^1))$, then $(\tilde{M}, \{\tilde{m}_j\})$ is a kit for*

$$(O, \tilde{g}; (n\beta_1)/1, \ldots, (n\beta_u)/1,\ \overbrace{\beta_{u+1}/\alpha_{u+1}, \ldots, \beta_{u+1}/\alpha_{u+1}}^{k_{u+1}\ \text{times}},\ \overbrace{\beta_{u+1}/\alpha'_{u+1}, \ldots, \beta_{u+1}/\alpha'_{u+1}}^{n_{u+1}\ \text{times}},$$
$$\ldots,\ \overbrace{\beta_{u+t}/\alpha_{u+t}, \ldots, \beta_{u+t}/\alpha_{u+t}}^{k_{u+t}\ \text{times}},\ \overbrace{\beta_{u+t}/\alpha'_{u+t}, \ldots, \beta_{u+t}/\alpha'_{u+t}}^{n_{u+t}\ \text{times}})$$

where n_i is the number of non-trivial orbits of $\omega(q_i)$, $\alpha'_i = \alpha_i/d_i$ $(i = u+1, \ldots, u+t)$, and $\tilde{g} = (2 - n\chi(F) + u(n-1) + \sum n_i(d_i - 1))/2$.

Remark. The number d_i in the statement of Lemma 2 is a divisor of α_i $(i = u+1, \ldots, u+t)$, for, since $\omega(h) = (1)$, the representation ω factors through the orbifold group $\langle a_1, \ldots, a_{2g}, q_{u+1}, \ldots, q_{u+t} : q_j^{\alpha_j} = 1, q_1 \cdots q_t = \prod [a_{2i-1}, a_{2i}] \rangle$.

Proof of Lemma 2. The preimage $\varphi^{-1}(h)$ has n components, then the restriction of the representation ω determines a covering $\psi : \tilde{F} \to F$ such that $\tilde{M} = \tilde{F} \times S^1$ and $\varphi = \psi \times 1 : \tilde{M} = \tilde{F} \times S^1 \to F \times S^1$.

Now since $(n, s_i) = 1$, $\tilde{q}_i = \varphi^{-1}(q_i)$ is connected, $i = 1, \ldots, u$; since $\omega(m_i) = \omega(q_i h^{\beta_i}) = \varepsilon^{s_i}$, also $\tilde{m}_i = \varphi^{-1}(m_i)$ is connected; we have $\varphi(\tilde{m}_i) = m_i^n = q_i^n h^{n\beta_i}$; if $\tilde{m}_i = \tilde{q}_i^{A_i}\tilde{h}^{B_i}$, then $\varphi(\tilde{m}_i) = q_i^{nA_i} h^{B_i}$; it follows that $A_i = 1$ and $B_i = n\beta_i$ $(i = 1, \ldots, u)$.

If $i \in \{u+1, \ldots, u+t\}$, then $\varphi^{-1}(q_i)$ has $d_i + k_i$ components $\tilde{q}_{i,1}, \ldots, \tilde{q}_{i,n_i}$, $\tilde{p}_{i,1}, \ldots, \tilde{p}_{i,k_i}$ where $\varphi| = \psi| : \tilde{q}_{i,j} \to q_i$ has degree d_i, and $\varphi| = \psi| : \tilde{p}_{i,j} \to q_i$ has degree 1. Therefore the component of $\varphi^{-1}(m_i)$ in $\tilde{p}_{i,j} \times S^1$ is the curve $\tilde{p}_{i,j}^{\alpha_i} \tilde{h}^{\beta_i}$, and a component of $\varphi^{-1}(m_i)$ in $\tilde{q}_{i,j} \times S^1$ is the curve $\tilde{q}_{i,j}^{\alpha_i/d_i} \tilde{h}^{\beta_i}$, for $\omega(m_i) = \omega(q_i^{\alpha_i} h^{\beta_i}) = \omega(q_i)^{\alpha_i} = (1)$.

The expression for $\tilde{g}$ is a rewriting of the Riemann–Hurwitz formula. □

COROLLARY. *With the numbers d_i, n_i and k_i as in Lemma 2,*

$$(O, \tilde{g};\ (n\beta_1)/1, \ldots, (n\beta_u)/1,\ \overbrace{\beta_{u+1}/\alpha_{u+1}, \ldots, \beta_{u+1}/\alpha_{u+1}}^{k_{u+1}\ \text{times}},\ \overbrace{\beta_{u+1}/\alpha'_{u+1}, \ldots, \beta_{u+1}/\alpha'_{u+1}}^{n_{u+1}\ \text{times}},$$
$$\ldots,\ \overbrace{\beta_{u+t}/\alpha_{u+t}, \ldots, \beta_{u+t}/\alpha_{u+t}}^{k_{u+t}\ \text{times}},\ \overbrace{\beta_{u+t}/\alpha'_{u+t}, \ldots, \beta_{u+t}/\alpha'_{u+t}}^{n_{u+t}\ \text{times}})$$

is an n-fold branched covering of $(O, g; \beta_1/1, \ldots, \beta_u/1, \beta_{u+1}/\alpha_{u+1}, \ldots, \beta_{u+t}/\alpha_{u+t})$ where $\alpha_i > 1$ for $i = u+1, \ldots, u+t$, $\alpha'_i = \alpha_i/d_i$ for $i = u+1, \ldots, u+t$, and $\tilde{g} = (2 - n\chi(F) + u(n-1) + \sum n_i(d_i - 1))/2$. □

4. *Cyclic coverings of torus knots.* Let $\tau_{\alpha_1,\alpha_2} \subset S^3$ be the α_1, α_2-torus knot $(2 \le \alpha_1 < \alpha_2)$. As an application of the results of Section 3 we will give a list of the cyclic branched coverings of S^3 branched along τ_{α_1,α_2}.

First the torus knot τ_{α_1,α_2} is an ordinary fiber of $(O, 0; \beta_1/\alpha_1, \beta_2/\alpha_2) (\cong S^3)$, where $\beta_1\alpha_2 + \alpha_1\beta_2 = -1$.

Let $(F \times S^1, \{m_0, m_1, m_2\})$ be a kit for $(O, 0; 0/1, \beta_1/\alpha_1, \beta_2/\alpha_2)$. If $\mu \subset S^3 - \tau_{\alpha_1,\alpha_2}$ is a meridian of τ_{α_1,α_2}, then

$$\begin{aligned} q_0 &= \mu^{-1} \\ q_1 &= a_1^{-\beta_1} \\ q_2 &= a_2^{-\beta_2} \\ \mu &= a_1^{-\beta_1} a_2^{-\beta_2} \\ h &= a_1^{\alpha_1} = a_2^{\alpha_2} \end{aligned}$$

where $\pi_1(S^3 - \tau_{\alpha_1,\alpha_2}) \cong \langle a_1, a_2 : a_1^{\alpha_1} = a_2^{\alpha_2} \rangle$.

The n-fold cyclic covering of S^3 branched along τ_{α_1,α_2} is induced by the representation $\omega : \pi_1(S^3 - \tau_{\alpha_1,\alpha_2}) \to S_n$ such that $\omega(\mu) = \varepsilon$. If $\omega(a_1) = \varepsilon^{s_1}$ and $\omega(a_2) = \varepsilon^{s_2}$, then $\varepsilon = \omega(\mu) = \omega(a_1^{-\beta_1} a_2^{-\beta_2}) = \varepsilon^{-s_1\beta_1 - s_2\beta_2}$; so we may assume $s_1 = \alpha_2$ and $s_2 = \alpha_1$. Therefore the n-fold cyclic covering of S^3 branched along τ_{α_1,α_2} is induced by the restriction $\omega : \pi_1(F \times S^1) \to S_n$ such that

$$\begin{array}{ll}\omega(q_0) = \varepsilon^{-1} & \omega(m_0) = \omega(q_0) = \varepsilon^{-1} \\ \omega(q_1) = \varepsilon^{-\alpha_2\beta_1} & \omega(m_1) = \omega(q_1^{\alpha_1} h^{\beta_1}) = (1) \\ \omega(q_2) = \varepsilon^{-\alpha_1\beta_2} & \omega(m_2) = \omega(q_1^{\alpha_2} h^{\beta_2}) = (1) \\ \omega(h) = \varepsilon^{\alpha_1\alpha_2}. & \end{array}$$

First case: $(\alpha_1\alpha_2, n) = 1$. Then Lemma 1 applies; here $r_0 = -1$, $r_1 = -\alpha_2\beta_1$, $r_2 = -\alpha_1\beta_2$, and $s = \alpha_1\alpha_2$. We have $A_i = \frac{n}{d_i}\alpha_i$ and $B_i = \frac{\beta_i + (\alpha_1\alpha_2)^* r_i \alpha_i}{d_i}$, where $d_i = (n, \beta_i(\alpha_1\alpha_2) + r_i\alpha_i) = (n, \beta_i + (\alpha_1\alpha_2)^* r_i \alpha_i)$ and $(\alpha_1\alpha_2)(\alpha_1\alpha_2)^* \equiv 1 \pmod{n}$. If k is an integer such that $(\alpha_1\alpha_2)(\alpha_1\alpha_2)^* = 1 - kn$, then $d_0 = (n, -(\alpha_1\alpha_2)^*) = 1$, $d_1 = (n, \beta_1 + (\alpha_1\alpha_2)^*\alpha_1\alpha_2\beta_1) = (n, \beta_1 kn) = n$, and $d_2 = (n, \beta_2 + (\alpha_1\alpha_2)^*\alpha_1\alpha_2\beta_2) = (n, \beta_2 kn) = n$. Therefore

$$\begin{array}{rr} A_0 = n, & B_0 = -(\alpha_1\alpha_2)^*, \\ A_1 = \alpha_1, & B_1 = \beta_1 k, \\ A_2 = \alpha_2, & B_2 = \beta_2 k. \end{array}$$

That is, in this case

$$\tilde{M} = (O, 0; \frac{-(\alpha_1\alpha_2)^*}{n}, \frac{\beta_1 k}{\alpha_1}, \frac{\beta_2 k}{\alpha_2})$$

is the n-fold cyclic covering of S^3 branched along τ_{α_1,α_2}.

Note that $|H_1(\tilde{M})| = |B_0A_1A_2 + A_0B_1A_2 + A_0A_1B_2| = |-(\alpha_1\alpha_2)^*\alpha_1\alpha_2 + n\beta_1 k\alpha_2 + n\alpha_1\beta_2 k| = |-(1-kn) + nk(\beta_1\alpha_2 + \alpha_1\beta_2)| = 1$. So that $\tilde{M}$ is a homology sphere.

Second case: $(\alpha_1\alpha_2, n) = n$. Notice that the order $o(\varepsilon^{-\alpha_2\beta_1}) = (\alpha_1, n)$ and $o(\varepsilon^{-\alpha_1\beta_2}) = (\alpha_2, n)$; therefore

$$\omega(q_1) = \varepsilon^{-\alpha_2\beta_1} = u_1 u_2 \cdots u_{n/(\alpha_1,n)}$$

and

$$\omega(q_2) = \varepsilon^{-\alpha_1\beta_2} = v_1 v_2 \cdots v_{n/(\alpha_2,n)}$$

where u_i is an (α_1, n)-cycle and v_j is an (α_2, n)-cycle for each i and j. Also $\omega(h) = (1)$, so that Lemma 2 applies. We compute $d_1 = (\alpha_1, n)$, $k_1 = 0$, $n_1 = n/(\alpha_1, n)$, $\alpha_1' = \alpha_1/(\alpha_1, n)$; $d_2 = (\alpha_2, n)$, $k_2 = 0$, $n_2 = n/(\alpha_2, n)$, $\alpha_2' = \alpha_2/(\alpha_2, n)$. The n-fold cyclic covering of S^3 branched along τ_{α_1,α_2} is

$$\tilde{M} = (O, \tilde{g}; \frac{0}{1}, \overbrace{\frac{\beta_1}{\alpha_1'}, \ldots, \frac{\beta_1}{\alpha_1'}}^{n_1 \text{ times}}, \overbrace{\frac{\beta_2}{\alpha_2'}, \ldots, \frac{\beta_2}{\alpha_2'}}^{n_2 \text{ times}}) = (O, \tilde{g}; \overbrace{\frac{\beta_1}{\alpha_1'}, \ldots, \frac{\beta_1}{\alpha_1'}}^{n_1 \text{ times}}, \overbrace{\frac{\beta_2}{\alpha_2'}, \ldots, \frac{\beta_2}{\alpha_2'}}^{n_2 \text{ times}}),$$

where

$$\begin{aligned}\tilde g &= (2-2n+(n-1)+n_1((\alpha_1,n)-1)+n_2((\alpha_2,n)-1))/2\\ &= (2-2n+n-1+n-\frac{n}{(\alpha_1,n)}-\frac{n}{(\alpha_2,n)})/2\\ &= (n+1-\frac{n}{(\alpha_1,n)}-\frac{n}{(\alpha_2,n)})/2\\ &= (n+1-(\alpha_1,n)-(\alpha_2,n))/2.\end{aligned}$$

Notice that $\tilde g = 0$ if and only if $n|\alpha_1$ or $n|\alpha_2$. If, say, $n|\alpha_1$, then the covering is $\tilde M = (O,0;\beta_1/\alpha_1', \overbrace{\beta_2/\alpha_2,\ldots,\beta_2/\alpha_2}^{n\text{ times}})$, and we compute$|H_1(\tilde M)| = |\beta_1\alpha_2^n + \overbrace{\alpha_1'\beta_2\alpha_2^{n-1}+\cdots+\alpha_1'\beta_2\alpha_2^{n-1}}^{n\text{ times}}| = |\beta_1\alpha_2^n + n\alpha_1'\beta_2\alpha_2^{n-1}| = |\beta_1\alpha_2\alpha_2^{n-1}+\alpha_1\beta_2\alpha_2^{n-1}| = \alpha_2^{n-1} > 1$, if $n>1$. That is, in this case $\tilde M$ is never a homology sphere.

Third case: $(\alpha_1,\alpha_2)=d$, with $d\notin\{1,n\}$. In this case $\omega(h)$ is not the identity nor an n-cycle, so Section 3 does not apply.

However we can proceed in a similar way as before. We need an extension of the Torus Lemma (whose proof is essentially the same plus some easy and elementary number theory tricks, and is omitted).

LEMMA. *Let T be a torus and let $a,b\subset T$ be a basis for $\pi_1(T)$. Let $\omega:\pi_1(T)\to S_n$ be the representation such that $\omega(a)=\varepsilon^\sigma$ and $\omega(b)=\varepsilon^\rho$. If $\varphi:\tilde T\to T$ is the covering space associated to ω then*

$\tilde T$ is connected if and only if $(n,\rho,\sigma)=1$,

and

a) Assume $(n,\rho,\sigma)=1$ and let $\tilde a\subset\varphi^{-1}(a)$ be a component.

1) If $(n,\sigma)=1$, then there exists $\tilde b\subset\tilde T$ a simple closed curve such that $\tilde a,\tilde b$ is a basis for $\pi_1(\tilde T)$, $\varphi(\tilde a)=a^n$, and $\varphi(\tilde b)=ba^{-\sigma^*\rho}$ where $\sigma\sigma^*\equiv 1 \pmod n$.

2) If $(n,\sigma)\neq 1,n$, then there exists $\tilde b\subset\tilde T$ a simple closed curve such that $\tilde a,\tilde b$ is a basis for $\pi_1(\tilde T)$, $\varphi(\tilde a)=a^{n/(n,\sigma)}$, and $\varphi(\tilde b)=b^{(n,\sigma)}a^{-(\sigma/(n,\sigma))^*\rho}$ where $\frac{\sigma}{(n,\sigma)}\left(\frac{\sigma}{(n,\sigma)}\right)^*\equiv 1 \pmod{\frac{n}{(n,\sigma)}}$.

b) Assume $(n,\rho,\sigma)=\zeta\neq 1$. Then $\tilde T$ has ζ components $\tilde T_1,\ldots,\tilde T_\zeta$. Let $\tilde a_i\subset\varphi^{-1}(a)\cap\tilde T_i$ be a component.

1) If $(n,\sigma)=\zeta$, then there exists $\tilde b_i \subset \tilde T_i$ a simple closed curve such that $\tilde a_i, \tilde b_i$ is a basis for $\pi_1(\tilde T_i)$, $\varphi(\tilde a_i)=a^{n/\zeta}$, and $\varphi(\tilde b_i)=ba^{-(\sigma/\zeta)^*(\rho/\zeta)}$ where $\frac{\sigma}{\zeta}\left(\frac{\sigma}{\zeta}\right)^* \equiv 1 \pmod{\frac{n}{\zeta}}$.

2) If $(n,\sigma)=\delta\neq\zeta$, then there exists $\tilde b_i \subset \tilde T_i$ a simple closed curve such that $\tilde a_i, \tilde b_i$ is a basis for $\pi_1(\tilde T_i)$, $\varphi(\tilde a_i)=a^{n/\delta}$, and $\varphi(\tilde b_i)=b^{\delta/\zeta}a^{-(\sigma/\delta)^*(\rho/\zeta)}$ where $\frac{\sigma}{\delta}\left(\frac{\sigma}{\delta}\right)^* \equiv 1 \pmod{\frac{n}{\delta}}$. $\square$

We are assuming $(\alpha_1\alpha_2,n)=d\neq 1,n$. If $\frac{\alpha_1\alpha_2}{d}\left(\frac{\alpha_1\alpha_2}{d}\right)^* \equiv 1 \pmod{\frac{n}{d}}$, let k be an integer such that $\frac{\alpha_1\alpha_2}{d}\left(\frac{\alpha_1\alpha_2}{d}\right)^* = 1-k\frac{n}{d}$. Call $T_i=q_i\times S^1$, $i=0,1,2$.

Notice that if $\tilde h$ is a component of $\varphi^{-1}(h)$, then $\varphi(h)=h^{n/d}$.

For T_0, since $\omega(q_0)=\varepsilon^{-1}$, $\tilde T_0$ is connected. By case (a.2) of the Lemma, we find $\tilde q_0\subset \tilde T_0$ such that $\varphi(\tilde q_0)=q_0^d h^{(\alpha_1\alpha_2/d)^*}$. Since $\omega(m_0)=\varepsilon^{-1}$, $\tilde m_0=\varphi^{-1}(m_0)$ is connected and $\varphi(\tilde m_0)=m_0^n$. If, say, $\tilde m_0=\tilde q_0^{A_0}\tilde h^{B_0}$, we compute $\varphi(\tilde m_0)= q_0^{A_0 d}h^{A_0(\alpha_1\alpha_2/d)^*+B_0(n/d)}$; then $A_0=\frac{n}{d}$ and $B_0=-\left(\frac{\alpha_1\alpha_2}{d}\right)^*$.

For T_1, the notation of the Lemma is $\sigma=\alpha_1\alpha_2$ and $\rho=-\alpha_2\beta_1$; then $(n,\rho,\sigma)=(n,-\alpha_2\beta_1,\alpha_1\alpha_2)=(n,(\alpha_2\beta_1,\alpha_1\alpha_2))=(n,\alpha_2(\beta_1,\alpha_1))=(n,\alpha_2)=\zeta_1$. Therefore $\tilde T_1$ is connected if and only if $\zeta_1=(n,\alpha_2)=1$. Similarly $\tilde T_2$ is connected if and only if $\zeta_2=(\alpha_1,n)=1$.

First sub–case: $\zeta_1=(\alpha_2,n)=1$. Then $\zeta_2=(\alpha_1,n)=d$. We have $\tilde T_1$ connected and $\tilde T_2$ with d components $\tilde T_{2,1},\dots,\tilde T_{2,d}$.

In $\tilde T_1$, by case (a.2) of the Lemma, there exists $\tilde q_1\subset\tilde T_1$ such that $\varphi(\tilde q_1)= q_1^d h^{(\alpha_1\alpha_2/d)^*\alpha_2\beta_1}$. Since $\omega(m_1)=(1)$, if $\tilde m_1$ is a component of $\varphi^{-1}(m_1)$, then $\varphi(\tilde m_1)= m_1=q_1^{\alpha_1}h^{\beta_1}$. If $\tilde m_1=\tilde q_1^{A_1}\tilde h^{B_1}$, then $\varphi(\tilde m_1)=q_1^{A_1 d}h^{A_1(\alpha_1\alpha_2/d)^*\alpha_2\beta_1+B_1(n/d)}$. Comparing the exponents we get $\alpha_1=A_1 d$ or $A_1=\alpha_1/d$, and $\beta_1=A_1(\alpha_1\alpha_2/d)^*\alpha_2\beta_1+B_1(n/d)=(\alpha_1\alpha_2/d)^*(\alpha_1\alpha_2/d)\beta_1+B_1(n/d)=(1-k(n/d))\beta_1+B_1(n/d)$, or $B_1=k\beta_1$.

Now in $\tilde T_{2,i}$, by case (b.2) of the Lemma, there exists $\tilde q_{2,i}\subset\tilde T_{2,i}$ such that $\varphi(\tilde q_{2,i})=q_2 h^{(\alpha_1\alpha_2/d)^*(\alpha_1\beta_2/d)}$. We have $\varphi(\tilde m_{2,i})=m_2=q_2^{\alpha_2}h^{\beta_2}$ for $\tilde m_{2,i}$ a component of $\varphi^{-1}(m_2)\cap\tilde T_{2,i}$; if $\tilde m_{2,i}=\tilde q_{2,i}^{A_2}\tilde h^{B_2}$, then $\varphi(\tilde m_{2,i})=q_2^{A_2}h^{A_2(\alpha_1\alpha_2/d)^*(\alpha_1\beta_2/d)+B_2(n/d)}$, that is, $A_2=\alpha_2$ and $B_2=\beta_2 k$.

By the Riemann–Hurwitz formula, $2-2\tilde g=2d-(d-1)-(d-1)-d(1-1)=2$; therefore

$$\tilde{M} = (O, 0; \frac{-(\alpha_1\alpha_2/d)^*}{(n/d)}, \frac{\beta_1 k}{(\alpha_1/d)}, \overbrace{\frac{\beta_2 k}{\alpha_2}, \dots, \frac{\beta_2 k}{\alpha_2}}^{d \text{ times}}).$$

Notice that $|H_1(\tilde{M})| = | - (\alpha_1\alpha_2/d)^*(\alpha_1/d)\alpha_2^d + (n/d)\beta_1 k\alpha_2^d + (n/d)(\alpha_1/d)\beta_2 k\alpha_2^{d-1}| = | - (1 - k(n/d))\alpha_2^{d-1} + k(n/d)\beta_1\alpha_2\alpha_2^{d-1} + k(n/d)\alpha_1\beta_2\alpha_2^{d-1}| = \alpha_2^{d-1} > 1$, if $d > 1$. Therefore $\tilde{M}$ is never a homology sphere.

Second sub–case: $\zeta_1 = (\alpha_2, n) = d$; then $\zeta_2 = (\alpha_1, n) = d$. This is symmetric to First Sub–case; then

$$\tilde{M} = (O, 0; \frac{-(\alpha_1\alpha_2/d)^*}{(n/d)}, \overbrace{\frac{\beta_1 k}{\alpha_1}, \dots, \frac{\beta_1 k}{\alpha_1}}^{d \text{ times}}, \frac{\beta_2 k}{(\alpha_2/d)}),$$

and $|H_1(\tilde{M})| = \alpha_1^{d-1} > 1$. Also $\tilde{M}$ is not a homology sphere.

Third sub–case: $\zeta_1 = (\alpha_2, n) = c_1 \neq 1, d$; then $\zeta_2 = (\alpha_1, n) = c_2 \neq 1, d$ and $d = c_1 c_2$. We see that $\tilde{T}_1$ has c_1 components, $\tilde{T}_{1,1}, \dots, \tilde{T}_{1,c_1}$ and $\tilde{T}_2$ has c_2 components $\tilde{T}_{2,1}, \dots, \tilde{T}_{2,c_2}$. By (b.2) of the Lemma, there exists $\tilde{q}_{1,i} \subset \tilde{T}_{1,i}$ such that $\varphi(\tilde{q}_{1,i}) = q_1^{d/c_1} h^{(\alpha_1\alpha_2/d)^*(\alpha_2\beta_1/c_1)}$, and there exists $\tilde{q}_{2,j} \subset \tilde{T}_{2,j}$ such that $\varphi(\tilde{q}_{2,j}) = q_2^{d/c_2} h^{(\alpha_1\alpha_2/d)^*(\alpha_1\beta_2/c_2)}$.

In $\tilde{T}_{1,i}$, if $\tilde{m}_{1,i}$ is a component of $\varphi^{-1}(m_1) \cap \tilde{T}_{1,i}$, then $\varphi(\tilde{m}_{1,i}) = m_1 = q^{\alpha_1} h^{\beta_1}$; if $\tilde{m}_{1,i} = \tilde{q}_{1,i}^{A_1} \tilde{h}^{B_1}$, then $\varphi(\tilde{m}_{1,i}) = q_1^{A_1 c_2} h^{A_1(\alpha_1\alpha_2/d)^*(\alpha_2\beta_1/c_1) + B_1(n/d)}$; we obtain $A_1 = \alpha_1/c_2 = \alpha_1/(n, \alpha_1)$, and $B_1 = \beta_1 k$. In $\tilde{T}_{2,j}$, similarly we get $A_2 = \alpha_2/c_1 = \alpha_2/(n, \alpha_2)$, and $B_2 = \beta_2 k$.

By the Riemann–Hurwitz formula $2 - 2\tilde{g} = 2d - (d-1) - c_1(c_2 - 1) - c_2(c_1 - 1) = -c_1c_2 + c_1 + c_2 + 1$; then $\tilde{g} = (c_1c_2 - c_1 - c_2 + 1)/2 = (c_1 - 1)(c_2 - 1)/2 \geq 1$.

Therefore

$$\tilde{M} = (O, \frac{(c_1 - 1)(c_2 - 1)}{2}; \frac{-(\alpha_1\alpha_2/d)^*}{(n/d)},$$

$$\overbrace{\frac{\beta_1 k}{(\alpha_1/(\alpha_1, n))}, \dots, \frac{\beta_1 k}{(\alpha_1/(\alpha_1, n))}}^{(\alpha_2, n) \text{ times}}, \overbrace{\frac{\beta_2 k}{(\alpha_2/(\alpha_2, n))}, \dots, \frac{\beta_2 k}{(\alpha_2/(\alpha_2, n))}}^{(\alpha_1, n) \text{ times}})$$

and $\tilde{M}$ is never a homology sphere.

We have proved

THEOREM 1. *Let* $\varphi : \tilde{M} \to S^3$ *be the* $n-$*cyclic branched covering along* τ_{α_1,α_2} *and assume* $\beta_1\alpha_2 + \alpha_1\beta_2 = -1$.

1) If $(\alpha_1\alpha_2, n) = 1$ *and* $\alpha_1\alpha_2(\alpha_1\alpha_2)^* = 1 - kn$, *then*

$$\tilde{M} = (O, 0; \frac{-(\alpha_1\alpha_2)^*}{n}, \frac{\beta_1 k}{\alpha_1}, \frac{\beta_2 k}{\alpha_2})$$

and $|H_1(\tilde{M})| = 1$.

2) If $(\alpha_1\alpha_2, n) = n$, *then*

$$\tilde{M} = (O, \tilde{g}; \overbrace{\frac{\beta_1}{\alpha_1'}, \ldots, \frac{\beta_1}{\alpha_1'}}^{n_1 \text{ times}}, \overbrace{\frac{\beta_2}{\alpha_2'}, \ldots, \frac{\beta_2}{\alpha_2'}}^{n_2 \text{ times}})$$

and

$$|H_1(\tilde{M})| = \begin{cases} \alpha_2^{n-1} & \textit{if } n|\alpha_1 \\ \alpha_1^{n-1} & \textit{if } n|\alpha_2 \\ \infty & \textit{otherwise} \end{cases}$$

where $\alpha_i' = \alpha_i/(n, \alpha_i)$, *and* $n_i = n/(n, \alpha_i)$, $i = 1, 2$, *and* $\tilde{g} = (n + 1 - (\alpha_1, n) - (\alpha_2, n))/2$.

3) If $(\alpha_1\alpha_2, n) = d \neq 1, n$ *and* $\frac{\alpha_1\alpha_2}{d}\left(\frac{\alpha_1\alpha_2}{d}\right)^* = 1 - k\frac{n}{d}$, *then*

a) If $(\alpha_2, n) = 1$, *then*

$$\tilde{M} = (O, 0; \frac{-(\alpha_1\alpha_2/d)^*}{(n/d)}, \frac{\beta_1 k}{(\alpha_1/d)}, \overbrace{\frac{\beta_2 k}{\alpha_2}, \ldots, \frac{\beta_2 k}{\alpha_2}}^{d \text{ times}}).$$

and $|H_1(\tilde{M})| = \alpha_2^{d-1}$.

b) If $(\alpha_2, n) = d$, *then*

$$\tilde{M} = (O, 0; \frac{-(\alpha_1\alpha_2/d)^*}{(n/d)}, \overbrace{\frac{\beta_1 k}{\alpha_1}, \ldots, \frac{\beta_1 k}{\alpha_1}}^{d \text{ times}}, \frac{\beta_2 k}{(\alpha_2/d)}),$$

and $|H_1(\tilde{M})| = \alpha_1^{d-1}$.

c) If $(\alpha_2, n) \neq 1, d$, *then*

$$\tilde{M} = (O, \frac{((\alpha_1, n) - 1)((\alpha_2, n) - 1)}{2}; \frac{-(\alpha_1\alpha_2/d)^*}{(n/d)},$$

$$\overbrace{\frac{\beta_1 k}{(\alpha_1/(\alpha_1,n))},\ldots,\frac{\beta_1 k}{(\alpha_1/(\alpha_1,n))}}^{(\alpha_2,n)\ \text{times}},\overbrace{\frac{\beta_2 k}{(\alpha_2/(\alpha_2,n))},\ldots,\frac{\beta_2 k}{(\alpha_2/(\alpha_2,n))}}^{(\alpha_1,n)\ \text{times}})$$

and $|H_1(\tilde{M})| = \infty$. $\square$

COROLLARY. *Let* $\varphi : \tilde{M} \to S^3$ *be the* $n-$*cyclic branched covering along* τ_{α_1,α_2}*, then*

$\tilde{M}$ *is a homology sphere if and only if* $(\alpha_1\alpha_2, n) = 1$. $\square$

5. *Proof of the main theorem.* Along this section $\tau_{p,q} \subset S^3$ will denote the p,q-torus knot with $2 \leq p < q$. If $k \subset S^3$ is a knot, the phrase '$\varphi : M \to (S^3, k)$ is a branched covering' will mean '$\varphi : M \to S^3$ is a branched covering branched along k'. We write $|X|$ for the number of components of the space X.

LEMMA 3 ([10]). *There exists* $\varphi : (O, 0; \pm 1/1) \to (S^3, \tau_{p,q})$ *a* pq*-fold branched covering with* $|\varphi^{-1}(\tau_{p,q})| = (p-1)(q-1)+1$. $\square$

Remark. In [10] one has $\varphi : (O, 0; 1/1) \to (S^3, \tau_{p,q})$ and $\bar{\varphi} : (O, 0; -1/1) \to (S^3, \bar{\tau}_{p,q})$ fiber preserving pq-fold branched coverings, where $\bar{\tau}_{p,q}$ is the mirror image of $\tau_{p,q}$; if $\eta : (S^3, \bar{\tau}_{p,q}) \to (S^3, \tau_{p,q})$ is a reflection, then $\eta \circ \bar{\varphi} : (O, 0; -1/1) \to (S^3, \tau_{p,q})$ gives us a (non-fiber preserving) pq-fold branched covering with $|\bar{\varphi}^{-1}(\bar{\tau}_{p,q})| = |(\eta \circ \bar{\varphi})^{-1}(\tau_{p,q})|$.

LEMMA 4. *If* $k > 0$ *and* μ *is a non-zero integer, then there exists a branched covering* $\varphi : (O, 0; \mu/1) \to (S^3, \tau_{p,q})$ *with* $|\varphi^{-1}(\tau_{p,q})| > k$.

Proof. Assume $\mu > 0$. By Lemma 3 we have $\varphi_1 : (O, 0; 1/1) \to (S^3, \tau_{p,q})$ a branched covering with $|\varphi_1^{-1}(\tau_{p,q})| \geq 3$ (the 'worst' case is $|\varphi_1^{-1}(\tau_{2,3})| = 3$).

Let $(F \times S^1, \{m_0, m_1, m_2\})$ be a kit for $(O, 0; 1/1, 0/1, 0/1)$, where the $\frac{0}{1}$-fibers $e_1, e_2 \subset (O, 0; 1/1)$ are preimages of $\tau_{p,q}$, and let $\omega_2 : \pi_1(F \times S^1) \to S_{\mu k}$ be the representation such that $\omega_2(q_0) = (1)$, $\omega_2(q_1) = \varepsilon$, $\omega_2(q_2) = \varepsilon^{-1}$, and $\omega_2(h) = (1)$. If $\varphi_2 : \tilde{M} \to (O, 0; 1/1)$ is the covering corresponding to ω_2, then, by Lemma 2, $\tilde{M} = (O, 0; \overbrace{1/1, \ldots, 1/1}^{\mu k\ \text{times}}, 0/1, 0/1) = (O, 0, (\mu k)/1)$ (for $\tilde{g} = (2 - 2\mu k + 2(\mu k - 1) + (1-1))/2 = 0$), and $|\varphi_2^{-1}(h)| = \mu k > k$ for any fiber $h \subset F \times S^1$ (in particular $|\varphi_2^{-1}(\varphi_1^{-1}(\tau_{p,q}))| > k$, for $e_3 \subset \varphi_1^{-1}(\tau_{p,q}) - e_1 \cup e_2$ is a fiber of $F \times S^1$). If $\omega_3 : \pi_1(O, 0; (\mu k)/1) \to S_k$ is the representation such that $\omega_3(q_0) = (1)$, and $\omega_3(h) = \varepsilon_k$, then $\varphi_3 : \tilde{N} \to (O, 0; (\mu k)/1)$, the associated covering of ω_3, is a covering space with $\tilde{N} = (O, 0; \mu/1)$. Also $|\varphi_3^{-1}(h)| = 1$ for any fiber h. Then $\varphi = \varphi_1\varphi_2\varphi_3 : (O, 0; \mu/1) \to (S^3, \tau_{p,q})$ is a branched covering with $|\varphi^{-1}(\tau_{p,q})| > k$. The case $\mu < 0$ is handled analogously. □

LEMMA 5. *If μ is a non-zero integer and $g \geq 0$, then there exists a branched covering $\psi : (O, g; \mu/1) \to (S^3, \tau_{p,q})$.*

Proof. Let $e_1, e_2, \ldots, e_{2(g+1)} \subset \varphi^{-1}(\tau_{p,q})$ be components, where the covering $\varphi : (O, 0; \mu/1) \to (S^3, \tau_{p,q})$ is as in Lemma 4.

Let $(F \times S^1, \{m_0, m_1, \ldots, m_{2(g+1)}\})$ be a kit for $(O, 0; \mu/1, 0/1, \ldots, 0/1)$, where the $\frac{0}{1}$–fibers are the curves $e_1, \ldots, e_{2(g+1)}$. Let $\omega_1 : F \times S^1 \to S_2$ be the representation such that $\omega_1(q_0) = (1)$, $\omega_1(q_i) = (1, 2)$ for $i = 1, \ldots, 2(g + 1)$, and $\omega_1(h) = (1)$, and let $\varphi_1 : \tilde{M} \to (O, 0; \mu/1)$ be the associated covering to ω_1. Then, by Lemma 2, $\tilde{M} = (O, \tilde{g}; \mu/1, \mu/1)$ with $\tilde{g} = (2 - 2 \cdot 2 + 2(g + 1)(2 - 1))/2 = g$; that is, $\tilde{M} = (O, g; (2\mu)/1)$. By taking the double covering of $\tilde{M}$ with representation ω_3 such that $\omega_3(q_0) = (1)$ and $\omega_3(h) = (1, 2)$, we obtain, after composing, a branched covering $\psi : (O, g; \mu/1) \to (S^3, \tau_{p,q})$ as desired. □

THEOREM 2. *If $e = \displaystyle\sum_{i=1}^{t} \frac{\beta_i}{\alpha_i} \neq 0$, then $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ is a branched covering of $(S^3, \tau_{p,q})$.*

Proof. By induction on t. Assume $t = 1$. If $\alpha_1 = 1$, then Lemma 5 gives us the desired covering. So assume $\alpha_1 > 1$, and let $(F \times S^1, \{m_0, m_1\})$ be a kit for $(O, g; 0/1, \beta_1/1)$. Let $\omega : \pi_1(F \times S^1) \to S_{\alpha_1}$ be the representation such that $\omega(q_0) = \varepsilon$, $\omega(q_1) = \varepsilon^{-1}$, and $\omega(h) = \varepsilon^{\beta_1^*}$, where $\beta_1\beta_1^* \equiv 1 \pmod{\alpha_1}$. If $\varphi : \tilde{M} \to (O, g; \beta_1/1)$ is the branched covering associated to ω, then, by Lemma 1, we have that $\tilde{M} = (O, g; B_0/A_0, B_1/A_1)$ with $A_0 = \alpha_1$, $B_0 = 0 + (\beta_1^*)^*(1)(1) = \beta_1$, and $A_1 = 1, B_1 = \beta_1 + (\beta_1^*)^*(-1) = 0$, that is $\tilde{M} = (O, g; \beta_1/\alpha_1, 0/1) = (O, g; \beta_1/\alpha_1)$. Notice that the branching of φ is along the $(\beta_1/1)$-fiber and the $\frac{0}{1}$-fiber which we may assume are preimages of $\tau_{p,q}$ in the covering of Lemma 5.

Assume that the theorem is true for t; that is, if $\sum_{i=1}^{t} \frac{\beta_i}{\alpha_i} \neq 0$, then the manifold $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ is a branched covering of $(S^3, \tau_{p,q})$.

Assume now that $e = \sum \frac{\beta_i}{\alpha_i} + \frac{\beta}{\alpha} \neq 0$ and let $i \in \{1, 2, \ldots, t\}$.

1. If $(\alpha_i, \alpha) = 1$, choose integers r_i, b_i such that $(\alpha\beta_i)/\alpha_i = r_i + (b_i/\alpha_i)$, and write $a_i = \alpha_i$.

2. If $(\alpha_i, \alpha) = d_i$ with $d_i \notin \{1, \alpha\}$, write $\alpha'_i = \alpha_i/d_i$, $\alpha' = \alpha/d_i$, and choose integers r_i, b_i such that $(\alpha'\beta_i)/\alpha'_i = r_i + (b_i/\alpha'_i)$. Note that $(\alpha'\beta_i, \alpha) = \alpha'$. Write also $a_i = \alpha'_i$.

3. If $(\alpha_i, \alpha) = \alpha$, write $d_i = \alpha_i/\alpha$, $r_i = 1$, $b_i = \beta_i - d_i$ and $a_i = d_i$.

4. Write $r_0 = -\sum_{i=1}^t r_i$, $b_0 = \beta + \sum_{i=1}^t r_i$, and $a_0 = 1$.

Let $(F \times S^1, \{m_0, m_1, \ldots, m_t\})$ be a kit for $(O, g; b_0/a_0, b_1/a_1, \ldots, b_t/a_t)$, and let $\omega : \pi_1(F \times S^1) \to S_\alpha$ be the representation such that $\omega(v_j) = (1)$ for $i = 1, \ldots, 2g$ and $\{v_j\}$ a basis for $\pi_1(F)$; $\omega(q_i) = \varepsilon^{r_i}$, $i = 0, 1, \ldots, t$; and $\omega(h) = \varepsilon$. By Lemma 1, if $\varphi : \tilde{M} \to (O, g; b_0/a_0, b_1/a_1, \ldots, b_t/a_t)$ is the associated branched covering of ω, then $\tilde{M} = (O, g; B_0/A_0, B_1/A_1, \ldots, B_t/A_t)$ where

1. If $(\alpha_i, \alpha) = 1$, then $(\alpha, b_i + r_i a_i) = (\alpha, \alpha\beta_1) = \alpha$ (by construction of b_i and r_i), so that $A_i = (\alpha/\alpha)a_i = \alpha_i$ and $B_i = (b_i + r_i a_i)/\alpha = (\alpha\beta_i)/\alpha = \beta_i$.

2. If $(\alpha_i, \alpha) = d_i$ with $d_i \notin \{1, \alpha\}$, then $(\alpha, b_i + r_i a_i) = (\alpha, \alpha'\beta_i) = \alpha'$, so that $A_i = (\alpha/\alpha')a_i = d_i(\alpha_i/d_i) = \alpha_i$ and $B_i = (b_i + r_i a_i)/\alpha' = (\alpha'\beta)/\alpha' = \beta_i$.

3. If $(\alpha_i, \alpha) = \alpha$, then $(\alpha, b_i + r_i a_i) = (\alpha, \beta_i - d_i + d_i) = (\alpha, \beta_i) = 1$, for $(\alpha_i, \beta_i) = 1$ and $\alpha|\alpha_i$; so that $A_i = (\alpha/1)a_i = \alpha(\alpha_i/\alpha) = \alpha_i$ and $B_i = b_i + r_i a_i = \beta_i - d_i + d_i = \beta_i$.

4. $A_0 = (\alpha/(\alpha, a_0))a_0 = \alpha$ and $B_0 = b_0 + (1)^* r_0 a_0 = (\beta + \sum r_i) - \sum r_i = \beta$.

Therefore $\tilde{M} = (O, g; \beta/\alpha, \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ is a branched covering of the manifold $(O, g; b_0/a_0, b_1/a_1, \ldots, b_t/a_t)$; notice that the branching which appears in this construction is (at most) in the (b_i/a_i)-fibers. One computes easily that (as it should be; see the last theorem of Section 2) $\sum_{i=0}^t \frac{b_i}{a_i} = \alpha(\sum_{i=1}^t \frac{\beta_i}{\alpha_i} + \frac{\beta}{\alpha}) \neq 0$; therefore, by induction, $(O, g; b_0/a_0, b_1/a_1, \ldots, b_t/a_t)$ is a branched covering of $(S^3, \tau_{p,q})$, and, with this, we conclude that also the manifold $(O, g; \beta/\alpha, \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ is a branched covering of $(S^3, \tau_{p,q})$. □

Example. Following the construction of the proof of Theorem 2 we see that $(O, 0; -1/2, 1/3, 1/5)$ is a branched covering of $(O, 0; -1/2, 2/3)$ with a 5-fold covering, for $\frac{5(-1)}{2} = -2 + \frac{-1}{2}$ and $\frac{5(1)}{3} = 1 + \frac{2}{3}$. Now $(O, 0; -1/2, 2/3)$ is a branched covering of $(O, 0; -1/2)$ with a 3-fold covering, for $\frac{3(-1)}{2} = -1 + \frac{-1}{2}$. And, by the first induction step, $(O, 0; -1/2)$ is a branched covering of $(O, 0; -1/1)$ with a 2-fold covering. By, say, Lemma 3, $(O, 0; -1/1)$ is a pq-fold covering of $(S^3, \tau_{p,q})$.

That is, the Poincaré homology sphere $(O, 0; -1/2, 1/3, 1/5)$ is a branched covering of $(S^3, \tau_{p,q})$ with a $30pq$-fold covering.

Notice that , for the trefoil knot, in the first step of this example we obtained a 5-fold covering (the 5-fold cyclic covering) of $(O, 0; -1/2, 1/3, 1/5)$ over $(S^3, \tau_{2,3})$; and following the proof of Theorem 2 we get a 180-fold covering of $(O, 0; -1/2, 1/3, 1/5)$ over $(S^3, \tau_{2,3})$.

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