Some universal Montesinos knots

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Cimat

Definition

Let $k \subset S^3$ be a link.
The link k is called universal if
each c.c.o. 3-manifold M
admits a branched covering

$$\varphi : M \to (S^3, k)$$

Strategy

Given a link $k \subset S^3$

find $\varphi : S^3 \to (S^3, k)$ such that

$\varphi^{-1}(k)$ contains a sublink

which is **universal**.

Problem

Given a link $k \subset S^3$ and
a branched covering $\varphi : S^3 \longrightarrow (S^3, k)$
compute the link type of $\varphi^{-1}(k)$ in S^3.

A related problem

Given a link $k \subset S^3$ and
a branched covering $\varphi : M \longrightarrow (S^3, k)$
compute the link type of $\varphi^{-1}(k)$ in M.

Let $(B, \{\alpha_i\}_{i=1}^n)$ be a trivial n-tangle.

That is:

B is a 3-ball, and

$\alpha_1, \ldots, \alpha_n \subset B$ are n properly embedded trivial arcs

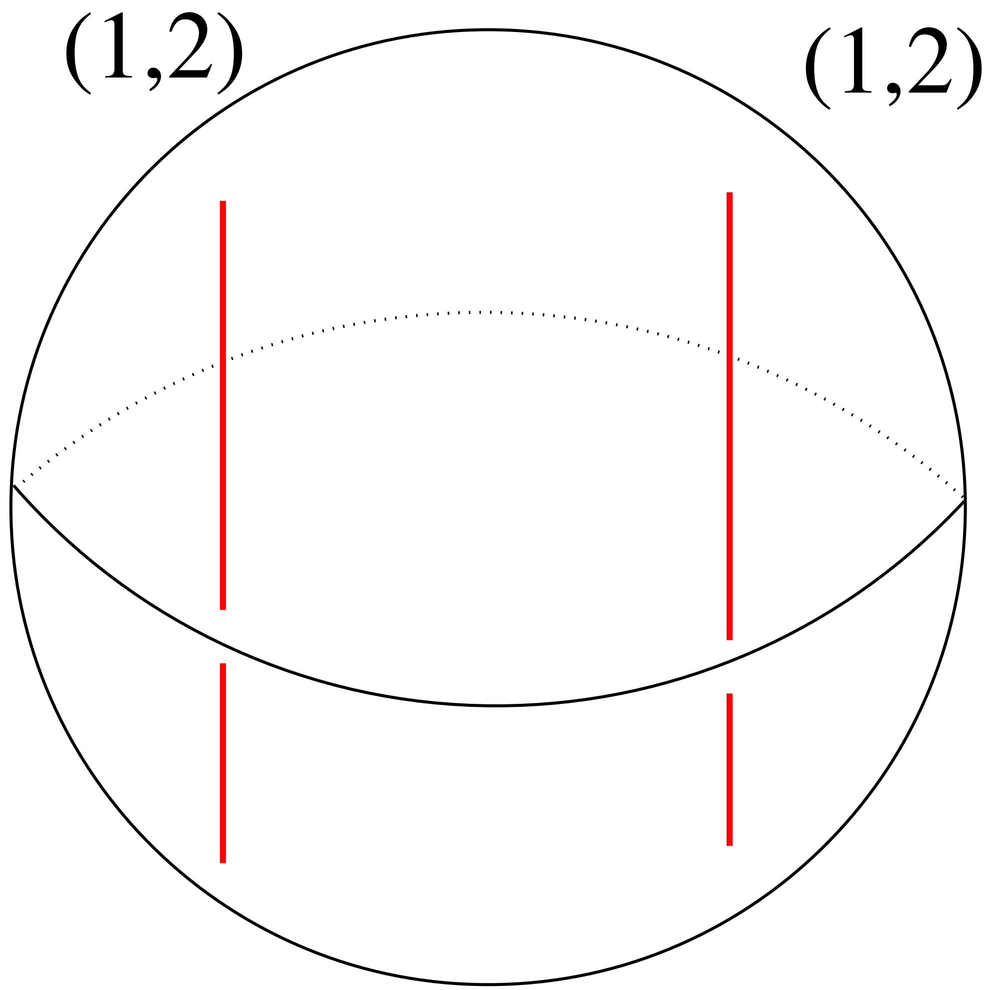
(there are n disjoint disks $D_1, \ldots, D_n \subset B$ such that $\partial D_i = \alpha_i \cup \beta_i$ with $\beta_i \subset \partial B$, and $\partial \alpha_i = \partial \beta_i$.)

Consider $\omega : \pi_1(B - \bigcup \alpha_i) \to S_d$ a representation into the symmetric group on d symbols.

We get a d-fold branched covering

$$\varphi_\omega : B_\omega \to (B, \sqcup \alpha_i).$$

Remark: B_ω is a handlebody.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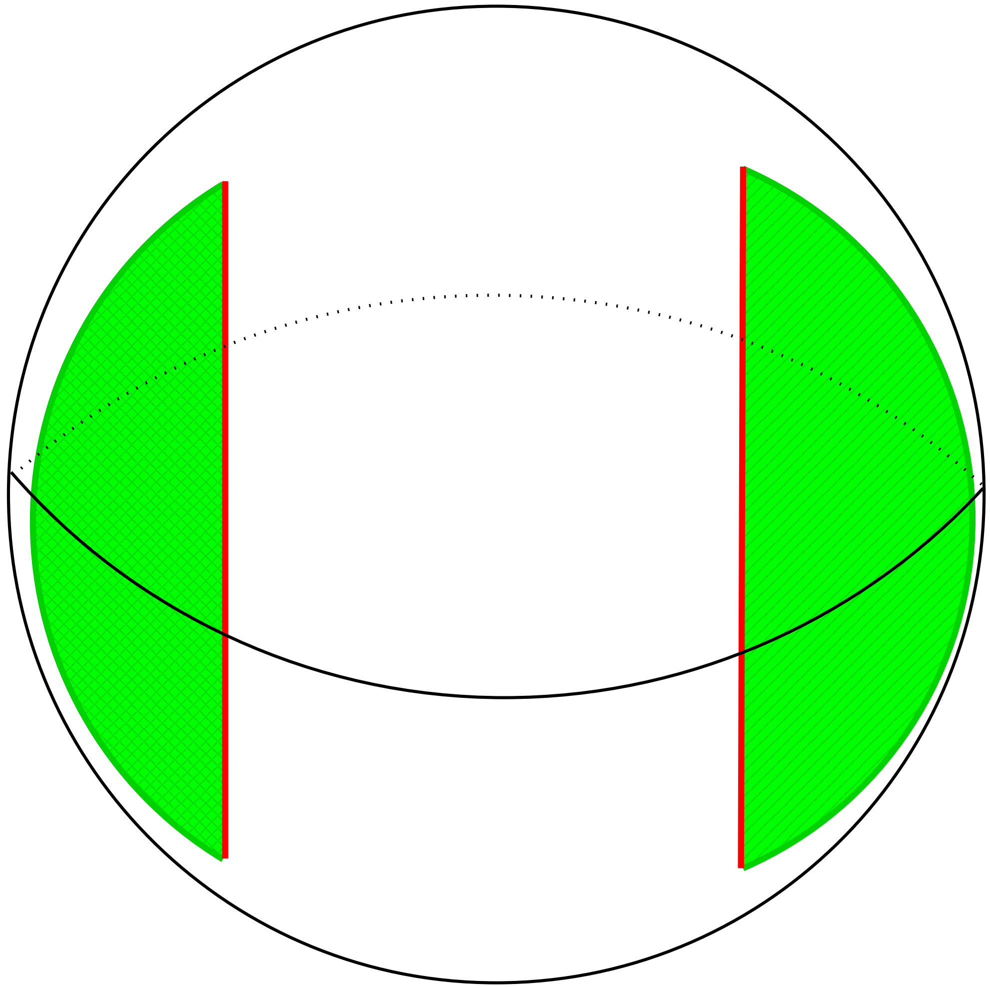
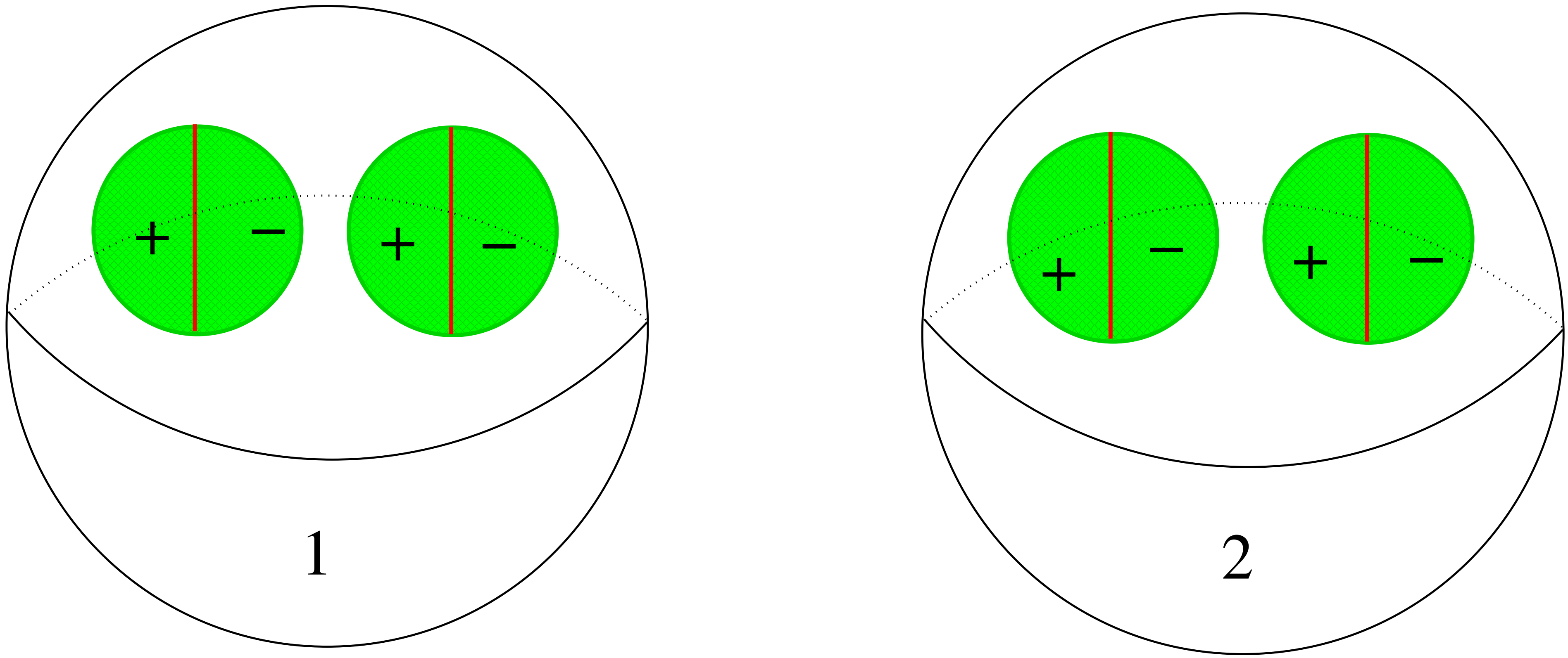
(1,2)

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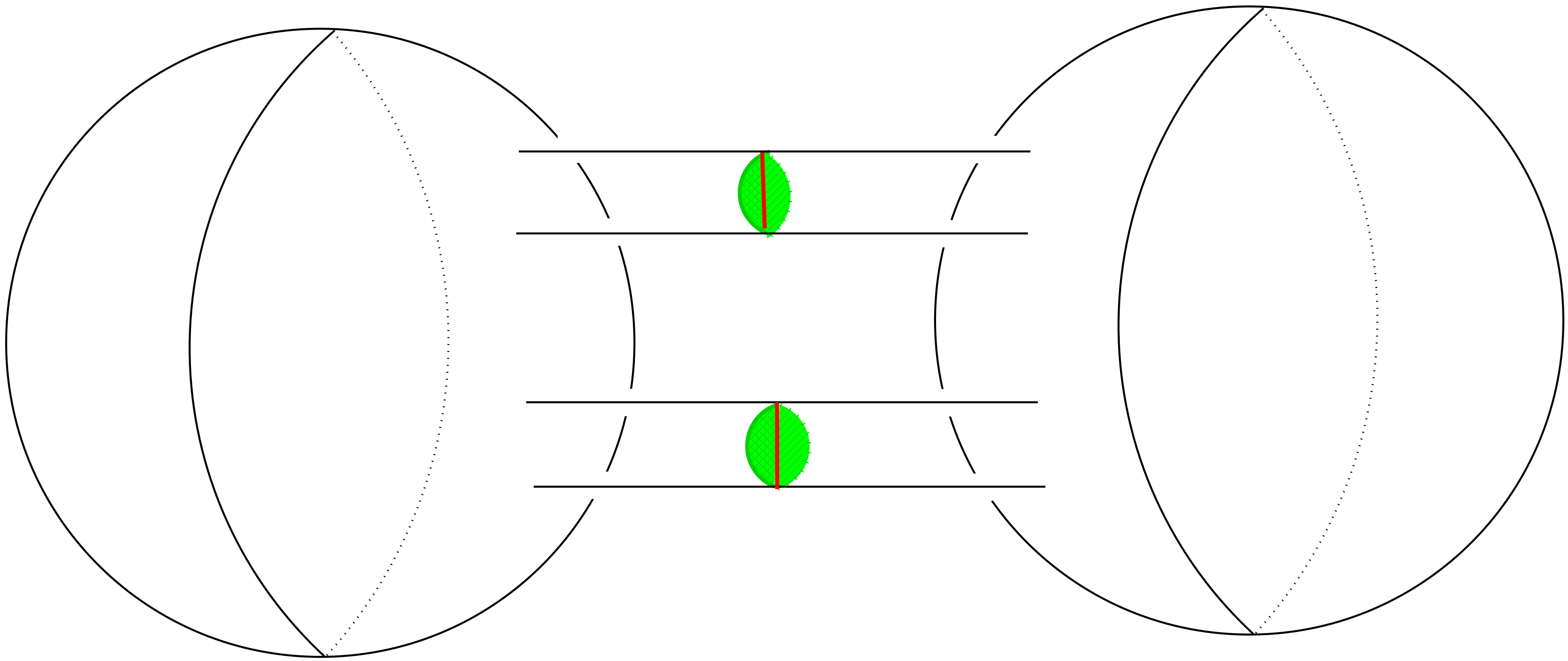

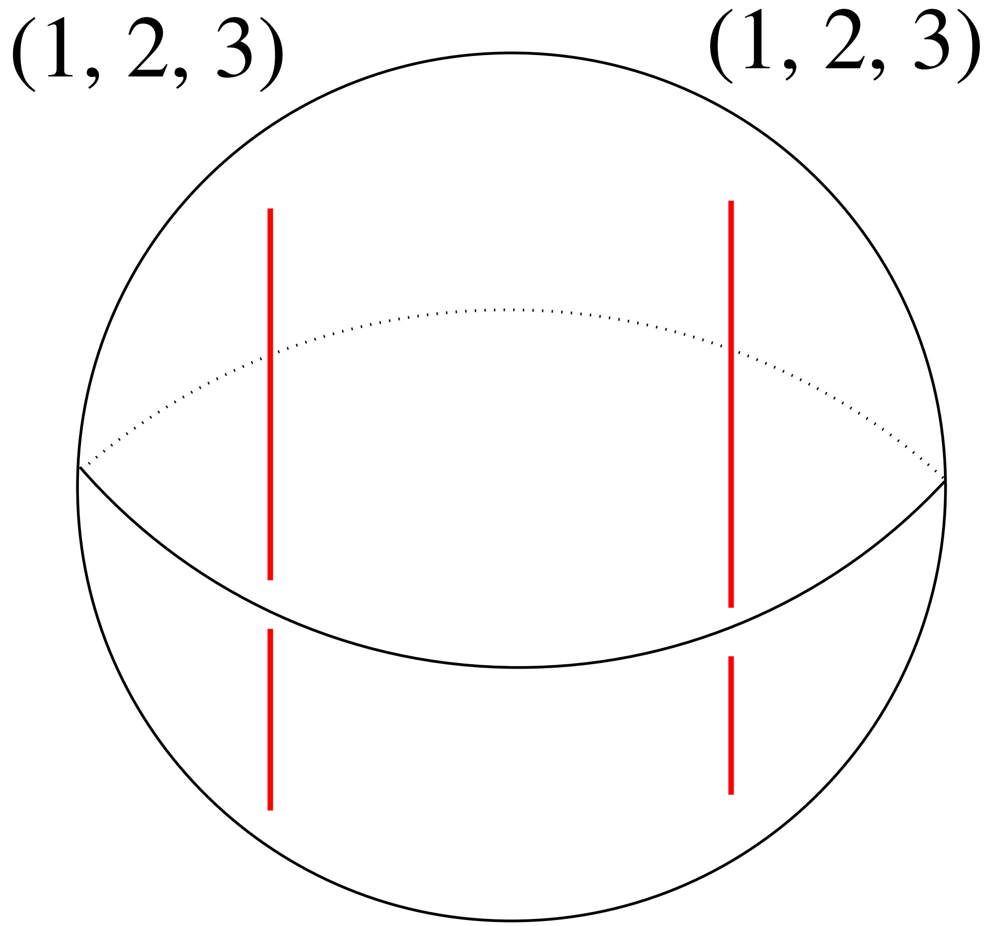
(1, 2, 3)
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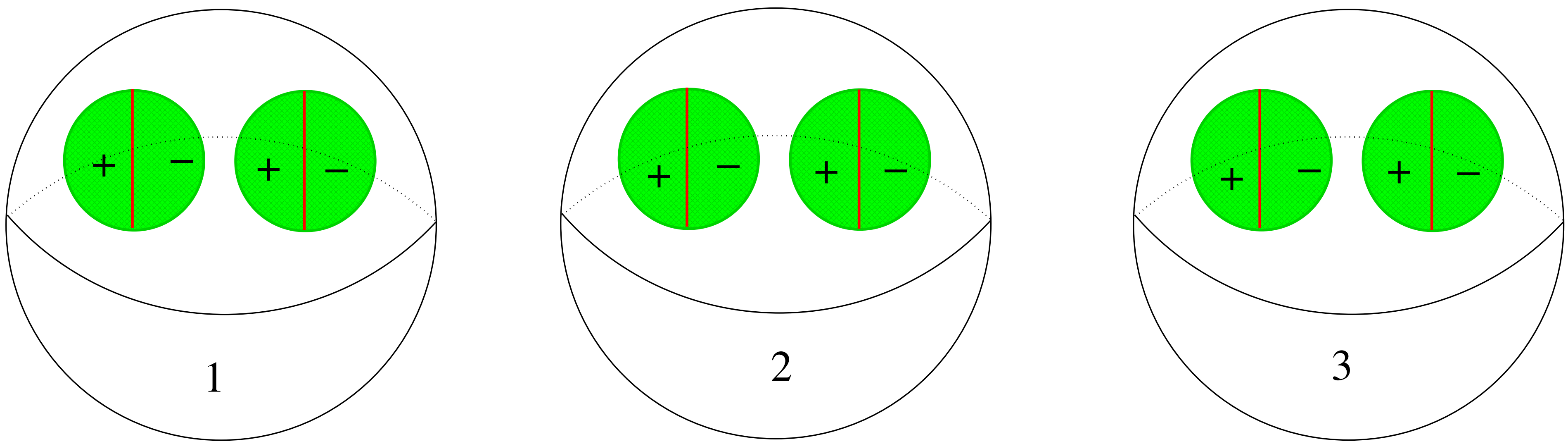
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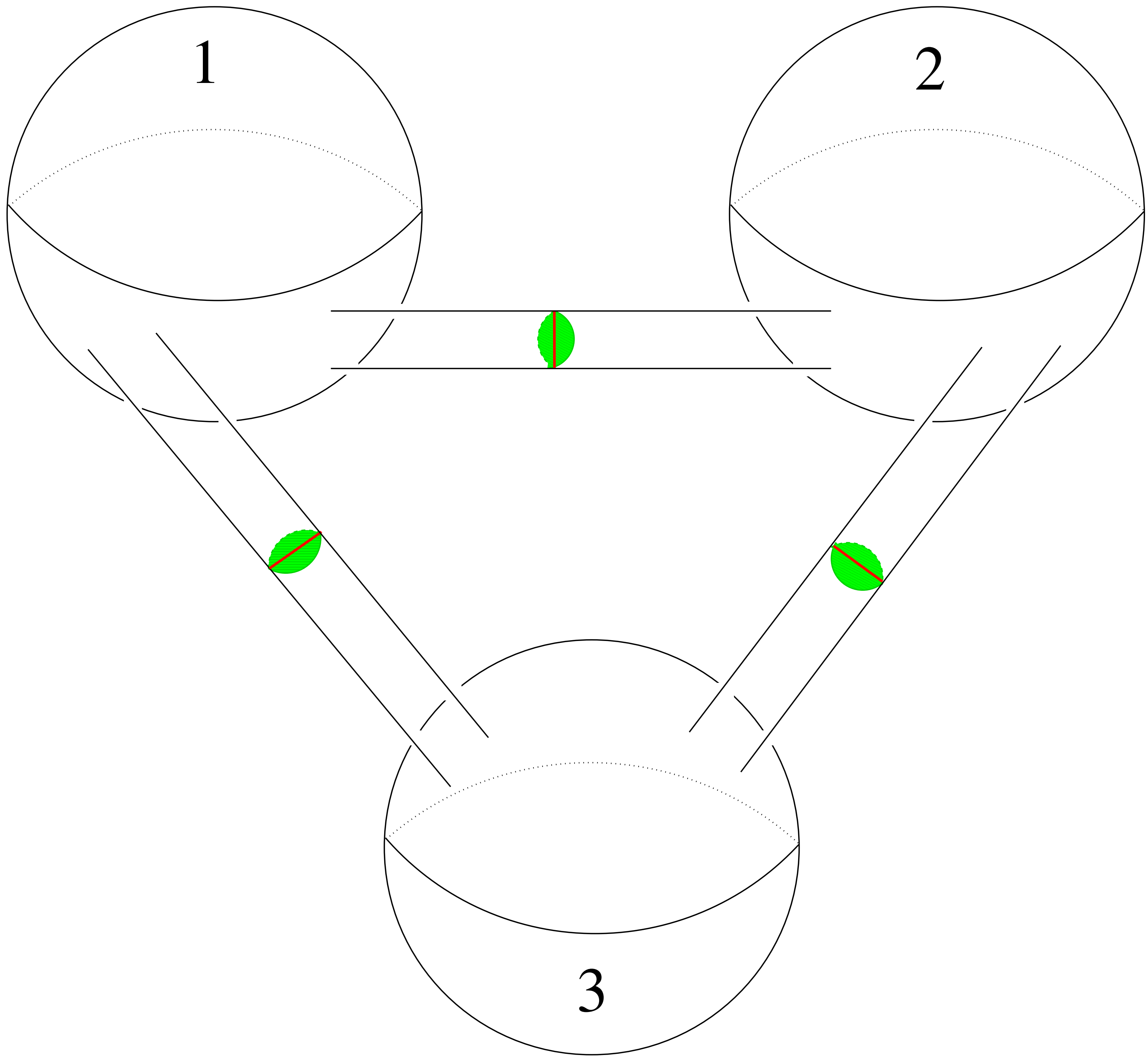
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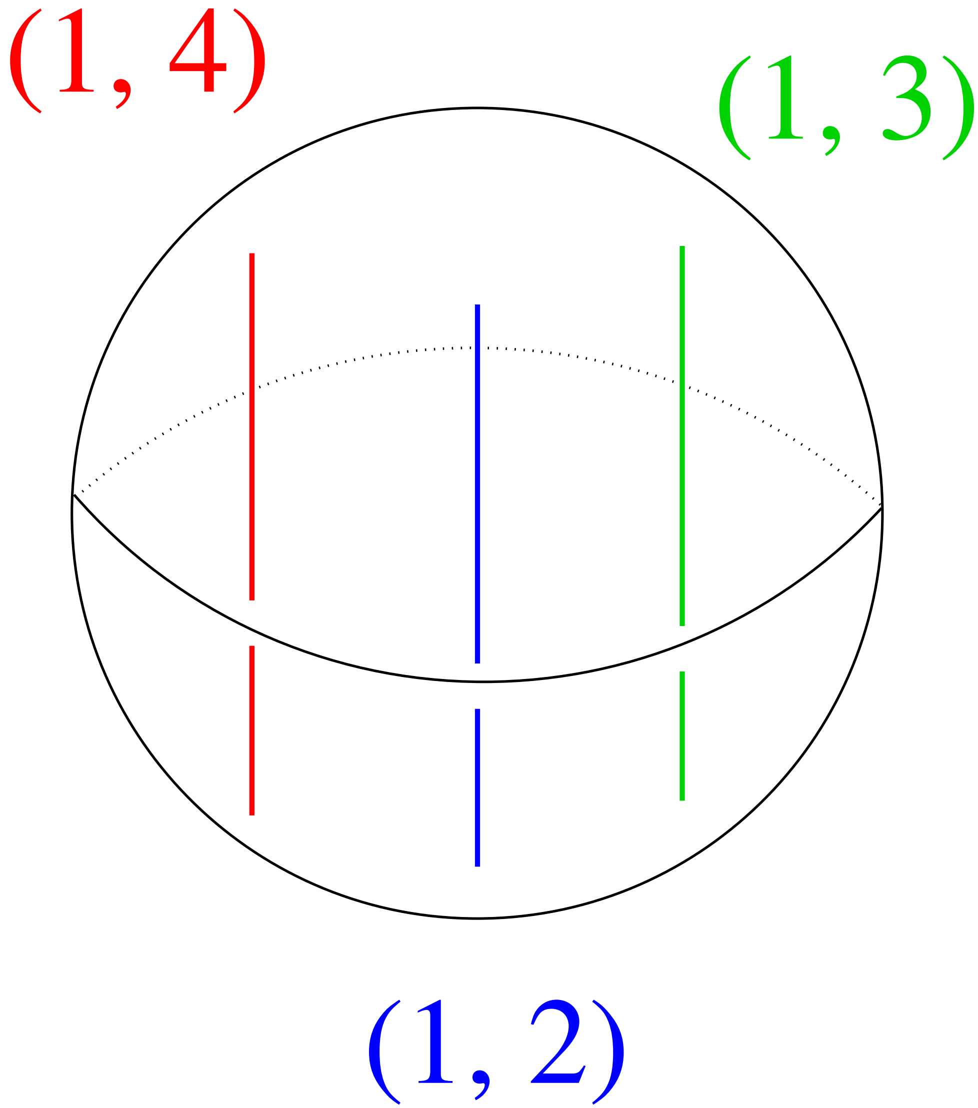
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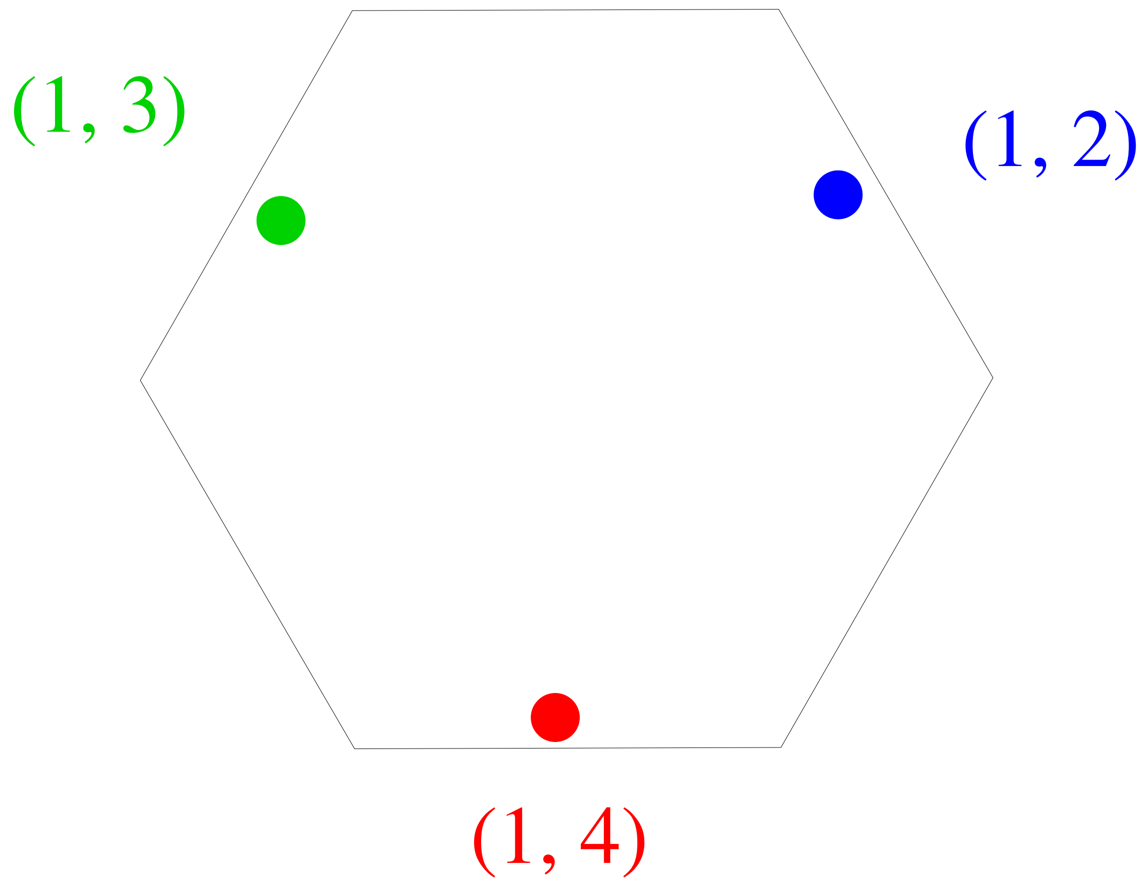
(1, 3)
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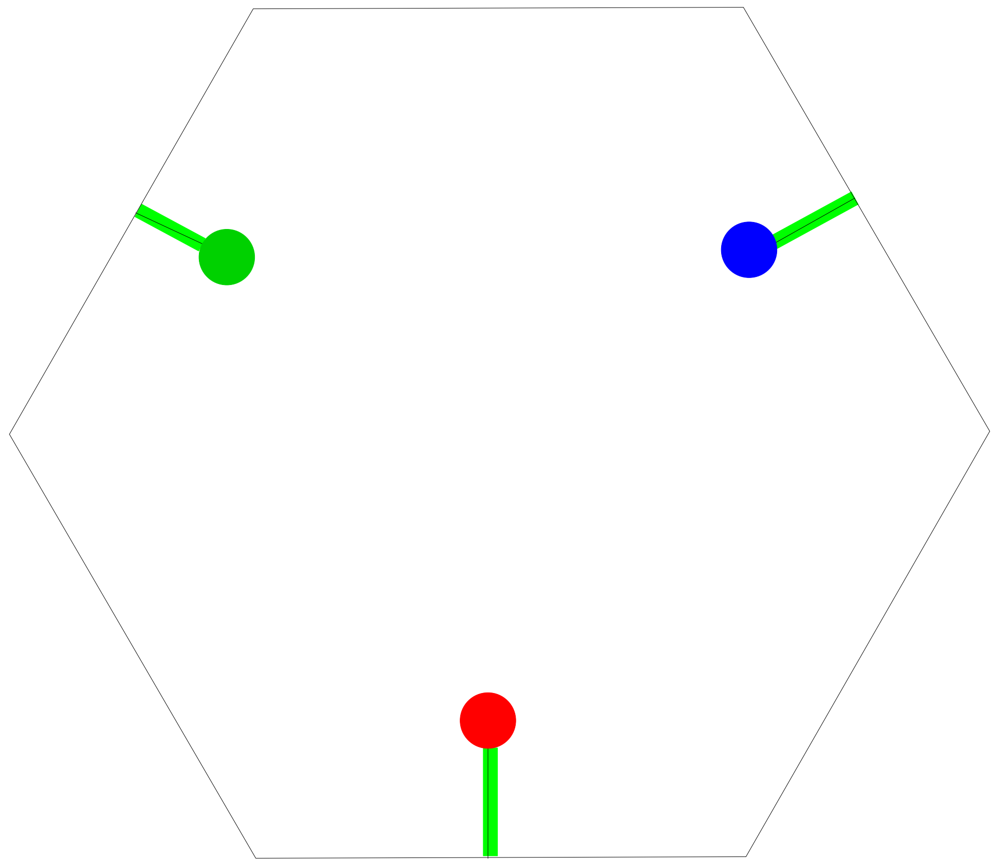

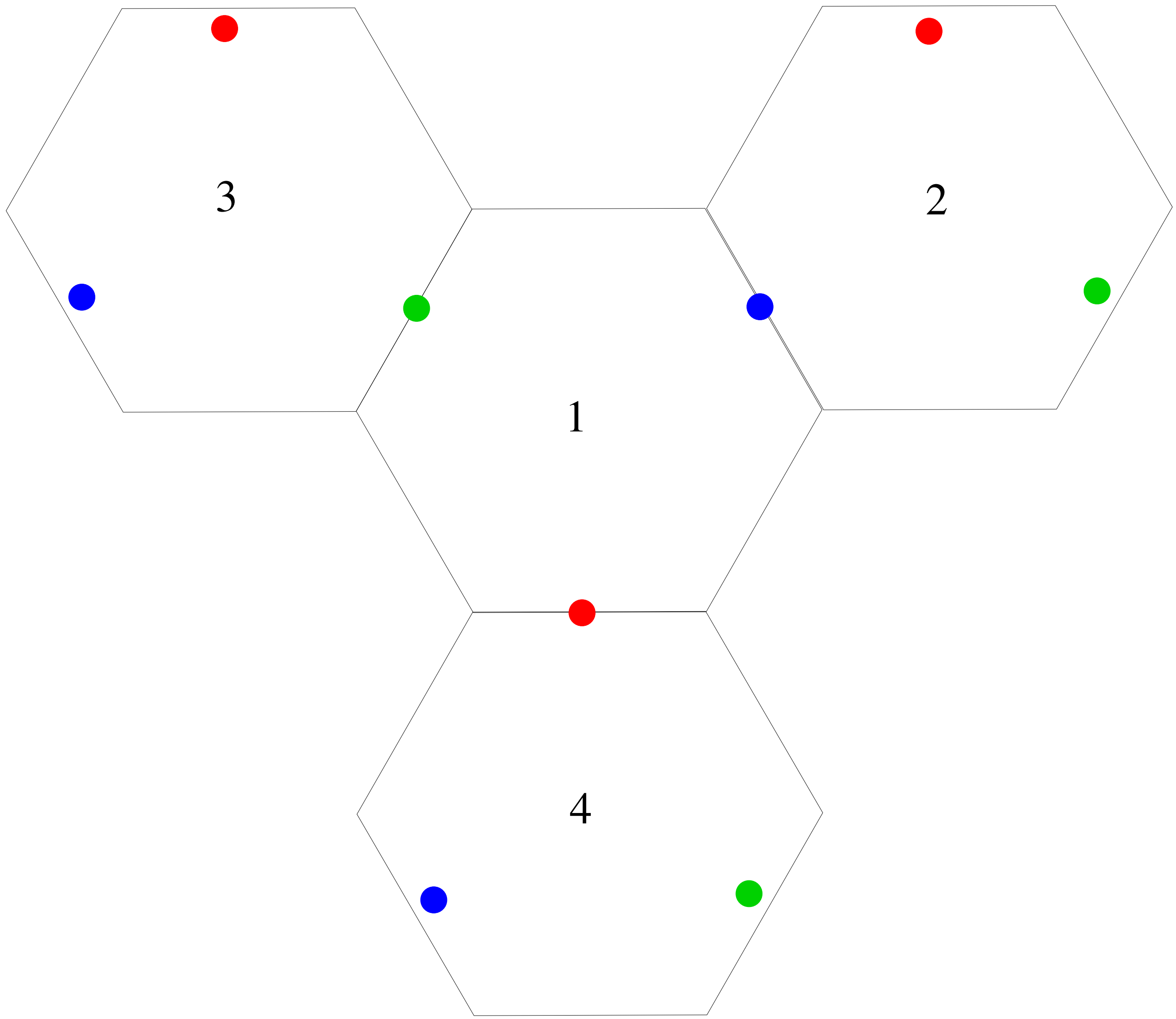
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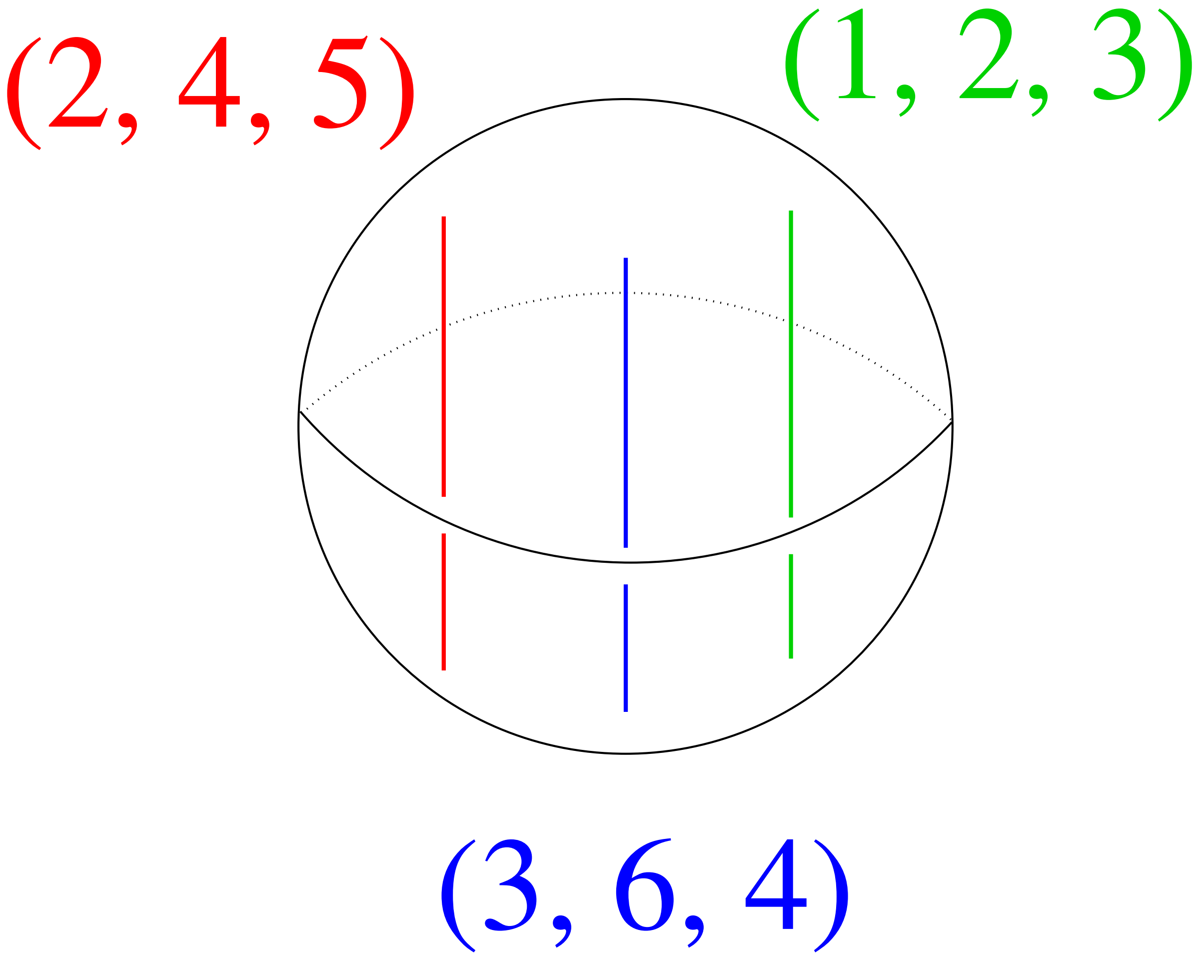
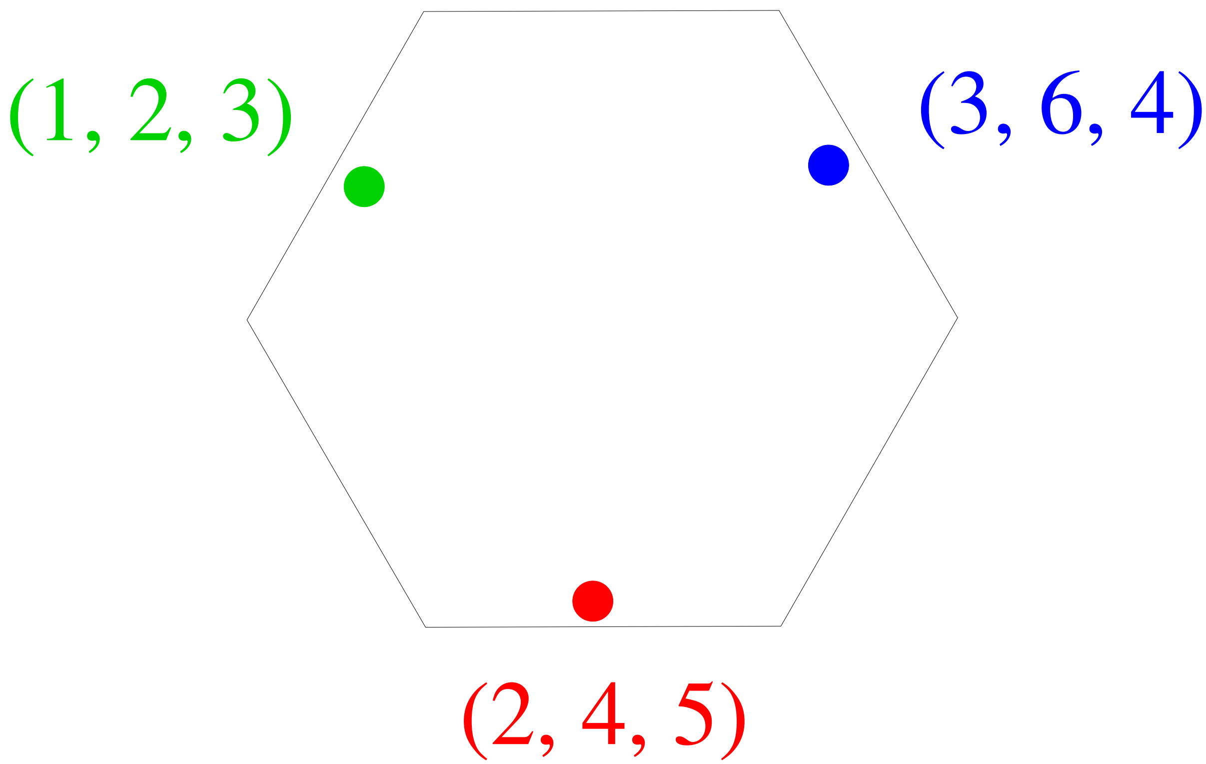
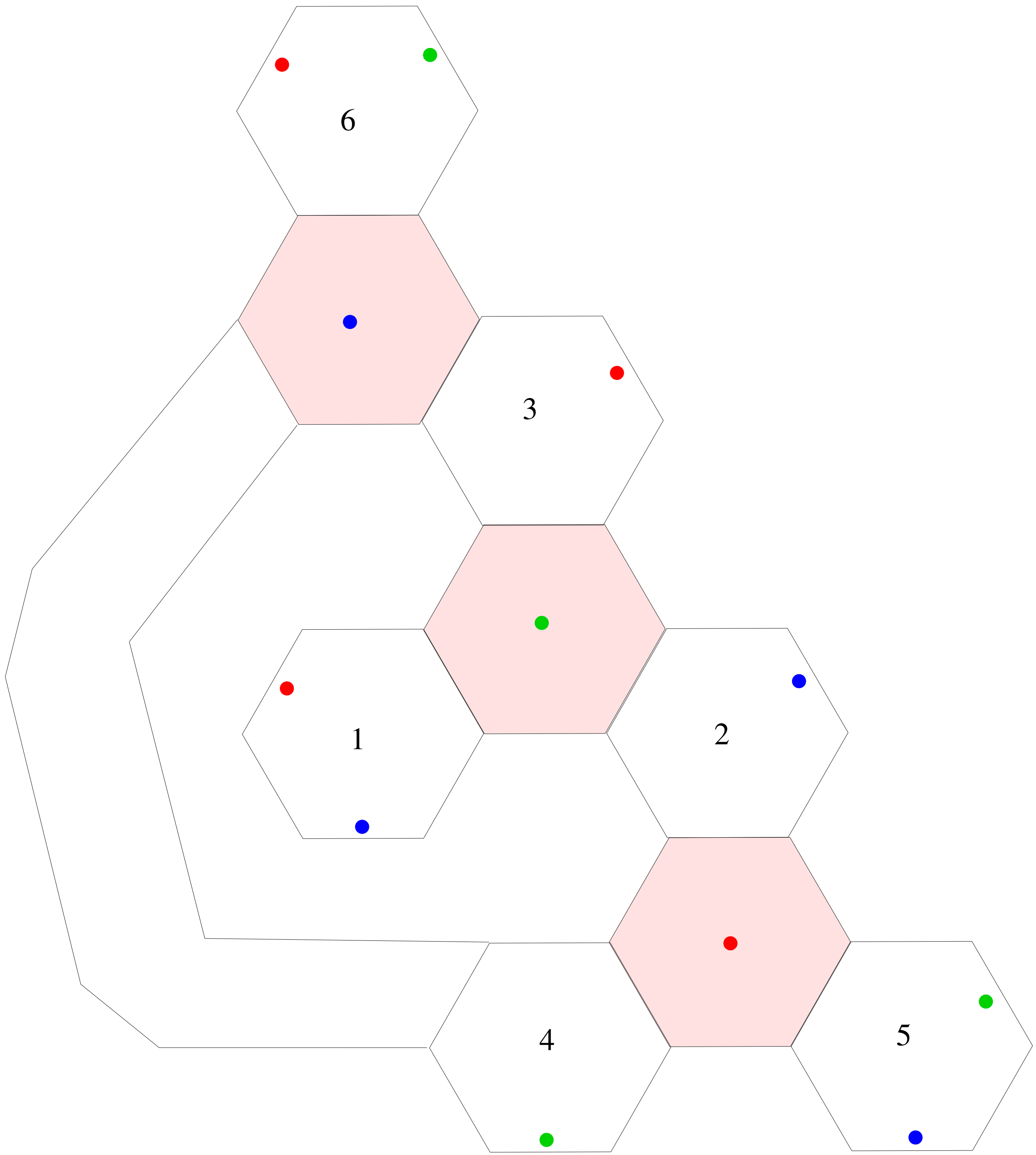

(2, 4, 5)
(1, 2, 3)
(3, 6, 4)


(1, 2, 3)
(3, 6, 4)
(2, 4, 5)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Let $k \subset S^3$ be a link in an n-bridge representation, that is,

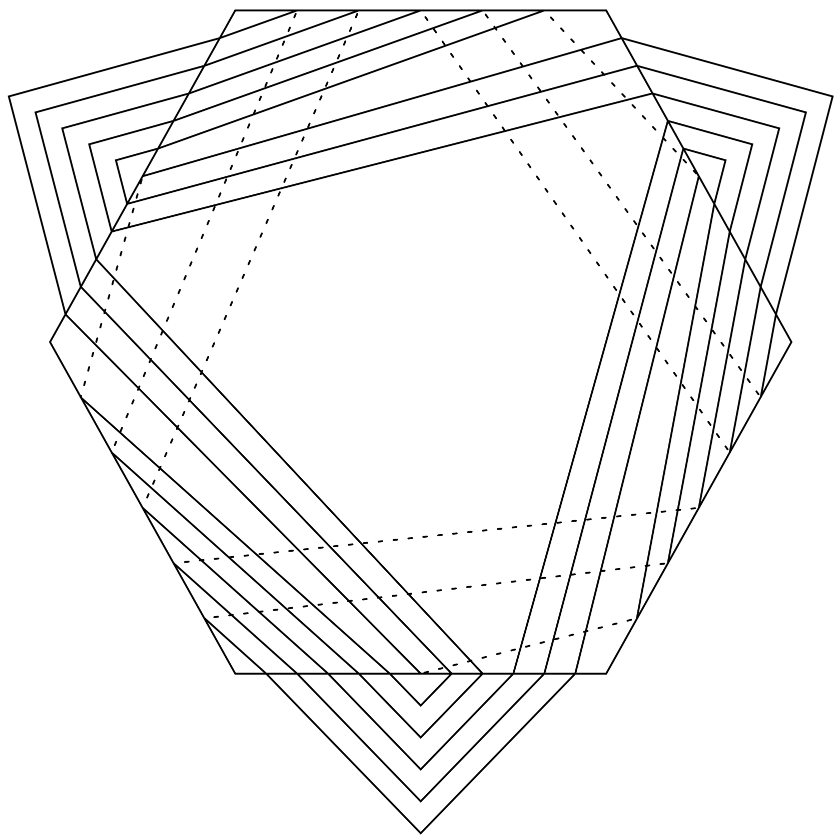
There are $(B, \{\alpha_i\})$ and $(B', \{\alpha'_i\})$ two trivial n-tangles such that

$$S^3 = B \cup_\partial B'$$

and

$$k = (\sqcup \alpha_1) \cup (\sqcup \alpha'_i)$$

We can push the arcs $\{\alpha_i'\}$ into ∂B and we get a $2n$-gonal pillowcase for k:



The knot $p(3,3,3) = m(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

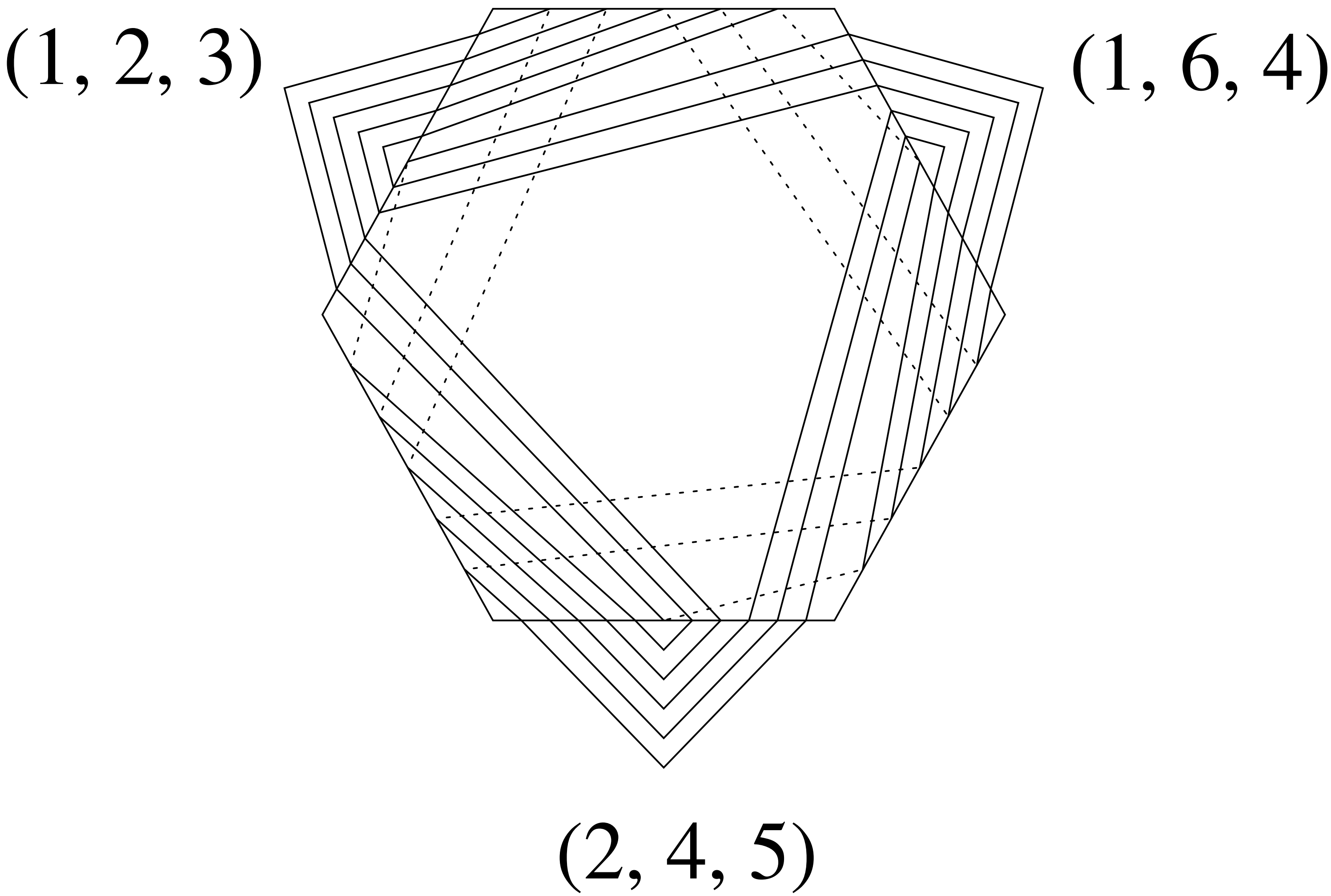
A $2n$-gonal pillowcase for k is
a 3-ball B with n trivial disjoint properly embedded arcs $\{\alpha_i\}_{i=1}^n$
and n disjoint arcs $\{\beta_i\}_{i=1}^n$ on ∂B

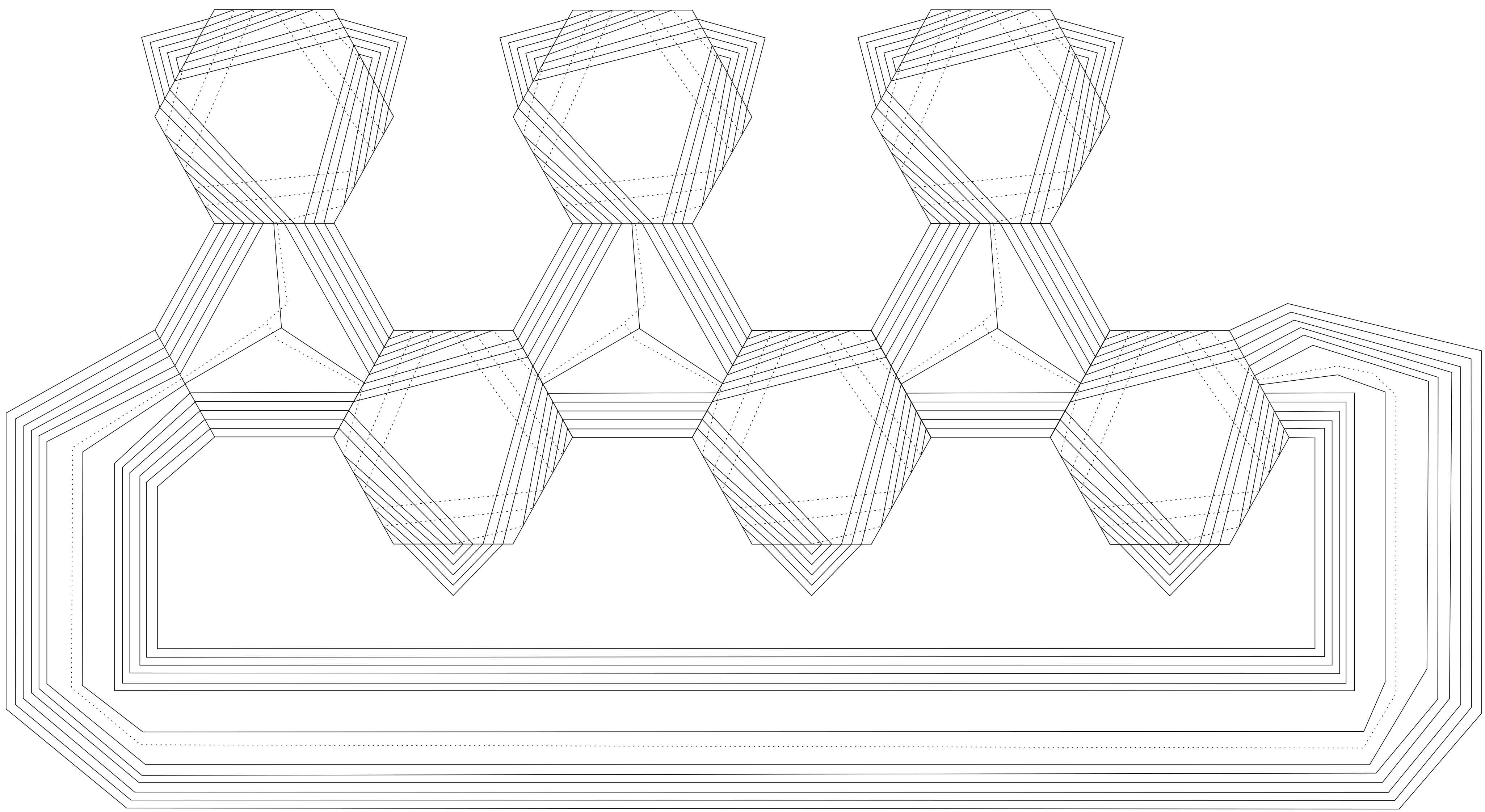
such that $k = (\sqcup \alpha_i) \cup (\sqcup \beta_i)$.

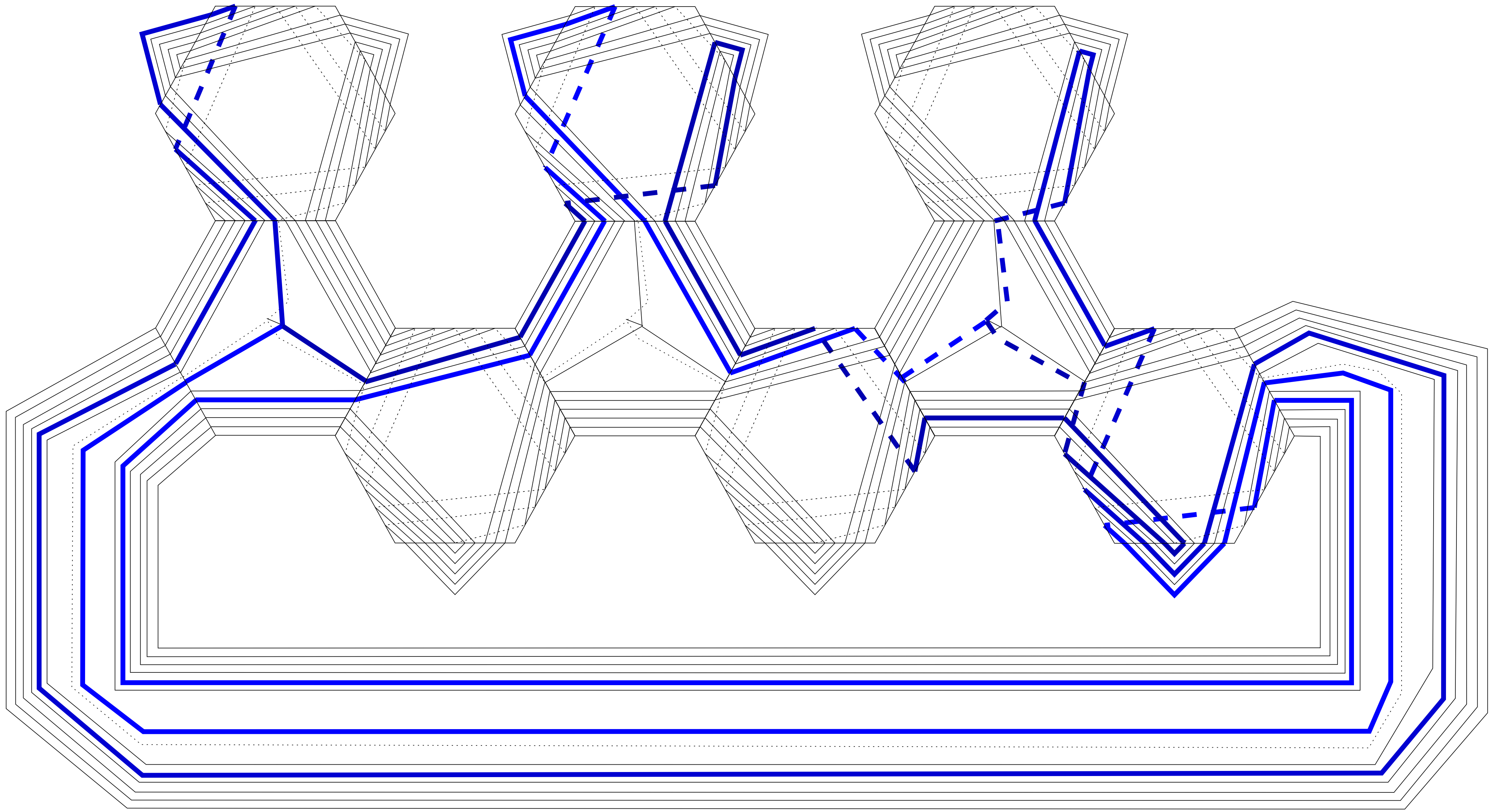
Now let $\omega : \pi_1(S^3 - k) \longrightarrow S_d$ be a transitive representation, and
let $\varphi : M \longrightarrow (S^3, k)$ be the d-fold branched covering associated to ω.

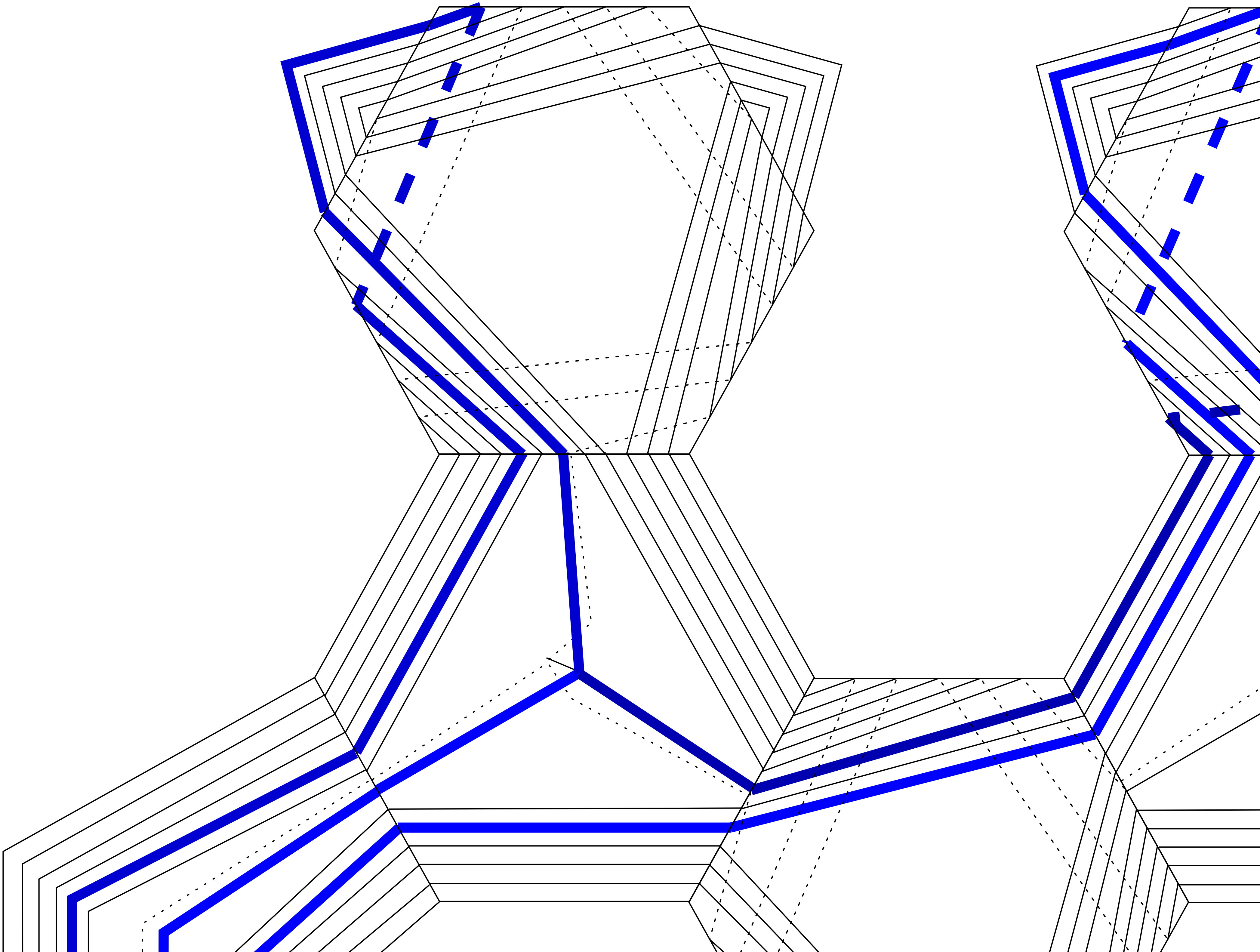
Examine $\varphi| : \varphi^{-1}(B) \to B$**.**

The preimage $\varphi^{-1}(\sqcup \alpha_i) \cup \varphi^{-1}(\sqcup \beta_i)$
is **not** a 1-manifold

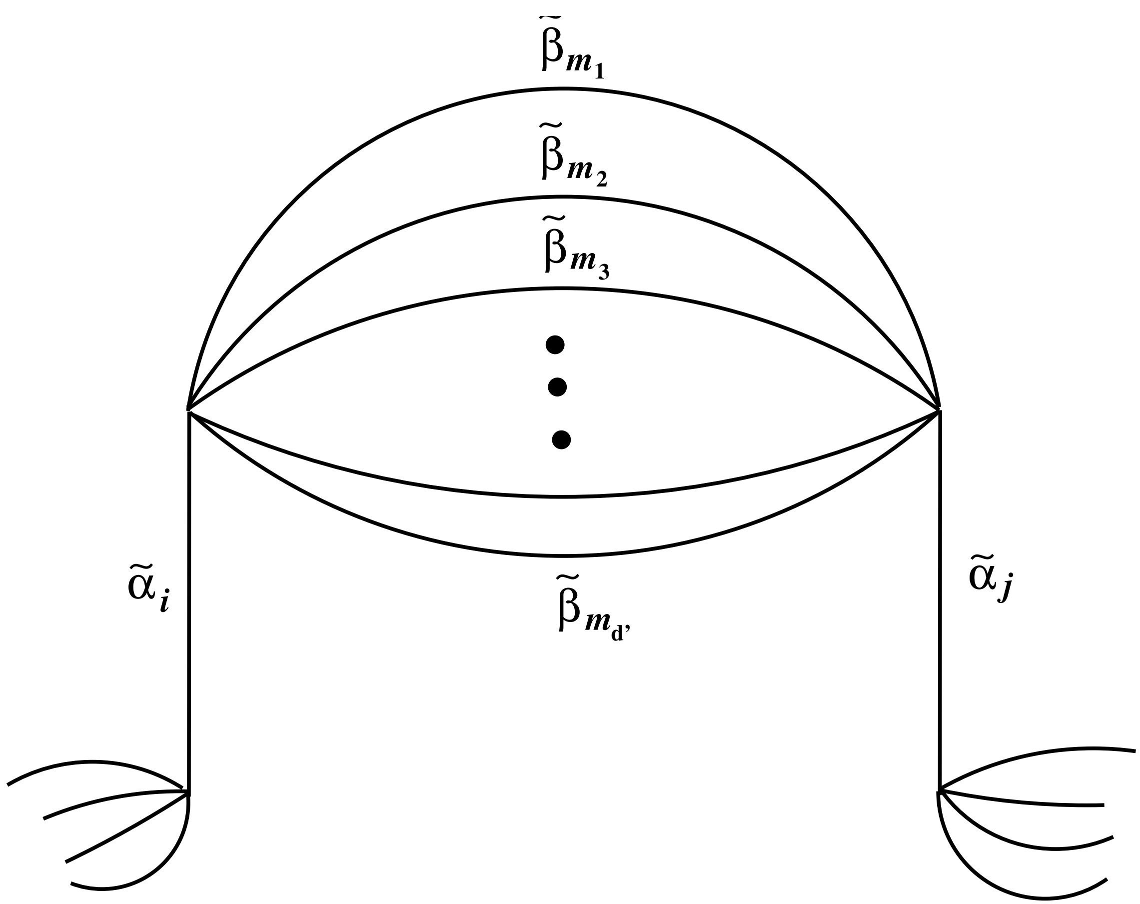

(1, 2, 3)
(1, 6, 4)
(2, 4, 5)





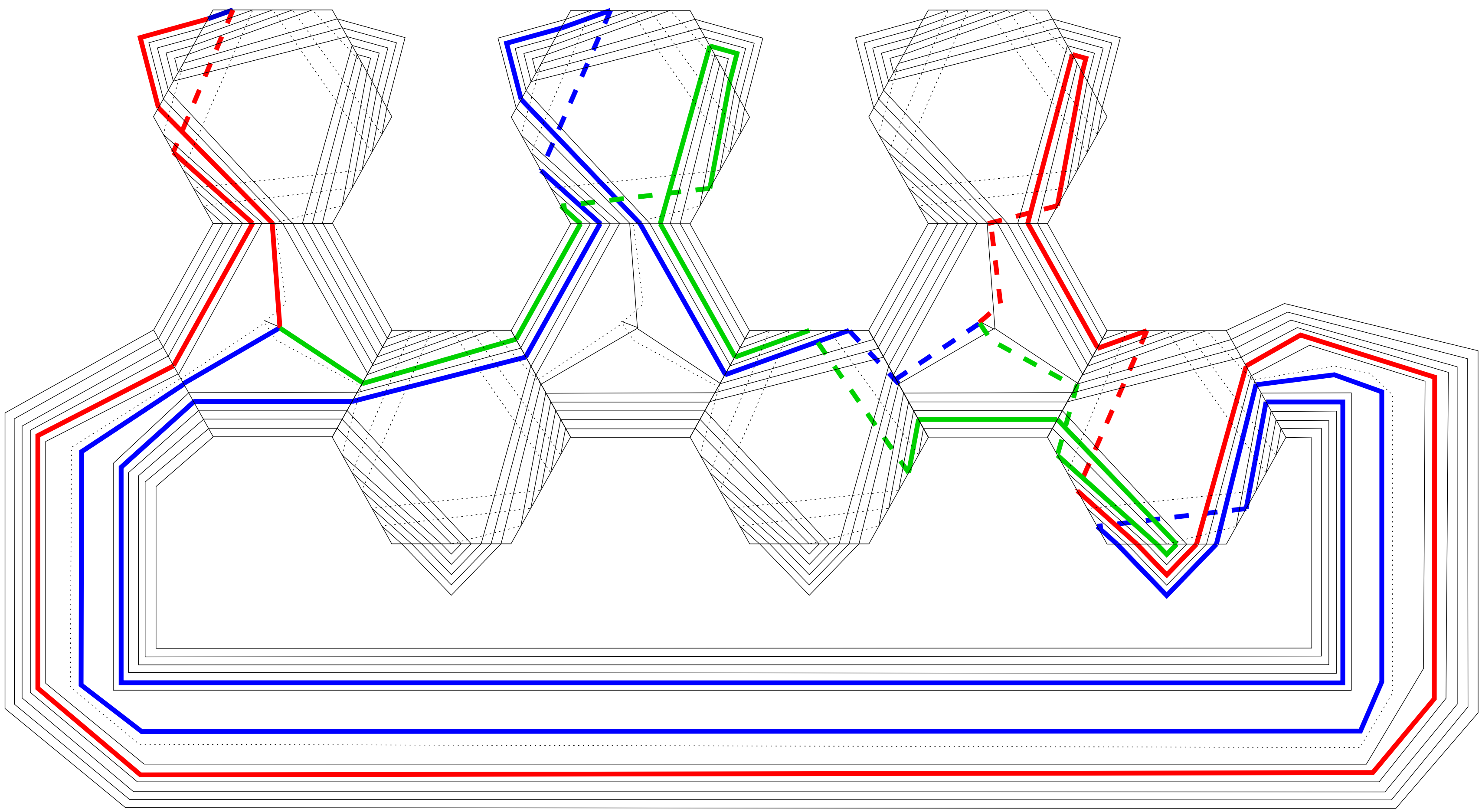


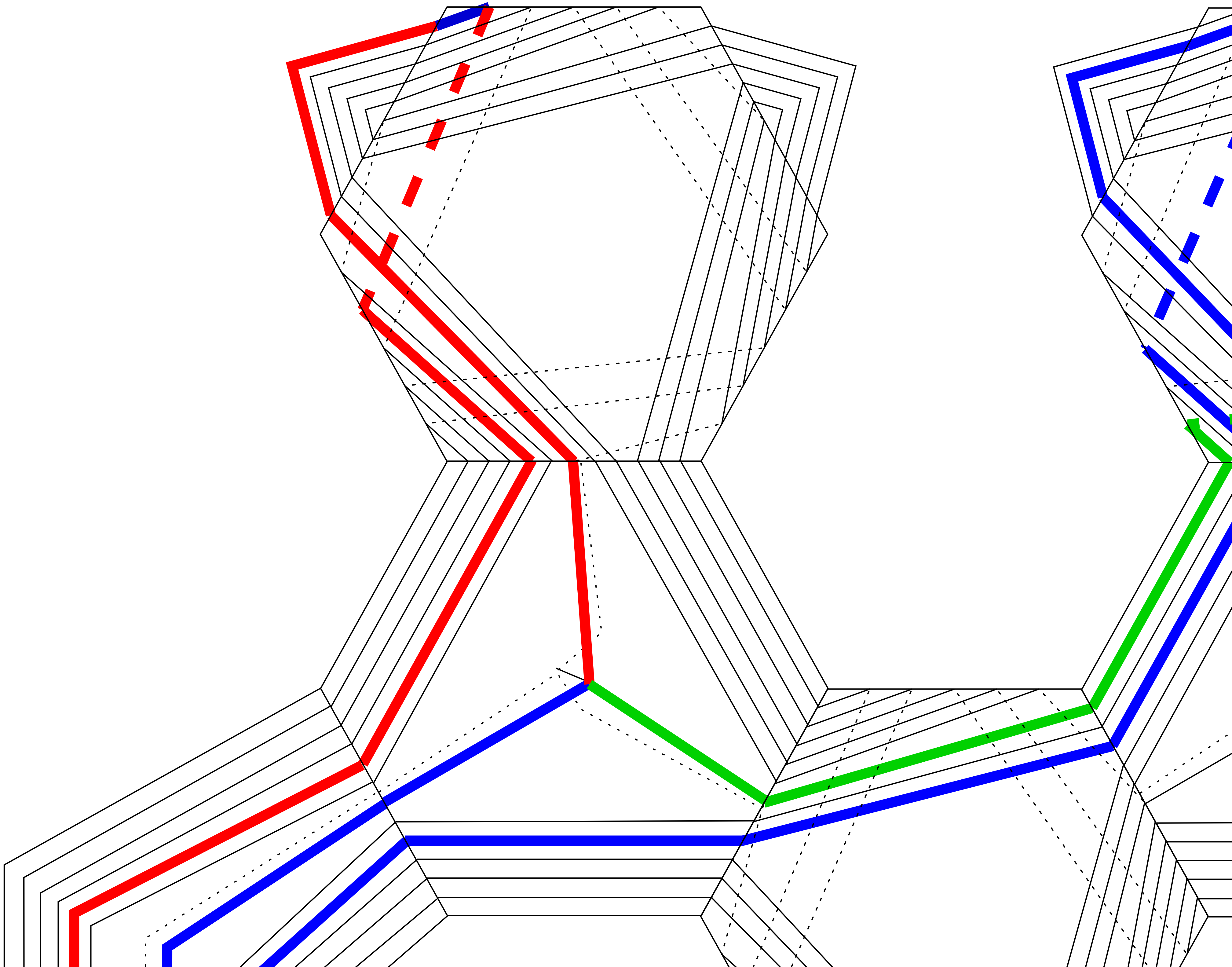
In general



$\varphi^{-1}(\beta_m)$ is a union of θ-graphs

Given an arc $\beta_m \subset \partial B$, a pair of consecutive arcs in $\varphi^{-1}(\beta_m)$ is called a <u>ramification cycle</u>





Now delete all arcs, except one, in each of the ramification cycles of $\varphi^{-1}(k)$.

Call the result

a <u>cleansing</u> of $\varphi^{-1}(k)$ on $\varphi^{-1}(B) \cong B_\omega$

THEOREM. (M. Jordán and V.) *Let $k \subset S^3$ be a link in an n-bridge representation and let (B, ℓ) be a $2n$-gonal pillowcase for k. Let $\omega : \pi_1(S^3 - k) \to S_d$ be a transitive representation, and let $\varphi : M \to (S^3, k)$ and $\psi : B_\omega \to (B, B \cap k)$ be the induced d-fold branched coverings.*

If there exists an embedding $\varepsilon : B_\omega \hookrightarrow M$ such that the ramification cycles on $\varepsilon(\partial B_\omega)$ bound disjoint 2-cells in $\overline{M - \varepsilon(B_\omega)}$, then any homeomorphism $\varepsilon(B_\omega) \cong \varphi^{-1}(B)$ can be extended to a homeomorphism of pairs $(M, \tilde{\ell}) \cong (M, \varphi^{-1}(k))$ for $\tilde{\ell}$ any cleansing of $\varepsilon(\psi^{-1}(\ell))$.

The pair $(\partial B_\omega, \text{ramification cycles})$ induces a Heegaard diagram for M.

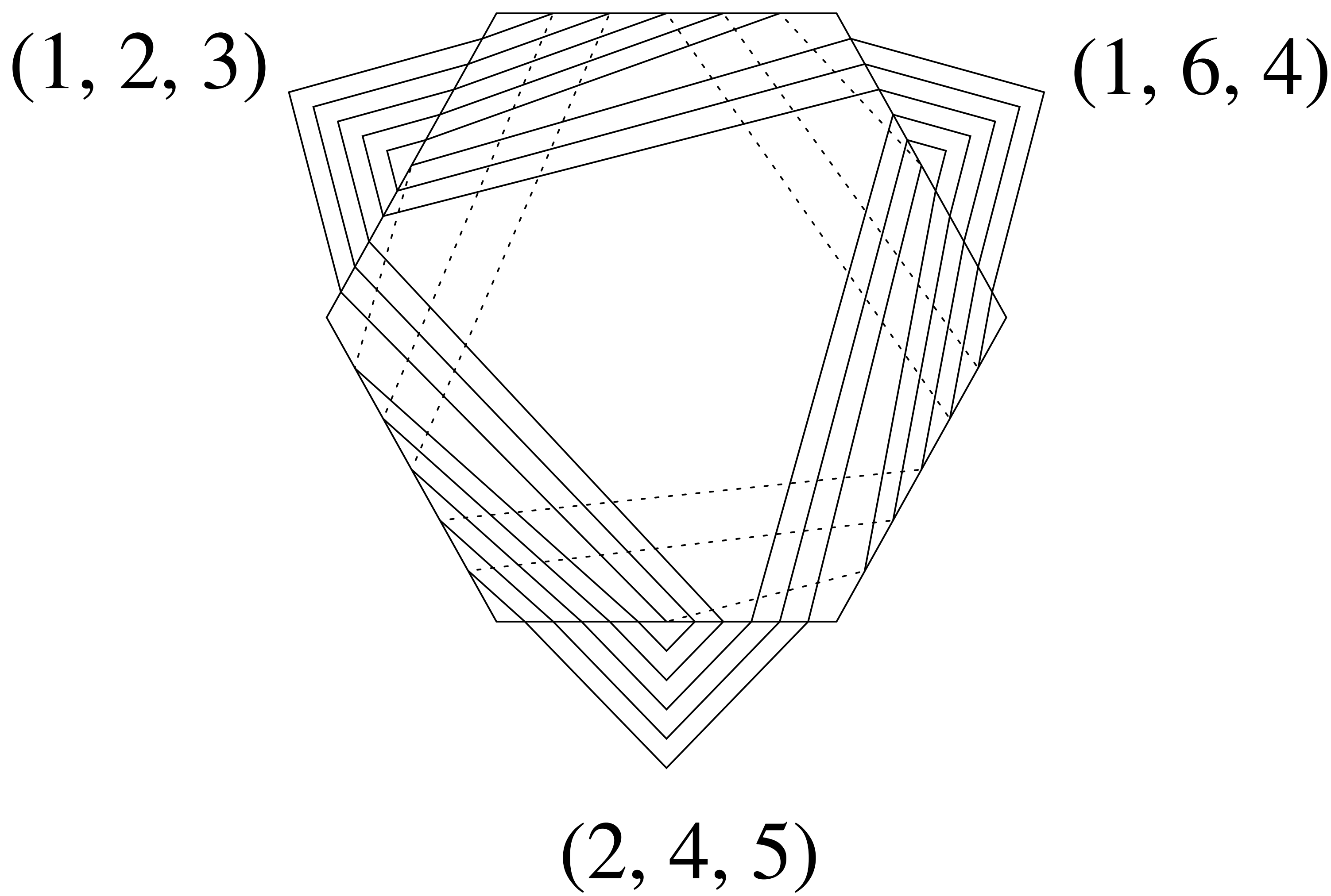
Montesinos knots

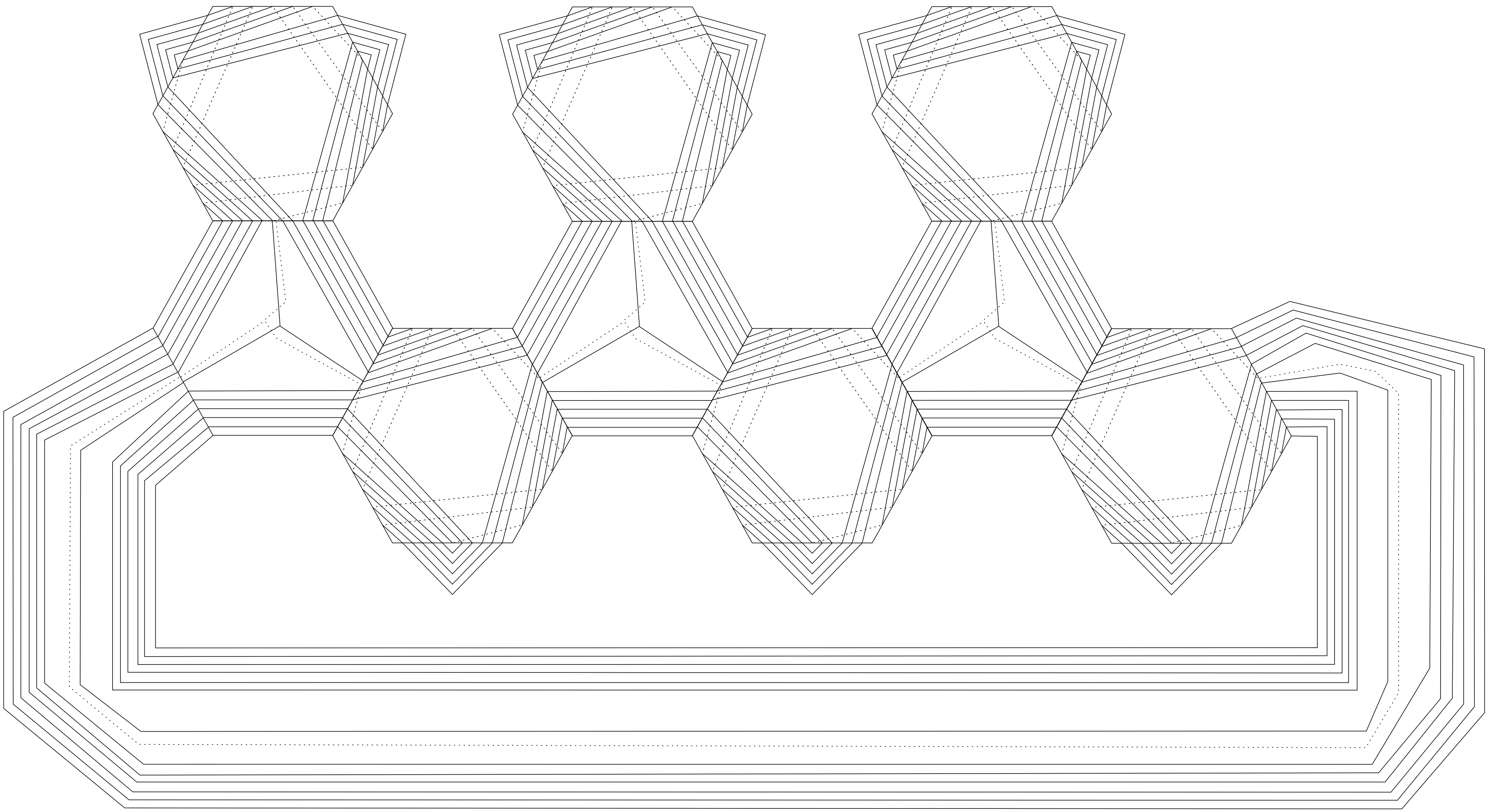
THEOREM (J. Rodríguez and V. '04). *All non-torus Montesinos knots of less than eleven crossings are universal, except for*

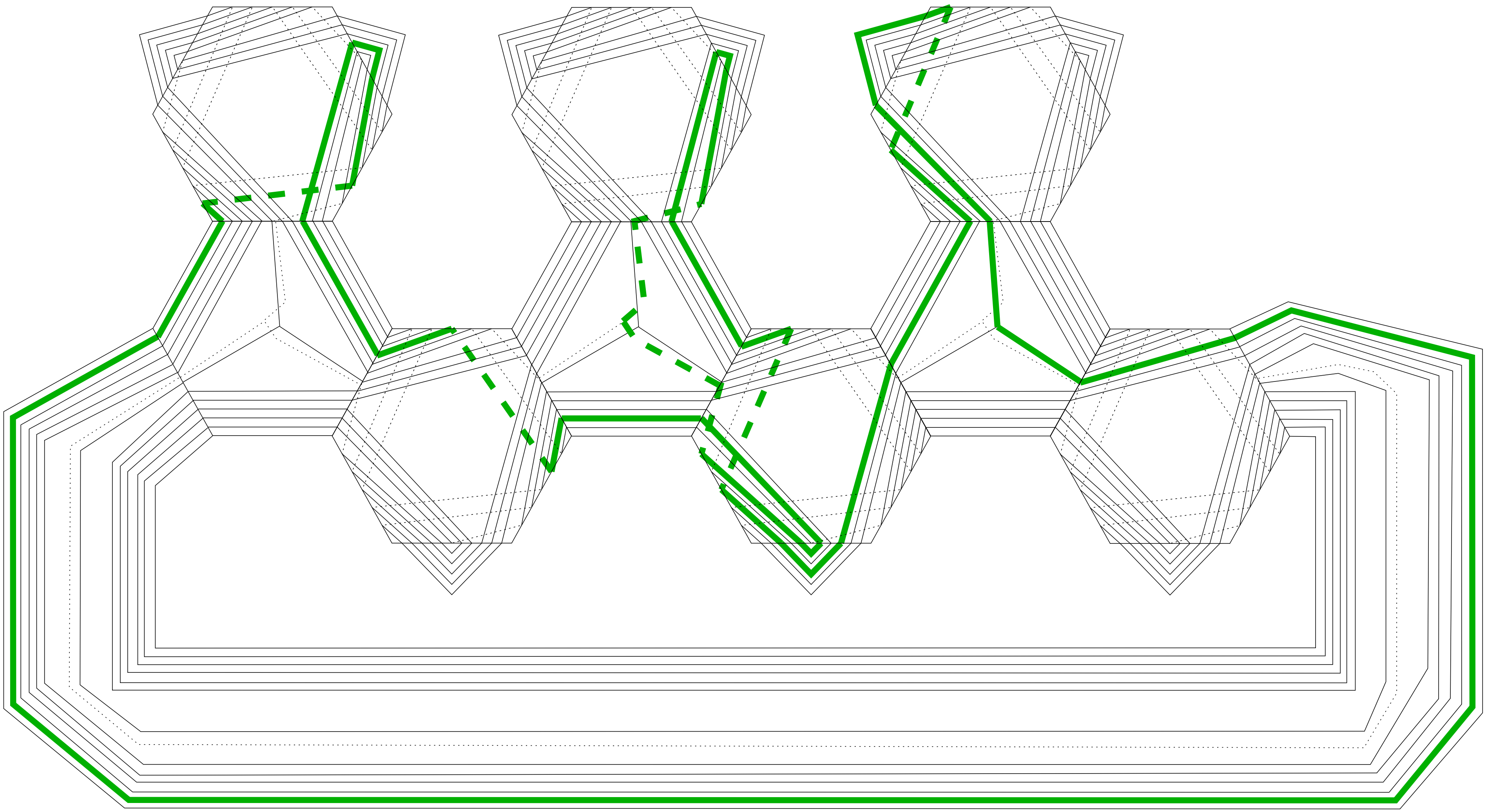
$$\begin{array}{ll} 9_{35} = m(1/3, 1/3, 1/3) & 9_{48} = m(2/3, 2/3, -1/3) \\ 10_{67} = m(2/5, 1/3, 2/3) & 10_{68} = m(3/5, 1/3, 1/3) \\ 10_{69} = m(3/5, 2/3, 2/3) & 10_{75} = m(2/3, 2/3, 5/3) \\ 10_{137} = m(2/5, 3/5, -1/2) & 10_{145} = m(2/5, 1/3, -2/3) \\ 10_{146} = m(2/5, 2/3, -1/3) & 10_{147} = m(3/5, 1/3, -1/3) \end{array}$$

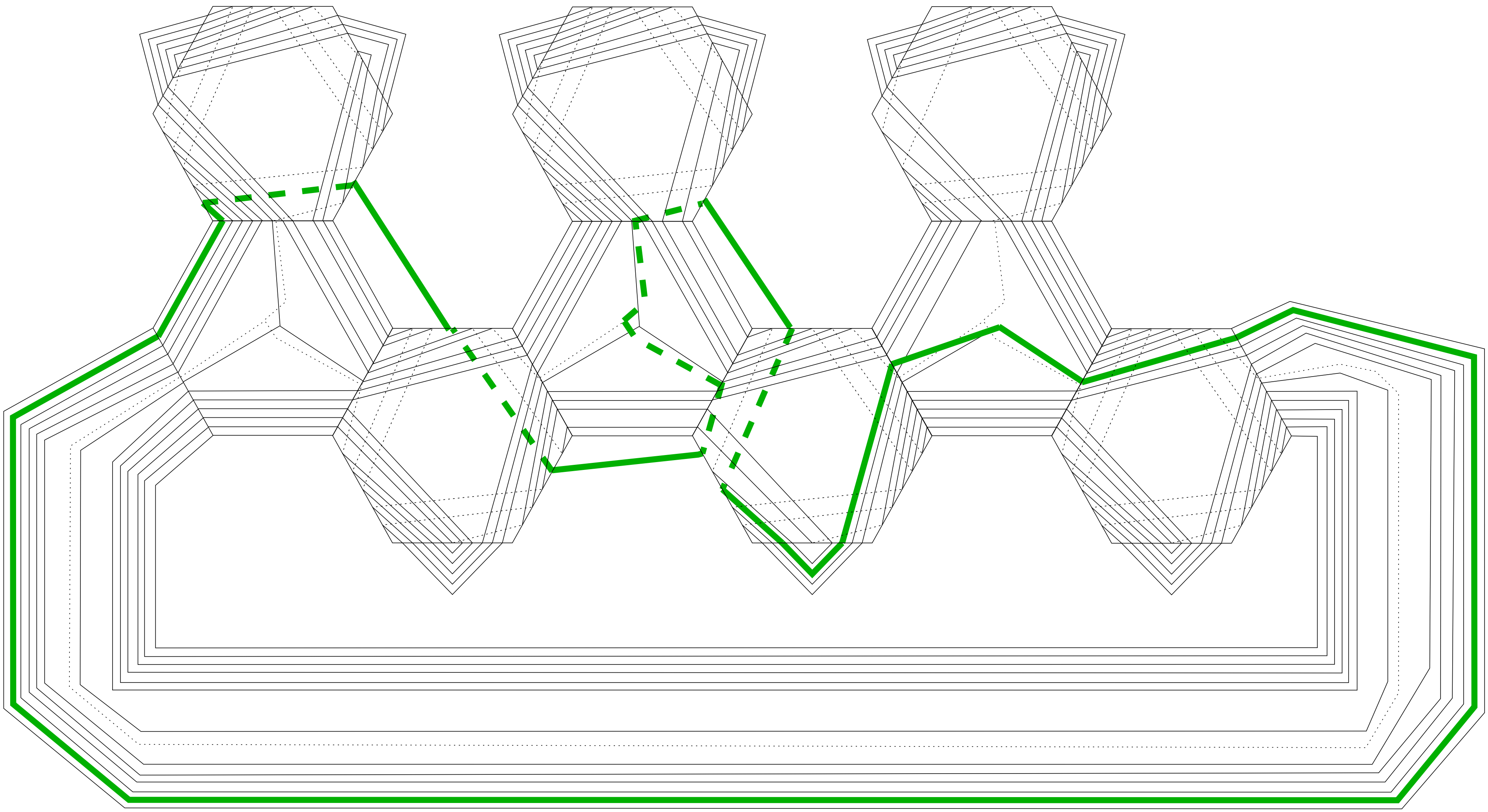
(We do not know).

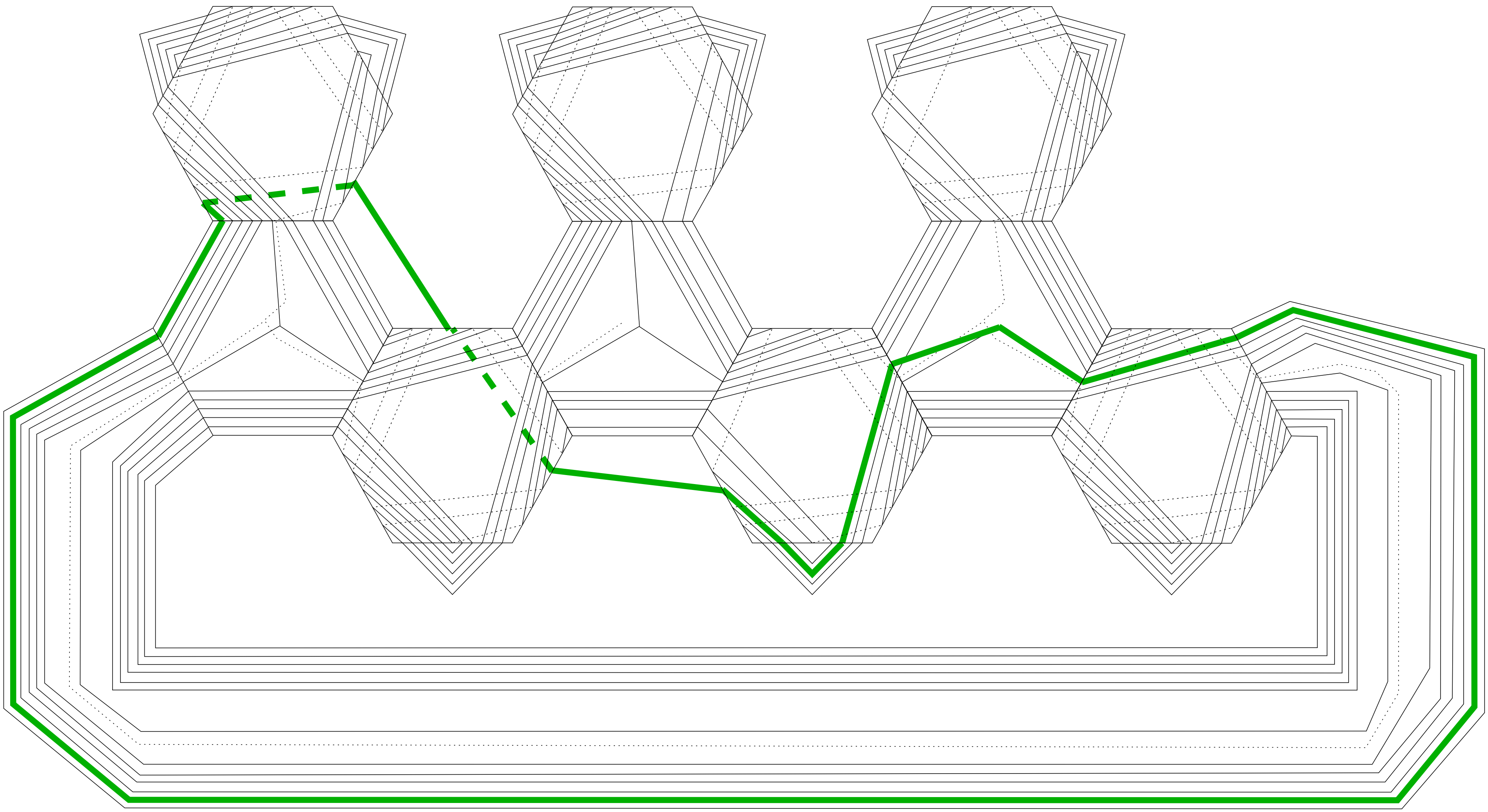
$$9_{35} = m(1/3, 1/3, 1/3)$$

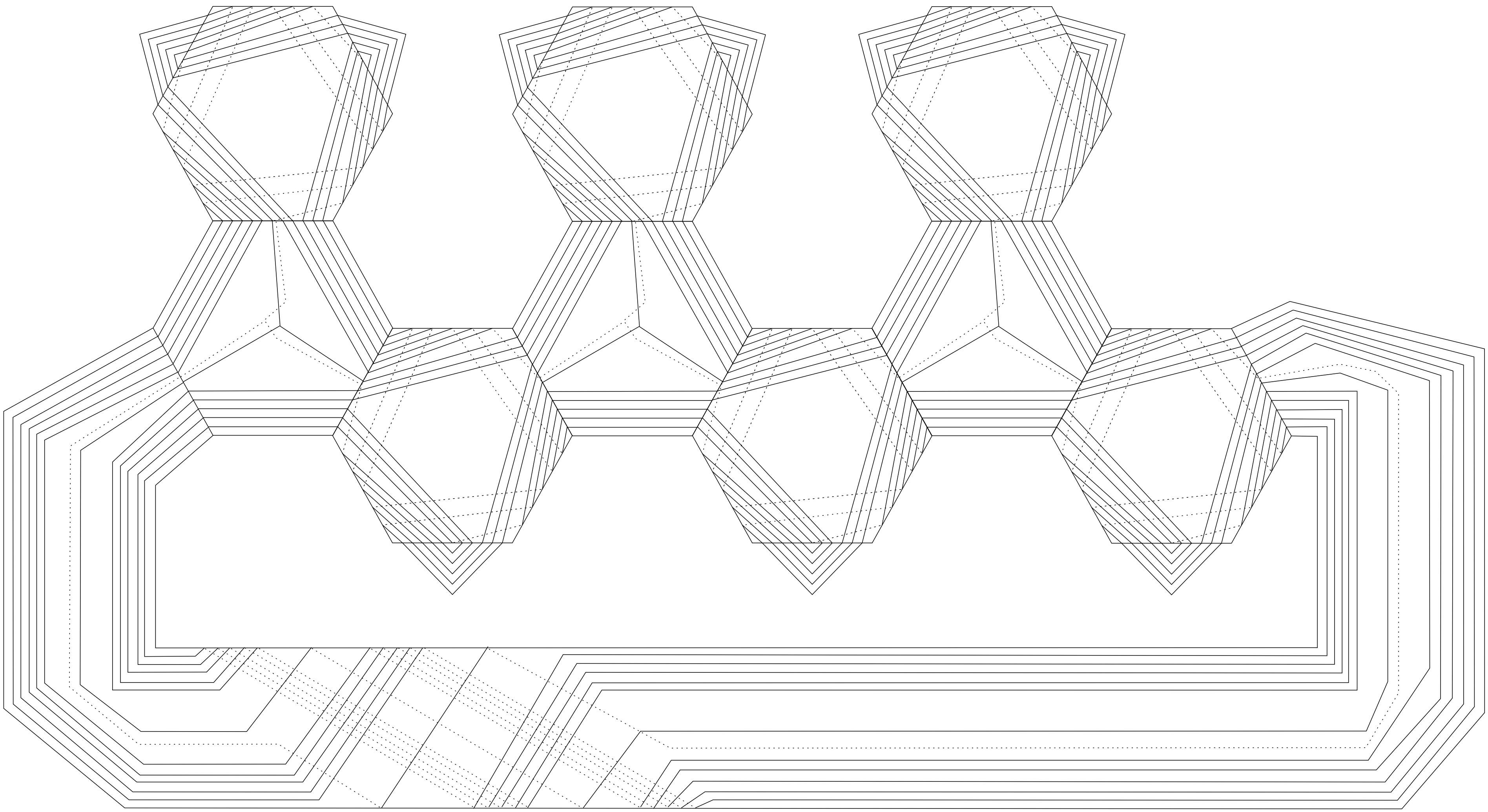


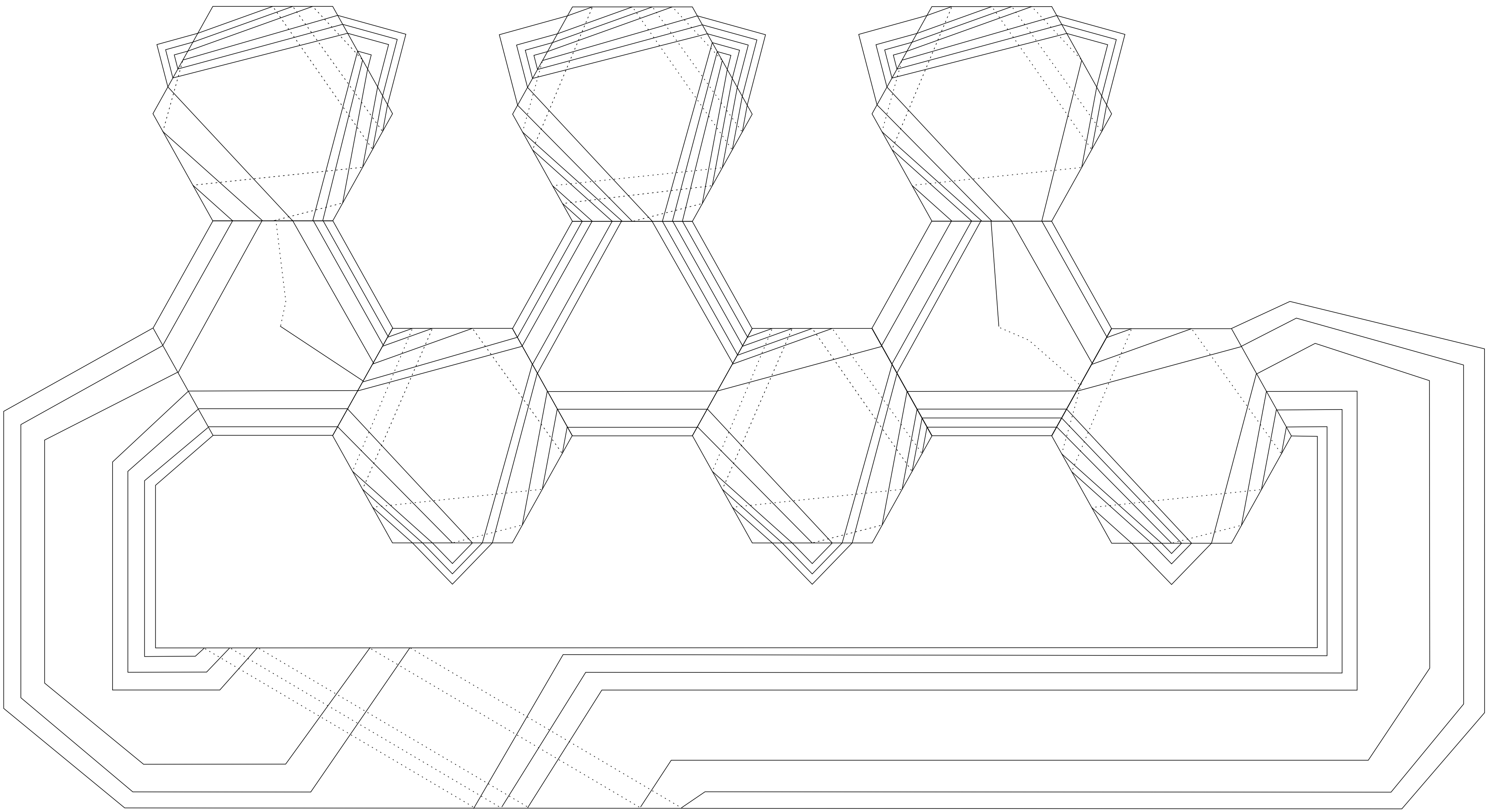


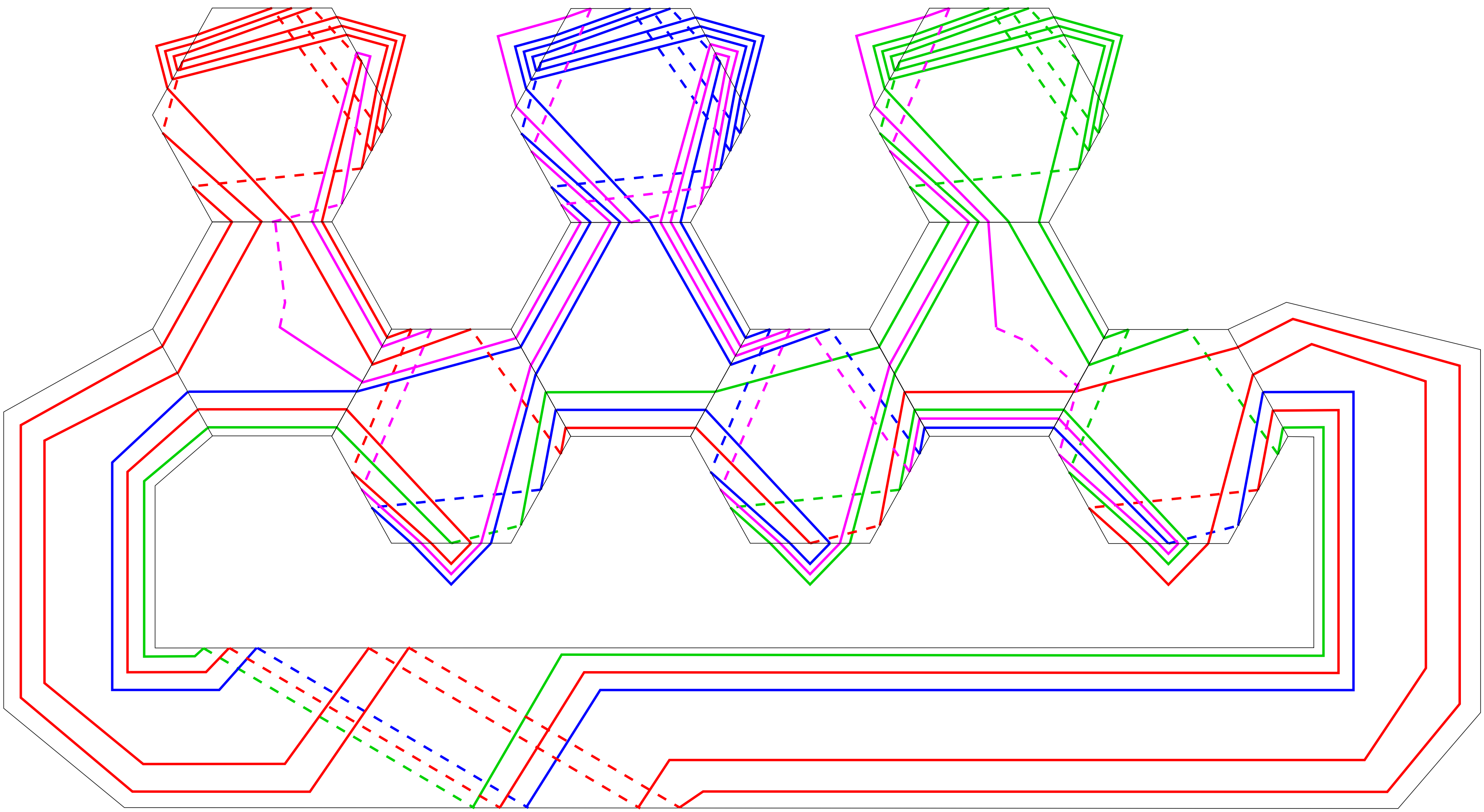


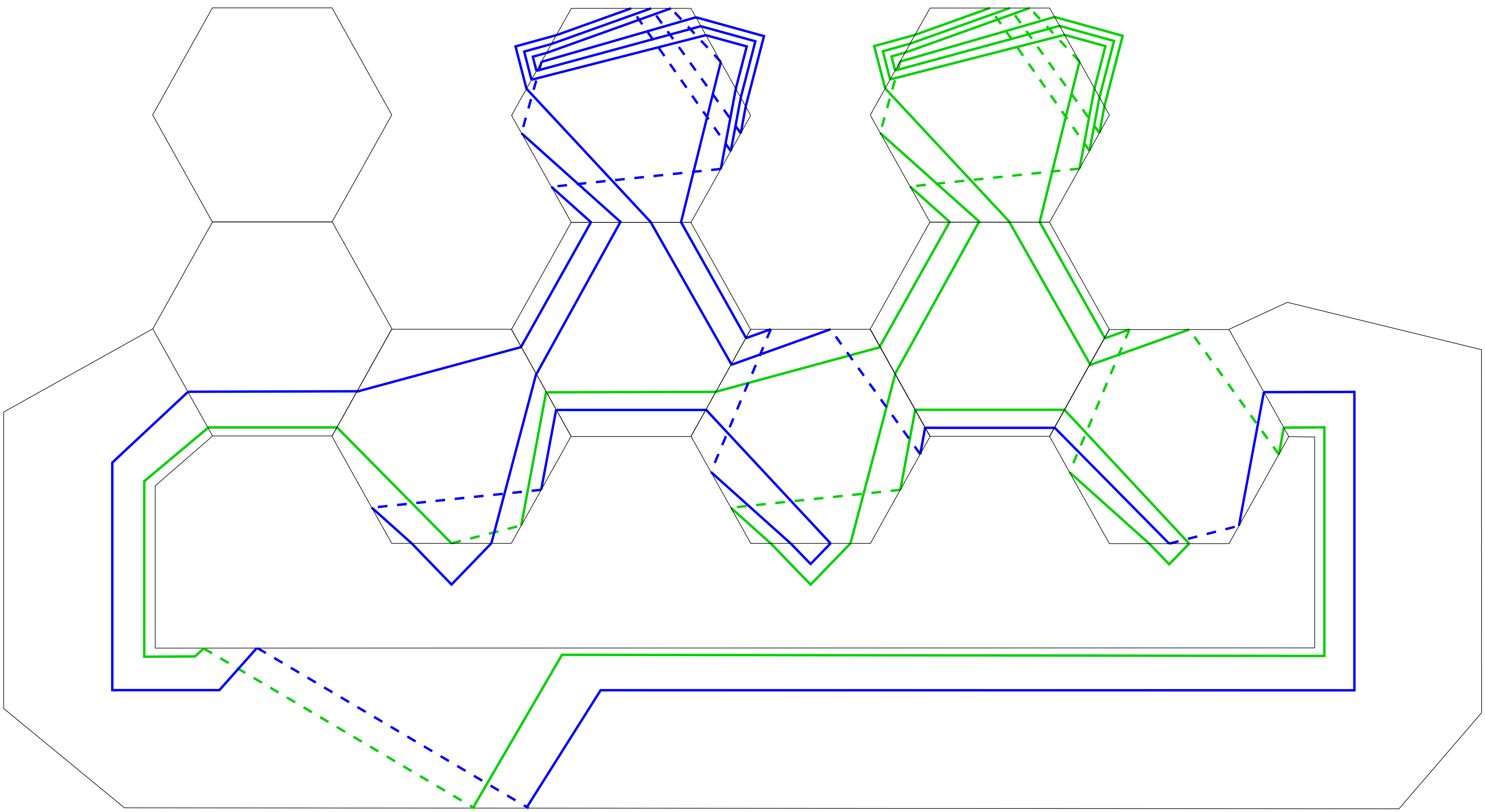


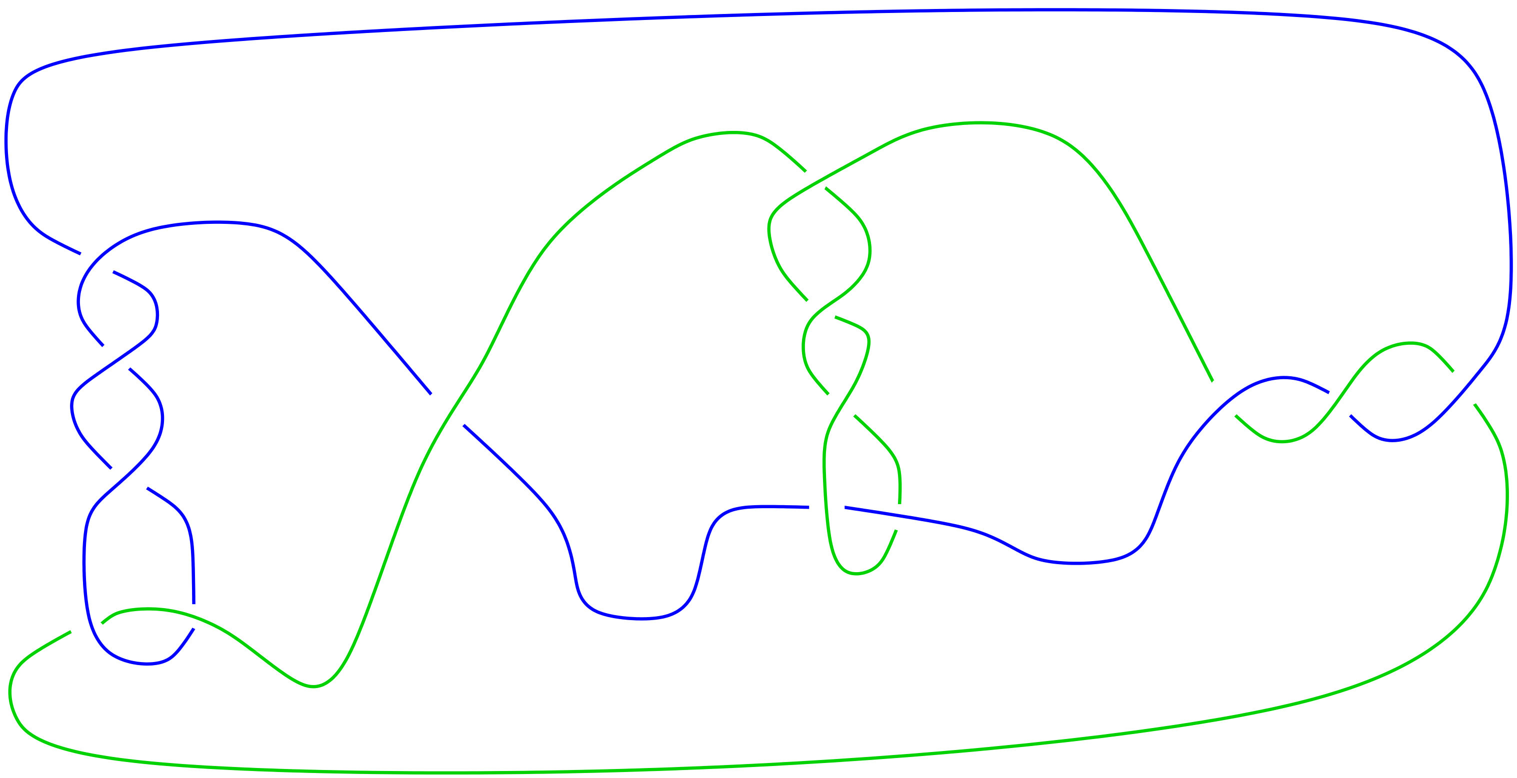












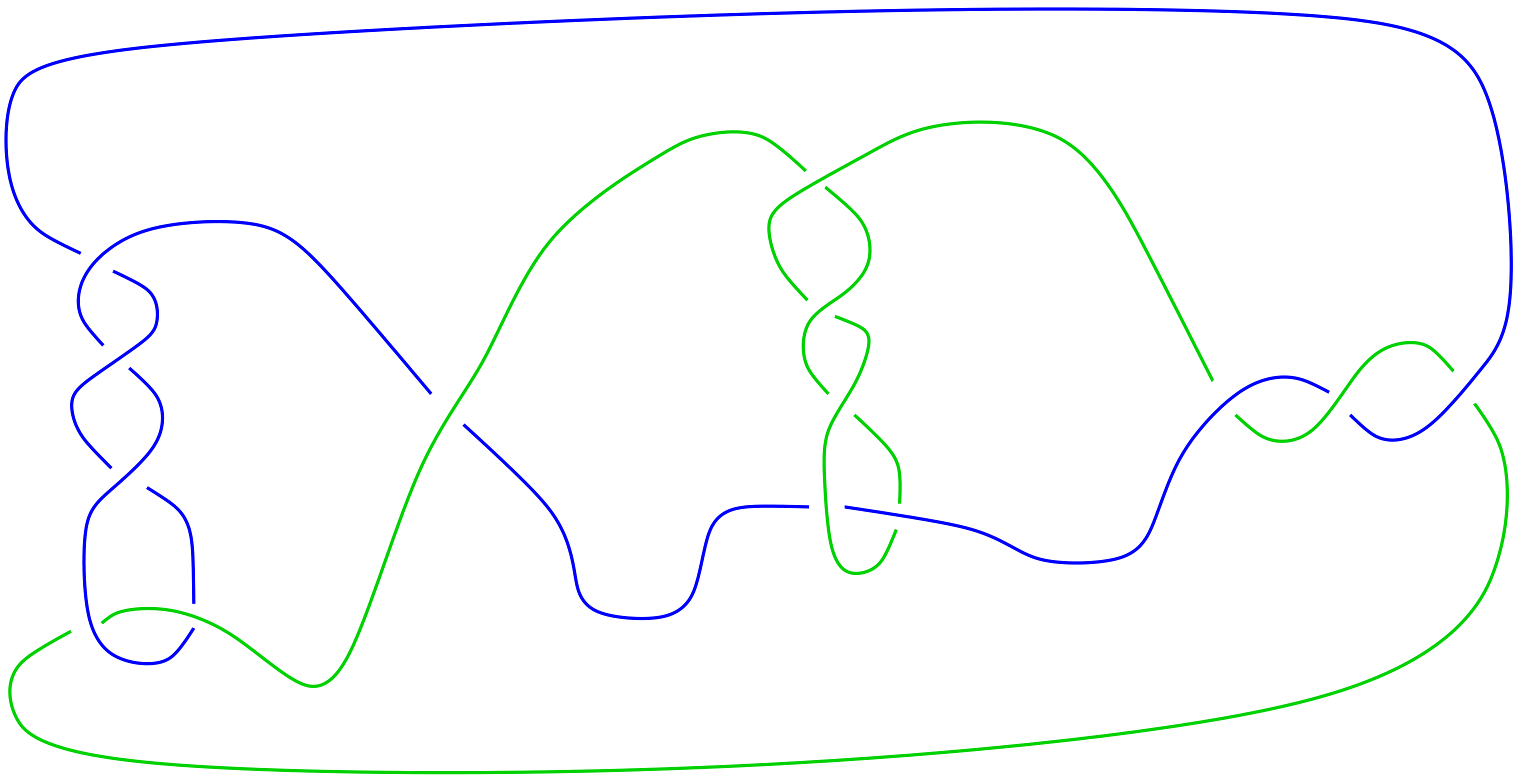
$$m(2/7, 1, 2/7, 3) \sim m(9/7, 23/7) \sim m(-224/97)$$

THEOREM (Hilden, Lozano and Montesinos). *All non-torus 2-bridged links are universal.*

(Also

$$m(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}) \sim m(-\frac{\alpha_1\beta_2 + \alpha_2\beta_1}{\alpha_2 r_1 + \beta_2 s_1})$$

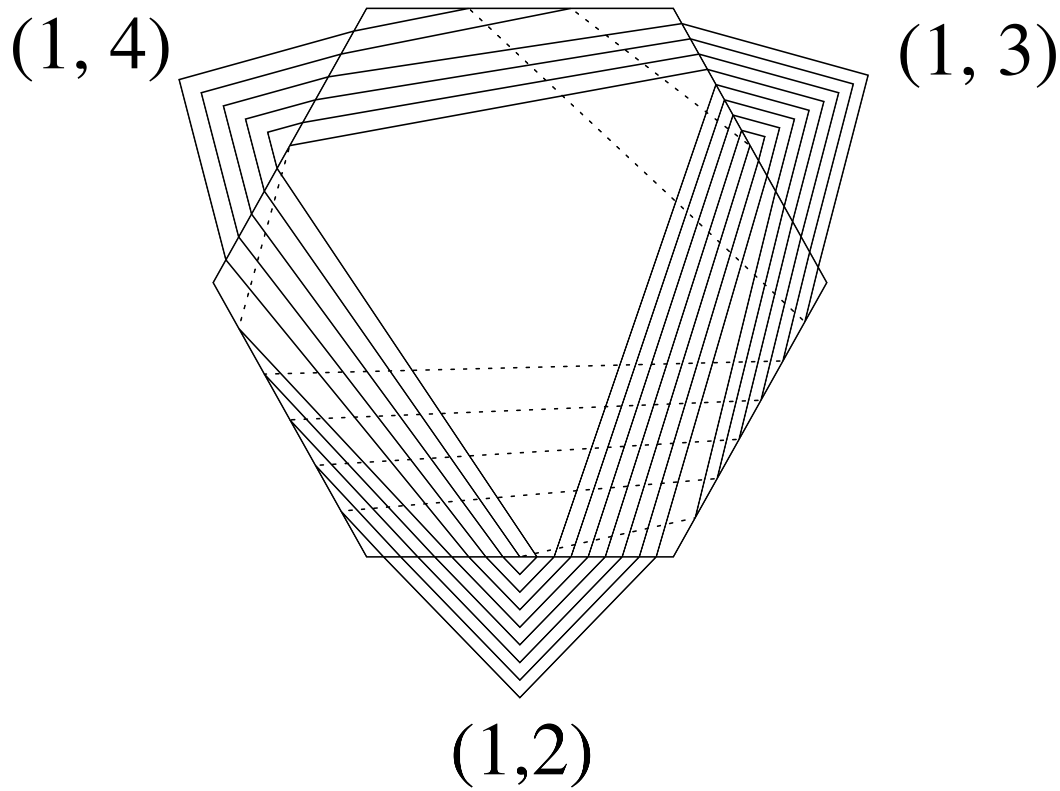
where $\alpha_1 r_1 - \beta_1 s_1 = 1$.)

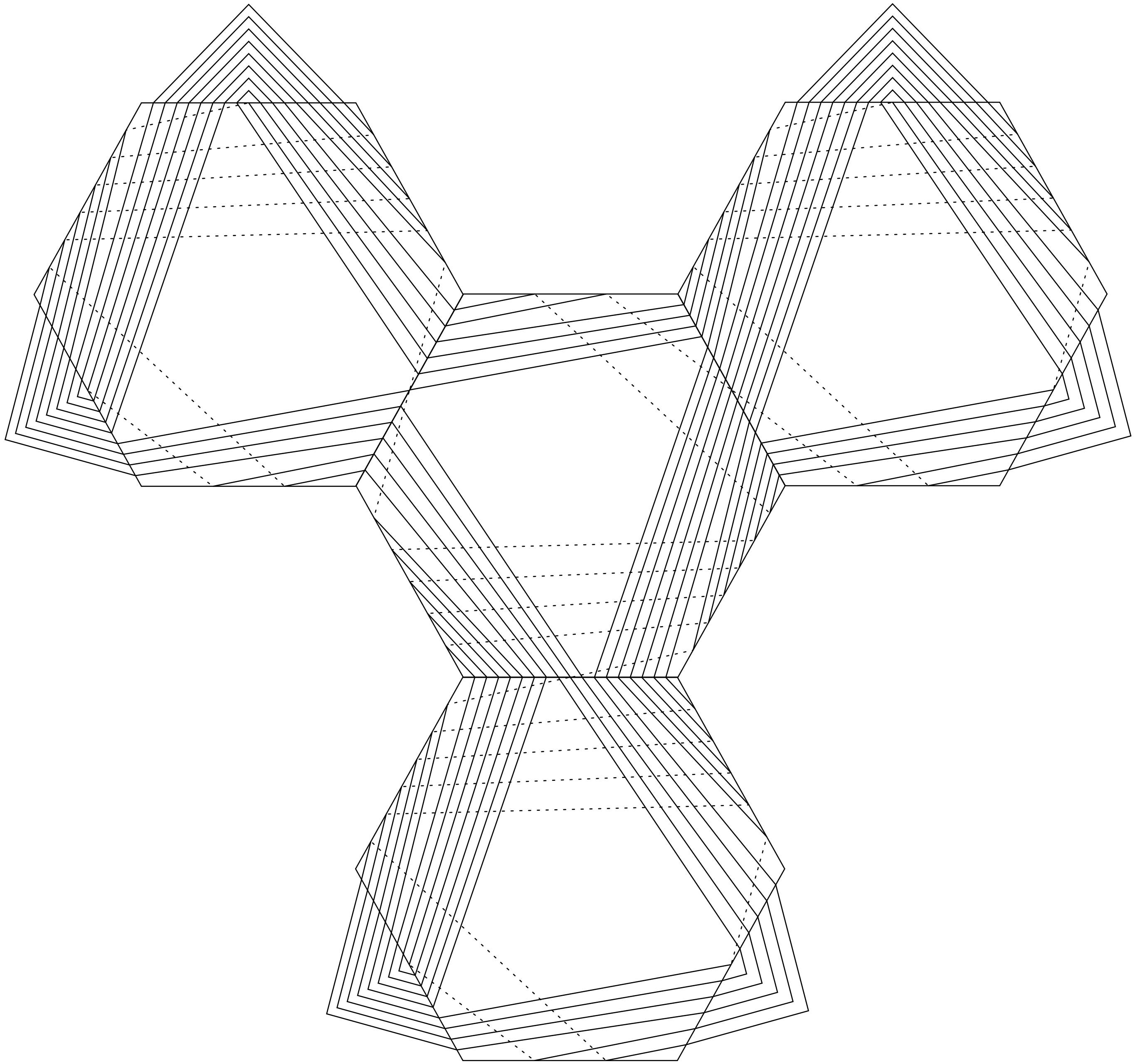


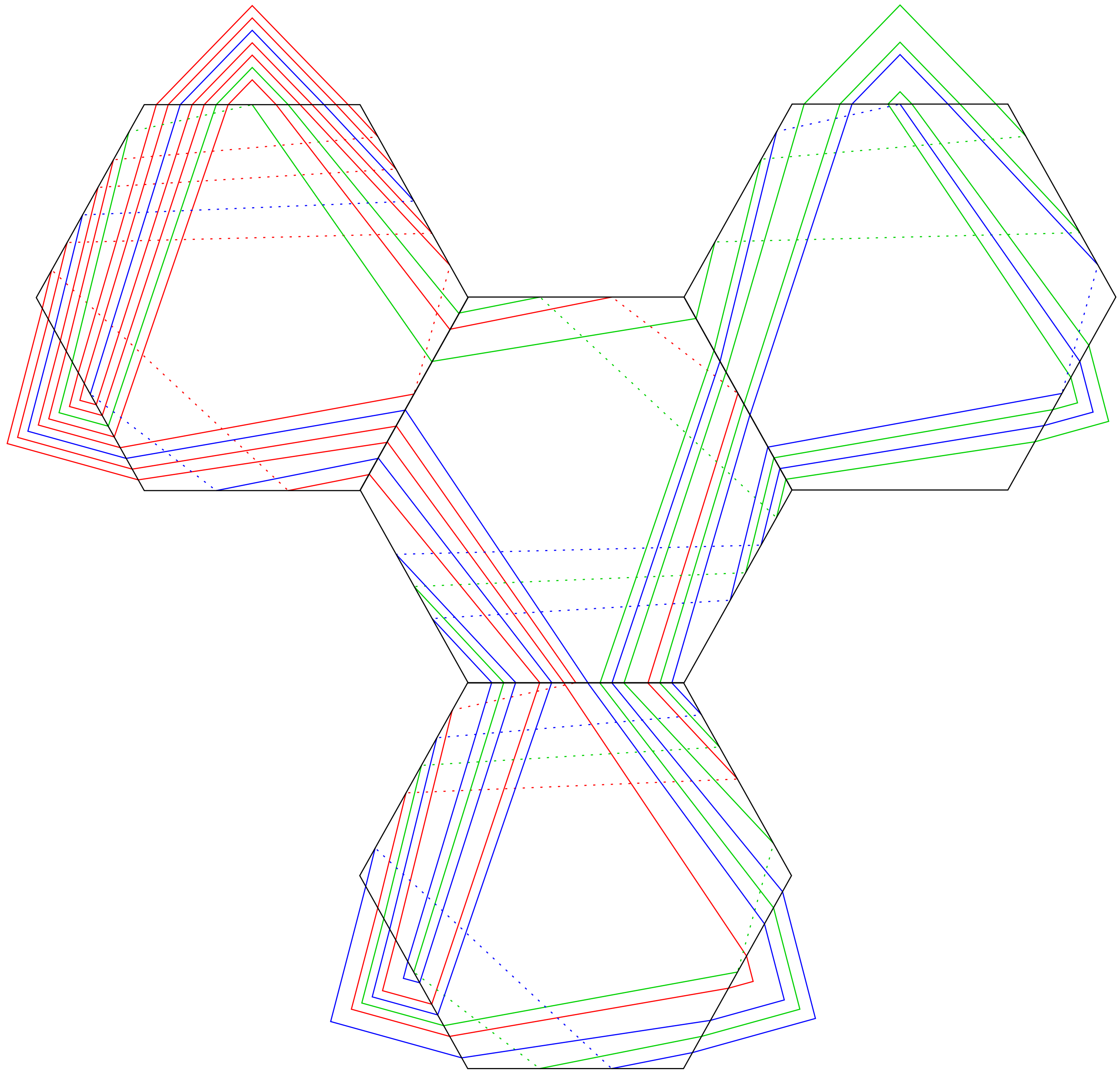
$$m(2/7, 1, 2/7, 3) \sim m(9/7, 23/7) \sim m(-224/97)$$

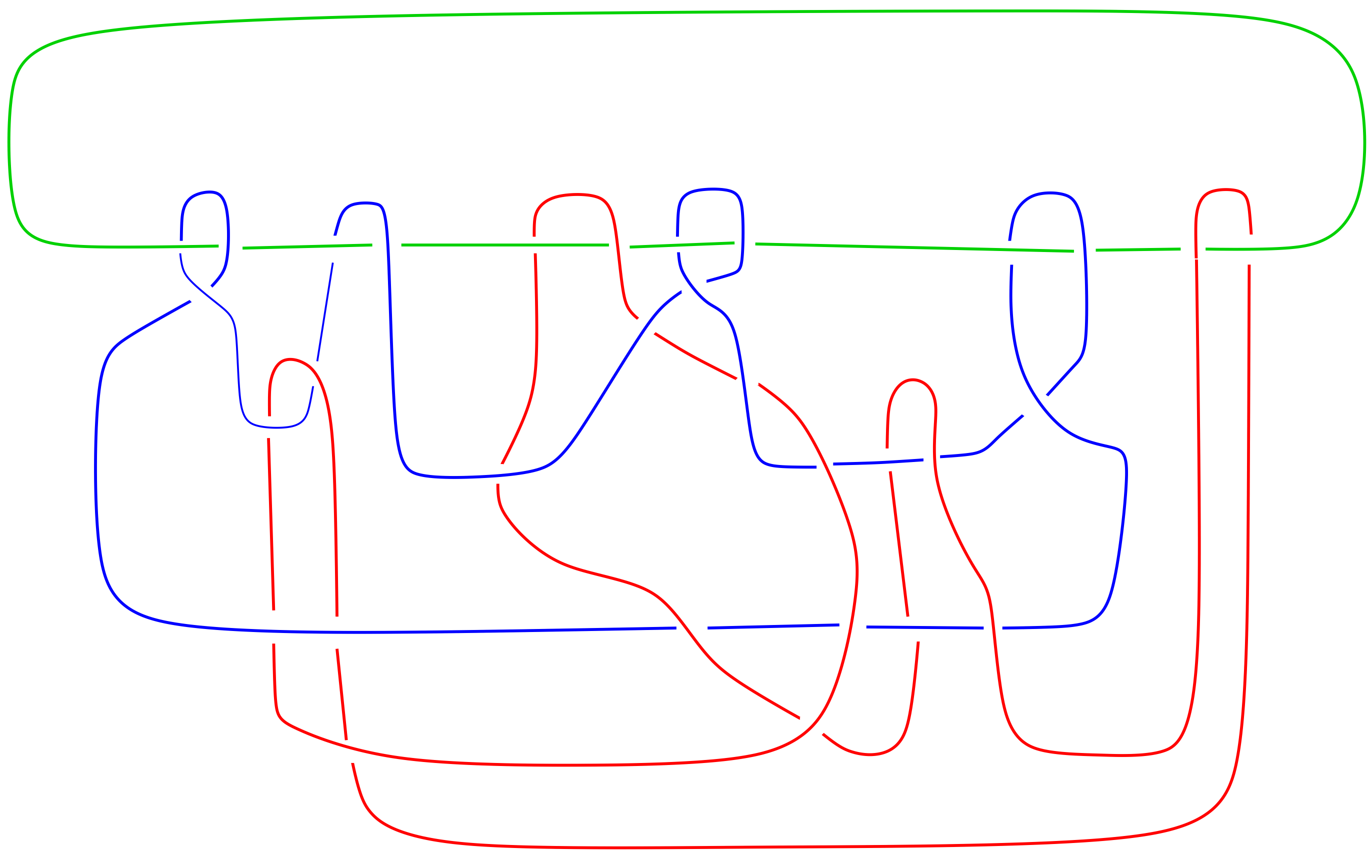
The knot 9_{35} *is universal.*

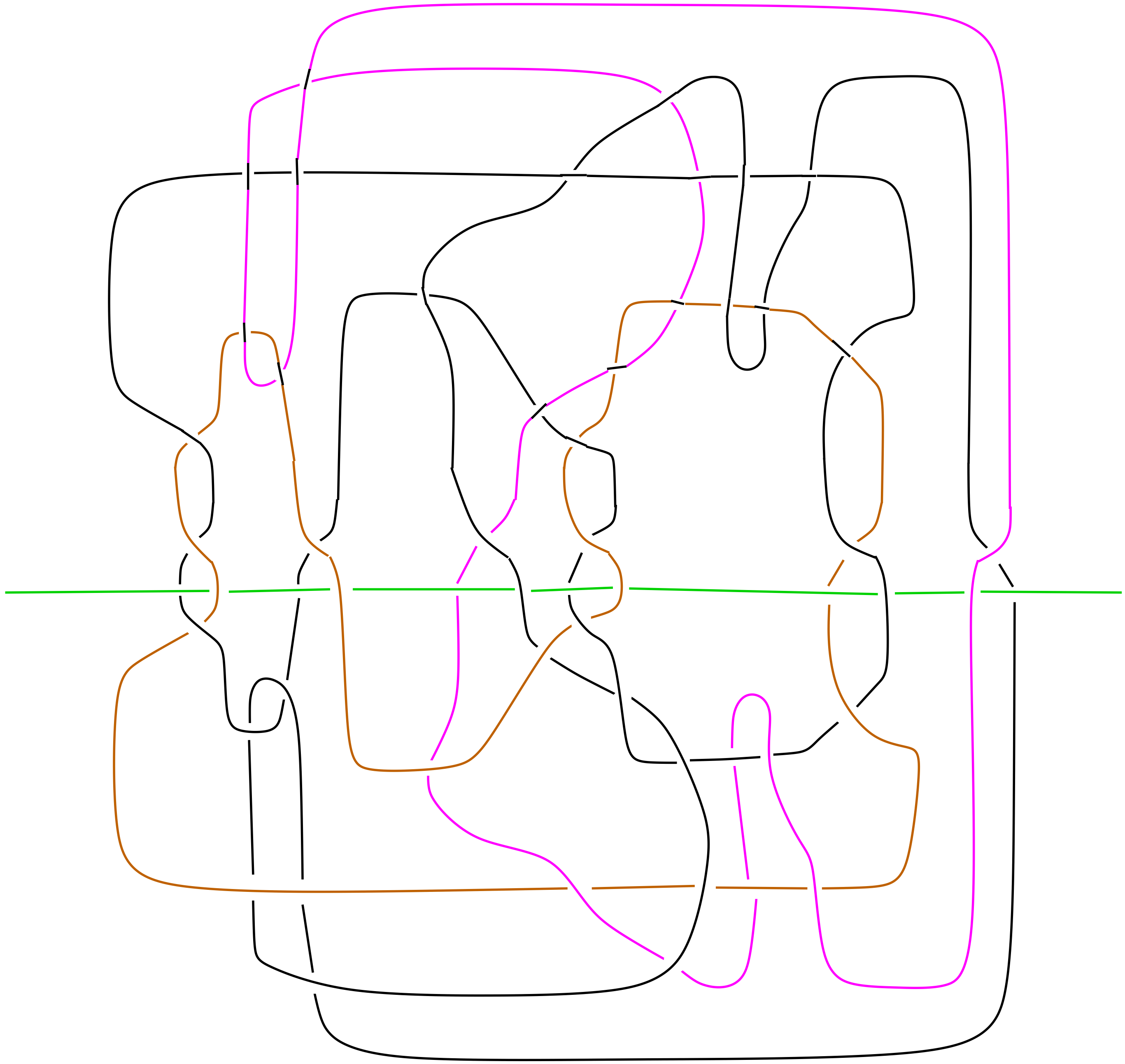
$$9_{48} = m(2/3, 2/3, -1/3)$$

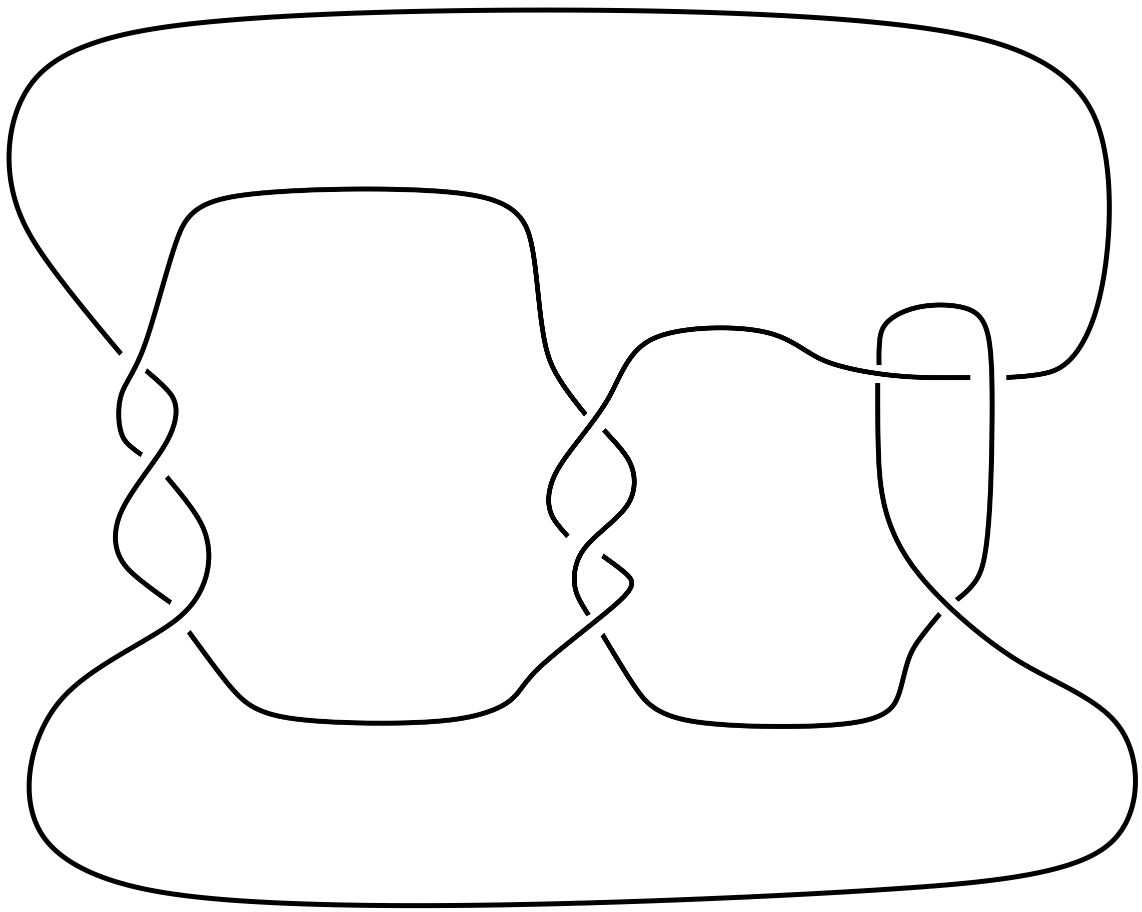












$$m(1/3, 1/3, 2/3) \sim m(4/3, 4/3, -4/3)$$

THEOREM (J. Rodríguez and V. '04). *For* $k = m(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_t}{\alpha_t})$ *write*

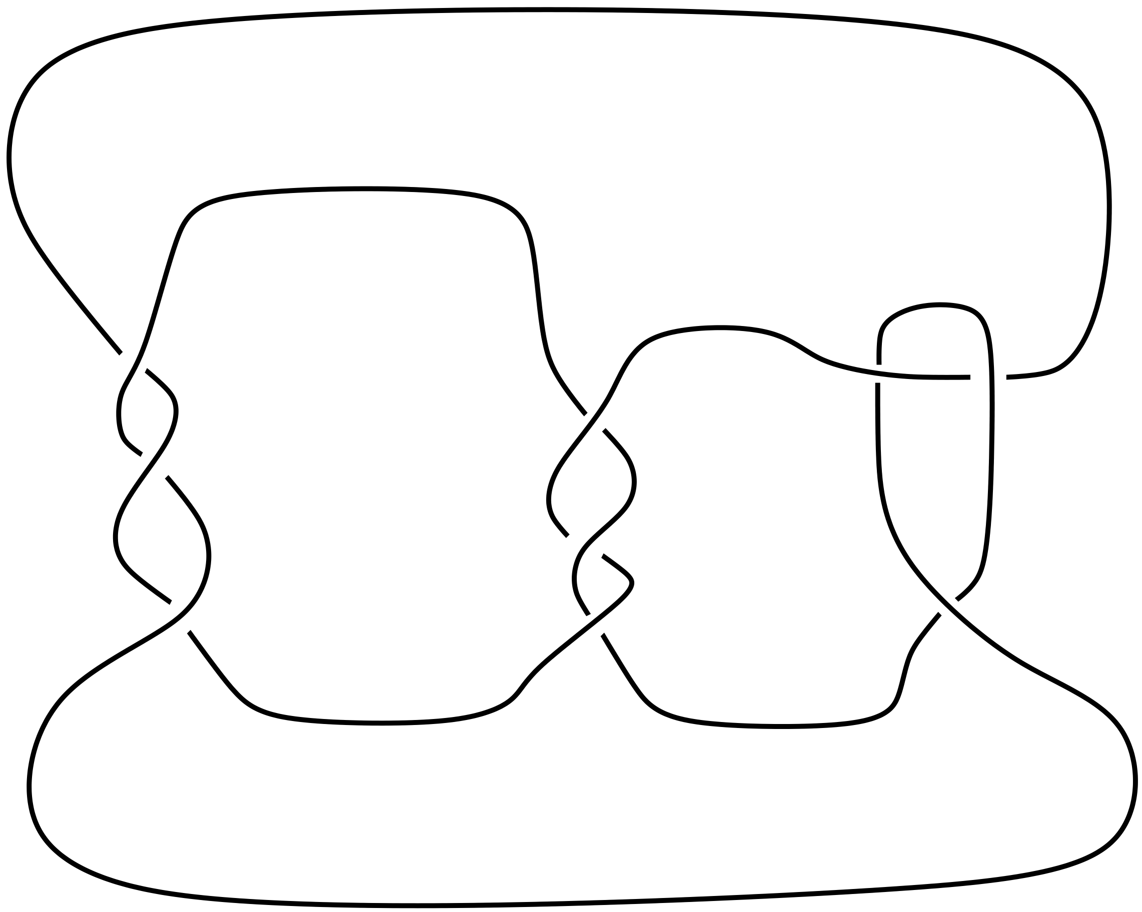
$$\Delta(k) = \beta_1\alpha_2\cdots\alpha_t + \alpha_1\beta_2\cdots\alpha_t + \cdots + \alpha_1\alpha_2\cdots\beta_t.$$

If n *is a positive divisor of* $\Delta(k)$ *and for each* i $(n, \alpha_i) = 1$, *then*

$$k \sim m(\frac{n \cdot b_1}{\alpha_1}, \ldots, \frac{n \cdot b_t}{\alpha_t}),$$

and there is a n*-fold branched covering* $\varphi : S^3 \to (S^3, k)$ *such that*

$$m(\frac{b_1}{\alpha_1}, \ldots, \frac{b_t}{\alpha_t}) \subset \varphi^{-1}(k).$$



$$m(1/3, 1/3, 2/3) \sim m(4/3, 4/3, -4/3) \leftarrow m(1/3, 1/3, -1/3)$$

The knot 9_{48} is universal

- $10_{68} = m(3/5, 1/3, 1/3) \sim m(-(19 \cdot 3/5, 19/3, 19/3) \leftarrow m(-3/5, 1/3, 1/3) \sim 10_{145}$

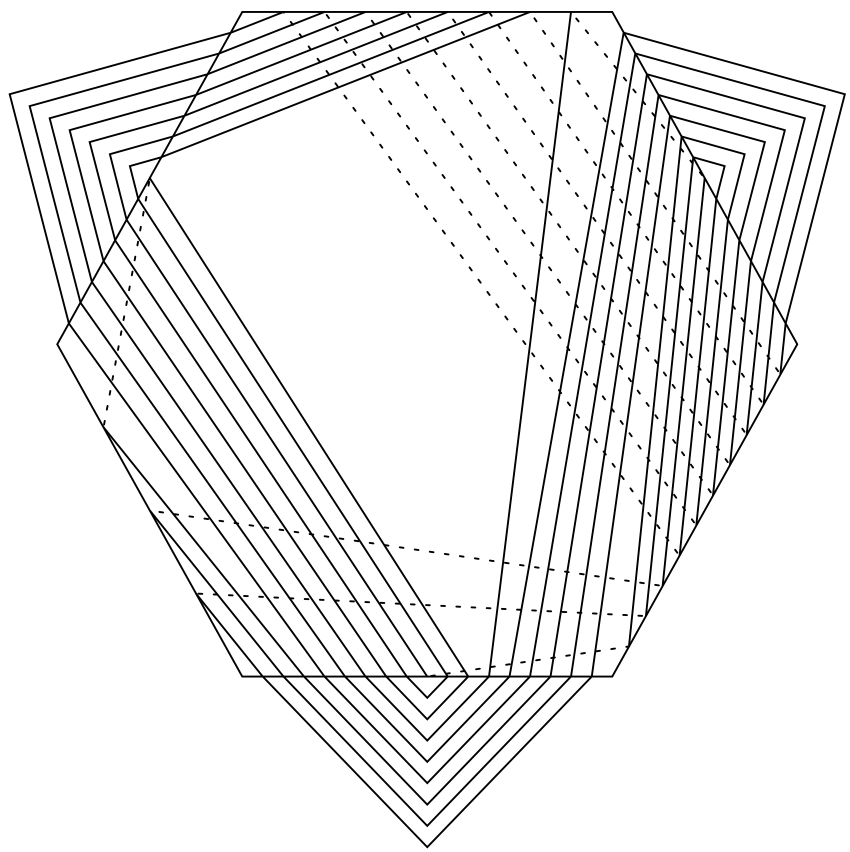
- $10_{69} = m(3/5, 2/3, 3/3) \sim m(-(29 \cdot 3)/5, 29/3, 29/3) \leftarrow m(-3/5, 1/3, 1/3) \sim 10_{145}$

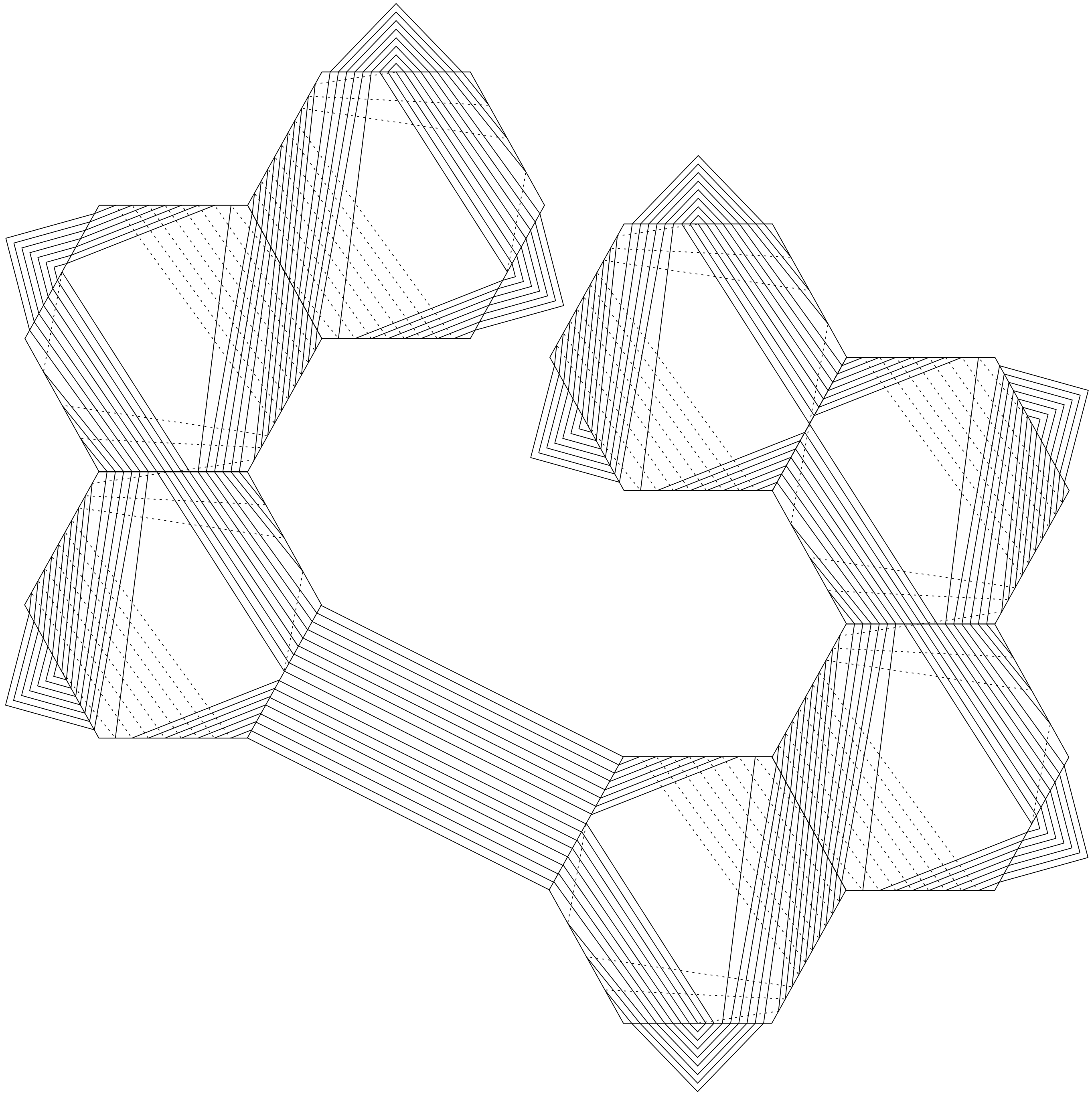
- $10_{146} = m(2/5, 2/3, -1/3) \sim m(-(11 \cdot 3)/5, 11/3, 11/3) \leftarrow m(-3/5, 1/3, 1/3) \sim 10_{145}$

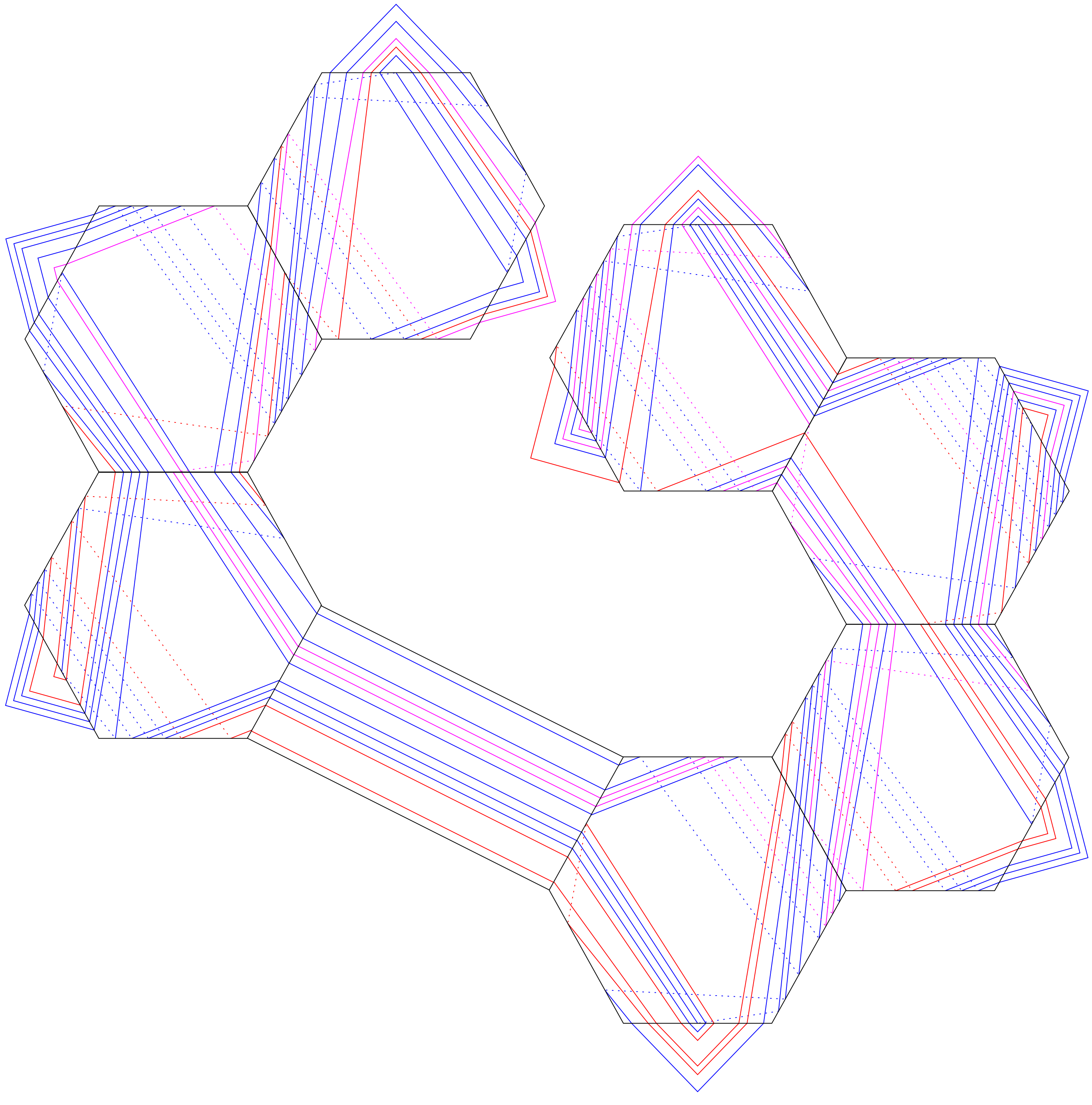
- $10_{75} = m(2/3, 2/3, 5/3) \leftarrow 10_{145}$

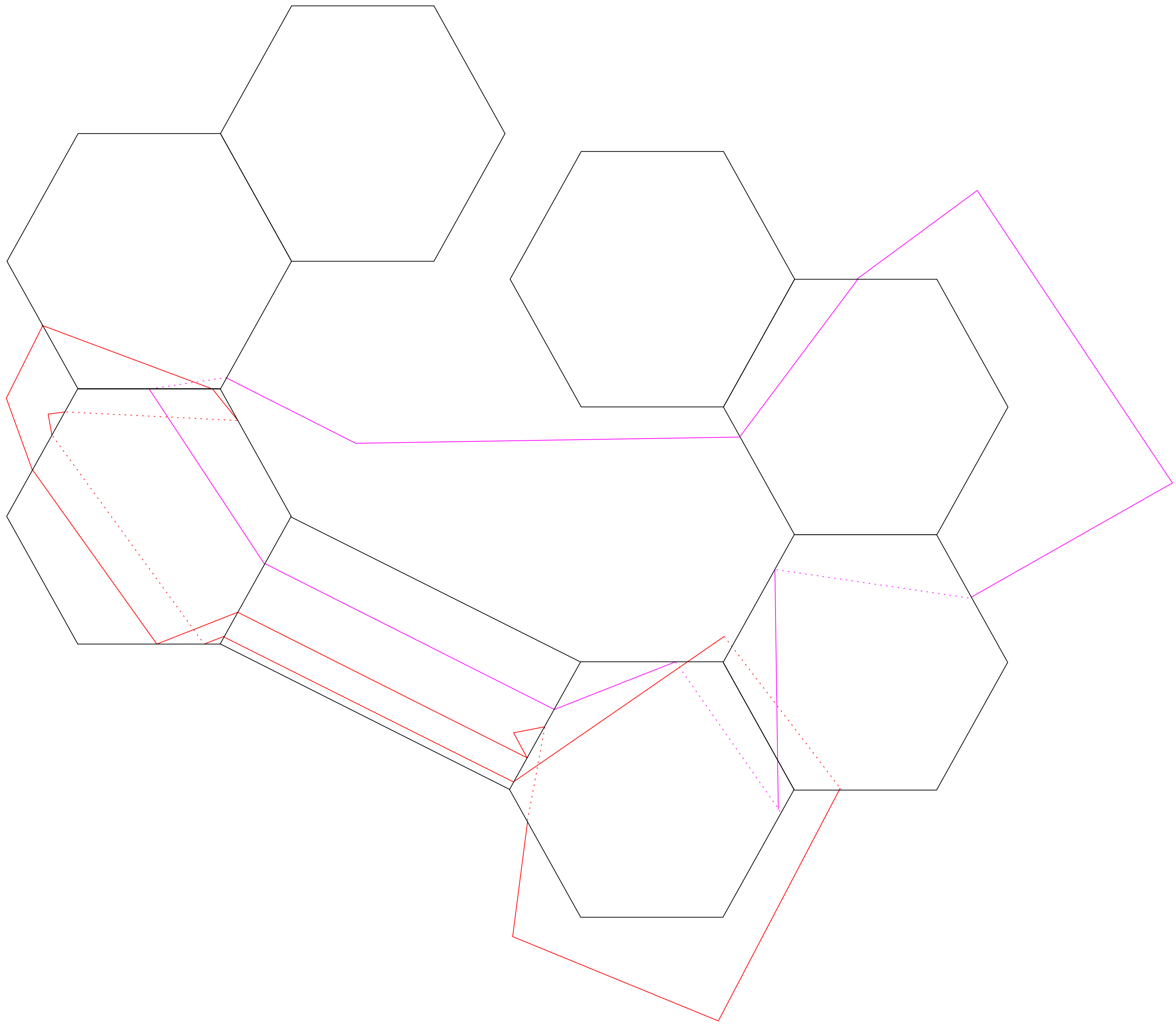
- $10_{147} = m(3/5, 1/3, -1/3) \leftarrow 10_{145}$

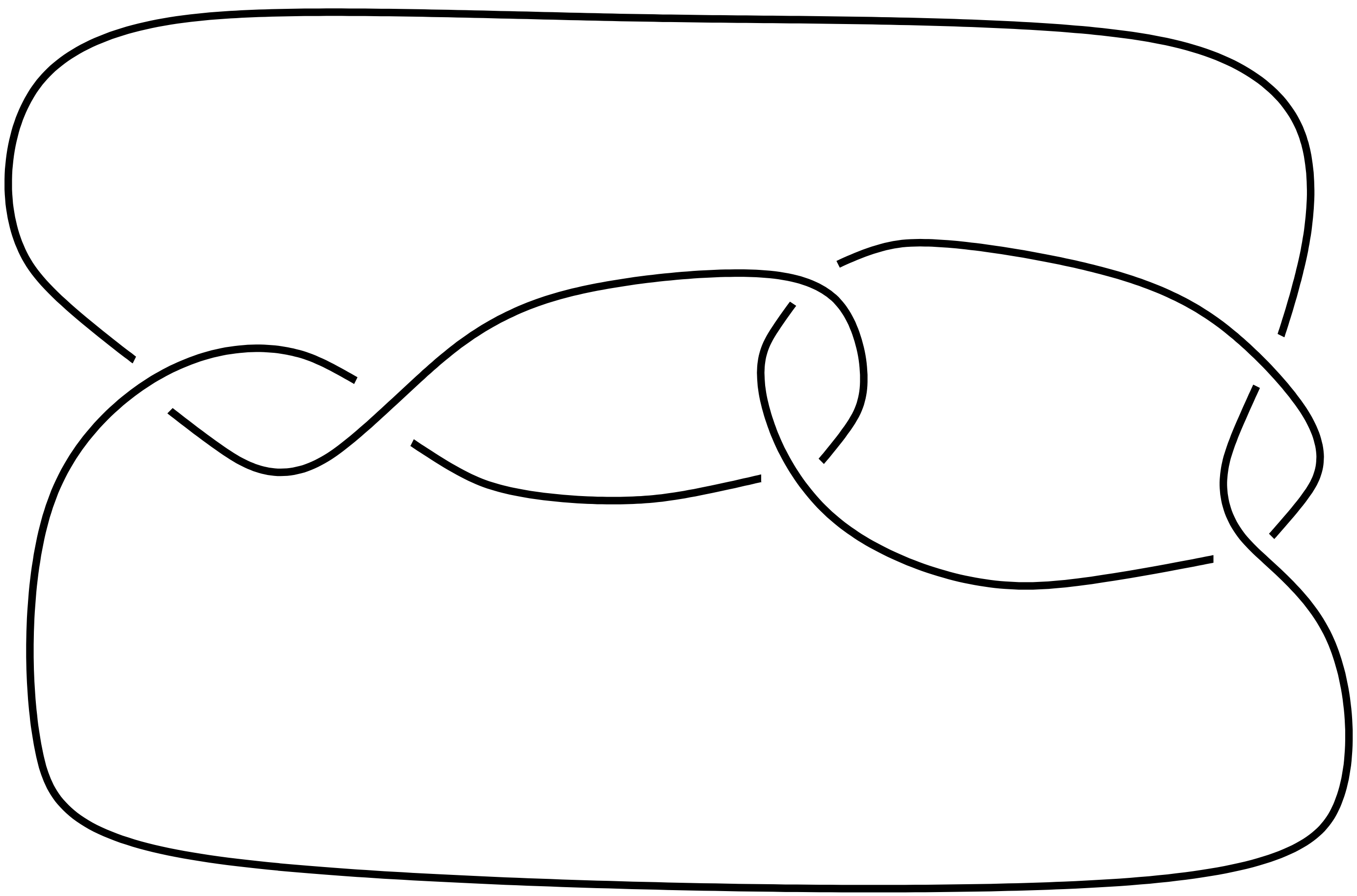
$$10_{145} = m(2/5, 2/3, -1/3)$$











$$m(2, -\frac{1}{2}, \frac{1}{2}) \sim m(-\frac{1}{2}, \frac{5}{2}) \sim m(\frac{8}{3})$$

The knot 10_{145} *is universal*

THEOREM. *All non-torus Montesinos knots of less than eleven crossings are universal, except for*

$$10_{67} = m(2/5, 1/3, 2/3) \quad \textit{and} \quad 10_{137} = m(2/5, 3/5, -1/2)$$

(We do not know).