A Short Voyage around the Three-Sphere

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Abstract

The Euclidean 3-sphere S^3 plays a paramount role in differential geometry. It is the unit sphere of the real 4-space $\mathbf{R}^4$, or the complex 2-space $\mathbf{C}^2$, or the space of quaternions $\mathbf{H}$. The Clifford decomposition of S^3 is the basic example of É. Cartan's family of isoparametric surfaces, and, topologically, it realizes S^3 as two solid tori glued together along their boundaries.
The Hopf map is a basic example of a Riemannian submerison, and topologically, it realizes S^3 as the total space of a fibre bundle with circles as fibres and the 2-sphere S^2 as the base. In addition, the Hopf map gives the non-trivial third homotopy class of S^2 (in striking contrast to homology and cohomology theories).
As a unit sphere in $\mathbf{C}^2$, the 3-sphere S^3 can be identified with the special unitary group $SU(2)$, the simplest and most important compact non-abelian Lie group. The (complex irreducible) representations W_p, $p \geq 0$, of $SU(2)$ beautifully line up in the (multiplicity 1) decomposition of the polynomial ring $\mathbf{C}[z, w]$ in two complex variables $z, w \in \mathbf{C}$. The Hopf map recurs as the $SU(2)$-orbit map of a polynomial in W_2 within its respective sphere.
The 2-sphere S^2 is the complex projective line $\mathbf{C}P^1$, and the Hopf map immediately generalizes to the bundle projection of the odd-sphere $S^{2m+1} \subset \mathbf{C}^{m+1}$ to the complex projective space $\mathbf{C}P^m$. On the lowest level, $\mathbf{C}P^1$ has a standard embedding into $\mathbf{C}P^2$ with image, in homogeneous coordinates $[X : Y : Z]$, the complex quadric given by $Y^2 = 2XZ$. On the respective circle bundles this embedding is induced by another classic, the complex Veronese map of S^3 to the 5-shere S^5. Once again, the complex Veronese map is an $SU(2)$-orbit of a polynomial in W_2 within its respective sphere.
What links the Hopf and the complex Veronese maps together is the fact that

their component functions give an orthogonal basis of the space $\mathcal{H}^2$ of spherical harmonics of order 2 on S^3. It is natural to ask if there are more examples of maps of S^3 into spheres with components in $\mathcal{H}^2$. In this talk we show that the space of such maps is a 10-dimensional compact convex body and describe its beautiful geometry.