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Stratification of the space of foliations on $\mathbb{CP}^2$

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ABSTRACT

We describe an algorithm for constructing an algebraic stratification of the space of holomorphic foliations on $\mathbb{CP}^2$ of degree d with respect to the action of $Aut(\mathbb{CP}^2)$ by change of coordinates. The strata are non-singular, locally-closed algebraic varieties. We show that these varieties parameterize foliations with certain type of degenerate singular points. We give the explicit form of the foliation in some strata. We also obtain the dimension of these varieties.

1. Introduction

There are many works on the study of holomorphic foliations on $\mathbb{CP}^2$, however there are still many unresolved classification problems on this topic. In this paper we use techniques from geometric invariant theory (GIT, see Fogarty et al., 1994) to parameterize foliations on $\mathbb{CP}^2$ of degree d with certain type of degenerate singularities. More specifically, we construct a GIT-stratification by locally closed subvarieties of the space of foliations and we give a geometric characterization of every stratum according to the type of degenerate singularities that the corresponding foliations have.

GIT states that, given the linear action of a reductive group on a projective variety, it is possible to construct a good quotient if we consider only the open set of semistable points. Non-semistable points are called unstable. We can construct a stratification of the variety where the unique open stratum is the set of semistable points and the other strata, whose union is the closed set of unstable points, are locally-closed algebraic varieties. The indexing set of the stratification is a subset of the virtual one parameter subgroups of the group, endowed with a partial ordering given by a norm. Using this order we can identify which unstable points are in the closure of other strata. This stratification is

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based on works of W.H. Hesselink (1979, 1978), G. Kempf (1978) and part II of F. Kirwan (1984). The algorithmic construction of this stratification can be found in Popov (2010) and Norbert A'Campo has an implementation of this in PARI/GP.

The aim of this paper is to construct this stratification for the space $\mathcal{F}_d$ of foliations on $\mathbb{CP}^2$ of degree d considering the action given by change of coordinates. We show how we can use it to describe foliations with certain dynamical properties. Following this we characterize the strata according to existence of invariant lines, degenerate singularities and existence of curves of singularities. For example, a complete analysis for foliations of degree 2 can be found in Alcántara (2013), a summary of this is in Table 1. Some results in this direction proved in this paper in the general case of foliations of degree d are the following:

Theorem 4.1. *The set:*

$$\begin{aligned} S = \{X \in \mathcal{F}_d : & X \text{ has only one singularity with multiplicity } d \\ & \text{and the } d\text{-jet is linearly equivalent to } z^d \frac{\partial}{\partial y}\} \\ = SL(3,\mathbb{C})\{(\alpha y^d + zP_{d-1}(y,z))\frac{\partial}{\partial x} + z^d \frac{\partial}{\partial y} : & P_{d-1}(y,z) \in \mathbb{C}_{d-1}[y,z],\ \alpha \in \mathbb{C}^*\}, \end{aligned}$$

and it is closed in the open set of foliations with isolated singularities. Moreover, S is an irreducible, non-singular subvariety of the closed set of unstable foliations in $\mathcal{F}_d$ and it has dimension $d+4$.

Theorem 4.2. *Let $\frac{d-1}{2} < j < d$, set:*

$$\begin{aligned} S = \{X \in \mathcal{F}_d : & X \text{ has a singularity with multiplicity } d, \\ & \text{Milnor number greater or equal than } d^2 + j + 1 \text{ and} \\ & \text{the } d\text{-jet is linearly equivalent to} \\ & z^j\Big((\sum_{k=j}^{d} \alpha_k z^{k-j} y^{d-k})\frac{\partial}{\partial y} + (\sum_{k=j}^{d-1} \beta_k z^{k-j+1} y^{d-k-1})\frac{\partial}{\partial z}\Big), \text{ where } \alpha_j \neq 0 \text{ or } \beta_j \neq 0\}. \end{aligned}$$

Then we have

$$\begin{aligned} S = SL(3,\mathbb{C})\{(\alpha y^d + zP_{d-1}(y,z))\frac{\partial}{\partial x} + (\sum_{k=j}^{d} \alpha_k z^k y^{d-k})\frac{\partial}{\partial y} + (\sum_{k=j}^{d-1} \beta_k z^{k+1} y^{d-k-1})\frac{\partial}{\partial z} : \\ P_{d-1}(y,z) \in \mathbb{C}_{d-1}[y,z], \alpha \in \mathbb{C}^* \text{ and } \alpha_j \neq 0 \text{ or } \beta_j \neq 0\}, \end{aligned}$$

and it is an irreducible, locally closed, non-singular subvariety of the closed set of unstable foliations in $\mathcal{F}_d$ of dimension $3d - 2j + 4$.

Theorem 4.3. *Let $\frac{4d-1}{6} < j \leq d-1$, set*

$$\begin{aligned} S = \{X \in \mathcal{F}_d : & X \text{ has a singularity with multiplicity } j, \text{ Milnor number } j(j+1) \\ & \text{and the } j\text{-jet in this point is linearly equivalent to } z^j \frac{\partial}{\partial y}\}. \end{aligned}$$

Then we have

$$S = SL(3,\mathbb{C})\Big\{P(y,z)\frac{\partial}{\partial x} + \Big(x^{d-j}z^j + \sum_{k=j+1}^{d} \alpha_k x^{d-k} z^k + \sum_{k=0}^{d-j-1} x^k Q_{d-k}(y,z)\Big)\frac{\partial}{\partial y} +$$

$$\Big(\sum_{k=j+1}^{d} \beta_k x^{d-k} y^k + \sum_{k=0}^{d-j-1} x^k R_{d-k}(y,z)\Big)\frac{\partial}{\partial z} :$$

$$\beta_{j+1} \neq 0,\ P \in \mathbb{C}_d[y,z],\ Q_{d-k}, R_{d-k} \in \mathbb{C}_{d-k}[y,z]\Big\},$$

and it is an irreducible, locally closed, non-singular subvariety of the closed set of unstable foliations in $\mathcal{F}_d$ of dimension $d^2 + 4d - j^2 - 3j + 4$.

This paper is organized as follows. In Section 2 we provide a brief background from the GIT-stratification of a projective variety with a linear action by a reductive group. And we give some results concerning holomorphic foliations on $\mathbb{CP}^2$ of degree d. In Section 3 we construct the stratification for the space of foliations on $\mathbb{CP}^2$ of degree d. Finally in Section 4 we give the geometric characterizations of the strata. And we describe an algorithm to determine, for a given foliation of degree d with isolated singularities, the stratum to which it belongs.

2. Preliminaries

2.1. Stratification of a projective variety

In this subsection we present a brief overview of the stratification that we will construct. The details may be consulted in Kirwan (1984) and Popov (2010).

Let $V \subset \mathbb{P}^n$ be a non-singular complex projective variety and consider a reductive group G acting linearly on V. We begin with the following basic definition.

Definition 2.1. Let $x \in V$, and consider $\bar{x}(\neq 0)$ in the affine cone of V such that $\bar{x} \in x$. Denote by $O(\bar{x})$ the orbit of $\bar{x}$, then:

1. x is **unstable** if $0 \in \overline{O(\bar{x})}$,
2. x is **semistable** if $0 \notin \overline{O(\bar{x})}$. The set of semistable points will be denoted by V^{ss}.
3. x is **stable** if it is semistable, the orbit of x, $O(x)$, is closed in V^{ss} and $\dim O(x) = \dim G$. The set of stable points will be denoted by V^s.

As we have said, the results presented in this paper are based on the construction of a stratification of the space of foliations on $\mathbb{CP}^2$ of degree d, as we shall see below this is a projective space of dimension $d^2 + 4d + 2$. We will consider in this variety the action of $Aut(\mathbb{CP}^2)$ by change of coordinates. The definition, existence and properties of this stratification are stated in the sequel.

Definition 2.2. A finite collection $\{S_\beta : \beta \in \mathcal{B}\}$ of subsets of V forms a **stratification of** V if V is the disjoint union of the strata $\{S_\beta : \beta \in \mathcal{B}\}$, and there is a partial order $>$ on the indexing set $\mathcal{B}$ such that:

$$\overline{S_\beta} \subset \bigcup_{\gamma \geq \beta} S_\gamma,$$

for every $\beta \in \mathcal{B}$.

Theorem 2.3. *(See Theorem 13.5 in Kirwan, 1984.) Let V be a non-singular projective variety with a linear action by a reductive group G. Then there exists a stratification*

$$\{S_\beta : \beta \in \mathcal{B}\}$$

of V such that the unique open stratum is V^{ss} and every stratum S_β in the set of unstable points is non-singular, locally-closed and isomorphic to $G \times_{P_\beta} Y_\beta^{ss}$, where Y_β^{ss} is a non-singular locally-closed subvariety of V and P_β is a parabolic subgroup of G.

Throughout the text we will use the same notation as in §12 of Kirwan (1984).

Definition 2.4. Let $Y(G)$ be the set of one parameter subgroups $\lambda : \mathbb{C}^* \to G$. Define in $Y(G) \times \mathbb{N}$ the equivalence relation: (λ_1, n_1) is related with (λ_2, n_2) if and only if $\lambda_1(t^{n_2}) = \lambda_2(t^{n_1})$ for all $t \in \mathbb{C}^*$. **A virtual one parameter subgroups of** G is an equivalence class of this relation, the set of these classes will be denoted by $M(G)$.

The indexing set $\mathcal{B}$ of the stratification is a finite subset of $M(G)$ and this may be described in terms of the weights of the representation of G which defines the action.

For the construction we must consider on $M(G)$ a norm q which is the square of an inner product $\langle\ ,\ \rangle$. This norm gives the partial order $>$ on $\mathcal{B}$.

On the other hand, the representation of $\mathbf{D}$ on $\mathbb{C}^{n+1}$, where $\mathbf{D}$ is a maximal torus of G, splits as a sum of scalar representations given by characters $\alpha_0, \ldots, \alpha_n$. These characters are elements of the dual of $M(\mathbf{D})$ but we can identify them with elements of $M(\mathbf{D})$ using $\langle\ ,\ \rangle$.

Once we have the indexing set $\mathcal{B}$ we can describe the objects that appear in Theorem 2.3. Letting $\beta \in \mathcal{B}$, we define:

$$Z_\beta = \{(x_0 : \ldots : x_n) \in V : x_j = 0 \text{ if } \langle \alpha_j, \beta\rangle \neq q(\beta)\},$$
$$Y_\beta = \{(x_0 : \ldots : x_n) \in V : x_j = 0 \text{ if } \langle \alpha_j, \beta\rangle < q(\beta)$$
$$\text{and } x_j \neq 0 \text{ for some } j \text{ with } \langle \alpha_j, \beta\rangle = q(\beta)\},$$

the map $p_\beta : Y_\beta \to Z_\beta$, $(x_0, \ldots, x_n) \mapsto (x'_0, \ldots, x'_n)$ as $x'_j = x_j$ if $\langle \alpha_j, \beta\rangle = q(\beta)$ and $x'_j = 0$ otherwise.

Consider $Stab(\beta)$ the stabilizer of β under the adjoint action of G. There exists a unique connected reductive subgroup G_β of $Stab_\beta$ such that $M(G_\beta) = \{\lambda \in M(Stab_\beta) : \langle \lambda, \beta\rangle = 0\}$ (see 12.21 in Kirwan, 1984). With this group we can define $Z^{ss}_\beta = \{x \in Z_\beta : x \text{ is semistable under the action of } G_\beta \text{ on } Z_\beta\}$ and $Y^{ss}_\beta = p_\beta^{-1}(Z^{ss}_\beta)$.

Finally the parabolic group of β is: if $x \in Y^{ss}_\beta$ then $P_\beta = \{g \in G : gx \in Y^{ss}_\beta\}$. And since S_β is isomorphic to $G \times_{P_\beta} Y^{ss}_\beta$, it has dimension $\dim Y^{ss}_\beta + \dim G - \dim P_\beta$.

2.2. Foliations on $\mathbb{CP}^2$ of degree d

Now we give a summary with general results about foliations on $\mathbb{CP}^2$ of degree d.

Definition 2.5. A holomorphic foliation X on $\mathbb{CP}^2$ of degree d is a non-trivial morphism of vector bundles:

$$X : \mathcal{O}_{\mathbb{CP}^2}(1-d) \to T\mathbb{CP}^2,$$

where $T\mathbb{CP}^2$ is the tangent bundle of $\mathbb{CP}^2$. We consider this morphism modulo multiplication by a nonzero scalar. Then the space of foliations of degree d is $\mathcal{F}_d := \mathbb{P}H^0(\mathbb{CP}^2, T\mathbb{CP}^2(d-1))$, where $d \geq 0$.

In what follows we will use the notation $\mathbb{C}_k[x, y, z]$ for the space of homogeneous polynomials of degree k in three variables. Due to Euler sequence we have the following:

Proposition 2.6. *Every foliation $X \in \mathcal{F}_d$ can be written as*

$$X = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z} = \begin{pmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{pmatrix}$$

where $P, Q, R \in \mathbb{C}_d[x, y, z]$, modulo multiplication by a nonzero scalar and if we consider the radial foliation

$$E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

then X and $X + F(x, y, z)E$ represent the same foliation for all $F \in \mathbb{C}_{d-1}[x, y, z]$. Therefore $\mathcal{F}_d$ is a projective space of dimension $d^2 + 4d + 2$.

Definition 2.7. A point $p = (a : b : c) \in \mathbb{CP}^2$ is **singular** for the above foliation X if

$$(P(a, b, c), Q(a, b, c), R(a, b, c)) = (ka, kb, kc),$$

for some $k \in \mathbb{C}$. The set of singular points of X will be denoted by $Sing(X)$.

Definition 2.8. Let $X \in \mathcal{F}_d$ and let p be an isolated singularity of X. Let

$$\begin{pmatrix} Q(y, z) = Q_m(y, z) + Q_{m+1}(y, z) + \dots \\ R(y, z) = R_n(y, z) + R_{n+1}(y, z) + \dots \end{pmatrix}$$

be a local generator of X in $p = (1 : 0 : 0)$, where Q_i, R_i are homogeneous of degree i, and Q_m, R_n are not identically zero. We define the **Milnor number of** p by $\mu_p(X) := \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^2,p}}{<Q,R>}$ and the **multiplicity of** p by $m_p(X) := min\{m, n\}$. We will say that the point p is a **degenerate singularity of** X if $\mu_p(X) > 1$.

As we have seen, the Milnor number of p is the intersection index of the algebraic curves Q and R. In general, we will use the notation $I_p(Q, R)$ for the intersection index of the curves Q and R at p.

The following is an important and well-known result for foliations with isolated singularities.

Proposition 2.9. *(See Theorem 2.3 in Jouanolou, 1979.) Let X be a foliation on $\mathbb{CP}^2$ of degree d with isolated singularities then*

$$d^2 + d + 1 = \sum_{p \in \mathbb{CP}^2} \mu_p(X).$$

From Lemma 1.2 in Gómez-Mont and Kempf (1989) we can deduce that

$$\{X \in \mathcal{F}_d : \text{there exists } p \in \mathbb{CP}^2 \text{ such that } \mu_p(X) \geq 2\}$$

is a divisor in $\mathcal{F}_d$, therefore we have the following.

Theorem 2.10. *The set $\{X \in \mathcal{F}_d$: every singular point for X has Milnor number $1\}$ is open and non-empty in $\mathcal{F}_d$.*

Finally we give the definition of algebraic leaf for a foliation.

Definition 2.11. A plane curve defined by a polynomial $F(x, y, z)$ is an **algebraic leaf for** X **or invariant by** X if and only if there exists a polynomial $H(x, y, z)$ such that:

$$P(x, y, z)\frac{\partial F(x, y, z)}{\partial x} + Q(x, y, z)\frac{\partial F(x, y, z)}{\partial y} + R(x, y, z)\frac{\partial F(x, y, z)}{\partial z} = FH.$$

Theorem 2.12. *(See Theorem 1.1, p. 158 in Jouanolou (1979) and Lins Neto and Soares (1996).) The set $\{X \in \mathcal{F}_d$: X has no algebraic leaves$\}$ is open and non-empty in $\mathcal{F}_d$.*

Generically a foliation on $\mathbb{CP}^2$ of degree d does not have degenerate singularities and does not have algebraic leaves. So it is important to classify foliations in the complement of these sets, this is the idea of this paper.

The group $PGL(3,\mathbb{C})$ of automorphisms of $\mathbb{CP}^2$ is a reductive group and it acts linearly on $\mathcal{F}_d$ by change of coordinates:

$$\begin{aligned} PGL(3,\mathbb{C}) \times \mathcal{F}_d &\to \mathcal{F}_d \\ (g, X) &\mapsto gX = DgX \circ (g^{-1}). \end{aligned}$$

In this paper we will use the group $SL(3,\mathbb{C})$ instead of $PGL(3,\mathbb{C})$. From the fact that any linear action of $\mathbb{C}^*$ is diagonalizable we have that:

Lemma 2.13. *Every 1-parameter subgroup (1-PS) in $Y(SL(3,\mathbb{C}))$ can be written as*

$$\begin{aligned} \mathbb{C}^* &\to SL(3,\mathbb{C}) \\ t &\mapsto g\lambda(t)g^{-1} = g\begin{pmatrix} t^{n_0} & 0 & 0 \\ 0 & t^{n_1} & 0 \\ 0 & 0 & t^{n_2} \end{pmatrix} g^{-1}, \end{aligned}$$

for some $g \in SL(3,\mathbb{C})$, where $n_0 \geq n_1 \geq n_2$ and $n_0 + n_1 + n_2 = 0$. We use the notation (n_0, n_1) for the above diagonal 1-PS, λ.

We will denote by $\mathbf{D}$ the subgroup of the diagonal matrices of $SL(3,\mathbb{C})$. We distinguish the following parabolic subgroups of $SL(3,\mathbb{C})$: $\mathbf{T}$ will be the subgroup of upper triangular matrices and

$$\mathbf{T}_1 = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & \alpha_{32} & \alpha_{33} \end{pmatrix} \in SL(3,\mathbb{C}) \right\},$$

$$\mathbf{T}_2 = \left\{ \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 0 & 0 & \alpha_{33} \end{pmatrix} \in SL(3,\mathbb{C}) \right\}.$$

3. Stratification of $\mathcal{F}_d$

In this section we describe how to obtain the indexing set of the stratification for the case of foliations on $\mathbb{CP}^2$ of degree d by change of coordinates.

The representation associated to the action of $SL(3,\mathbb{C})$ on $\mathcal{F}_d$ is the kernel of the contraction map:

$$\begin{aligned} i : Sym^d(\mathbb{C}^3)^* \otimes \mathbb{C}^3 &\to Sym^{d-1}(\mathbb{C}^3)^* \\ P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z} &\mapsto \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}, \end{aligned}$$

this kernel is the irreducible representation Γ_d (see §13.2 of Fulton and Harris, 1991) and $Sym^d(\mathbb{C}^3)^* \otimes \mathbb{C}^3 = \Gamma_d \bigoplus Sym^{d-1}(\mathbb{C}^3)^*$. Therefore our representation is:

$$\begin{aligned} \Gamma_d = \{X = P\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + R\frac{\partial}{\partial z} : \\ Div(X) := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0\}. \end{aligned}$$

According to Example 2, p. 16 of Popov (2010), to obtain the indexing set $\mathcal{B}$ for the stratification, we must consider the set of weights of the action by the torus $\mathbf{D}$ on Γ_d as a subset of $\mathbb{Q}^2$ and we must take the lines $L \subset \mathbb{Q}^2$, up to the action by the Weyl group of $SL(3,\mathbb{C})$ (i.e. the symmetric group S_3), such that:

1. L does not pass through 0,
2. L passes at least through one weight,

3. the end of the perpendicular dropped from 0 onto L, is contained in a segment representing a convex hull of the weights lying on L,
4. if L is parallel to no root or if L is parallel to a certain root but L does not contain two different single weights then we must take this line L.

In $(M(\mathbf{D}))^*$ we take the S_3-invariant inner product given by $\langle L_i, L_j\rangle = \delta_{ij} - \frac{1}{3}$, where L_i is defined by:

$$L_i : M(\mathbf{D}) \to \mathbb{Q}, \quad \left(t \mapsto \begin{pmatrix} t^{a_1} & 0 & 0 \\ 0 & t^{a_2} & 0 \\ 0 & 0 & t^{a_3} \end{pmatrix}\right) \mapsto a_i,$$

for $1 \le i \le 3$, every element in $(M(\mathbf{D}))^*$ is a linear combination of these characters. We have $q(\beta) = \langle \beta, \beta\rangle$ for $\beta \in (M(\mathbf{D}))^*$. We identify $M(\mathbf{D})^*$ with $M(\mathbf{D})$ using this inner product.

According to 12.8 of Kirwan (1984), $\mathcal{B}$ is the set of the closest points to zero in the above lines and in our chosen Weyl chamber $\{b_1L_1 - b_3L_3 : b_1, b_3 \ge 0\}$.

The package HNC written by Norbert A'Campo and Vladimir Popov, allows to obtain the indexing set and the dimension of every stratum. Before running hnc.gp you need to have PARI/GP and LiE on the system. Now try the command:

```
> hnc(GROUP, REPRESENTATION)
```

GROUP is a semisimple group given by the rank and the REPRESENTATION is given by a highest weight vector. In the case of the action of $SL(3, \mathbb{C})$ in $\mathcal{F}_d$ the input is:

```
> hnc(A2,"X1,d")
```

The output is the list of strata. For example, in the following cases we have in the first entry the dimension of the stratum and in the second entry the virtual one parameter subgroup which defines the stratum.

Degree 3 (16 strata):

$$[3, (5/3, 2/3)], [4, (5/3, -1/3)], [5, (3/2, 0)], [5, (5/3, -2/3)],$$
$$[6, (55/42, -11/42)], [7, (7/6, -2/6)], [7, (2/3, 2/3)], [8, (1, 0)],$$
$$[9, (20/21, -4/21)], [9, (2/3, 1/6)], [12, (1/2, 0)], [13, (2/3, -1/3)],$$
$$[13, (1/6, 1/6)], [14, (5/21, -1/21)], [14, (2/21, 1/42)], [15, (7/78, -2/78)].$$

Degree 4 (26 strata):

$$[3, (2, 1)], [5, (2, 0)], [6, (2, -1)], [6, (25/14, -5/14)], [7, (21/13, -6/13)],$$
$$[7, (1, 1)], [8, (3/2, -1/2)], [8, (3/2, 0)], [9, (10/7, -2/7)], [9, (8/7, 2/7)],$$
$$[10, (35/26, -10/26)], [12, (15/14, -3/14)], [12, (1, 0)], [13, (1/2, 1/2)],$$
$$[13, (16/19, -2/19)], [14, (4/7, 1/7)], [15, (5/7, -1/7)], [15, (5/13, 2/13)],$$
$$[16, (1, -1/2)], [17, (1/2, 0)], [17, (2/7, 1/14)], [17, (7/13, -2/13)],$$
$$[18, (7/38, 1/38)], [18, (3/7, -1/7)], [19, (5/14, -1/14)], [20, (7/26, -2/26)].$$

The following lemma will be used in the description of the strata, the proof is valid for any irreducible representation of $SL(3, \mathbb{C})$. We will use the notation $P_{\beta_j} = P_j$, $S_{\beta_j} = S_j$, $Z_{\beta_j} = Z_j$ and $Y_{\beta_j} = Y_j$.

Lemma 3.1. *Let $X \in Z_j$ be such that $Stab(\beta_j) = \mathbf{D}$, i.e. the virtual one parameter subgroup (n_0, n_1) corresponding to β_j satisfies $n_0 > n_1 > n_2$. Then $X \in Z_j^{ss}$ if and only if β_j is the closest point to zero in C_X with respect to $\mathbf{D}$, where C_X is the convex hull formed with the weights of X.*

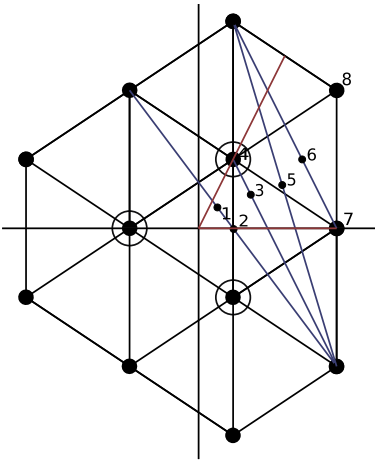

Fig. 1. Representation of foliations of degree 2.

Proof. We follow the notation of 12.5 in Kirwan (1984). We have that β_j is the closest point to zero in C_X with respect to $\mathbf{D}$ if and only if $\Lambda_{\mathbf{D}}(X) = \Lambda_{Stab(\beta_j)}(X) = \{\frac{\beta_j}{|\beta_j|}\}$. Since $Stab\,\beta_j$ is optimal for X then $\Lambda_{\mathbf{D}}(X) = M(\mathbf{D}) \cap \Lambda_G(X)$. Therefore $\Lambda_{\mathbf{D}}(X) = \{\frac{\beta_j}{|\beta_j|}\}$ if and only if $\frac{\beta_j}{|\beta_j|} \in \Lambda_G(X)$, and it happens if and only $X \in Z_j^{ss}$. $\square$

In Proposition 4.2 of Hesselink (1979) the author proves the following result that we will use for our construction.

Lemma 3.2. *S_j is irreducible and it is open in its closure.*

3.1. Example: the characterization of the strata of $\mathcal{F}_2$

In the case of foliations of degree 2 the output of Norbert A'Campo's implementation of Popov's algorithm is:

$$[3, [5/3, 4/3]], [4, [2/3, 4/3]], [5, [1, 1]], [6, [2/3, 5/6]]$$

$$[7, [2/3, 1/3]], [8, [1/2, 1/2]], [9, [5/21, 4/21]], [10, [1/6, 1/3]].$$

Fig. 1 shows the weights of the representation, the weights in the central triangle have to be taken twice, the number j represents the virtual one parameter subgroup β_j in the indexing set.

In Table 1 we summarize the results obtained from the analysis of the strata of $\mathcal{F}_2$, the proofs can be consulted in Alcántara (2013). The second column gives the characterization of the foliations belonging to each stratum. The third column is the dimension of each substratum and the last one is its closure. Here m and μ denote the multiplicity and Milnor number of a singular point, respectively.

To have a complete description of the table we need the following lemmas. In the first one we characterize the foliations with isolated singularities in the unique open stratum S_0, which is the set of semistable foliations. In the second one we characterize the semistable foliations of degree 1, the proof of this can be found in Alcántara (2011).

Theorem 3.3. *Let $X \in \mathcal{F}_2$ with isolated singularities, then X is semistable if and only if:*

i) the leaves of X are not lines, or
ii) if every singularity of X has multiplicity 1 and if L is an invariant line then L has no singularities with Milnor number 5.

Table 1
Stratification of $\mathcal{F}_2$.

	Characterization	Dim	Closure
S_0	semistable foliations (see 3.3)	14	$\mathcal{F}_2$
S_1	invariant line with a unique singularity with $m=1$ and $\mu=5$	9	$\bigcup_{j\geq 1,\, j\neq 2} S_j$
S_2	1) singular point with $m=2$ and $\mu=4$	10	$\bigcup_{j\geq 2} S_j$
	2) singular point with $m=2$ and $\mu=5$	(9)	
	3) singularity with $m=2$, $\mu=6$, the 2-jet has no double tangents	(8)	
	4) $\{(L,X)\in(\mathbb{CP}^2)^*\times\mathcal{F}_1^{ss} : L\cap Sing(X)=\{p\}\}$	(8)	
	5) $\{(L_1,L_2,p)\in(\mathbb{CP}^2)^*\times(\mathbb{CP}^2)^*\times\mathbb{CP}^2 : L_1\neq L_2,\ p\notin L_1\cup L_2\}$	(6)	
S_3	1) singularity with $m=2$, $\mu=6$, the 2-jet has double tangents	8	$\bigcup_{j\geq 3} S_j$
	2) invariant line with a singular point with $m=2$ and $\mu=7$	(7)	
	3) a non-singular conic with a point in this conic	(6)	
	4) $\{(L,X)\in(\mathbb{CP}^2)^*\times\mathcal{F}_1 : X$ has a unique singularity p, L is not invariant by X, $p\in L\}$	(6)	
S_4	$\{(L,X)\in(\mathbb{CP}^2)^*\times\mathcal{F}_1^{ss} : L$ has at least two singularities of $X\}$	7	$\bigcup_{j\geq 4,\, j\neq 5} S_j$
S_5	unique singularity, rational first integral	6	$\bigcup_{j\geq 5} S_j$
S_6	$\{(L,X)\in(\mathbb{CP}^2)^*\times\mathcal{F}_1 : X$ has a unique singularity, L is the unique invariant line of $X\}$	5	$\bigcup_{j\geq 6} S_j$
S_7	$\{(L_1,L_2)\in(\mathbb{CP}^2)^*\times(\mathbb{CP}^2)^* : L_1\neq L_2\}$	4	$S_7\cup S_8$
S_8	$\{(L,p)\in(\mathbb{CP}^2)^*\times\mathbb{CP}^2 : p\in L\}$	3	S_8

Proof. For the proof see Theorem 4 of Alcántara (2011). □

Lemma 3.4. *Let X be a linear foliation or a foliation of degree 1 on $\mathbb{CP}^2$. Then X is semistable if and only if X has 2 or 3 different singularities, or X is a line of singularities L and a point p such that $p\notin L$.*

4. Foliations of degree *d* with degenerate singularities

In this section we obtain the characterization of the strata according to existence of certain type of degenerate singular points. For that we must focus our attention on the generators as linear space of the spaces Y_β^{ss}. We have the following results.

Theorem 4.1. *The set:*

$$\begin{aligned} S &= \{X\in\mathcal{F}_d : X \text{ has one singularity with multiplicity } d \\ &\quad \text{and the } d\text{-jet is linearly equivalent to } z^d\frac{\partial}{\partial y}\} \\ &= SL(3,\mathbb{C})\{(\alpha y^d + zP_{d-1}(y,z))\frac{\partial}{\partial x} + z^d\frac{\partial}{\partial y} : P_{d-1}(y,z)\in\mathbb{C}_{d-1}[y,z],\ \alpha\in\mathbb{C}^*\}, \end{aligned}$$

and it is closed in the open set of foliations with isolated singularities. Moreover, S is an irreducible, non-singular subvariety of the closed set of unstable foliations in $\mathcal{F}_d$ and it has dimension $d+4$.

Proof. Let S_β be the stratum corresponding to β, the closest point to zero in the line defined by the weights $(d+1)L_1+dL_3$ and $-L_1+(-1-d)L_3$. The corresponding one parameter subgroup is $(2d+1, 1-d)$. We can see that:

$$\begin{aligned} Z_\beta^{ss} &= \{\alpha y^d\frac{\partial}{\partial x} + z^d\frac{\partial}{\partial y} : \alpha\in\mathbb{C}^*\}, \\ Y_\beta^{ss} &= \{(\alpha y^d + zP_{d-1}(y,z))\frac{\partial}{\partial x} + z^d\frac{\partial}{\partial y} : P_{d-1}(y,z)\in\mathbb{C}_{d-1}[y,z],\ \alpha\in\mathbb{C}^*\}. \end{aligned}$$

The foliations in Y_β^{ss} have only one singular point in $(1:0:0)$, with multiplicity d and with *d-jet*, $z^d\frac{\partial}{\partial y}$. The parabolic subgroup P_β of β is $\mathbf{T}$, and

$$S_\beta = SL(3,\mathbb{C}) \times_{P_\beta} Y^{ss}_\beta,$$

therefore $S_\beta \subset S$ is a locally closed, $SL(3,\mathbb{C})$-invariant, non-singular subvariety of $\mathcal{F}_d$ and $\dim S_\beta = \dim Y^{ss}_\beta + 3 = d+4$.

On the other hand, let $X \in S$. Then up to change of coordinates, $X = P(y,z)\frac{\partial}{\partial x} + z^d \frac{\partial}{\partial y}$. Since the singular point has Milnor number d^2+d+1 then $P(y,z) = \alpha y^d + zP^{d-1}(y,z)$ where $P_{d-1}(y,z) \in \mathbb{C}_{d-1}[y,z]$ and $\alpha \in \mathbb{C}^*$. We conclude that $S = S_\beta$ and with 4.2 of Hesselink (1979), we have that S is irreducible in $\mathcal{F}^{un}_d$.

We know by Theorem 2.3 that $\overline{S_\beta} \subset \bigcup_{\gamma \geq \beta} S_\beta$. If $\gamma > \beta$ then a foliation $X \in Y^{ss}_\gamma$ has the form $X = zP_{d-1}\frac{\partial}{\partial x}$ or $X = zP_{d-1}\frac{\partial}{\partial x} + z^d\frac{\partial}{\partial y}$, where $P_{d-1}(y,z) \in \mathbb{C}_{d-1}[y,z]$. In both cases the foliation has $z=0$ as a curve of singularities. Therefore there are no foliations in $\overline{S_\beta} - S_\beta$ with isolated singularities. This says that S_β is closed in the open set of foliations with isolated singularities.

The stabilizer of $X^d_1 = y^d\frac{\partial}{\partial x} + z^d\frac{\partial}{\partial y}$ is

$$\left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix} : a = a^d b^{2d+1} \right\},$$

then $\dim O(X^d_1) = 6$ and the generic foliation in $Y^{ss}_\beta - Z^{ss}_\beta$ has as stabilizer the group $(\mathbb{C},+)$. $\square$

If we take a generic foliation in Y^{ss}_β and we blow up the unique singular point p we obtain the foliation:

$$(w_1^2 P(1,w_2) - w_1 w_2^d)dw_2 - w_2^{d+1}dw_1 = 0,$$

where $w_1 = 0$ is the exceptional divisor. This foliation is a Bernoulli differential equation and we can obtain its solutions (see p. 104 in Hille, 1976).

Theorem 4.2. *Let $\frac{d-1}{2} < j < d$, set:*

$$S = \{X \in \mathcal{F}_d : X \text{ has a singularity with multiplicity } d,$$
$$\text{Milnor number greater or equal than } d^2+j+1 \text{ and}$$
$$\text{the } d\text{-jet is linearly equivalent to}$$
$$z^j\Big(\Big(\sum_{k=j}^{d} \alpha_k z^{k-j} y^{d-k}\Big)\frac{\partial}{\partial y} + \Big(\sum_{k=j}^{d-1} \beta_k z^{k-j+1} y^{d-k-1}\Big)\frac{\partial}{\partial z}\Big), \text{ where } \alpha_j \neq 0 \text{ or } \beta_j \neq 0\}.$$

Then we have

$$S = SL(3,\mathbb{C})\{(\alpha y^d + zP_{d-1}(y,z))\frac{\partial}{\partial x} + \Big(\sum_{k=j}^{d} \alpha_k z^k y^{d-k}\Big)\frac{\partial}{\partial y} + \Big(\sum_{k=j}^{d-1} \beta_k z^{k+1} y^{d-k-1}\Big)\frac{\partial}{\partial z} :$$
$$P_{d-1}(y,z) \in \mathbb{C}_{d-1}[y,z], \alpha \in \mathbb{C}^* \text{ and } \alpha_j \neq 0 \text{ or } \beta_j \neq 0\},$$

and it is an irreducible, locally closed, non-singular subvariety of the closed set of unstable foliations in $\mathcal{F}_d$ of dimension $3d-2j+4$.

Proof. We must consider β the closest point to zero in the line defined by the weights $(d+1)L_1 + dL_3$ and $(d-j-1)L_1 + (d-2j-1)L_3$. If $\frac{d-1}{2} < j$ then β is in the indexing set of the stratification of $\mathcal{F}_d$ and

$$Y^{ss}_\beta = \{(\alpha y^d + zP_{d-1}(y,z))\frac{\partial}{\partial x} + \Big(\sum_{k=j}^{d} \alpha_k z^k y^{d-k}\Big)\frac{\partial}{\partial y} + \Big(\sum_{k=j}^{d-1} \beta_k z^{k+1} y^{d-k-1}\Big)\frac{\partial}{\partial z} :$$
$$P_{d-1}(y,z) \in \mathbb{C}_{d-1}[y,z],\ \alpha \in \mathbb{C}^* \text{ and } \alpha_j \neq 0 \text{ or } \beta_j \neq 0\},$$

the generic foliation in Y_β^{ss} has a singular point in $(1:0:0)$ with multiplicity d and Milnor number d^2+j+1. But in this space we can get foliations with a singular point in $(1:0:0)$ with multiplicity d and Milnor number greater than d^2+j+1. For example $y^d\frac{\partial}{\partial x}+(\alpha y^{d-j}z^j+\alpha_1 z^d)\frac{\partial}{\partial y}+\alpha y^{d-j-1}z^{j+1}\frac{\partial}{\partial z}\in Y_\beta^{ss}$ where $\alpha,\alpha_1\in\mathbb{C}^*$, has Milnor number d^2+d+1 in $(1:0:0)$.

The d-jet is $z^j\big((\sum_{k=j}^d\alpha_k z^{k-j}y^{d-k})\frac{\partial}{\partial y}+(\sum_{k=j}^{d-1}\beta_k z^{k-j+1}y^{d-k-1})\frac{\partial}{\partial z}\big)$, where $\alpha_j\neq 0$ or $\beta_j\neq 0$. The dimension of this stratum is $3d-2j+4$.

Now, suppose that $X\in\mathcal{F}_d$ has a singularity in $(1:0:0)$ with multiplicity d, Milnor number greater than or equal to d^2+j+1 and suppose that the d-jet is $z^j\big((\sum_{k=j}^d\alpha_k z^{k-j}y^{d-k})\frac{\partial}{\partial y}+(\sum_{k=j}^{d-1}\beta_k z^{k-j+1}y^{d-k-1})\frac{\partial}{\partial z}\big)$, where $\alpha_j\neq 0$ or $\beta_j\neq 0$. Then the local generator of X in $p=(1:0:0)$ has the form:

$$\begin{pmatrix}\sum_{k=j}^d\alpha_k z^k y^{d-k}-yP(y,z)\\ \sum_{k=j}^{d-1}\beta_k z^{k+1}y^{d-k-1}-zP(y,z)=z(\sum_{k=j}^{d-1}\beta_k z^k y^{d-k-1}-P(y,z))\end{pmatrix},$$

where $P(y,z)\in\mathbb{C}_d[y,z]$ and the Milnor number is

$$\begin{aligned}&I(z,\sum_{k=j}^d\alpha_k z^k y^{d-k}-yP(y,z))+\\ &I(\sum_{k=j}^{d-1}(\alpha_k-\beta_k)z^k y^{d-k}+\alpha_d z^d,\sum_{k=j}^{d-1}\beta_k z^k y^{d-k-1}-P(y,z))=\\ &I(z,\sum_{k=j}^d\alpha_k z^k y^{d-k}-yP(y,z))+\\ &I(z^j\big(\sum_{k=j}^{d-1}(\alpha_k-\beta_k)z^{k-j}y^{d-k}+\alpha_d z^{d-j}\big),\sum_{k=j}^{d-1}\beta_k z^k y^{d-k-1}-P(y,z)),\end{aligned}$$

and this expression is greater than or equal to $d+1+dj+(d-j)(d-1)=d^2+j+1$ if and only if $P(y,z)=\alpha y^d+zP_{d-1}(y,z)$ with $\alpha\neq 0$ and $P_{d-1}(y,z)\in\mathbb{C}_{d-1}[y,z]$. If $\alpha_j\neq 0$ or $\beta_j\neq 0$ then $X\in Y_\beta^{ss}$ and for certain coefficients, which satisfy an open condition, we have that the Milnor number of $(1:0:0)$ is d^2+j+1. □

Theorem 4.3. *Let $\frac{4d-1}{6}<j\leq d-1$, set*

$$S=\{X\in\mathcal{F}_d: X \text{ has a singularity with multiplicity } j, \text{Milnor number } j(j+1)$$
$$\text{and the } j\text{-jet in this point is linearly equivalent to } z^j\frac{\partial}{\partial y}\}.$$

Then we have

$$\begin{aligned}S=SL(3,\mathbb{C})\Big\{&P(y,z)\frac{\partial}{\partial x}+\Big(x^{d-j}z^j+\sum_{k=j+1}^d\alpha_k x^{d-k}z^k+\sum_{k=0}^{d-j-1}x^kQ_{d-k}(y,z)\Big)\frac{\partial}{\partial y}+\\ &\Big(\sum_{k=j+1}^d\beta_k x^{d-k}y^k+\sum_{k=0}^{d-j-1}x^kR_{d-k}(y,z)\Big)\frac{\partial}{\partial z}:\\ &\beta_{j+1}\neq 0,\ P\in\mathbb{C}_d[y,z],\ Q_{d-k},R_{d-k}\in\mathbb{C}_{d-k}[y,z]\Big\},\end{aligned}$$

and it is an irreducible, locally closed, non-singular subvariety of the closed set of unstable foliations in $\mathcal{F}_d$ of dimension d^2+4d-j^2-3j+4.

Proof. It is easy to see that if $\frac{4d-1}{6} < j$ then the closest point β to zero in the line defined by the weights $(2j+2-d)L_1 + (j+2)L_3$ and $(-d-j-1)L_1 + (-j-1)L_3$ is in the indexing set of the stratification. We will prove that the stratum corresponding to β is S.

For this β we have:

$$Y_\beta^{ss} = \{P(y,z)\frac{\partial}{\partial x} + \Big(x^{d-j}z^j + \sum_{k=j+1}^{d} \alpha_k x^{d-k}z^k + \sum_{k=0}^{d-j-1} x^k Q_{d-k}(y,z)\Big)\frac{\partial}{\partial y} +$$
$$\Big(\sum_{k=j+1}^{d} \beta_k x^{d-k}y^k + \sum_{k=0}^{d-j-1} x^k R_{d-k}(y,z)\Big)\frac{\partial}{\partial z}$$
$$: \beta_{j+1} \neq 0,\ P \in \mathbb{C}_d[y,z],\ Q_{d-k}, R_{d-k} \in \mathbb{C}_{d-k}[y,z]\}.$$

The foliations in Y_β^{ss} satisfy $m_{(1:0:0)} = j$ and $\mu_{(1:0:0)} = j(j+1)$ and the *j-jet* in this point is $z^j\frac{\partial}{\partial y}$. The dimension of the stratum is $d+1+2(d-j)+d(d+1)-j(j+1)+3 = d^2+4d-j^2-3j+4$ because $P_\beta = \mathbf{T}$.

If we take a foliation X of degree d such that the point $(1:0:0)$ has multiplicity j, Milnor number $j(j+1)$ and the *j-jet* is $z^j\frac{\partial}{\partial y}$, it is easy to see that $X \in Y_\beta^{ss}$. $\square$

In Ferrer and Vainsencher (2013) the authors obtain the dimension of the subspace of foliations of degree d with at least one singular point with multiplicity greater than or equal to j, they use techniques from the theory of jet bundles. With this stratification we can obtain the same result when $j > \frac{2d+1}{3}$.

Proposition 4.4. *Let j be an integer number such that $\frac{2d+1}{3} < j \leq d$. Then the space*

$$M_j = \{X \in \mathcal{F}_d : X \text{ has a singular point of multiplicity} \geq j\},$$

is a closed subvariety of $\mathcal{F}_d$ of codimension $j(j+1)-2$.

Proof. Let j be a nonnegative integer. The first observation is that a foliation X with $m_{(1:0:0)}(X) = j$ has the following form

$$X = P(y,z)\frac{\partial}{\partial x} + \big(x^{d-j}Q_j(y,z) + \sum_{k=j+1}^{d} x^{d-k}Q_k(y,z)\big)\frac{\partial}{\partial y} +$$
$$\big(x^{d-j}R_j(y,z) + \sum_{k=j+1}^{d} x^{d-k}R_k(y,z)\big)\frac{\partial}{\partial z},$$

where $P \in \mathbb{C}_d[y,z]$, $Q_k, R_k \in \mathbb{C}_k[y,z]$ and $Q_j \neq 0$ or $R_j \neq 0$.

Consider the foliations $x^{d-j}z^j\frac{\partial}{\partial y}$ and $x^{d-j}y^j\frac{\partial}{\partial z}$, the corresponding weights in the representation are $(j-d-1)L_1 - (j+1)L_3$ and $(2j-d)L_1 + (j+1)L_3$, respectively. The closest point to zero in the line defined by these weights is $\beta_j = \frac{3j-2d-1}{2}L_1$. Therefore β_j is in the indexing set of $\mathcal{F}_d$ if and only if $\frac{2d+1}{3} < j \leq d$ because our chosen Weyl chamber is $\{b_1L_1 - b_3L_3 : b_1, b_3 \geq 0\}$.

Suppose that $\frac{2d+1}{3} < j \leq d$, we must consider:

$$Y_{\beta_j} = \Big\{P(y,z)\frac{\partial}{\partial x} + \big(x^{d-j}Q_j(y,z) + \sum_{k=j+1}^{d} x^{d-k}Q_k(y,z)\big)\frac{\partial}{\partial y} +$$
$$\big(x^{d-j}R_j(y,z) + \sum_{k=j+1}^{d} x^{d-k}R_k(y,z)\big)\frac{\partial}{\partial z} :$$

$$P \in \mathbb{C}_d[y,z],\ Q_k, R_k \in \mathbb{C}_k[y,z],\ Q_j \neq 0 \text{ or } R_j \neq 0\Big\}$$

as we defined in Section 2.1. It is easy to see that:

$$\overline{Y_{\beta_j}} = \{X \in \mathcal{F}_d : m_{(1:0:0)}(X) \geq j\}.$$

By 12.6 of Kirwan (1984) we have that $M_j = SL(3,\mathbb{C})\overline{Y_{\beta_j}}$ is closed. Since $P_{\beta_j} = \mathbf{T}_1$ is the stabilizer of $(1:0:0)$ then $\overline{Y_{\beta_j}}$ is invariant by P_{β_j}, therefore M_j has the same dimension as S_{β_j} which is $\dim Y_{\beta_j} + \dim SL(3,\mathbb{C}) - \dim P_{\beta_j} = 2\sum_{k=j}^{d} k + 2(d-j+1) + d + 2 = d^2 + 4d + 2 - (j(j+1) - 2)$. □

In these final lines we are going to say how we can identify, given a foliation, the stratum to which it belongs. We need the following lemma.

Lemma 4.5. *Let $\mathcal{B}$ be the indexing set of the stratification of $\mathcal{F}_d$. If $\beta \in \mathcal{B}$ and $X \in Y_\beta^{ss}$ has isolated singularities, then $(1:0:0)$ is the singular point of X with the maximum multiplicity.*

Proof. We know that if $X \in Y_\beta^{ss}$ then $(1:0:0)$ is a singular point for X, because in our Weyl chamber we can have neither $x^d \frac{\partial}{\partial y}$ nor $x^d \frac{\partial}{\partial z}$. Suppose that $(1:0:0)$ has multiplicity m. Let q be a singular point for X with multiplicity $n > m$.

On the other hand, one of the following happens, there exists $g \in P_\beta$ such that $g(q) = (0:1:0)$ or $g(q) = (0:0:1)$. Since $g \in P_\beta$ then $gX \in Y_\beta^{ss}$. The fact that $n > m$, analyzing the diagram of weights, says us that gX has a curve of singularities, $y = 0$ in the first case and $z = 0$ in the second one. □

Let X be a foliation on $\mathbb{CP}^2$ of degree d with isolated singularities. We know by Theorem 1.1 of Alcántara (2010) that if every singularity of X has multiplicity $\leq \frac{d-1}{3}$ then X is semistable.

Take X such that the singular point with maximal multiplicity is $(1:0:0)$ and consider the local generator of X as

$$\begin{pmatrix} Q(y,z) = Q_m(y,z) + Q_{m+1}(y,z) + \dots \\ R(y,z) = R_m(y,z) + R_{n+1}(y,z) + \dots \end{pmatrix},$$

with the following condition if $n_0 = \max\{n : z^n | Q_m, z^n | R_m\}$ and $n_1 = \max\{n : y^n | Q_m, y^n | R_m\}$ then $n_0 \geq n_1$. We can get this foliation with the change of coordinates given by:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

If we consider X in this way, then X (again, by the choice of the Weyl chamber) is unstable if and only if the closest point in the convex hull of weights of X to zero is in $\mathcal{B}$. Moreover, $X \in Y_\beta^{ss}$ if and only if

$$q(\beta) = \min\{q(\gamma) : \gamma \in \mathcal{B},\ m(X,\gamma) \geq 1\}$$

see 12.1 and 12.5 in Kirwan (1984). If $m(X,\gamma) < 0$ for every $\gamma \in \mathcal{B}$ then X is semistable.

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