Invariant measures,

ergodic theory,

dynamical systems as stochastic processes.

Invariant measures.

The basic goal of the study of dynamical systems is to understand the effect of time evolution on the state of the system, and primarily the long time behaviour of the state (although transient behaviours are sometimes very interesting and important in applications).

There are basically two approaches to this study.

The first one can be called topological or geometric and studies objects like periodic orbits, attractors, bifurcations etc.

The second approach can be called ergodic and studies the behavior of trajectories from a measure theoretic (statistical) point of view.

This is mostly interesting when there is a large variety of complex dynamical behaviours depending on the initial condition.

One can for example discuss average quantities whose interest often come from the geometrical approach.

In this course we will mostly deal with this second more statistical approach.

Nevertheless, as we will often see below the interplay between the two approaches (geometric and ergodic) is particularly important and fruitful.

The main contact between dynamical systems and probabilities is through measures. We will consider normalised (probability) measures on the phase space Ω of the dynamical system. Let μ be such a measure, we can think of it as a distribution on initial conditions.

It is natural to ask what is the distribution of the points in phase space after n time steps. This distribution is the image measure or the push-forward of μ by T^n, namely

$$T_*^n\mu(A) = \mu(T^{-n}(A)) \ .$$

In this formula, when T is not invertible, T^{-1} denotes the set inverse map, namely

$$T^{-1}(A) = \big\{\omega \in \Omega \,\big|\, T(\omega) \in A\big\} \ .$$

${T^n}_*\mu = \mu \circ T^{-n}$ is a probability measure on Ω and we can 'test' this distribution by using random variables (observables). For example, the probability that at time n a trajectory is in the subset A of Ω is

$$\int_A d{T^n}_*\mu = \int_\Omega \chi_A d{T^n}_*\mu = \int_\Omega \chi_A \circ T^n(\omega)\, d\mu(\omega)$$

$$= \int_\Omega \chi_A\big(T^n(\omega)\big)\, d\mu(\omega)\ .$$

Given a probability measure μ on the phase space Ω and the time evolution map T (or a (semi)-flow), we are exactly in the setting of stochastic processes.

For example, if we have a preferred observable g, namely a real measurable function g on Ω (for example the characteristic function of a measurable subset), we can look at the stochastic process (X_n) given by

$$X_n = g \circ T^n \ .$$

This is indeed a sequence of real random variables on the probability space (Ω, μ), namely a real valued stochastic process.

When a probability measure μ is selected on Ω, we will use the standard probabilistic notation $\mathbb{P}$, $\mathbb{E}$ for the probability and expectation with respect to μ.

The point which may look a little unusual for a $\mathbb{P}$robabilist is that the probability is given on the initial condition, and there is no randomness appearing in the time evolution.

However the points on the phase space completely characterize the orbits, and we can think of μ as a probability measure on the orbits, the time evolution being the shift.

In that sense, any stochastic process is a dynamical system (a not very useful remark in practice).

Recall that we will be mostly interested in the long time behaviour of the orbits, and in particular in the statistical description of complicated long time evolution.

The tools for this kind of approach were developed under the name of ergodic theory starting from the works of Boltzmann.

Ludwig Boltzmann (1844-1906).

To introduce in more details the ergodic approach, let us consider the following question which is historically one of the first motivation of ergodic theory.

Let A be subset of the phase space Ω (describing the states with a property of interest, for example that all the molecules in this room are in the left half).

One would like to know for example in a long time interval $[0, N]$ (N large) how much time the system has spent in A, namely how often the state has the property described by A?

Assume for simplicity we have a discrete time evolution.
If χ_A denotes the characteristic function of the set A, the average time the system has spent in A over an interval $[0, N]$ starting in the initial state x_0 is given by

$$\mathcal{A}_N(x_0, A) = \frac{1}{N+1} \sum_{j=0}^{N} \chi_A\big(T^j(x_0)\big) \ . \tag{1}$$

It is natural to ask if this quantity has a limit when N tends to infinity.
The answer may of course depend on A and x_0, but we can already make two important remarks. Assume the limit exists and denote it by $\mu_{x_0}(A)$.

First remark: it is easy to check that the limit also exists for $T(x_0)$ and also for any $y \in T^{-1}(x_0) = \big\{z \,|\, T(z) = x_0\big\}$.
Moreover we have

$$\mu_{T(x_0)}(A) = \mu_y(A) = \mu_{x_0}(A)\ . \tag{2}$$

Second remark: the limit also exists if A is replaced by $T^{-1}(A)$ $(= \{\omega \,|\, T(\omega) \in A\})$ and has the same value, namely

$$\mu_{x_0}(T^{-1}(A)) = \mu_{x_0}(A)\ . \tag{3}$$

If one assumes that $\mu_{x_0}(A)$ does not depend on x_0 at least for Borel sets A (or some other sigma algebra but we will mostly consider the Borel sigma algebra below), one is immediately lead to the notion of invariant measure.

Definition 1

A measure μ on a sigma-algebra $\mathscr{B}$ is invariant by the measurable map T if for any measurable set A

$$\mu\big(T^{-1}(A)\big) = \mu(A)\ . \tag{4}$$

A similar definition holds for (semi-)flows.

Unless otherwise stated, when speaking below of an invariant measure we will assume it is a probability measure. For simplicity we will also assume that the phase space is compact, although this condition is in many cases not necessary.

We will denote by $(\Omega, T, \mathscr{B}, \mu)$ the dynamical system with state space Ω, discrete time evolution T, $\mathscr{B}$ is a sigma-algebra on Ω such that T is measurable with respect to $\mathscr{B}$ and μ is a measure on $\mathscr{B}$ invariant by T.

As mentioned above $\mathscr{B}$ will most often be the Borel sigma-algebra and we will often not mention it.

Exercise 1

Let (Ω, T) be a dynamical system and assume Ω is a metric space. Assume T is continuous, and let μ be a Borel measure. Show that μ is invariant if and only if

$$\int g \circ T \, d\mu = \int g \, d\mu \,,$$

for any bounded continuous function g.

The goal of ergodic theory is to study systems $(\Omega, T, \mathscr{B}, \mu)$ and in particular their large time evolution.

Assume now we have an observable g which we recall is a measurable function on the phase space Ω.

We have seen above one such observable, namely the function χ_A which takes the value one if the state is in A (has the property described by A) and takes the value zero otherwise.

$(\Omega, \mathscr{B}, \mu)$ is a probability space and therefore g is a random variable on this probability space.

More generally $(X_n) = \left(g \circ T^n\right)$ is a discrete time, stationary, stochastic process. If $g = \chi_A$, this process describes the occurrences of a certain event (property) as times goes on.

Therefore we can apply all the ideas and results of the theory of stochastic process to dynamical systems equipped with an invariant measure. The goal of the statistical approach to dynamical systems is to study these “particular” processes.

As mentioned above and as we will see in more details below, this will be particularly interesting when done in conjunction with questions and concepts coming from the geometric approach.

In other words, dynamical systems are stochastic processes with more structure(s) on the space of trajectories.

Although any (stationary) stochastic process can be considered as a dynamical system (with phase space the set of all possible trajectories and transformation given by the time shift), there are however some important differences to mention.

First of all, it is often the case that a dynamical system has many invariant measures.

In other words, given a dynamical system (Ω, T), there may (and often will) exist many invariant measures.

This raises the question of whether there is a more natural (most beautiful) invariant measure.

This is indeed the case from a Physical point of view (the so called Physical measure when it exists, will be defined later).

However other invariant measures can be interesting from different point of views (measure of maximal entropy for example).

In other words, in dynamical system theory, a set of measure zero for an invariant measure may be important from some other point of view (for example of full measure for another invariant measure).

There are other concepts like Hausdorff dimension and Hausdorff measure which can be interesting but may involve sets of measure zero.

It is also important to mention that some sets of measure zero can indeed be “observed” (like for example in the multifractal formalism).

It is worth recalling (see the beginning of this lecture) that the choice of a particular invariant measure is related to the choice of an initial condition for a trajectory. We will come back to this point when discussing the ergodic theorem.

The conclusion is that in dynamical systems, sets of measure zero should not be disregarded as systematically as in probability theory.

Exercise 2

If μ_1 is an invariant measure for the dynamical system (Ω_1, T_1) and the map Φ is measurable, then the measure $\mu_2 = \mu_1 \circ \Phi^{-1}$ is an invariant measure for the dynamical system (Ω_2, T_2) conjugated by Φ.

We now discuss some simple examples of invariant measures.

The simplest situation is when a system has a fixed point, namely a point in phase space which does not move through time evolution.

If we have a discrete time system given by a map T, this is a point ω of phase space such that $T(\omega) = \omega$.

If we have a continuous time system given by a vector field $\vec{X}$, a fixed point (also called a stationary state) is a point ω of phase space such that $\vec{X}(\omega) = \vec{0}$.

Such a point satisfies $\varphi_t(\omega) = \omega$ for any $t \in \mathbb{R}$ where φ_t is the flow integrating $\vec{X}$.

It is easy to verify that if a system has a fixed point, the Dirac mass in this fixed point is an invariant measure, namely this measure satisfies equation (4) $\mu\big(T^{-1}(A)\big) = \mu(A)$ (one can use also exercise 1).

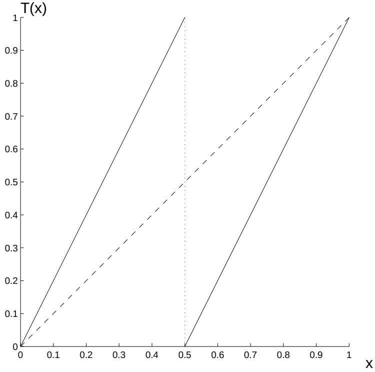
More generally, if we have a periodic orbit for a discrete time evolution, namely a (finite) sequence of points in phase space $(\omega_0, \ldots, \omega_{n-1})$ such that $T(\omega_k) = \omega_{k+1 \pmod{n}}$, the average of the Dirac masses at the points of the orbit is an invariant measure.

Exercise 3

Find an infinite sequence of periodic orbits for the map $x \mapsto 2x \pmod 1$.

The Lebesgue measure is also invariant by the map $x \mapsto 2x$ (mod 1).

To prove this we compute $T^{-1}(A)$ for each measurable set $A \subset [0,1]$.



It is easy to verify that (if $1 \notin A$)

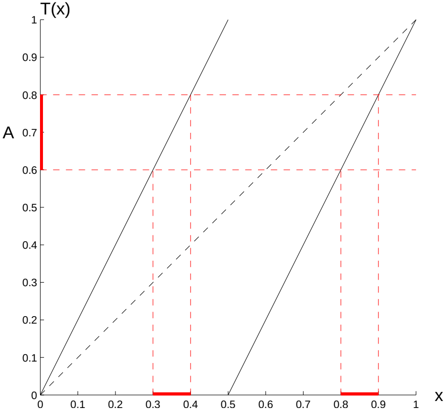
$$T^{-1}(A) = \big(A/2\big) \cup \big(1/2 + A/2\big)$$

where

$$\big(A/2\big) = \big\{x \in [0, 1/2] \,\big|\, 2x \in A\big\} \ ,$$

and

$$\big(1/2 + A/2\big) = \big\{x \in [1/2, 1] \,\big|\, 2x - 1 \in A\big\} \ .$$


T(x)
A
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

If λ denotes the Lebesgue measure, we have

$$\lambda\big(A/2\big) = \lambda\big(1/2 + A/2\big) = \lambda(A)/2$$

and since the intersection of the two sets $\big(A/2\big)$ and $\big(1/2 + A/2\big)$ is empty or reduced to the point $1/2$, equality (4) follows for $\mu = \lambda$ and any measurable set A.

We will see below a generalisation of this idea.

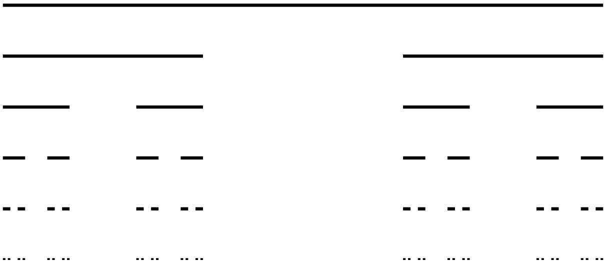
Exercise 4

Prove that the Lebesgue measure on $[-1, 1]$ is invariant for the map $T(x) = 1 - 2|x|$. Use exercise 3 to prove that the measure $dx/\sqrt{1-x^2}$ is invariant by the map $T_2(x) = 1 - 2x^2$.

Exercise 5

Consider the middle third triadic Cantor set K. Recall that $K = \cap K_n$ where each K_n is a disjoint union of 2^n intervals, the intervals of K_{n+1} being obtained from those of K_n by dropping the (open) middle third subinterval.

Show that this is the set of points whose triadic representation does not contain any 1. Define a measure ν by giving the weight 2^{-n} to any interval in K_n. Show that ν is invariant by the map $3x$ (mod 1).



Recursive construction of a Cantor set.

Let p be a Markov transition matrix on a finite alphabet $\mathcal{A}$.

Recall that this is a $\left|\mathcal{A}\right| \times \left|\mathcal{A}\right|$ matrix with non negative elements and such that for any $a \in \mathcal{A}$ we have

$$\sum_{b \in \mathcal{A}} p_{b,a} = 1 \ .$$

Such matrices are often denoted by $p(b|a)$.

Let q be an eigenvector with non negative entries and eigenvalue one, namely for any $b \in \mathcal{A}$

$$q_b = \sum_{a \in \mathcal{A}} p_{b,a} q_a \ . \tag{5}$$

Let $\Omega = \mathcal{A}^{\mathbb{Z}}$ and consider the dynamical system on this phase space given by the shift.

Given a finite sequence $x_r, \ldots, x_p$ $(r \leq p \in \mathbb{Z})$ of elements of $\mathcal{A}$, often denoted by the short hand notation x_r^p, we denote by $C(x_r^p)$ the cylinder subset of Ω given by

$$C(x_r^p) = \left\{ \underline{y} \in \mathcal{A}^{\mathbb{Z}} \,\middle|\, y_j = x_j \;\; r \leq j \leq p \right\} \, .$$

We now define a measure μ on Ω by its value on any (finite) cylinder set:

$$\mu\big(C(x_r^p)\big) = q_{x_r} \prod_{j=r}^{p-1} p_{x_{j+1},x_j} \, .$$

It is easy to verify that this defines a measure on Ω, which is invariant by the shift.

This is of course nothing but a stationary Markov chain with transition probability p.

In the particular case where for any $b \in \mathcal{A}$ the numbers $p_{b,a}$ do not depend on a we get a sequence of i.i.d. random variables on $\mathcal{A}$. The shift equipped with this invariant measure is called a Bernoulli shift (or a Bernoulli scheme). If $\mathcal{A}$ is a set of two symbols this is the game of head or tales.

This construction can be generalised in various ways and in particular to the construction of Gibbs measures on sub-shifts of finite type.

Andrei Markov (1856-1922).

Exercise 6

Let $p \in]0,1[$ and $q = 1 - p$.

Consider the phase space $\Omega = \{0,1\}^N$ equipped with the shift operator $\mathcal{S}$.

Let μ_p be the infinite product measure with $\mu_p(0) = p$ (coin flipping).

Using the conjugation of the shift and the map $T(x) = 2x$ (mod 1) (except for a countable set), show that one obtains an invariant measure (also denoted μ_p) for the map T.

Show that the measure of any dyadic interval of length 2^{-n} is of the form $p^r q^{n-r}$ where $0 \le r \le n$ is given by the dyadic coding of the interval.

Show that for $p = q = 1/2$ one gets the Lebesgue measure.

Exercise 7

Prove that the two dimensional Lebesgue measure is invariant for the baker's map, the cat map and the standard map defined in the preceding chapter. Hint: compute the Jacobian.

Show that there is an invariant measure for the dissipative baker map which can be identified as the product of the Lebesgue measure (in the horizontal direction) with the Bernoulli (product) measure of parameters $(1/2, 1/2)$ (in the vertical direction).

The Perron-Frobenius operator.

It is natural at this point to ask how one finds the invariant measures of a dynamical system.

Solving the invariance condition (equation 4)

$$\mu\big(T^{-1}(A)\big) = \mu(A)$$

is in general a non trivial task, in particular it is often the case that dynamical systems have an uncountable number of (inequivalent) invariant measures.

One can then try to determine the invariant measures with some special properties.

Consider the particular case of a map T of the unit interval $[0, 1]$ and look for the invariant probability measures which are absolutely continuous with respect to the Lebesgue measure (a.c.i.p.m. for short).

In other words, we are looking for measures $d\mu = hdx$, with h a non negative integrable function (of integral one) satisfying the invariance condition (equation 4). This is equivalent to

$$\int_0^1 g(x)h(x)dx = \int_0^1 g(T(x))h(x)dx \ , \tag{6}$$

for any continuous function g.

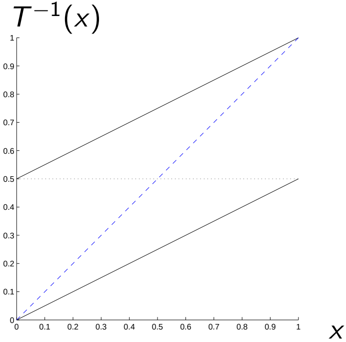
Assume T is piecewise monotone with finitely many pieces, namely there is a sequence $a_0 = 0 < a_1 < \ldots < a_k = 1$ such that on each interval $[a_j, a_{j+1}]$ the map T is monotone and continuous.

We have seen the case of piecewise expanding maps of the interval, but this is also the case for the quadratic family $x \mapsto 1 - \mu x^2$ and more generally for unimodal maps and for regular maps of the interval with finitely many critical points.

Let ψ_j denote the inverse of the map $T\Big|_{[a_j, a_{j+1}]}$.

This inverse exists since T is monotone on $[a_j, a_{j+1}]$, it is also a monotone map from $[T(a_j), T(a_{j+1})]$ to $[a_j, a_{j+1}]$.

The two inverse branches ψ_1 and ψ_2 of the map $T(x) = 2x$ mod 1.



If the inverse branches ψ_j are differentiable, we can perform a change of variables $y = T(x)$ in the right hand side of the above identity (6) for the density h of an invariant measure.
More precisely

$$\int_0^1 g\bigl(T(x)\bigr)h(x)dx = \sum_{j=0}^{k-1} \int_{a_j}^{a_{j+1}} g(T(x))h(x)dx$$

$$= \sum_{j=0}^{k-1} \int_{T(a_j)}^{T(a_{j+1})} g(y)\bigl|\psi_j'(y)\bigr|h\bigl(\psi_j(y)\bigr)dy \ . \tag{7}$$

Since the invariance relation

$$\int_0^1 g(x)h(x)dx = \int_0^1 g(T(x))h(x)dx \ ,$$

should hold for any bounded and measurable function g, we conclude that h is the density of an a.c.i.p.m. for the map f if and only if $Ph = h$ where P is the so called Perron-Frobenius operator given by

$$Pg(x) = \sum_j \chi_{[T(a_j), T(a^{j+1})]}(x) g\left(\psi_j(x)\right) \left|\psi_j'(x)\right| . \tag{8}$$

In other words, h is the density of an a.c.i.p.m. for the map T if and only if it is a nonnegative eigenvector of eigenvalue one for the operator P (in other words $Ph = h$).

Using the fact that the functions ψ_j are local inverses of T, the Perron-Frobenius operator is also given by

$$Pg(x) = \sum_{y,\, T(y)=x} \frac{g(y)}{\left|T'(y)\right|} \tag{9}$$

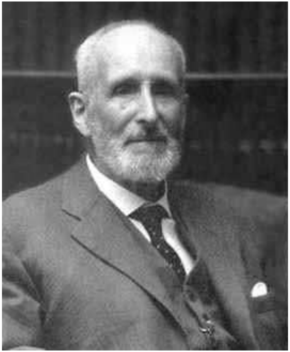
An immediate consequence of this formula is that for any measurable functions u and g,

$$P\big(u\, g \circ T\big) = g\ P\big(u\big)\ . \tag{10}$$

The above calculation can be generalised to the problem of finding an invariant measure absolutely continuous with respect to (a non invariant) reference measure.

In the case where the map is invertible (diffeomorphism for example) the above ideas do not help. If there is an attractor one can project along the transverse stable manifolds and work with a non invertible transformation on the unstable manifolds (see below for these notions).

Perron and Frobenius studied the simple case of iteration of matrices with nonnegative entries. The above operator is a vast generalisation of this case (see below for Markov chains).



Oskar Perron (1880-1975), Georg Frobenius (1849-1917) .

We mention without proof an important result for piecewise expanding maps of the interval.

Before we state this result we recall that a function g on the interval is of bounded variation if there is a finite number $C > 0$ such that for any strictly increasing sequence $0 \leq b_0 < b_1 < \ldots < b_n \leq 1$ we have

$$\sum_{j=0}^{n-1} \left| g(b_j) - g(b_{j+1}) \right| \leq C \; .$$

The space of functions of bounded variations equipped with a norm given by the infimum of all these numbers C plus the sup norm is a Banach space denoted below by **BV**.

Theorem 2

Any piecewise expanding map of the interval has at least one absolutely continuous invariant probability measure with density in the space of functions of bounded variation.

Recall that a piecewise expanding map T of the interval satisfies $|T'| > \alpha > 1$ (or more generally there is an integer n such that $|T^{n\prime}| > \alpha > 1$).

This Theorem due to Lasota and Yorke is proved by investigating the spectral properties of the Perron-Frobenius operator.

We refer to the literature for the details, references and consequences. We will also give later some complements to this result.

The equation $Ph = h$ for the density of an a.c.i.p.m. where P is the Perron-Frobenius operator is very reminiscent of equation (5) for the invariant measures of a Markov chain

$$q_b = \sum_{a \in \mathcal{A}} p_{b,a} q_a \ .$$

The next exercise develops this analogy.

Exercise 8

Consider a piecewise expanding Markov map of the interval with affine pieces. Show that there are a.c.i.p.m. with piecewise constant densities. Using the coding of exercise I.7, show that this system is conjugated to a stationary Markov chain. Show that any stationary Markov chain with a finite number of states can be realised by a piecewise expanding Markov map of the interval with affine pieces.

Exercise 9

Let $p_1,\ p_2,\dots$ be a sequence of positive numbers summing to one. Define an infinite sequence of intervals $I_1 = [a_2, a_1]$, $I_2 = [a_3, a_2]$ …, $I_j = [a_{j+1}, a_j]$ … where $a_1 = 1$ and

$$a_j = \sum_{l=j}^{\infty} p_l \ .$$

Define a map of the unit interval into itself by

$$T(x) = \begin{cases} \frac{x-a_1}{1-a_1} & \text{if } x \in I_1 \ , \\ a_j + (a_{j-1} - a_j)\frac{x-a_{j+1}}{a_j-a_{j+1}} & \text{if } x \in I_j \quad \text{for} \quad j > 1 \ . \end{cases}$$

Choose an initial condition at random with respect to the Lebesgue measure on the interval I_1 and generate the trajectory. Show that this dynamical system can be interpreted as a renewal process.

There is another version of the invariance relation (6) $\int g \circ T \, d\mu = \int g \, d\mu$ that will be useful later on.
Let U be the Koopman operator defined on measurable functions by

$$Ug(x) = g(T(x)) \,. \tag{11}$$

It is easy to verify that if μ is an invariant measure, U is an isometry of $L^2(d\mu)$.
Equation (6) can now be written in the following form

$$\int_0^1 g_2 \; U\big(g_1\big) dx = \int_0^1 P\big(g_2\big) \; g_1 dx \tag{12}$$

for any pair g_1, g_2 of square integrable functions.

Although we have worked here explicitly with the Lebesgue measure, we mentioned already that similar relations can be obtained for other reference measures.

An important case is the case of Gibbs states on sub-shift of finite type. Recall that for a sub-shift of finite type, the phase space Ω is a shift invariant subset of $\Theta = \mathcal{A}^{\mathbb{Z}}$ where $\mathcal{A}$ is a finite alphabet. For two elements $\underline{x}$ and $\underline{y}$ of Θ, denote by $\delta\big(\underline{x}, \underline{y}\big)$ the nearest position to the origin where these two sequences differ, namely

$$\delta\big(\underline{x}, \underline{y}\big) = \min\left\{ |q| \,\middle|\, x_q \neq y_q \right\} \ .$$

For a given number $0 < \zeta < 1$ we define a distance d_ζ (denoted simply by d when there is no ambiguity in the choice of ζ) by

$$d_\zeta = \zeta^{\delta\left(\underline{x}, \underline{y}\right)} \ . \tag{13}$$

Exercise 10

Prove that d_ζ is a distance (even ultrametric) and that Θ is compact in this topology. Show that the phase space Ω of any sub-shift of finite type with alphabet $\mathcal{A}$ is closed in Θ. Prove that the shift map $\mathcal{S}$ on Ω is continuous (and even Hölder), with a continuous inverse.

Theorem 3

Let ϕ be a real valued Hölder continuous function on Ω the phase space of sub-shift of finite type. Assume the incidence matrix $\mathcal{M}$ of the sub-shift is irreducible and aperiodic (in other words, there is an integer r such that all the entries of the matrix $\mathcal{M}^r$ are non zero).

Then there is a unique probability measure μ invariant by the shift and a positive constant $\Gamma > 1$ such that for any cylinder set $C\big(x_q^p\big)$ $(q \le p)$ and for any $\underline{y} \in C\big(x_q^p\big)$

$$\Gamma^{-1} \le \frac{\mu\Big(C\big(x_q^p\big)\Big)}{e^{-(p-q+1)P_\phi}\, e^{\sum_{j=q}^{p} \phi\left(\mathcal{S}^j(\underline{x})\right)}} \le \Gamma\ , \qquad (14)$$

where P_ϕ is the pressure defined by

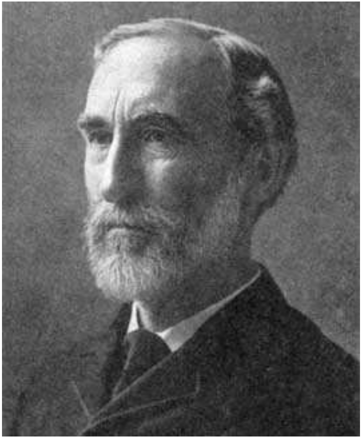
$$P_\phi = \lim_{n\to\infty} \frac{1}{2n+1} \log \left(\sum_{x_{-n}^{n}\,,\, M_{x_j, x_{j+1}}=1} e^{\sum_{j=-n}^{n} \phi\left(\mathcal{S}^j(\underline{x})\right)} \right) \,. \qquad (15)$$

In these formulas, $\underline{x}$ denote any point of Ω belonging to the cylinder set $C\left(x_{-n}^{n}\right)$.

Exercise 11

Prove that the pressure does not depend on the choice of the point $\underline{x}$ in the cylinder set $C\left(x_q^p\right)$ (use the Hölder continuity of ϕ).

This measure is a particular case of a Gibbs state of statistical mechanics with potential ϕ.



Josiah Willard Gibbs, 1839 - 1903

We refer to the books by Bowen or Ruelle for a detailed proof of this result using a Perron-Frobenius operator and the relation with Gibbs states in statistical mechanics.

We just give here the expression of the Perron-Frobenius operator in this case (assuming $\phi(\underline{x})$ depends only on (x_0^∞))

$$P\psi(x_0^\infty) = \lambda^{-1} \sum_{y \in \mathscr{A}\,,\, \mathcal{M}_{y,x_0}=1} e^{\phi(yx_0^\infty)} \psi(yx_0^\infty)$$

where yx_0^∞ denotes the sequence $y, x_0, x_1, \ldots$ and λ is a suitable positive constant ($\lambda = \exp(-P_\phi)$).

Exercise 12

Show that when $\phi(\underline{x})$ depends only on x_0 and x_1 one gets all the Markov chains with finite states (start by constructing the incidence matrix).