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LAGRANGIAN GRAPHS, MINIMIZING MEASURES AND MAÑÉ'S CRITICAL VALUES

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Abstract

Let $\mathbb{L}$ be a convex superlinear Lagrangian on a closed connected manifold N. We consider critical values of Lagrangians as defined by R. Mañé in [M3]. We show that the critical value of the lift of $\mathbb{L}$ to a covering of N equals the infimum of the values of k such that the energy level k bounds an exact Lagrangian graph in the cotangent bundle of the covering. As a consequence, we show that up to reparametrization, the dynamics of the Euler-Lagrange flow of $\mathbb{L}$ on an energy level that contains supports of minimizing measures with non-zero rotation vector can be reduced to Finsler metrics. We also show that if the Euler-Lagrange flow of $\mathbb{L}$ on the energy level k is Anosov, then k must be strictly bigger than the critical value $c_u(\mathbb{L})$ of the lift of L to the universal covering of N. It follows that given $k < c_u(\mathbb{L})$, there exists a potential ψ with arbitrarily small C^2-norm such that the energy level k of $\mathbb{L}+\psi$ possesses conjugate points. Finally we show the existence of weak KAM solutions for coverings of N and we explain the relationship between Fathi's results in [F1,2] and Mañé's critical values and action potentials.

1 Introduction

Let N be a closed connected smooth manifold and let $\mathbb{L}\colon TN \to \mathbb{R}$ be a smooth convex superlinear Lagrangian. This means that $\mathbb{L}$ restricted to each T_xN has positive definite Hessian and that for some Riemannian metric we have that

$$\lim_{|v|\to\infty} \frac{\mathbb{L}(x,v)}{|v|} = \infty \,,$$

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uniformly on $x \in N$. Let $\mathbb{H}\colon T^*N \to \mathbb{R}$ be the Hamiltonian associated to $\mathbb{L}$ and let $\mathcal{L}\colon TN \to T^*N$ be the Legendre transform $(x,v) \mapsto \partial\mathbb{L}/\partial v(x,v)$. Since N is compact, the extremals of $\mathbb{L}$ give rise to a complete flow $\phi_t\colon TN \to TN$ called the Euler-Lagrange flow of the Lagrangian. Using the Legendre transform we can push forward ϕ_t to obtain another flow ϕ_t^* which is the Hamiltonian flow of $\mathbb{H}$ with respect to the canonical symplectic structure of T^*N. Recall that the *energy* $\mathbb{E}\colon TN \to \mathbb{R}$ is defined by

$$\mathbb{E}(x,v) = \frac{\partial\mathbb{L}}{\partial v}(x,v).v - \mathbb{L}(x,v)\,.$$

Since $\mathbb{L}$ is autonomous, $\mathbb{E}$ is a first integral of the flow ϕ_t.

A very interesting aspect of the dynamics of the Euler-Lagrange flows is given by those orbits or invariant measures that satisfy some global variational properties. Research on these special orbits goes back to M. Morse [Mo] and G.A. Hedlund [H] and has reappeared in recent years in the work of V. Bangert [B1,2], M.J. Dias Carneiro [Di], A. Fathi [F1,2], R. Mañé [M3,4] and J. Mather [Ma1,2]. For autonomous systems, like the ones we are considering, these distinguished orbits and measures have the remarkable property of living on certain energy levels related to minimal values of the action. This link was discovered by Dias Carneiro [Di] and later exploited and enhanced by Mañé in his unfinished manuscript [M3] (for proofs of Mañé's results in [M3] we refer to [CoDI], [CoI]).

Recall that the action of the Lagrangian $\mathbb{L}$ on an absolutely continuous curve $\gamma\colon [a,b] \to N$ is defined by

$$A_{\mathbb{L}}(\gamma) = \int_a^b \mathbb{L}\big(\gamma(t), \dot\gamma(t)\big)\,dt\,.$$

Given two points, x and y in N and $T > 0$ denote by $\mathcal{C}_T(x,y)$ the set of absolutely continuous curves $\gamma\colon [0,T] \to N$, with $\gamma(0) = x$ and $\gamma(T) = y$. For each $k \in \mathbb{R}$ we define the *action potential* $\Phi_k\colon N \times N \to \mathbb{R}$ by

$$\Phi_k(x,y) = \inf\left\{A_{\mathbb{L}+k}(\gamma) : \gamma \in \cup_{T>0}\mathcal{C}_T(x,y)\right\}.$$

The *critical value of* $\mathbb{L}$, which was introduced by Mañé in [M3], is the real number $c(\mathbb{L})$ defined as the infimum of $k \in \mathbb{R}$ such that for some $x \in N$, $\Phi_k(x,x) > -\infty$. Since $\mathbb{L}$ is convex and superlinear and N is compact such a number exists and it has various important properties that we review in section 2. We briefly mention a few of them since we shall need them below. For any $k \geq c(\mathbb{L})$, the action potential Φ_k is a Lipschitz function that satisfies a triangle inequality. In general the action potential is not symmetric but if we define $d_k\colon N \times N \to \mathbb{R}$ by setting

$$d_k(x,y) = \Phi_k(x,y) + \Phi_k(y,x)\,,$$

then d_k is a distance function for all $k > c(\mathbb{L})$ and a pseudo-distance for $k = c(\mathbb{L})$. In [M3], [CoDI], the critical value is characterized in other ways relating it to minimizing measures (cf. section 2) or to the existence of Tonelli minimizers with fixed energy between two points.

We can also consider the critical value of the lift of the Lagrangian $\mathbb{L}$ to a covering of the compact manifold N. Suppose that $p\colon M \to N$ is a covering space and consider the Lagrangian $L\colon TM \to \mathbb{R}$ given by $L := \mathbb{L} \circ dp$. For each $k \in \mathbb{R}$ we can define an action potential Φ_k in $M \times M$ just as above and similarly we obtain a critical value $c(L)$ for L. It can be easily checked that if M_1 and M_2 are coverings of N such that M_1 covers M_2, then

$$c(L_1) \leq c(L_2)\,, \tag{1}$$

where L_1 and L_2 denote the lifts of the Lagrangian $\mathbb{L}$ to M_1 and M_2 respectively.

Among all possible coverings of N there are two distinguished ones; the universal covering which we shall denote by $\widetilde{N}$, and the abelian covering which we shall denote by $\overline{N}$. The latter is defined as the covering of N whose fundamental group is the kernel of the Hurewicz homomorphism $\pi_1(N) \mapsto H_1(N,\mathbb{R})$. When $\pi_1(N)$ is abelian, $\widetilde{N}$ is a finite covering of $\overline{N}$.

The universal covering of N gives rise to the critical value

$$c_u(\mathbb{L}) \stackrel{\text{def}}{=} c(\text{lift of } \mathbb{L} \text{ to } \widetilde{N})\,,$$

and the abelian covering of N gives rise to the critical value

$$c_a(\mathbb{L}) \stackrel{\text{def}}{=} c(\text{lift of } \mathbb{L} \text{ to } \overline{N})\,.$$

From inequality (1) it follows that

$$c_u(\mathbb{L}) \leq c_a(\mathbb{L})\,,$$

but in general the inequality may be strict, as shown in [PPa2].

One of our aims in this paper is to give new characterizations of the critical value. In section 3 we shall prove:

Theorem A. *If M is any covering of the closed manifold N, then*

$$\begin{aligned} c(L) &= \inf_{f \in C^\infty(M,\mathbb{R})} \sup_{x \in M} H(x, d_x f) \\ &= \inf\left\{k \in \mathbb{R} : \text{there exists } f \in C^\infty(M,\mathbb{R}) \text{ such that } H(df) < k\right\}, \end{aligned}$$

where H is the Hamiltonian associated with L.

Hence the critical value can be seen as the infimum of the values of $k \in \mathbb{R}$ for which there exist smooth solutions of the Hamilton-Jacobi inequality $H(df) < k$.

Recall that a smooth one form ω in M is a section of the bundle $T^*M \mapsto M$. Let $G_\omega \subset T^*M$ be the graph of ω. It is well known that G_ω is a Lagrangian submanifold of T^*M if and only if ω is closed. When ω is exact we shall say that G_ω is an *exact Lagrangian graph.* Theorem A could be restated by saying that $c(L)$ is the infimum of the values of $k \in \mathbb{R}$ for which $H^{-1}(-\infty, k)$ contains an exact Lagrangian graph. This is a very geometric way of describing the critical value.

Let $\alpha\colon H^1(N,\mathbb{R}) \to \mathbb{R}$ be the convex dual to Mather's minimal action function $\beta\colon H_1(N,\mathbb{R}) \to \mathbb{R}$ (cf. [Ma1] and section 2). Using the relationship between critical values and the function α discovered by Mañé in [M3], [CoDI] (cf. section 2), in section 3 we also derive the following corollary.

Corollary 1. $\alpha(q) = \inf_{[\omega]=q} \sup_{x\in N} \mathbb{H}(x,\omega(x))$.

Following Mañé in [M3] let us define the *strict critical value* of $\mathbb{L}$ as

$$c_0(\mathbb{L}) := \min_{q\in H^1(N,\mathbb{R})} \alpha(q)\,.$$

It was shown in [PPa2] that $c_0(\mathbb{L}) = c_a(\mathbb{L})$. It follows right away from Corollary 1 that $c_0(\mathbb{L})$ can be characterized as the infimum of the values of $k \in \mathbb{R}$ such that $\mathbb{H}^{-1}(-\infty, k)$ contains a Lagrangian graph. The strict critical value is particularly interesting since it is the cut off value for the existence of energy levels containing minimizing measures. Indeed it follows from a result of Dias Carneiro [Di] that a minimizing measure has support contained in a fixed energy level k with $k \geq c_0(\mathbb{L})$ and the minimizing measure has non-zero rotation vector iff $k > c_0(\mathbb{L})$. Hence an energy level $\mathbb{E}^{-1}(k)$ contains the support of a minimizing measure with non-zero rotation vector iff $\mathbb{H}^{-1}(-\infty, k)$ contains a Lagrangian graph.

Theorem A has also the following interesting corollary whose proof will also be given in section 3. Let $E\colon TM \to \mathbb{R}$ be the energy function of L.

Corollary 2. *If $k > c(L)$, then it is possible to see the dynamics of $\phi_t|_{E^{-1}(k)}$ as the reparametrization of the geodesic flow on the unit tangent bundle of an appropriately chosen Finsler metric on M.*

Observe that the last corollary implies that if we take $k > c_0(\mathbb{L})$ then it is possible to see the dynamics of $\phi_t|_{\mathbb{E}^{-1}(k)}$ as the reparametrization of the geodesic flow on the unit tangent bundle of an appropriately chosen Finsler metric on N. Simply apply the corollary to the Lagrangian $\mathbb{L} - \omega$ where ω is a closed one form such that $c_0(\mathbb{L}) = \alpha([\omega])$. Therefore the dynamics of the Euler-Lagrange flow on an energy level that contains supports of minimizing measures with non-zero rotation vector can be reparametrized as a Finsler geodesic flow. Being able to reparametrize the Euler-Lagrange flow on a fixed energy level as a Finsler geodesic flow has its importance. It means, for example, that we can change the speed of the flow to make it of contact type; this is not always possible as we explain below. Several results from Riemannian geometry involving properties that are invariant under reparametrizations (eg. [LyFe], [P1]) extend without significant changes to Finsler geometry. Using them and Corollary 2 one can show, for example, the existence of closed orbits on prescribed energy levels or positivity of topological entropy for Euler-Lagrange flows. Finally we would like to mention that in [A] M.C. Arnaud obtains results related to Corollary 2, particularly about the existence of closed orbits with weaker hypotheses on the Hamiltonian.

In our next theorem we turn to a different subject but still related to Theorem A. Let $\pi\colon TN \to N$ denote the canonical projection and, if $(x,v) \in TN$, let $V(x,v)$ denote the vertical fibre at (x,v) defined as usual as the kernel of $d\pi_{(x,v)}\colon T_{(x,v)}TN \to T_xN$. Let us set

$$e = \max_{x\in M} \mathbb{E}(x,0) = -\min_{x\in N} \mathbb{L}(x,0)\,.$$

Note that the energy level $\mathbb{E}^{-1}(k)$ projects *onto* the manifold N if and only if $k \geq e$ and for any $k > e$, the energy level $\mathbb{E}^{-1}(k)$ is a smooth closed hypersurface of TN that intersects each tangent space T_xN in a sphere containing the origin in its interior. It is quite easy to check that the inequality $e \leq c_u(\mathbb{L})$ always holds, but in general the inequality may be strict (cf. [PPa2]). An *Anosov energy level* is a regular energy level on which the flow ϕ_t is an Anosov flow. G.P. Paternain and M. Paternain showed in [PPa1] that Anosov energy levels are free of conjugate points and that they must project onto the whole manifold thus generalizing a well-known result of Klingenberg [K] for geodesic flows (cf. also [M1]). Conjugate points, means, as usual, pair of points $(x_1,v_1) \neq (x_2,v_2) = \phi_t(x_1,v_1)$ such that $d\phi_t(V(x_1,v_1))$ intersects $V(x_2,v_2)$ non-trivially. Moreover in [PPa2] they showed that if there exists k such that for all $k' \geq k$, the energy level $\mathbb{E}^{-1}(k')$ is Anosov, then $k > c_u(\mathbb{L})$. In section 4 we shall complete these results by showing:

Theorem B. *If the energy level* $\mathbb{E}^{-1}(k)$ *is Anosov, then*

$$k > c_u(\mathbb{L})\,.$$

For the sake of simplicity, from now on whenever we say "energy level

k" we shall be referring to $\mathbb{E}^{-1}(k)$ or $E^{-1}(k)$.

In [PPa2], G.P. Paternain and M. Paternain gave examples of Anosov energy levels k, with $k < c_0(\mathbb{L})$ on surfaces of genus greater than or equal to two. These examples are Lagrangians of the form kinetic energy plus a magnetic field. The Riemannian metric has negative Gaussian curvature and the magnetic field is chosen so that it acts only on appropriately chosen regions of constant negative curvature. The intensity of the magnetic field is small enough to preserve the Anosov property of the flow but strong enough to produce curves homologous to zero and with negative $\mathbb{L} + k$-action. These examples gave a negative answer to a question raised by Mañé, and Theorem B finally settles the issue of which critical value bounds from below the energy of an Anosov level. Passing by, we would like to point out a remarkable feature of these examples. Since $k > c_u(\mathbb{L})$, Corollary 2 assures that the energy level can be reparametrized as a Finsler geodesic flow on the universal covering. In particular, since Finsler geodesic flows are contact, one can thus reparametrize the Euler-Lagrange flow to make it of contact type on the universal covering. However they *cannot* be reparametrized as Finsler geodesic flows on N itself: Proposition 4.2 in [P2] shows that if an Anosov energy level k on a surface can be reparametrized to make it of contact type, then $k > c_0(\mathbb{L})$. This last result is closely related to the regularity of the strong stable and unstable bundles.

We also have the following corollary of Theorem B whose proof will also be given in section 4.

COROLLARY 3. *Given a convex superlinear Lagrangian $\mathbb{L}$, $k < c_u(\mathbb{L})$ and $\varepsilon > 0$ there exists a smooth function $\psi\colon N \to \mathbb{R}$ with $|\psi|_{C^2} < \varepsilon$ and such that the energy level k of $\mathbb{L} + \psi$ possesses conjugate points.*

We remark that if k is a regular value of the energy such that $k < e$, then the energy level k always contains conjugate points (cf. Proposition 8 in section 4), therefore in the light of the previous discussion it is natural to pose the following:

PROBLEM. Is it true that if $k < c_u(\mathbb{L})$, then the energy level k possesses conjugate points?

After writing the first draft of this paper and having send it to Prof. Albert Fathi, we learned from him that he had also discovered Corollary 1. His proof was similar to ours and relied in his weak KAM theorem proved in [F1]. Our proof, as we explain below, is based on the related notion of action potential. In what follows we shall try to clarify the relationship between Fathi's approach and Mañé's, and at the same time we shall prove the existence of weak KAM solutions also for non compact coverings M. Let us begin with a few definitions. Let $c = c(L)$. We say that an absolutely continuous curve $\gamma\colon [a,b] \to M$ is *semistatic* if

$$A_{L+c}(\gamma|_{[s,t]}) = \Phi_c\big(\gamma(s), \gamma(t)\big)$$

for all $a \le s \le t \le b$ and that γ is *static* if in addition

$$A_{L+c}(\gamma|_{[s,t]}) = -\Phi_c\big(\gamma(t), \gamma(s)\big)\,,$$

equivalently, if γ is semistatic and $d_c(\gamma(s), \gamma(t)) = 0$ for all $a \le s \le t \le b$. Denote by $x_v\colon \mathbb{R} \to M$ the solution of the Euler-Lagrange equation of L such that $\dot{x}_v(0) = v$. Let

$$\Sigma^+ := \big\{\, v \in TM \,|\, x_v : [0, +\infty) \to M \text{ is semistatic } \big\}\,,$$
$$\widehat{\Sigma} := \big\{\, v \in TM \,|\, x_v : \mathbb{R} \to M \text{ is static } \big\}\,.$$

Define an equivalence relation on $\pi(\widehat{\Sigma})$ by saying that $x_1 \equiv x_2$ if $d_c(x_1, x_2) = 0$. Since the projection π restricted to $\widehat{\Sigma}$ is injective (see Theorem 12 in section 5) we can define an equivalence relation on $\widehat{\Sigma}$ as well simply by saying that $v_1 \equiv v_2$ if $d_c(\pi(v_1), \pi(v_2)) = 0$. We shall call the classes defined by these relations *static classes*. An important fact is that the ω-limit of a semistatic vector $v \in \Sigma^+$ is contained in $\widehat{\Sigma}$ [CoDI]. A lemma in section 5 ensures that Σ^+ is never empty. When M is compact the ω-limit of a semistatic vector is never empty, hence $\widehat{\Sigma} \neq \emptyset$, but in general it could be empty if M is not compact.

Given a continuous function $u\colon M \to \mathbb{R}$, we shall write $u \prec L + c$ whenever $u(x) - u(y) \le \Phi_c(y, x)$ for all $x, y \in M$. Let us define the sets

$$\Gamma_0^+(u) := \big\{\, v \in \Sigma^+ \,|\, u(x_v(t)) - u(x_v(0)) = \Phi_c(x_v(0), x_v(t))\ \forall t > 0\big\}\,,$$
$$\Gamma^+(u) := \bigcup_{t>0} \phi_t\big(\Gamma^+(u)\big)\,,$$

where ϕ_t is the Euler-Lagrange flow on TM.

We shall say that a continuous function $u\colon M \to \mathbb{R}$ is a *weak KAM solution* if u satisfies the following three conditions:

1. u is Lipschitz;
2. $u \prec L + c$;
3. $\pi(\Gamma_0^+(u)) = M$.

It is important to point out that using the action potentials it is quite simple to show the existence of a function u that satisfies only properties 1 and 2 above. Take any point $p \in M$ and set $u(x) = \Phi_c(p, x)$. Elementary properties of the action potential (cf. section 2) show that u satisfies 1 and 2.

This is used in the proof of Theorem A which is based on a convolution argument that smooths out a function u that satisfies 1 and 2.

Fathi shows in [F1] that weak KAM solutions exist assuming that M is compact. His proof is based on applying the Banach fixed point theorem to a certain semigroup of operators defined on the space of continuous functions on M divided by the constant functions and, as presented, it cannot be applied when M is not compact. In our next, and last, theorem we show the existence of weak KAM solutions for M an arbitrary covering of a compact manifold. In fact we give more information than just the existence of a weak KAM solution and we explicitly show the form of the solution in terms of the action potential hoping that this will clarify the relationship between Fathi's approach and Mañé's.

Given a semistatic vector $w \in \Sigma^+$, let $\gamma(t) = x_w(t)$ and define $u\colon M \to \mathbb{R}$ by

$$u(x) = \sup_{t>0} \left[\Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t))\right]. \tag{2}$$

The function u thus defined clearly resembles the Busemann functions from Riemannian geometry. In fact, the supremum in (2) is a limit as we shall see in section 5, thus if ω-limit $\omega(w) \neq \emptyset$, then $u(x) = u(p) - \Phi_c(x, p)$ for all $x \in M$ and any p in $\pi(\omega(w))$.

Theorem C. *The function $u(x)$ in (2) is a weak KAM solution and at the points $x \in M$ where u is differentiable it satisfies $H(x, d_x u) = c$. Moreover $\pi|_{\Gamma^+(u)}$ is injective with Lipschitz inverse, u is differentiable on the set $\pi(\Gamma^+)$ and the derivative $d_x u$ is the image of $(\pi|_{\Gamma^+(u)})^{-1}(x)$ under the Legendre transform. In particular the vectors in Σ^+ have energy c.*

Given $v \in \Gamma_0^+(u)$ if the ω-limit $\omega(v) \neq \emptyset$, then $u(x) = u(p) - \Phi_c(x, p)$ for all $x \in M$ and any p in the static class of $\pi(\omega(v))$.

Hence when M is compact, since $\omega(w) \neq \emptyset$ for any semistatic vector w, the function $u(x) = -\Phi_c(x, p)$, where p is any point in the static class of $\pi(\omega(w))$, is a weak KAM solution.

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2 Critical Values and Mather's α and β functions

Let N be a closed connected manifold and $p\colon M \to N$ a covering map. Given a convex superlinear Lagrangian $\mathbb{L}\colon TN \to \mathbb{R}$ let $L := \mathbb{L} \circ dp$ be the lift of $\mathbb{L}$ to M.

The action of the Lagrangian L on an absolutely continuous curve $\gamma\colon [a,b] \to M$ is defined by

$$A_L(\gamma) = \int_a^b L\big(\gamma(t), \dot\gamma(t)\big)\,dt\,.$$

Given two points, x and y in M and $T > 0$ denote by $\mathcal{C}_T(x,y)$ the set of absolutely continuous curves $\gamma\colon [0,T] \to M$, with $\gamma(0) = x$ and $\gamma(T) = y$. For each $k \in \mathbb{R}$ we define the *action potential* $\Phi_k\colon M \times M \to \mathbb{R}$ by

$$\Phi_k(x,y) = \inf\left\{A_{L+k}(\gamma):\ \gamma \in \cup_{T>0}\mathcal{C}_T(x,y)\right\}.$$

Theorem 4 (Basic properties of the critical value [M3], [CoDI]). *There exists $c(L) \in \mathbb{R}$ such that*

1. *if $k < c(L)$, then $\Phi_k(x_1,x_2) = -\infty$, for all x_1 and x_2 in M;*
2. *if $k \geq c(L)$, then $\Phi_k(x_1,x_2) > -\infty$ for all x_1 and x_2 in M and Φ_k is a Lipschitz function;*
3. *if $k \geq c(L)$, then*
$$\Phi_k(x_1,x_3) \leq \Phi_k(x_1,x_2) + \Phi_k(x_2,x_3)\,,$$
for all x_1, x_2 and x_3 in M and
$$\Phi_k(x_1,x_2) + \Phi_k(x_2,x_1) \geq 0\,,$$
for all x_1 and x_2 in M;
4. *if $k > c(L)$, then for $x_1 \neq x_2$ we have*
$$\Phi_k(x_1,x_2) + \Phi_k(x_2,x_1) > 0\,.$$

Observe that in general the action potential Φ_k is *not* symmetric, however defining $d_k\colon M \times M \to \mathbb{R}$ by

$$d_k(x,y) = \Phi_k(x,y) + \Phi_k(y,x)\,,$$

Theorem 4 says that d_k is a metric for $k > c(L)$ and a pseudometric for $k = c(L)$. The number $c(L)$ is called the *critical value* of L.

Using the theorem it is straightforward to check that if M_1 and M_2 are coverings of M such that M_1 covers M_2, then

$$c(L_1) \leq c(L_2)\,, \tag{3}$$

where L_1 and L_2 denote the lifts of the Lagrangian $\mathbb{L}$ to M_1 and M_2 respectively. Note that if M_1 is a *finite* covering of M_2 then $c(L_1) = c(L_2)$.

As we mentioned in the introduction among all possible coverings of N there are two distinguished ones; the universal covering which we shall denote by $\widetilde{N}$, and the abelian covering which we shall denote by $\overline{N}$. The latter is defined as the covering of N whose fundamental group is the kernel of the Hurewicz homomorphism $\pi_1(N) \mapsto H_1(N,\mathbb{R})$. When $\pi_1(N)$ is abelian, $\widetilde{N}$ is a finite covering of $\overline{N}$.

The universal covering of N gives rise to the critical value

$$c_u(\mathbb{L}) \stackrel{\text{def}}{=} c(\text{lift of } \mathbb{L} \text{ to } \widetilde{N}) \,,$$

and the abelian covering of N gives rise to the critical value

$$c_a(\mathbb{L}) \stackrel{\text{def}}{=} c(\text{lift of } \mathbb{L} \text{ to } \overline{N}) \,.$$

From inequality (3) it follows that

$$c_u(\mathbb{L}) \leq c_a(\mathbb{L}) \,,$$

but in general the inequality may be strict as it was shown in [PPa2].

Let us recall now the main concepts introduced by Mather in [Ma1]. Let $\mathcal{M}(\mathbb{L})$ be the set of probabilities on the Borel σ-algebra of TN that have compact support and are invariant under the Euler-Lagrange flow ϕ_t. Let $H_1(N,\mathbb{R})$ be the first real homology group of N. Given a closed one-form ω on N and $\rho \in H_1(N,\mathbb{R})$, let $\langle \omega, \rho \rangle$ denote the integral of ω on any closed curve in the homology class ρ. If $\mu \in \mathcal{M}(\mathbb{L})$, its *rotation vector* is defined as the unique $\rho(\mu) \in H_1(N,\mathbb{R})$ such that

$$\big\langle \omega, \rho(\mu) \big\rangle = \int \omega \, d\mu \,,$$

for all closed one-forms on N. The integral on the right-hand side is with respect to μ with ω considered as a function $\omega \colon TN \to \mathbb{R}$. The function $\rho \colon \mathcal{M}(\mathbb{L}) \to H_1(N,\mathbb{R})$ is surjective [Ma1]. The rotation vector of an invariant measure is the projection of Schwartzman's asymptotic cycle [S].

The *action* of $\mu \in \mathcal{M}(\mathbb{L})$ is defined by

$$A_{\mathbb{L}}(\mu) = \int \mathbb{L} \, d\mu \,.$$

Finally we define the function $\beta \colon H_1(N,\mathbb{R}) \to \mathbb{R}$ by

$$\beta(\gamma) = \inf \big\{ A_{\mathbb{L}}(\mu) : \rho(\mu) = \gamma \big\} \,.$$

The function β is *convex* and *superlinear* and the infimum can be shown to be a *minimum* [Ma1] and the measures at which the minimum is attained are called *minimizing measures*. In other words, $\mu \in \mathcal{M}(\mathbb{L})$ is a minimizing measure iff

$$\beta(\rho(\mu)) = A_{\mathbb{L}}(\mu) \,.$$

Let us recall how the convex dual $\alpha\colon H^1(N,\mathbb{R})\to\mathbb{R}$ of β is defined. Since β is convex and superlinear we can set

$$\alpha([\omega]) = \max\left\{\langle\omega,\gamma\rangle - \beta(\gamma) : \gamma\in H_1(N,\mathbb{R})\right\},$$

where ω is any closed one-form whose cohomology class is $[\omega]$. The function α is also convex and superlinear. It is not difficult to see that [Ma1]

$$\alpha([\omega]) = -\min\left\{\int(\mathbb{L}-\omega)\,d\mu : \mu\in\mathcal{M}(\mathbb{L})\right\}. \tag{4}$$

Mañé [M3], [CoDI] established a connection between the critical values of a Lagrangian and α, the convex dual of Mather's β function. He showed that

$$c(\mathbb{L}) = -\min\left\{\int\mathbb{L}\,d\mu : \mu\in\mathcal{M}(\mathbb{L})\right\}, \tag{5}$$

and therefore combining (4) and (5) we obtain the remarkable equality

$$c(\mathbb{L}-\omega) = \alpha([\omega]), \tag{6}$$

for any closed one-form ω whose cohomology class is $[\omega]$.

Finally, Mañé defined the *strict critical value* of $\mathbb{L}$ as

$$c_0(\mathbb{L})\overset{\text{def}}{=}\min_{q\in H^1(N,\mathbb{R})}\alpha(q) = \min\left\{c(\mathbb{L}-\omega) : [\omega]\in H^1(N,\mathbb{R})\right\} = -\beta(0).$$

It was shown in [PPa2] that the strict critical value of $\mathbb{L}$ equals the critical value of the lift of $\mathbb{L}$ to the abelian covering of N, that is, $c_a(\mathbb{L}) = c_0(\mathbb{L})$.

3 Proof of Theorem A and Corollaries 1 and 2

Theorem A will be an immediate consequence of Lemma 5 and Proposition 7 below.

LEMMA 5. *If there exists a C^1 function $f\colon M\to\mathbb{R}$ such that $H(df)<k$, then $k\geq c(L)$.*

Proof. Recall that

$$H(x,p) = \max_{v\in T_xM}\left\{p(v) - L(x,v)\right\}.$$

Since $H(df)<k$ it follows that for all $(x,v)\in TM$,

$$d_xf(v) - L(x,v) < k.$$

Therefore, if $\gamma\colon[0,T]\to M$ is any absolutely continuous closed curve with $T>0$, we have

$$\int_0^T\big(L(\gamma,\dot\gamma)+k\big)dt = \int_0^T\big(L(\gamma,\dot\gamma)+k-d_\gamma f(\dot\gamma)\big)dt > 0,$$

and thus $k\geq c(L)$. □

LEMMA 6. *Let $k \geq c(L)$. If $f\colon M \to \mathbb{R}$ is differentiable at $x \in M$ and satisfies*

$$f(y) - f(x) \leq \Phi_k(x,y)$$

for all y in a neighbourhood of x, then $H(x, d_x f) \leq k$.

Proof. Let $\gamma(t)$ be a differentiable curve on M with $(\gamma(0), \dot\gamma(0)) = (x,v)$. Then

$$\limsup_{t\to 0^+} \frac{f(\gamma(t)) - f(x)}{t} \leq \liminf_{t \to 0^+} \frac{1}{t}\int_0^t \big[L(\gamma,\dot\gamma) + k\big]\,ds\,.$$

Hence $d_x f(v) \leq L(x,v) + k$ for all $v \in T_x M$ and thus

$$H(x, d_x f) = \max_{v \in T_x M} \big\{ d_x f(v) - L(x,v)\big\} \leq k\,. \qquad \square$$

PROPOSITION 7. *For any $k > c(L)$ there exists $f \in C^\infty(M,\mathbb{R})$ such that $H(df) < k$.*

Proof. We shall explain first how to prove the proposition in the case of $M = N$ and then we will lift the construction to an arbitrary covering M.

Set $c = c(L)$. Fix $q \in M$ and let $u(x) := \Phi_c(q,x)$. By the triangle inequality, we have that

$$u(y) - u(x) \leq \Phi_c(x,y) \quad \text{for all} \quad x,y \in M\,.$$

By the previous lemma, $H(d_x u) \leq c$ at any point $x \in M$ where $u(x)$ is differentiable.

We proceed to regularize u. We can assume that $M \subseteq \mathbb{R}^N$. Let U be a tubular neighbourhood of M in $\mathbb{R}^N$, and $\rho\colon U \to M$ a C^∞ projection along the normal bundle. Extend $u(x)$ to U by $\overline{u}(z) = u\big(\rho(z)\big)$. Then $\overline{u}(z)$ is also Lipschitz.

Extend the Lagrangian to U by $\overline{L}(z,v) = L(\rho(z), d_z\rho(v)) + \frac{1}{2}|v - d_z\rho(v)|^2$. Then the corresponding Hamiltonian satisfies $\overline{H}(z, p\circ d_z\rho) = H(\rho(z),p)$ for $p \in T^*_{\rho(z)}M$. At any point of differentiability of $\overline{u}$, we have that $d_z\overline{u} = d_{\rho(z)}u \circ d_z\rho$, and $\overline{H}(d_z\overline{u}) = H(d_{\rho(z)}u) \leq c$.

Let $\varepsilon > 0$ be such that

(a) The 3ε-neighbourhood of M in $\mathbb{R}^N$ is contained in U.

(b) If $x \in M$, $(y,p) \in T^*\mathbb{R}^N = \mathbb{R}^{2N}$, $\overline{H}(y,p) \leq c$ and $d_{\mathbb{R}^N}(x,y) < \varepsilon$ then $\overline{H}(x,p) < k$.

Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\psi(x) \geq 0$, $\operatorname{support}(\psi) \subset (-\varepsilon,\varepsilon)$ and $\int_{\mathbb{R}^N} \psi(|x|)\,dx = 1$. Let $K\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be $K(x,y) = \psi(|x-y|)$. Let N_ε be the ε-neighbourhood of M in $\mathbb{R}^N$. Define $f\colon N_\varepsilon \to \mathbb{R}$ by

$$f(x) = \int_{\mathbb{R}^N} \overline{u}(y) K(x,y)\,dy\,.$$

Then f is C^∞ on N_ε.

Observe that $\partial_x K(x,y) = -\partial_y K(x,y)$. Since $\overline{u}(y)$ is Lipschitz, it is differentiable at Lebesgue almost every point of U (Rademacher's Theorem, cf. [EG]). Moreover it is weakly differentiable (cf. [EG, Section 4.2.3]), that is, for any C^∞ function $\varphi\colon U \to \mathbb{R}$ with compact support

$$\int_{\mathbb{R}^N} (\varphi\, d\overline{u} + \overline{u}\, d\varphi) dx = 0\,.$$

Hence

$$-\int_{\mathbb{R}^N} \overline{u}(y)\partial_y K(x,y)dy = \int_{\mathbb{R}^N} K(x,y) d_y\overline{u}\, dy\,.$$

Now, since

$$d_x f = \int_{\mathbb{R}^N} \overline{u}(y)\partial_x K(x,y)dy\,,$$

we obtain

$$d_x f = \int_{\mathbb{R}^N} K(x,y) d_y\overline{u}\, dy\,.$$

From the choice of $\varepsilon > 0$ we have that $\overline{H}(x, d_y\overline{u}) < k$ for almost every $y \in \operatorname{supp} K(x,\cdot)$ and $x \in M$. Since $K(x,y)\,dy$ is a probability measure, by Jensen's inequality

$$H(d_x f) \le \overline{H}(d_x f) \le \int_{\mathbb{R}^N} \overline{H}(x, d_y\overline{u}) K(x,y) dy < k\,.$$

for all $x \in M$.

Now, suppose that M is a covering of a compact manifold N with covering projection p. Assume that $N \subseteq \mathbb{R}^N$. Fix $q \in M$ and set $u(x) := \Phi_{c(L)}(q,x)$. We can regularize our function u similarly as we shall now explain. For $\widehat{x} \in M$ let x be the projection of $\widehat{x}$ to N and let μ_x be the Borel probability measure on N defined by

$$\int_N \varphi\, d\mu_x = \int_{\mathbb{R}^n} (\varphi \circ \rho)(y) K(x,y) dy\,,$$

for any continuous function $\varphi\colon N \to \mathbb{R}$. Then the support of μ_x satisfies

$$\operatorname{supp}(\mu_x) \subset \big\{ y \in N : d_N(x,y) < \varepsilon \big\}\,.$$

Let $\widehat{\mu}_{\widehat{x}}$ be the Borel probability measure on M uniquely defined by the conditions: $\operatorname{supp}(\widehat{\mu}_{\widehat{x}}) \subset \{\widehat{y} \in M : d_M(\widehat{x},\widehat{y}) < \varepsilon\}$ and $p_*\widehat{\mu}_{\widehat{x}} = \mu_x$. Then we have

$$\frac{d}{d\widehat{x}} \int_M \varphi\, d\widehat{\mu}_{\widehat{x}} = \int_M d_{\widehat{y}}\varphi\, d\widehat{\mu}_{\widehat{x}}(\widehat{y})\,,$$

for any weakly differentiable function $\varphi\colon M \to \mathbb{R}$. The same arguments as above show that

$$f(\widehat{x}) = \int_M u(\widehat{y}) d\widehat{\mu}_{\widehat{x}}(\widehat{y})\,,$$

satisfies $H(d_{\widehat{x}}f) < k$. $\square$

Let us prove Corollary 1. Let us fix a closed one form ω_0 such that $[\omega_0] = q$. By equality (6) we have that $\alpha(q) = c(\mathbb{L} - \omega_0)$. Hence, it suffices to show that

$$c(\mathbb{L} - \omega_0) = \inf_{[\omega]=q} \sup_{x \in N} \mathbb{H}\big(x, \omega(x)\big)\,. \tag{7}$$

It is straightforward to check that the Hamiltonian associated with $\mathbb{L} - \omega_0$ is $\mathbb{H}(x, p + \omega_0(x))$. Since all the closed one forms in the class q are given by $\omega_0 + df$ where f ranges among all smooth functions, equality (7) is now an immediate consequence of Theorem A. $\square$

Let us prove now Corollary 2. If $k > c(L)$, then $H^{-1}(-\infty, k)$ contains an exact Lagrangian graph. This means that there exists a smooth function $f\colon M \to \mathbb{R}$ such that $H(x, d_x f) < k$ for all $x \in M$. Therefore, the new Hamiltonian $H_{df}(x,p) \stackrel{\text{def}}{=} H(x, p + d_x f)$ is such that $H_{df}^{-1}(-\infty, k)$ contains the zero section of T^*M. Let $\varphi\colon T^*M \to T^*M$ be the map $\varphi(x,p) = (x, p + d_x f)$. Observe that the Hamiltonian flow ϕ_t^* of H and the Hamiltonian flow ψ_t of H_{df} are related by $\psi_t^* \circ \varphi = \varphi \circ \phi_t$. Define now a new Hamiltonian G on T^*M minus the zero section such that G takes the value one on $H_{df}^{-1}(k)$ and such that $G(x, \lambda p) = \lambda^2 G(x,p)$ for all positive λ. Since G is positively homogeneous of degree two and convex in p, it follows that the Legendre transform $\mathcal{L}_G$ associated to G is a diffeomorphism from TM minus the zero section to T^*M minus the zero section. Therefore the Hamiltonian G induces a Finsler metric on M simply by taking $G \circ \mathcal{L}_G$.

Since by definition $G^{-1}(1) = H_{df}^{-1}(k)$ it follows that the Hamiltonian flows of G and $H_{df}^{-1}(k)$ coincide up to reparametrization on the energy level $G^{-1}(1) = H_{df}^{-1}(k)$ and therefore the Euler-Lagrange solutions of L with energy k are reparametrizations of unit speed geodesics of $G \circ \mathcal{L}_G$. $\square$

4 Proof of Theorem B and Corollary 3

Suppose that the energy level k is Anosov and set $\Sigma \stackrel{\text{def}}{=} \mathbb{H}^{-1}(k)$. Let $\pi : T^*N \to N$ denote the canonical projection. G.P. Paternain and M. Paternain proved in [PPa1] that Σ must project onto the whole manifold N and that the weak stable foliation $\mathcal{W}^s$ of ϕ_t^* is *transverse* to the fibres of the fibration by $(n-1)$-spheres given by

$$\pi|_\Sigma : \Sigma \to N\,.$$

Let $\widetilde{N}$ be the universal covering of N. Let $\widetilde{\Sigma}$ denote the energy level k of the lifted Hamiltonian H. We also have a fibration by $(n-1)$-spheres

$$\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{N}\,.$$

Let $\widetilde{\mathcal{W}^s}$ be the lifted foliation which is in turn a weak stable foliation for the Hamiltonian flow of H restricted to $\widetilde{\Sigma}$. The foliation $\widetilde{\mathcal{W}^s}$ is also transverse to the fibration $\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{N}$. Since the fibres are compact a result of Ehresman (cf. [CL]) implies that for every $(x,p) \in \widetilde{\Sigma}$ the map

$$\widetilde{\pi}|_{\widetilde{\mathcal{W}^s}(x,p)} : \widetilde{\mathcal{W}^s}(x,p) \to \widetilde{N}\,,$$

is a covering map. Since $\widetilde{N}$ is simply connected, $\widetilde{\pi}|_{\widetilde{\mathcal{W}^s}(x,p)}$ is in fact a diffeomorphism and $\widetilde{\mathcal{W}^s}(x,p)$ is simply connected. Consequently, $\widetilde{\mathcal{W}^s}(x,p)$ intersects each fibre of the fibration $\widetilde{\pi}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \widetilde{N}$ at just one point. In other words, each leaf $\widetilde{\mathcal{W}^s}(x,p)$ is the graph of a one form. On the other hand it is well known that the weak stable leaves of an Anosov energy level are Lagrangian submanifolds. Since any closed one form in the universal covering must be exact, it follows that each leaf $\widetilde{\mathcal{W}^s}(x,p)$ is an exact Lagrangian graph. The theorem now follows from Lemma 5 and the fact that by Structural Stability there exists $\varepsilon > 0$ such that for all $k' \in (k-\varepsilon, k+\varepsilon)$ the energy level k' is Anosov. □

Let us prove now Corollary 3. Suppose now that there exists $\epsilon > 0$ such that for every ψ with $|\psi|_{C^2} < \epsilon$, the energy level k of $\mathbb{L}+\psi$ has no conjugate points. The main result in [CoIS] says that in this case the energy level k of $\mathbb{L}$ must be Anosov thus contradicting Theorem B. □

PROPOSITION 8. *If k is a regular value of the energy such that $k < e$, then the energy level k has conjugate points.*

Proof. If an orbit does not have conjugate points then there exist along it two subbundles called the *Green subbundles*. They have the following properties: they are invariant, Lagrangian and they have dimension $n = \dim N$. Moreover, they are contained in the same energy level as the orbit and they do not intersect the vertical subbundle (cf. [CoI]). If k is a regular value of the energy with $k < e$, then $\pi(\mathbb{E}^{-1}(k))$ is a manifold with boundary and at the boundary the vertical subspace is completely contained in the energy level. Therefore the orbits that begin at the boundary must have conjugate points, because at the boundary two n-dimensional subspaces contained in the energy level (which is $(2n-1)$-dimensional) must intersect. □

5 Proof of Theorem C

We begin by recalling a few definitions from the introduction. Let $c = c(L)$. We say that an absolutely continuous curve $\gamma\colon [a,b] \to M$ is *semistatic* if

$$A_{L+c}(\gamma|_{[s,t]}) = \Phi_c\big(\gamma(s), \gamma(t)\big)$$

for all $a \le s \le t \le b$ and that γ is *static* if

$$A_{L+c}(\gamma|_{[s,t]}) = -\Phi_c\big(\gamma(t), \gamma(s)\big)\,,$$

equivalently, if γ is semistatic and $d_c(\gamma(s), \gamma(t)) = 0$ for all $a \le s \le t \le b$. Denote by $x_v\colon \mathbb{R} \to M$ the solution of the Euler-Lagrange equation of L such that $\dot{x}_v(0) = v$. Let

$$\Sigma^+ := \big\{ v \in TM \bigm| x_v : [0, +\infty) \to M \text{ is semistatic} \big\}\,,$$
$$\widehat{\Sigma} := \big\{ v \in TM \bigm| x_v : \mathbb{R} \to M \text{ is static} \big\}\,.$$

Lemma 9. $\Sigma^+ \neq \emptyset$.

We need the following lemma which is stated in [CoDI] for compact M but it holds for any covering M of a compact manifold. This remark applies as well to Theorem 12 and Lemma 13 below.

Lemma 10 [CoDI, Corollary 1.4]. *There exists $A > 0$ such that if $p, q \in M$ and $x \in \mathcal{C}_T(p,q)$ satisfy*

(a) $A_L(x) = \min\{A_L(y) \mid y \in \mathcal{C}_T(p,q)\}$, *that is, x is a Tonelli minimizer.*
(b) $A_{L+c}(x) < \Phi_c(p,q) + d_M(p,q)$.

Then $|\dot{x}(t)| < A$ for all $t \in [0,T]$.

Lemma 10 is an easy corollary (cf. [CoDI]) of the following lemma due to Mather [Ma1, p. 182] (cf. also [M2, Theorem 3.3]) that is stated and proved for the abelian covering of the compact manifold N. Its proof holds for any covering M of N.

Lemma 11. *For all $C > 0$, there exists $A = A(C) > 0$ such that if $T > 0$, $p, q \in M$ and $x \in \mathcal{C}_T(p,q)$ satisfy*

(a) $A_L(x) = \min\{A_L(y) \mid y \in \mathcal{C}_T(p,q)\}$, *that is, x is a Tonelli minimizer.*
(b) $A_L(x) < CT$.

Then $|\dot{x}(t)| < A$ for all $t \in [0,T]$.

Proof. In the autonomous case Lemma 11 has a considerably simpler proof that we now indicate for the sake of completeness. Since the Lagrangian is convex and superlinear and N is compact there exist positive constants B and D such that for all $(x,v) \in TM$ we have $L(x,v) \ge B|v| - D$. The hypothesis $A_L(x) < CT$ implies that there exists a $t_0 \in [0,T]$ such that

$|\dot{x}(t_0)| < \frac{D+C}{B}$. The conservation of energy ensures now the existence of another constant depending only on C and the Lagrangian that bounds uniformly the speed of the minimizer in all $[0,T]$. $\square$

Proof of Lemma 9. If M is compact then there exists a minimizing measure μ and the results in [CoDI] imply that $\emptyset \neq \mathrm{supp}(\mu) \subseteq \widehat{\Sigma} \subset \Sigma^+$.

Assume that M is not compact. Then there is a sequence $\{q_n\} \subset M$ such that $d_M(q_0, q_n) \to +\infty$. Let $x_n \colon [0, T_n] \to M$ be a Tonelli minimizer such that $x_n(0) = q_0$, $x_n(T_n) = q_n$ and

$$A_{L+c}(x_n) \leq \Phi_c(q_0, q_n) + \tfrac{1}{n}. \tag{8}$$

Since for any $x \colon \mathbb{R} \to M$, the function $\delta(t) = A_{L+c}(x|_{[0,t]}) - \Phi_c(x(0), x(t))$ is non-decreasing, inequality (8) implies that

$$A_{L+c}(x_n|_{[s,t]}) \leq \Phi_c\big(x_n(s), x_n(t)\big) + \tfrac{1}{n} \tag{9}$$

for all $0 \leq s \leq t \leq T_n$.

By Lemma 10, $|\dot{x}_n(t)| < A$ for all n large enough, $0 \leq t \leq T_n$. Let $v_n = \dot{x}_n(0)$ and v a density point of $\{v_n\}$. We can assume that $v_n \to v$. Since $d_M(q_0, q_n) \to +\infty$, then $T_n \to +\infty$. Since $x_n|_{[0,t]} \xrightarrow{C^1} x_v|_{[0,t]}$ for all $t > 0$, from (9) we obtain that $x_v \colon [0, +\infty) \to M$ is semistatic. $\square$

Before beginning with the proof of Theorem C we state the following important graph properties. Set

$$\Sigma^\varepsilon := \big\{ v \in TM \bigm| x_v : [0, \varepsilon) \to M \text{ or } x_v : (-\varepsilon, 0] \to M \text{ is semistatic} \big\}.$$

Theorem 12 (Graph Properties [M3], [CoDI])**.** (a) *If $\gamma(t)$, $t \geq 0$ is an orbit in $\Sigma^+(L)$, then, denoting the canonical projection by $\pi \colon TM \to M$, the map $\pi|_{\{\dot{\gamma} | t>0\}}$ is injective with Lipschitz inverse.*

(b) *For all $p \in \pi(\widehat{\Sigma})$ there exists a unique $\xi(p) \in T_pM$ such that $(p, \xi(p)) \in \Sigma^\varepsilon$, in particular $(p, \xi(p)) \in \widehat{\Sigma}$ and $\widehat{\Sigma} = graph(\xi)$. Moreover, the map $\xi \colon \pi(\widehat{\Sigma}) \to \widehat{\Sigma}$ is Lipschitz.*

Let us begin now with the proof of Theorem C. Our candidate for a weak KAM solution is defined as follows. Given a semistatic vector $w \in \Sigma^+$, let $\gamma(t) = x_w(t)$ and define $u \colon M \to \mathbb{R}$ by

$$u(x) = \sup_{t>0} \big[\Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t))\big].$$

By the triangle inequality, for $x \in M$ and $t > 0$,

$$\begin{aligned} \Phi_c\big(\gamma(0), \gamma(t)\big) - \Phi_c\big(x, \gamma(t)\big) &\leq \Phi_c\big(\gamma(0), x\big) + \Phi_c\big(x, \gamma(t)\big) - \Phi_c\big(x, \gamma(t)\big) \\ &= \Phi_c\big(\gamma(0), x\big). \end{aligned}$$

Hence $u(x) \leq \Phi_c(\gamma(0),x) < +\infty$. Moreover, the function $\delta(t) := \Phi_c(\gamma(0),\gamma(t)) - \Phi_c(x,\gamma(t))$ is increasing in t because if $0 \leq s \leq t$, then

$$\begin{aligned}\delta(t)-\delta(s) &= \Phi_c\big(\gamma(0),\gamma(t)\big) - \Phi_c\big(x,\gamma(t)\big) - \Phi_c\big(\gamma(0),\gamma(s)\big) + \Phi_c\big(x,\gamma(s)\big)\\ &= \Phi_c\big(\gamma(0),\gamma(s)\big) + \Phi_c\big(\gamma(s),\gamma(t)\big) - \Phi_c\big(x,\gamma(t)\big) - \Phi_c\big(\gamma(0),\gamma(s)\big)\\ &\qquad + \Phi_c\big(x,\gamma(s)\big) \geq 0\,.\end{aligned}$$

In the last inequality we used the triangle inequality for the triangle $(x,\gamma(s),\gamma(t))$. Hence the supremum in the definition of u is a limit. For $x,y \in M$, we have that

$$\begin{aligned}u(x)-u(y) &= \lim_{t\to+\infty}\big[\Phi_c(\gamma(0),\gamma(t))-\Phi_c(x,\gamma(t))-\Phi_c(\gamma(0),\gamma(t))+\Phi_c(y,\gamma(t))\big]\\ &= \lim_{t\to+\infty}\big[\Phi_c(y,\gamma(t)) - \Phi_c(x,\gamma(t))\big] \qquad (10)\\ &\leq \lim_{t\to+\infty}\big[\Phi_c(y,x) + \Phi_c(x,\gamma(t)) - \Phi_c(x,\gamma(t))\big]\\ &\leq \Phi_c(y,x)\,.\end{aligned}$$

Hence $u \prec L+c$. This property implies that

$$\big|u(x)-u(y)\big| \leq \max\big\{\,|\Phi_c(x,y)|\,,|\Phi_c(y,x)|\,\big\}\,,$$

and hence $u(x)$ is Lipschitz, with the same Lipschitz constant as Φ_c.

We show now that $M \setminus \pi(\widehat{\Sigma}) \subseteq \pi(\Gamma_0^+(u))$, where $\Gamma_0^+(u)$ was defined in the introduction. Let $x \in M\setminus\pi(\widehat{\Sigma})$ and let $x_{v_n}\colon [0,T_n] \to M$ be a Tonelli minimizer such that $x_{v_n}(0)=x$, $x_{v_n}(T_n)=\gamma(n)$ and

$$\Phi_c\big(x,\gamma(n)\big) \leq A_{L+c}(x_{v_n}|_{[0,T_n]}) \leq \Phi_c\big(x,\gamma(n)\big) + \tfrac{1}{n}\,.$$

The same argument as in inequality (9) shows that

$$A_{L+c}(x_{v_n}|_{[s,T_n]}) - \tfrac{1}{n}\cdot \leq \Phi_c\big(x_{v_n}(s),\gamma(n)\big) \leq A_{L+c}(x_{v_n}|_{[s,T_n]})\,,$$

for all $0 \leq s \leq T_n$, and then

$$\big|\Phi_c(x,\gamma(n)) - \Phi_c(x_{v_n}(t),\gamma(n)) - A_{L+c}(x_{v_n}|_{[0,t]})\big| \leq \tfrac{1}{n}\,. \qquad (11)$$

By Lemma 10, $|\dot{x}_{v_n}(t)| < A$ for all n large enough and $0 \leq t \leq T_n$. We prove below that $T_n \to +\infty$, then the same arguments as in Lemma 9 show that any limit point of $\{v_n\}$ is in Σ^+ so we may assume that $v_n \to v \in \Sigma^+$. Using the triangle inequality we get

$$\begin{aligned}\Phi_c\big(x_v(t),\gamma(n)\big) - \Phi_c\big(x_v(t),x_{v_n}(t)\big) &\leq \Phi_c\big(x_{v_n}(t),\gamma(n)\big)\\ &\leq \Phi_c\big(x_v(t),\gamma(n)\big) + \Phi_c\big(x_{v_n}(t),x_v(t)\big)\end{aligned}$$

and then

$$\big|\Phi_c(x_{v_n}(t),\gamma(n)) - \Phi_c(x_v(t),\gamma(n))\big| \leq K\, d_M\big(x_{v_n}(t),x_v(t)\big)\,, \qquad (12)$$

where K is the Lipschitz constant of Φ_c. Combining (12) and (11) we obtain

$$\big|\Phi_c(x,\gamma(n))-\Phi_c(x_v(t),\gamma(n))-A_{L+c}(x_{v_n}|_{[0,t]})\big| \le \tfrac{1}{n}+K\,d_M\big(x_{v_n}(t),x_v(t)\big)\,. \tag{13}$$

Therefore

$$\begin{aligned} u(x_v(t))-u(x) &= \lim_{n\to+\infty}\big[\Phi_c(x,\gamma(n))-\Phi_c(x_v(t),\gamma(n))\big] \\ &= \lim_{n\to+\infty} A_{L+c}(x_{v_n}|_{[0,t]}) \\ &= A_{L+c}(x_v|_{[0,t]}) \\ &= \Phi_c\big(x,x_v(t)\big)\,, \end{aligned}$$

because $x_{v_n}|_{[0,t]} \xrightarrow{C^1} x_v|_{[0,t]}$ and x_v is semistatic.

Now we prove that $\lim_n T_n=+\infty$. Suppose that this is not the case. Then there exists a subsequence that we still denote by $\{T_n\}$ such that $\lim_n T_n=T_0<+\infty$. Hence the speed $|\dot x_{v_n}|$ is uniformly bounded in $[0,T_0]$ and therefore we can assume that $\{v_n\}$ converges to a vector v, $\lim\dot\gamma(n)=(p,w_1)\in\widehat\Sigma$, $x_{v_n}|_{[0,T_0]}\xrightarrow{C^1} x_v|_{[0,T_0]}$ and that $x_v|_{[0,T_0]}$ is semistatic. Note that $\dot x_v(T_0)$ has the form (p,w_2). Since $x_v|_{[0,T_0]}$ is semistatic, then $\dot x_v(T_0)$ belongs to Σ^ε for any ε sufficiently small and therefore the graph property (b) in Theorem 12 implies that $w_1=w_2$. Since $\widehat\Sigma$ is invariant, then $x\in\pi(\widehat\Sigma)$. This contradicts the hypothesis $x\in M\setminus\pi(\widehat\Sigma)$.

Now let $(x,v)\in\widehat\Sigma$ and $t>0$. Let $p=x_v(t)$ and $y\in M$. Since $d_c(x,p)=0$, then

$$\begin{aligned} \Phi_c(x,y) &= \Phi_c(x,p)+\Phi_c(p,x)+\Phi_c(x,y) \\ &\ge \Phi_c(x,p)+\Phi_c(p,y)\ge\Phi_c(x,y)\,. \end{aligned}$$

Hence $\Phi_c(x,y)=\Phi_c(x,p)+\Phi_c(p,y)$. For $y=\gamma(s)$ (and $p=x_v(t)$), we have that

$$\begin{aligned} u\big(x_v(t)\big)-u(x) &= \lim_{s\to+\infty}\big[\Phi_c(x,\gamma(s))-\Phi_c(p,\gamma(s))\big]=\Phi_c\big(x,x_v(t)\big) \\ &= A_{L+c}(x_v|_{[0,t]})\,. \end{aligned}$$

We now prove the graph properties for $\Gamma^+(u)$. A proof of the following lemma due to Mather can be found in [Ma1] or [M2].

Lemma 13 (Crossing Lemma). *Given $A>0$ there exists $K>0$, $\varepsilon_1>0$ and $\delta>0$ with the following property: if $|v_i|<A$, $(x_i,v_i)\in TM$, $i=1,2$ satisfy $d(x_1,x_2)<\delta$ and $d((x_1,v_1),(x_2,v_2))\ge K^{-1}d(x_1,x_2)$ and denoting by $x_{v_i}\colon\mathbb{R}\to M$, $i=1,2$ the solution of the Euler-Lagrange equation such*

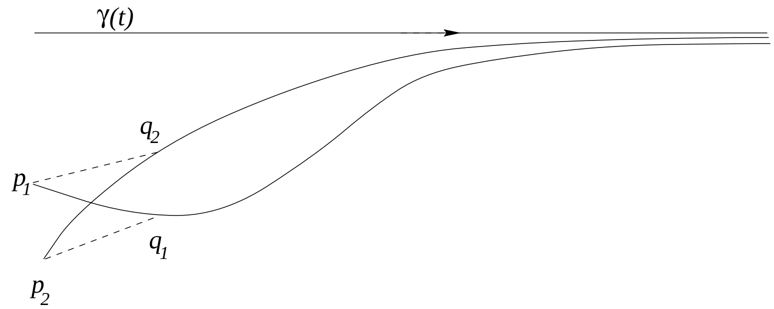

Figure 1.

that $x_{v_i}(0) = x_i$ *and* $\dot{x}_{v_i}(0) = v_i$, *then there exist solutions* $\eta_i\colon [-\varepsilon, \varepsilon] \to M$ *of* L *with* $0 < \varepsilon < \varepsilon_1$, *satisfying*

$$\eta_1(-\varepsilon) = x_{v_1}(-\varepsilon) \quad , \quad \eta_1(\varepsilon) = x_{v_2}(\varepsilon)\,,$$
$$\eta_2(-\varepsilon) = x_{v_2}(-\varepsilon) \quad , \quad \eta_2(\varepsilon) = x_{v_1}(\varepsilon)\,,$$
$$A_L(x_{v_1}|_{[-\varepsilon,\varepsilon]}) + A_L(x_{v_2}|_{[-\varepsilon,\varepsilon]}) > A_L(\eta_1) + A_L(\eta_2)\,. \tag{14}$$

Let $(x_1,v_1),(x_2,v_2)\in\Gamma^+(u)$ and suppose that $d_{TM}(v_1,v_2)>K d_M(x_1,x_2)$, where $K > 0$ is given by Lemma 13 and the $A > 0$ that we input in Lemma 13 is given by Lemma 10. Let $p_i = x_{v_i}(-\varepsilon)$ and $q_i = x_{v_i}(+\varepsilon)$, $i = 1, 2$, where $\varepsilon > 0$ is such that $\phi_{[-\varepsilon,\varepsilon]}(v_i) \subset \Gamma^+(u)$, $i = 1, 2$ (cf. Figure 1). Let η_1, η_2 be as in Lemma 13.

Since $A_{L+c}(x_{v_i}|_{[-\varepsilon,\varepsilon]}) = \Phi_c(p_i, q_i)$, $i = 1, 2$, then inequality (14) implies that

$$\Phi_c(p_1, q_2) + \Phi_c(p_2, q_1) < \Phi_c(p_1, q_1) + \Phi_c(p_2, q_2)\,.$$

Writing $\overline{pq} = \Phi_c(p, q)$, for $t > 0$ we have that

$$\overline{p_1q_2} + \overline{p_2q_1} - \overline{p_2\gamma(t)} + \overline{q_2\gamma(t)} < \overline{p_1q_1} + \overline{p_2q_2} - \overline{p_2\gamma(t)} + \overline{q_2\gamma(t)}\,.$$

Using the triangle inequality we obtain

$$\overline{p_1\gamma(t)} - \overline{q_1\gamma(t)} < \overline{p_1q_1} + \overline{p_2q_2} - \overline{p_2\gamma(t)} + \overline{q_2\gamma(t)}\,.$$

Letting $t \to +\infty$ and using (10) we get that

$$u(q_1) - u(p_1) < \overline{p_1q_1} + \overline{p_2q_2} + u(p_2) - u(q_2)\,.$$

Since $\phi_{[-\varepsilon,\varepsilon]}(v_1) \subset \Gamma_0^+(u)$, then $u(q_1) - u(p_1) = \overline{p_1q_1}$. Then

$$\overline{p_1q_1} < \overline{p_1q_1} + \overline{p_2q_2} + u(p_2) - u(q_2)\,.$$

This implies that $u(q_2) - u(p_2) < \Phi_c(p_2, q_2)$, which contradicts the fact that $\phi_{[-\varepsilon,\varepsilon]}(v_2) \subset \Gamma^+(u)$. Hence $\pi|_{\Gamma^+(u)}$ is injective and has Lipschitz inverse.

If $(x, v) \in \Gamma_0^+(u)$ and $\omega(v) \neq \emptyset$, let $p \in \pi(\omega(v))$ and $t_n \to +\infty$ be such that $\pi\phi_{t_n}(v) \to p$. Then by definition of $\Gamma_0^+(u)$, $u(x) = u(\pi\phi_{t_n}(v)) -$

$\Phi_c(x, \pi\phi_{t_n}(v))$. Letting $t \to +\infty$ we get $u(x) = u(p) - \Phi_c(x,p)$. Hence the last equality holds for any p in the static class of $\pi(\omega(v))$.

The remaining claims in Theorem C are proved by Fathi in [F1] and they do not rely on any compactness assumption; they just use the fact that u is a weak KAM solution and that $\pi|_{\Gamma^+(u)}$ is injective with Lipschitz inverse. $\square$

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