Lyapunov minimizing measures for expanding maps of the circle

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Abstract

We consider the set of maps $f \in \mathcal{F}_{\alpha+} = \cup_{\beta>\alpha}\mathcal{C}^{1+\beta}$ of the circle which are coverings of degree D, expanding , $\min_{x\in S^1} f'(x) > 1$ and orientation preserving. We are interested in characterizing the set of such maps f which admit a unique f-invariant probability measure μ minimizing $\int \ln f'\, d\mu$ over all f-invariant probability measures. We show there exists a set $\mathcal{G}_+ \subset \mathcal{F}_{\alpha+}$, open and dense in the $\mathcal{C}^{1+\alpha}$-topology, admitting a unique minimizing measure supported on a periodic orbit. We also show that every f not in $\mathcal{G}_+$ is a limit in the $\mathcal{C}^{1+\alpha}$-topology of maps admitting a unique minimizing measure supported on a strictly ergodic set of positive topological entropy.

We use in an essential way a sub-cohomological equation to produce the perturbation. In the context of Lagrangian Systems, the analog equation was introduced by R. Mañé and extended to a sub-cohomological equation to all the configuration space by A. Fathi.

We will also present some results for measures maximizing (or minimizing) a general potential A.

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1 Introduction

We consider the space $\mathcal{F}_\alpha$ of $\mathcal{C}^{1+\alpha}$ maps of the circle $f : S^1 \to S^1$, (where $\alpha < 1$) which are covering maps of degree D, orientation-preserving and expanding:

$$\lambda(f) = \min_{x\in S^1} f'(x) > 1.$$

We denote by $\text{Höl}_\alpha(A)$, the α-Hölder constant of a function $A : S^1 \to \mathbb{R}$:

$$\text{Höl}_\alpha(A) = \sup_{0<d(x,y)<\frac{1}{2}} \{\frac{|A(x) - A(y)|}{d(x,y)^\alpha}\}$$

by $\|A\|_0$, the uniform norm and by $\|A\|_\alpha$, the α-Hölder norm of A:

$$\|A\|_\alpha = \text{Höl}_\alpha(A) + \|A\|_0.$$

We note $\mathcal{C}^\alpha$, the set of α-Hölder functions and $\mathcal{C}^{\alpha+}$ the set $\cup_{\beta>\alpha}\mathcal{C}^\beta$ equipped with the $\mathcal{C}^\alpha$ topology. The set of $\mathcal{C}^{1+\alpha}$ maps of the circle is endowed with the distance

$$d(f,g) = \|\overline{f} - \overline{g}\|_0 + \|f' - g'\|_\alpha$$

where $\overline{f}$, $\overline{g}$ denote the covering maps of f, g to $\mathbb{R}$ fixing 0. The space $\mathcal{C}^{1+\alpha}$ becomes a complete space and $\mathcal{F}_\alpha$ an open set of $\mathcal{C}^{1+\alpha}$. We denote by $\mathcal{F}_{\alpha+}$ the set $\cup_{\beta>\alpha}\mathcal{F}_\beta$ equipped with the $\mathcal{C}^{1+\alpha}$-topology.

Definition. 1 Let $K(f)$ be the set of f-invariant probability measures,

$$J(f) = \inf\{\exp(\int \ln f' d\nu) \mid \nu \in K(f)\}$$

$$\mathcal{M}(f) = \{\mu \in K(f) \mid \int \ln f'\, d\mu = \ln J(f)\}.$$

A measure in $\mathcal{M}(f)$ is called *Lyapunov minimizing measure.*

The purpose of this paper is to show the following theorem. We recall first that strictly ergodic means minimal and uniquely ergodic.

Theorem. 2 *Let* $\alpha < 1$.

(i) The set $\mathcal{G}_+$ of maps f in $\mathcal{F}_{\alpha+}$ having a unique Lyapunov minimizing measure supported on a periodic orbit and satisfies the property of continuously varying support is open and dense in $\mathcal{F}_{\alpha+}$.

(ii) Any map f in $\mathcal{F}_{\alpha+} \setminus \mathcal{G}_+$ is a limit in the $\mathcal{C}^{1+\alpha}$-topology of maps (f_n) in $\mathcal{F}_{\alpha+}$ admitting a unique Lyapunov minimizing measure μ_n such that f_n restricted to $\mathrm{supp}(\mu_n)$ is strictly ergodic and has positive topological entropy.

The property of continuously varying support has first been introduced by Mañé in the Lagrangian setting. It can be formulated in our setting in the following way:

Definition. 3 Suppose $f \in \mathcal{F}_\alpha$ admits a unique Lyapunov minimizing measure μ. We say that f satisfies the property of continuously varying support, if for any sequence (f_n) of $\mathcal{F}_\alpha$ converging to f in the $\mathcal{C}^{1+\alpha}$-topology, for any Lyapunov minimizing measures μ_n for f_n, the sequence (μ_n) converges to μ in the weak topology and the sequence of compact sets $(\mathrm{supp}(\mu_n))$ converges to $\mathrm{supp}(\mu)$ in the Hausdorff topology.

The problem we consider here is in some sense analogous (though we do not consider the homological position) to the problems considered in the Aubry-Mather theory (see [18], [6]) for Lagrangian flows. A recent result of Mañé (see [18] and also [6], [5]) on Lagrangian flows shows that generically on the Lagrangian there is a unique measure minimizing action. The main difference of these results to our setting is that the minimization in our theorem is among invariant probabilities and in in Mañé's Theorem, the minimization can be done among a fixed set of probabilities (not necessarilly invariant) independent of the Lagrangian L. Therefore, in the setting considered here, if we change the map, the set of invariant probabilities will also change and this is one of the main differences from [18] to our proof. The above theorem is the analogous result for the setting of expanding maps of a conjecture for Lagrangians proposed by Mañé in [18] , [19]. E. Ott also proposed a related conjecture (see [12]). In the Lagrangian case, the conjecture whether, generically on the Lagrangian, the unique maximizing measure is supported on a periodic orbit is still open. Lyapunov minimizing measures have been previously considered by [16], [22], [12], [13], [14] and [15].

In our setting, the set $\mathcal{M}(f)$ may not be reduced to a single probability and the more pathological case is when $\ln f'$ is cohomologous to a constant

J, that is, when there exists a positive function $h : S^1 \to \mathbb{R}$ such that $f' = J(h \circ f/h)$. In that case, $J(f) = J$ and any invariant measure is minimizing. Nevertheless, $\ln f'$ always satisfies a sub-cohomological equation which is one of the key ingredients of the proof of theorem 2.

Theorem. 4 *For any $f \in \mathcal{F}_\alpha$, there exists a α-Hölder function $h : S^1 \to]0, 1]$ such that $f' \geq J(f)(h \circ f/h)$ everywhere on S^1 and $f' = J(f)(h \circ f/h)$ on the support of any Lyapunov minimizing measures.*

The proof of theorems 2 depends actually on similar statements where f is fixed and $A = -\log f'$ varies among all Hölder functions with zero-pressure. We therefore develop in section 2 a theory for maximizing a general potential A.

Definition. 5 For any $\mathcal{C}^1$-expanding map f and any Hölder function defined on S^1, $A : S^1 \to \mathbb{R}$, we call

$$m(A, f) = \sup\{\int A d\nu \mid \nu \in K(f)\}$$

$$\mathcal{M}(A, f) = \{\mu \in K(f) \mid \int A d\mu = m(A, f)\}.$$

Any measure in $\mathcal{M}(A, f)$ will be called a maximizing measure for (A, f).

Theorem. 6 *Let f be a $\mathcal{C}^1$-expanding map and $\alpha > 0$. Then the set of α-Hölder functions A admiting a unique maximizing measure for (A, f) is generic in $\mathcal{C}^\alpha$. For such functions A, the map f is strictly ergodic on the support of its unique maximizing measure.*

If A is $\mathcal{C}^0$ and has a unique maximizing measure μ, if (A_n) converges to A in the $\mathcal{C}^0$-topology and μ_n is choosen in $\mathcal{M}(A_n, f)$ for all n, then (μ_n) converges to μ in the weak topology (see step one in the proof of Proposition 16). In the following definition we ask a stronger property:

Definition. 7 Let f be a $\mathcal{C}^1$-expanding map and A be a α-Hölder function admitting a unique maximizing measure μ. We say A satisfies the property of continuously varying support, if for any sequence (A_n) of α-Hölder functions converging to A in the α-Hölder topology and for any measure μ_n choosen in $\mathcal{M}(A_n, f)$, the sequence of measures (μ_n) converges weakly to μ and the sequence of its compact supports $(\operatorname{supp} \mu_n)$ converges to $\operatorname{supp} \mu$ in the Hausdorff topology.

Theorem. 8 *Let f be a $\mathcal{C}^1$-expanding map.*

(i) the set of $A \in \mathcal{C}^\alpha$ having a unique maximizing measure μ_A satisfying the property of continuously varying support is dense in $\mathcal{C}^\alpha$.

(ii) The set $\mathcal{G}$ of $A \in \mathcal{C}^\alpha$ having a unique maximizing measure supported on a periodic orbit and satisfying the property of continuously varying support is open in $\mathcal{C}^\alpha$ and $\mathcal{G}_+ = \mathcal{G} \cap \mathcal{C}^{\alpha+}$ is open and dense in $\mathcal{C}^{\alpha+}$.

(iii) There is a dense set in $\mathcal{C}^{\alpha+} \setminus \mathcal{G}_+$ of functions A having a unique maximizing measure which has in addition positive entropy, is supported on a strictly ergodic compac set and satisfies the property of continuously varying support.

As before, the main tool to prove Theorem 8 is the following sub-cohomological equation.

Theorem. 9 *Let f be a $\mathcal{C}^1$-expanding map and $A \in \mathcal{C}^\alpha$. Then there exists $V \in \mathcal{C}^\alpha$ such that $A \leq V \circ f - V + m(A, f)$. In particular A is cohomologous to $m(A, f)$ on the support of any maximizing measure $\mu \in \mathcal{M}(A, f)$, that is, $A = V \circ f - V + m(A, f)$ on the support of μ.*

The function V should be called sub-action in analogy with Lagrangian Mechanics, KAM Theory and with [8], because it corresponds, for the discrete time version, to a sub-solution of Hamilton-Jacobi equation:

$$S(\gamma(1)) - S(\gamma(0)) \leq \int_0^1 L\,dt - E,$$

where L is the Lagrangian restricted to $\gamma(t), t \in (0,1)$ and E is the energy. The function A plays the role of the Lagrangian, $m(A, f)$ plays the role of energy, f plays the role of the flow and the inequality is inverted because we are maximizing and not minimizing.

The strategy of the proof of the above results is the following: in section 2 we prove theorems 6, 8 and 9 for a fixed expanding maps $f \in \mathcal{F}_0(\lambda, \Lambda)$. We then transfer in section 3 the above properties to prove theorems 2, 4 for varying functions f. In that case, the potential A is linked to the the function by the formula : $A = -\log f'$ and Thermodynamic Formalism is used. Analogous results for the one-sided shift in D-symbols are also true.

2 Maximizing measures for Hölder potentials

We begin by proving that, generically, there exists a unique maximizing measure. This comes mainly from the fact that, for a compact convex set in $\mathbb{R}^N$, among the set of hyperplanes which support the convex set, the set of those hyperplanes having an intersection reduced to a single point is generic. Nontheless the proof has to be carried in infinite dimension and requires more details.

We first recall some definitions. We say that a point p is an extremal point of a compact convex set C of $\mathbb{R}^n$ if p is not the mid point of a segment totally included in C. We say that p is strictly extremal if there exists a linear form which attains its maximum at the point p only. A classical result (see [21]) states that C is equal to the closed convex hull of its strictly extremal points. Theorem 6 is a direct consequence of the one where $K = K(f)$ and $\mathcal{H} = \mathcal{C}^\alpha$.

Proposition. 10 *Let K be a compact convex subset of the set of probability measures on S^1 and $(\mathcal{H}, \|.\|_\mathcal{H})$ be a dense Banach space in $\mathcal{C}^0(S^1, \mathbb{R})$ which embeds continuously in $\mathcal{C}^0(S^1, \mathbb{R})$. Then there exists a residual set $\mathcal{R}$ in $\mathcal{H}$ (for the $\|.\|_\mathcal{H}$)-topology) such that, for all $A \in \mathcal{R}$, if*

$$\mathcal{M}(A) \stackrel{\text{def}}{=} \{\mu \in K \mid \int A\, d\mu = m(A)\} and m(A) \stackrel{\text{def}}{=} \max\{\int A\, d\mu \mid \mu \in K\}$$

then $\mathcal{M}(A)$ contains a unique measure.

Proof. Let $\{H_n\}_{n>1}$ be a dense subset of the unit ball of $\mathcal{H}$. Since $\mathcal{H}$ is dense

$$d(\mu, \mu') \stackrel{\text{def}}{=} \sum_{n\geq 1} \frac{1}{2^n} |\int H_n\, d\mu - \int H_n\, d\mu'|$$

defines a metric on K compatible with the weak topology. Let us call

$$\mathcal{R}_\epsilon \stackrel{\text{def}}{=} \{A \in \mathcal{C}^0(S^1, \mathbb{R}) \text{ s.t. } \operatorname{diam} \mathcal{M}(A) < \epsilon\}.$$

We claim that $\mathcal{R}_\epsilon$ is open in $\mathcal{C}^0(S^1, \mathbb{R})$ and $\mathcal{R}_\epsilon \cap \mathcal{H}$ is dense in $\mathcal{H}$ for the $\|.\|_\mathcal{H}$-topology. The desired residual set will be $\mathcal{R} = \cap_\epsilon \mathcal{R}_\epsilon \cap \mathcal{H}$.

We show by contradiction that $\mathcal{R}_\epsilon$ is open. If not, one can find B_n in $\mathcal{C}^0(S^1, \mathbb{R})$ and (μ_n, ν_n) in $\mathcal{M}(A + B_n)$ such that $\|B_n\|_0$ converges to 0 and

$d(\mu_n,\nu_n)\geq\epsilon$ for all n. We may assume by taking a subsequence that (μ_n) and (ν_n) converge to $\bar\mu$ and $\bar\nu$. Let us prove that $\bar\mu\in\mathcal{M}(A)$: indeed for every $\mu\in K$

$$\int(A+B_n)d\mu\leq\int(A+B_n)d\mu_n\leq\int A\,d\mu_n+\|B_n\|_0$$

and $\int A\,d\mu\leq\int A\,d\bar\mu$ by taking the limit on n. For the same reason $\bar\nu$ belongs to $\mathcal{M}(A)$. We have obtain a contradiction since $d(\bar\mu,\bar\nu)\geq\epsilon$.

We now show that $\mathcal{R}_\epsilon\cap\mathcal{H}$ is dense in $\mathcal{H}$. Let $A_0\in\mathcal{H}$ and $K_0=\mathcal{M}(A_0)$. The continuous projections $\pi_n:K\to\mathbb{R}^n$, $\pi_n(\mu)=(\int H_1 d\mu,\ldots,\int H_n d\mu)$ sends K_0 to a compact convex set $\pi_n(K_0)$ which admits a strictly extremal point p_n. We first notice that $\operatorname{diam}\pi_n^{-1}(p_n)\leq 2^{-n}$ and we choose once for all, n large enough so that $2^{-n}<\epsilon$. By definition of $p_n=(p^1,\ldots,p^n)$ there exists $(a^1,\ldots,a^n)\in\mathbb{R}^n$ such that

$$\sum_{i=1}^n a^i p^i>\sum_{i=1}^n a^i q^i\quad\forall\, q=(q^1,\ldots,q^n)\in\pi_n(K_0),q\neq p.$$

In particular if $A_1=\sum_{i=1}^n a^i H_i$, $m_0(A_1)=\max\{\int A_1 d\mu\mid\mu\in K_0\}$ and

$$\mathcal{M}_0(A_1)\stackrel{\text{def}}{=}\{\mu\in K_0\mid\int A_1 d\mu=m_0(A_1)\},$$

then $\mathcal{M}_0(A_1)=\pi_n^{-1}(p_n)$ has diameter less than ϵ. We show that, for small enough $\zeta>0$, $A_\zeta=(1-\zeta)A_0+\zeta A_1\in\mathcal{R}_\epsilon$. More precisely we show that, for any open set $\mathcal{U}\supset\mathcal{M}_0(A_1)$, for any ζ sufficiently small, $\mathcal{M}(A_\zeta)\subset\mathcal{U}$. By contradiction, there exists a sequence $\mu_n\in\mathcal{M}(A_{\zeta_n})\backslash\mathcal{U}$ for some ζ_n converging to 0. We may assume that (μ_n) converges to $\bar\mu\in K\setminus\mathcal{U}$. We first show that $\bar\mu\in K_0=\mathcal{M}(A_0)$: for every $\mu\in K$

$$\int A_{\zeta_n}d\mu\leq\int A_{\zeta_n}d\mu_n\leq\int A_0\,d\mu_n+\zeta_n\|A_1-A_0\|_0$$

and by taking a limit in n: $\int A_0 d\mu\leq\int A_0 d\bar\mu$. We then show that $\bar\mu$ belongs to $\mathcal{M}_0(A_1)$: for every $\mu\in K_0$

$$\int A_{\zeta_n}d\mu=(1-\zeta_n)\int A_0\,d\mu+\zeta_n\int A_1\,d\mu\leq(1-\zeta_n)\int A_0\,d\mu_n+\zeta_n\int A_1\,d\mu_n.$$

Since $\int A_0 \, d\mu_n \leq \int A_0 \, d\mu$, we obtain $\int A_1 \, d\mu \leq \int A_1 \, d\mu_n$ and $\int A_1 \, d\mu \leq \int A_1 \, d\bar{\mu}$. We have obtain a contradiction since $\bar{\mu} \notin \mathcal{U}$. ■

We now prove the cocycle theorem 9. We show that any $A \in \mathcal{C}^\alpha$ is sub-cohomologous to a constant $m(A, f)$ with a unique minimal coboundary V_A. The main tool is the shadowing lemma. We say that two points x and y belong to the same inverse branch of length n if there exists a close interval I containing x and y such that f^n restricted to $\mathrm{int}(I)$ is one-to-one and onto S^1.

Proposition. 11 *Let f be a $\mathcal{C}^1$-expanding map and $A \in \mathcal{C}^\alpha$. Then there exists a unique minimal non-negative α-Hölder function $V_A : S^1 \to \mathbb{R}$ such that*

(i) $A \leq V_A \circ f - V_A + m(A, f)$ on S^1,

(ii) for any non-negative function $W : S^1 \to \mathbb{R}$ (not necessarily Hölder) satisfying $A \leq W \circ f - W + m(A, f)$ on S^1, we have $V_A \leq W$ on S^1.

Moreover $\mathrm{Höl}_\alpha(V_A) \leq \mathrm{Höl}_\alpha(A)/(\lambda^\alpha - 1)$ and $A = V_A \circ f - V_A + m(A, f)$ everywhere on $\mathrm{supp}(\mu)$ *of any maximizing measure μ for (A, f).*

Proof. We may assume $m(A, f) = 0$. We recall that $S_n A$ denotes the Birkhoff sum $\sum_{k=0}^{n-1} A \circ f^k$ and that $S_n A = 0$ when $n = 0$. Let us define

$$V_A(x) = \sup\{S_n A(y) \mid n \geq 0 \text{ and } f^n(y) = x\}.$$

We show that V_A is well defined. On the one hand, for any periodic point p of period n, $S_n A(p) \leq 0$. On the other hand, for any $n \geq 0$ and any $y \in S^1$ there exists a periodic point p in the same inverse branch of length n as y such that $f^n(p) = p$. Moreover if N is large enough, $\lambda^N > 2$, for all $k < n - N$, $d(f^k(y), f^k(p)) < \frac{1}{2}$ and

$$|S_n A(y) - S_n A(p)| \leq \frac{\mathrm{Höl}_\alpha(A)}{\lambda^\alpha - 1} + 2N\|A\|_0.$$

We now show that V_A is α-Hölder. If $f^n(y) = x$, $f^n(y') = x'$ and y and y' are in the same inverse branch of length n, if $d(x, x') < \frac{1}{2}$ then

$$|S_n A(y) - S_n A(y')| \leq \frac{\mathrm{Höl}_\alpha(A)}{\lambda^\alpha - 1} d(x, x')^\alpha$$

and by taking supremum over y and n, we finally obtain

$$|V_A(x) - V_A(x')| \le \frac{\text{Höl}_\alpha(A)}{\lambda^\alpha - 1} d(x,x')^\alpha.$$

By definition of V, we have $V_A \circ f \ge A + V_A$ on S^1. If W is non-negative and satisfies $W \circ f \ge A + W$ on S^1, then for every $n \ge 0$, (x,y) such that $f^n(y) = x$, $W(x) \ge S_n A(y) + W(y)$ and therefore $W(x) \ge V_A(x)$. If μ is a maximizing measure, $\int (V_A \circ f - V_A - A) d\mu = 0$ and $V_A \circ f - V_A = A$ on $\text{supp}(\mu)$. ■

Remark. 12

(i) If we choose in Proposition 11 the following definition of $\tilde{V}_A$:

$$\tilde{V}_A(x) = \lim_{n\to\infty} \sup_{k>n} \sup_{f^k(y)=x} \{S_k(A(y) - km(A,f) =\},$$

we obtain again a sub-coboundary which satisfies in addition the functional equation considered by Th. Bousch [2] and A. Fathi [8]:

$$\forall x \in S^1, \quad \tilde{V}_A(x) = \max_{f(y)=x} \{A(y) - m(A,f) + \tilde{V}_A(y)\}.$$

In particular, for any opposite points x, x' (i.e. $f(x) = f(x')$), one of them belongs to the set $\{A - m(A,f) = \tilde{V}_A \circ f - \tilde{V}_A\}$ which has therefore always a non empty interior.

(ii) If A is not equal to a coboundary modulo a constant then the set of α-Hölder functions W solution of $A \le W \circ f - W + m(A,f)$ is always a large set. Indeed let $R = V_A \circ f - V_A - A + m(A,f)$ and $1 \ge \alpha_0 \ge \alpha_1 \cdots \ge \alpha_n \ge 0$, α-Hölder functions, then

$$W = V_A + \alpha_0 R + \alpha_1 \circ f R \circ f + \cdots + \alpha_n \circ f^n R \circ f^n$$

is such a solution.

(iii) Proposition 11 shows that, in the case A admits a unique maximizing measure μ_A, then f is uniquely ergodic on $\text{supp}(\mu_A)$. This follows from the fact that any measure ν with support contained in the support of μ satisfies $\int A d\nu = m(A,f)$. A maximizing measure (on S^1) for A may not be unique: the more pathological case is when A is of the form $A = m + V \circ f - V$, where any invariant measure is maximizing.

In order to prove the property of continuously varying support, we introduce a vocabulary close to what is used in the setting of Lagrangians flows.

Definition. 13 Let A be a continuous function and $m = m(A, f)$.

(i) A function W is called a "sub-action" for (A, f) if it satisfies the inequality $A - m \leq W \circ f - W$ everywhere. We then call the set

$$\{A - m = W \circ f - W\}$$

the W-action set (we could have called it the Aubry-set).

(ii) If W is a sub-action and (x, y) is a pair of points of S^1, we say that the points (x, y) are W-connected and we write $x \overset{W}{\to} y$ if, for every $\epsilon > 0$, there exist $z \in S^1$ and $n \geq 1$ such that $d(z, x) < \epsilon$, $f^n(z) = y$ and

$$|S_n(A - m)(z) - (W(y) - W(x))| < \epsilon.$$

We say that (x, y) are equivalent if $x \overset{W}{\to} y$ and $y \overset{W}{\to} x$.

(iii) A point $x \in S^1$ is said to be recurrent with respect to (A, f) if, for any $\epsilon > 0$, there exist $z \in S^1$ and $n \geq 1$ such that

$$d(z, x) < \epsilon,\ f^n(z) = x \text{ and } |S_n(A - m)(z)| < \epsilon.$$

We denote by $\Omega(A, f)$, the set of non-wandering points with respect to (A, f).

(iv) If W is a sub-action, a compact invariant set K, (i.e. $f(K) = K$), is said to be W-irreducible if any two points (x, y) of K are W-equivalent.

We first give elementary properties. The main tool is the standard shadowing lemma. We recall that, for a complete orbit $\underline{x} = (x_n)_{n \in \mathbb{Z}}$ in S^1 ($f(x_n) = x_{n+1}$ for all $n \in \mathbb{Z}$), $\omega(\underline{x})$ denotes the compact invariant set of accumulation points of $(x_n)_{n \geq 0}$ and $\alpha(\underline{x})$ the compact invariant set of accumulation points of $(x_{-n})_{n \geq 0}$. We also recall that an ϵ-pseudo orbit with M jumps from x to y is a finite sequence of points $\{x_0 \ldots x_n\}$ such that $x_0 = x$, $x_n = y$, $d(f(x_k), x_{k+1}) < \epsilon$ for all $0 \leq k < n$ and the cardinality of the set of indices $0 \leq k < n$ satisfying $f(x_k) \neq x_{k+1}$ is bounded by M.

Lemma. 14 *Let W be a Hölder sub-action.*

(i) Let (x, y) be a pair of points and $M > 0$ an integer. Then $x \overset{W}{\to} y$ if and only if, for every $\epsilon > 0$ there exists a ϵ-pseudo orbit with at most M jumps from x to y such that

$$|\sum_{k=0}^{n-1}(A - m(A, f))(x_k) - (W(y) - W(x))| < \epsilon.$$

(ii) If $x \overset{W}{\to} y$ and $y \overset{W}{\to} z$ then $x \overset{W}{\to} z$.

(iii) Any W-irreducible compact invariant set K is included in $\Omega(A, f)$.

(iv) If $\underline{x} = (x_n)_{n\in\mathbb{Z}}$ is a complete orbit in the W-action set, then $\alpha(\underline{x})$ and $\omega(\underline{x})$ are compact invariant W-irreducible sets and therefore belong to $\Omega(A, f)$. For any $\alpha \in \alpha(\underline{x})$ and $\omega \in \omega(\underline{x})$, $\alpha \overset{W}{\to} x \overset{W}{\to} \omega$

The main ingredients for proving Theorem 8 are given by the following two propositions.

Proposition. 15 *Let A be a Hölder function. Then :*

(i) $\Omega(A, f)$ is a compact invariant set included in the W-action set of any continuous sub-action W.

(ii) For any invariant measure μ, $\mu \in \mathcal{M}(A, f)$ if and only if $\mathrm{supp}(\mu) \subset \Omega(A, f)$. *In particular (A, f) admits a unique maximizing measure if and only if $\Omega(A, f)$ is uniquely ergodic. If μ is an ergodic maximizing measure,* $\mathrm{supp}(\mu)$ *is W-irreducible for any continuous sub-action W.*

(iii) If $\Omega(A, f)$ is W-irreducible for some Hölder sub-action W, then $\Omega(A, f)$ contains all compact invariant set included in the W-action set.

(iv) If (A, f) admits a unique maximizing measure, then $\Omega(A, f)$ is W-irreducible for any Hölder sub-action W.

Proof. *Part (i).* If $(x_i)_{i\geq 0}$ is a sequence of points of $\Omega(A, f)$ converging to x and $\epsilon > 0$, then $d(x_i, x) < \frac{1}{2}\epsilon$ for some i and there exist y_i, $n > 0$ such that $d(y_i, x_i) < \frac{1}{2}\epsilon$, $d(f^n(y_i), x_i) < \frac{1}{2}\epsilon$ and $|S_n(A - m(A, f))(y_i)| < \epsilon$. Then $d(y_i, x) < \epsilon$ and $d(f^n(y_i), x) < \epsilon$. We have proved $\Omega(A, f)$ is closed.

Let x be a point of $\Omega(A,f)$ and $\epsilon > 0$. We choose, using the continuity of f and A, $\eta > 0$ so that $f(B(x,\eta)) \subset B(f(x),\epsilon)$ and the oscilation of A on $B(x,\eta)$ is bounded by ϵ. Then there exist y and$n > 0$ such that $d(y,x) < \eta$, $d(f^n(y),x) < \eta$ and $|S_n(A - m(A,f))(y)| < \epsilon$. Using the definition of η, we obtain $d(f(y),f(x)) < \eta$, $d(f^{n+1}(y),f(x)) < \eta$ and

$$|S_n(A - m(A,f))(f(y))| = |S_n(A - m(A,f))(y) - (A - A \circ f^n)(y)| < 2\epsilon.$$

We have proved $f(\Omega(A,f)) \subset \Omega(A,f)$.

If $x' \in \Omega(A,f)$, $(y_i')_{i \geq 0}$ is a sequence of points in S^1 converging to x', $(n_i)_{i\geq 0}$ an increasing sequence of integers such that $f^{n_i}(y_i')$ converges to x' and $|S_{n_i}(A - m(A,f))(y_i')|$ converges to 0, by taking a subsequence, we may assume $f^{n_i-1}(y_i')$ converges to some x satisfying $f(x) = x'$. For any large i, y_i' admits a unique pre-image y_i close to x and as above $|S_{n_i}(A - m(A,f))(y_i)|$ converges to 0. We have proved $f(\Omega(A,f)) = \Omega(A,f)$.

To prove that $\Omega(A,f)$ is included in the W-action set, we introduce the non-negative function :

$$R = W \circ f - W + m(A,f) - A.$$

If $x \in \Omega(A,f)$, $(y_i)_{i\geq 0}$ converges to x and $(n_i)_{i\geq 0}$ is choosen so that $f^{n_i}(y_i)$ converges to x and $S_{n_i}(A - m(A,f))(y_i)$ converges to 0, then

$$S_{n_i}(A - m(A,f))(y_i) + S_{n_i}R(y_i) = W \circ f^{n_i}(y_i) - W(y_i)$$

converges to 0 and therefore $R(x) = 0$.

Part (ii). The fact that the support of any maximizing measure μ is included in $\Omega(A,f)$ follows from Atkinson's theorem [1] since, by definition of μ, $\int (A - m(A,f))d\mu = 0$. If, in addition μ is ergodic, for any two points (x,y), there exists a path $\{z,\ldots,f^n(z)\}$ included in the support of μ which connects as close as we want (x,y). Since that path is included in the W-action set, we just have shown that supp(μ) is W-irreducible.

Part (iii). We now assume $\Omega(A,f)$ is W-irreducible for some Hölder sub-action W. Let K be a compact invariant set included in the W-action set. We pick $\underline{z} = (z_n)_{n\in\mathbb{Z}}$, a complet orbit in K passing through z (i.e. $z_0 = z$), $\alpha \in \alpha(\underline{z})$ and $\omega \in \omega(\underline{z})$. Then $\alpha \stackrel{W}{\to} z \stackrel{W}{\to} \omega$ and by irreducibility of $\Omega(A,f)$, $\omega \stackrel{W}{\to} \alpha$. Therefore $z \stackrel{W}{\to} z$ and $z \in \Omega(A,f)$.

Part (iv). The proof is very similar to *Part (iii)*. Let μ be the unique maximizing measure, x, y, points of $\Omega(A,f)$, $\underline{x} = (x_n)_{n\in\mathbb{Z}}$, $\underline{y} = (y_n)_{n\in\mathbb{Z}}$

complete orbits passing through x, y respectively. We define two sequences of probability measures :

$$\omega_n = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{x_k} \quad \text{and} \quad \alpha_n = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{y_{-k}}.$$

By taking subsequences, $(\omega_{n'})$ and $(\alpha_{n'})$ converge to the same measure μ and in particular one can find $\omega \in \omega(\underline{x}) \cap \operatorname{supp}(\mu)$ and $\alpha \in \alpha(\underline{y}) \cap \operatorname{supp}(\mu)$. Since $x \overset{W}{\to} \omega$, $\alpha \overset{W}{\to} y$ and $\omega \overset{W}{\to} \alpha$ by ergodicity of μ, we obtain $x \overset{W}{\to} y$ for all $(x, y) \in \Omega(A, f)$: $\Omega(A, f)$ is W-irreducible. ■

Proposition. 16 *Let $A : S^1 \to \mathbb{R}$ be a α-Hölder function.*

(i) If $\Omega(A, f)$ is minimal then, for any sequence $(A_n)_{n\geq 0}$ of α-Hölder functions converging to A in the $\mathcal{C}^\alpha$-topology, the sequence of compact sets $(\Omega(A_n, f))_{n\geq 0}$ converges to $\Omega(A, f)$ in the Hausdorff topology.

(ii) If $\Omega(A, f)$ is strictly ergodic, A satisfies the property of continuously varying support.

Proof. Step one. Using only the convergence of $(A_n)_{n\geq 0}$ to A in the $\mathcal{C}^0$-topology, we show that $(m(A_n, f))_{n\geq 0}$ converges to $m(A, f)$ and that, if μ_n is some maximizing measure for (A_n, f), any weak limite of $(\mu_n)_{n\geq 0}$ is again a maximizing measure for (A, f). Indeed we have :

$$|m(A, f) - m(B, f)| \leq ||A - B||_0$$

for all $\mathcal{C}^0$ A, B and for any sub-sequence $(\mu_{n'})_{n'}$ converging to some measure μ

$$\int A d\mu = \lim_{n'} \int A_n d\mu_{n'} = \lim_{n'} m(A_{n'}, f) = m(A, f).$$

Step two. Let $K_n = \Omega(A_n, f)$ and K be a limit set in the Hausdorff topology of some sub-sequence $(K_{n'})_{n'}$. For each n, there exists a α-Hölder sub-action V_n for (A_n, f) such that K_n is included in the V_n-action set :

$$A_n - m(A_n, f) = V_n \circ f - V_n \quad \text{on} \quad K_n.$$

We may assume that V_n is normalized so that $V_n = 0$ for some point in K_n. Since $(A_n)_n$ converges to A in the $\mathcal{C}^\alpha$-topology, using Proposition 11, we may also assume that the α-Hölder norm of V_n is uniformly bounded. By Ascoli's Theorem, we can extract a sub-sequence $(V_{n'})_{n'}$ which converges uniformly to a α-Hölder function V. We thus have obtained a sub-action $V \in \mathcal{C}^\alpha$ for (A, f) and a compact invariant set K, (i.e. $f(K) = K$) included in the V-action set. Since $\Omega(A, f)$ is irreducible by transitivity, K has to be included in $\Omega(A, f)$ and therefore has to be equal to $\Omega(A, f)$ by minimality.

Step three. We assume now that $\Omega(A, f)$ is strictly ergodic and is equal to the support of the unique ergodic measure μ. Let μ_n be some maximizing measure for (A_n, f) and $K_n = \mathrm{supp}(\mu_n)$. Since K_n is included in $\Omega(A_n, f)$, any accumulation set of $(K_n)_{n\geq 0}$ is contained in $\Omega(A, f)$ and equals $\Omega(A, f)$ by minimality. Any weak-accumulation measure has a support included in $\Omega(A, f)$ and is therefore equal to μ by unique ergodicity. ■

We are now able to prove the first part of Theorem 8

Proof of Theorem 8, part (i). If A is α-Hölder function and admits a unique maximizing measure μ, we can find $\phi \in \mathcal{C}^\alpha$, with small $\|.\|_\alpha$-norm, such that $\phi = 0$ on $\mathrm{supp}(\mu)$ and $\phi > 0$ everywhere else. Then $m(A-\phi, f) = m(A, f)$ and $A-\phi$ admits the same unique maximizing measure μ. Moreover V_A is again a sub-action for $A - \phi$ and

$$\{A - \phi - m(A - \phi, f) = V_A \circ f - V_A\} = \mathrm{supp}(\mu).$$

Therefore $\Omega(A-\phi, f) = \mathrm{supp}(\mu)$ is uniquely ergodic and according to Proposition 16, $A - \phi$ satisfies the property of continuously varying support.

The strategy of the proof of Theorem 8, *part (ii)* is to find a periodic orbit well distributed and closed to $\Omega(A, f)$. We begin by proving a theorem of approximation in the $\mathcal{C}^\alpha$-topology for functions having a better regulartity $\mathcal{C}^\beta$, $\beta > \alpha$.

Lemma. 17 *Let $0 < \alpha < \beta < 1$. For any β-Hölder functions A:*

$$H\ddot{o}l_\alpha(A) \leq 2H\ddot{o}l_\beta(A)^{\frac{\alpha}{\beta}} \|A\|_0^{1-\frac{\alpha}{\beta}}.$$

In particular, if (A_n) is a sequence of β-hölder functions uniformly bounded in the $\mathcal{C}^\beta$-topology, if (A_n) converges to 0 in the $\mathcal{C}^0$-topology, then (A_n) converges to 0 in the $\mathcal{C}^\alpha$-toplogy.

Proof. The following two estimates

$$|A(x) - A(y)| \le \text{Höl}_\beta(A) d(x,y)^\beta \le \text{Höl}_\beta(A) d(x,y)^{\beta-\alpha} d(x,y)^\alpha,$$
$$|A(x) - A(y)| \le 2\|A\|_0 \le \frac{2\|A\|_\infty}{d(x,y)^\alpha} d(x,y)^\alpha,$$

shows that A is α-Hölder with

$$\text{Höl}_\alpha(A) \le 2 \sup_{x \ne y} \min\left(\frac{\|A\|_0}{d(x,y)^\alpha}, \text{Höl}_\beta(A) d(x,y)^{\beta-\alpha}\right).$$

The minimum is reached at the intersection of the decreasing graph $t \mapsto \|A\|_0 t^{-\alpha}$ and the increasing graph $t \mapsto \text{Höl}_\beta(A) t^{\beta-\alpha}$. ∎

Although it is a very standard lemma, we need a precise estimate of the shadowing constant in the proof of Lemma 19 and we give a proof.

Lemma. 18 *Let f be $\mathcal{C}^1$-expanding, $\lambda = \lambda(f)$ and N such that $\lambda^N > 2$. Then for any $x \in S^1$ such that $d(x, f^N(x))$ is less than $\frac{1}{4}$, there exists a periodic point p of period N such that*

$$\forall\, 0 \le k \le N \qquad d(f^k(p), f^k(x)) \le \frac{\lambda^N}{\lambda^N - 1} \frac{d(x, f^N(x))}{\lambda^{N-k}}.$$

Notice that $\lambda^N/(\lambda^N - 1) \to 1$ when $N \to +\infty$.

Proof. We call $x_1 = f^N(x)$ and $x_0 = x$. We assume $x_1 \ne x_0$ (otherwise we choose $p = x_0$) and we denote by $]a, b[$ the smallest unordered interval joining a and b. Let x_0' be the closest preimage of x_1, $f^N(x_0') = x_1$, such that f^N restricted to $]x_0', x_0[$ is an homeomorphism onto $S^1 \setminus \{x_1\}$ and such that $]x_0', x_0[$ is disjoint from $]x_0, x_1[$ (we use the fact that $\lambda^N > 2$). Let x_1 be the unique preimage of x_0 in $]x_0', x_0[$ by the map f^N, then $f^N(]x_{-1}, x_0[) =]x_0, x_1[$ (we use the fact that f^N preserves the orientation). By induction we obtain a sequence (x_{-k}) of points $]x_0', x_0[$ such that $(]x_{-k}, x_{-k+1}[)$ are pairwise disjoint and such that f^N maps $]x_{-k-1}, x_{-k}[$ onto $]x_{-k}, x_{-k+1}[$. The point p, limit of (x_{-k}) is therefore a periodic point of period N and the length of the interval $]p, x_1[$ is bounded by

$$d(x_0, x_1)[1 + \frac{1}{\lambda^N} + \frac{1}{\lambda^{2N}} + \dots] \le \frac{\lambda^N}{\lambda^N - 1} d(x_0, x_1).$$

In particular, this length is bouded by $\frac{1}{2}$, f^N maps $]p, x_0[$ onto $]p, x_1[$ and

$$\forall\, 0 \leq k \leq N \qquad d(k^k(p), f^k(x_0)) \leq \frac{1}{\lambda^{N-k}} d(p, x_1).$$

The two inequalities combined end the proof. ■

Lemma. 19 *Let Ω be an invariant compact set, $f(\Omega) = \Omega$, without periodic orbit. Let $\eta = \frac{1}{2}(1 - \frac{1}{\lambda})$. Then for any $\epsilon > 0$ sufficiently small, there exists a periodic orbit $\{p, f(p), \ldots, f^N(p) = p\}$ such that*

$$d(f^i(p), \Omega) < \epsilon \quad \text{and} \quad d(f^i(p), f^j(p)) > \eta\epsilon,$$

for all $0 \leq i < j < N$.

Proof. For each $K > 0$ we define

$$d(K) = \min\{d(f^i(x), f^j(x)) \mid x \in \Omega, 0 \leq i < j \leq K\}.$$

Since no periodic orbit belongs to Ω, $d(K) > 0$ and tends to 0 when K tends to ∞. We now choose K so that $\lambda^K > 5$ and

$$1 - \frac{\lambda^K}{\lambda^K - 1}\left(\frac{1}{\lambda} + \frac{1}{\lambda^K}\right) > \eta = .$$

Let ϵ be small enough such that $\epsilon < \frac{1}{3} d(K)$. We first exibit an almost closed orbit $\{x, f(x), \ldots, f^N(x)\}$ in Ω satisfying

$$d(x, f^N(x)) < \epsilon \quad \text{and} \quad d(f^i(x), f^j(x)) \geq \epsilon$$

for all $0 \leq i < j \leq N$ except $i = 0$ and $j = N$. This can be achieved by choosing an orbit $\{y, f(y), \ldots, f^n(y)\}$ in Ω such that $d(y, f^n(y)) < \epsilon$ and a sub-orbit $\{x, f(x), \ldots, f^N(x)\}$ with the property that $d(x, f^N(x)) < \epsilon$ and N is minimal (x is a point in the orbit of y). Since $\epsilon < d(K)$, N has to be bigger than K. By the shadowing lemma 18 there exists a periodic point p of period N such that

$$d(f^i(p), f^i(x)) < \frac{\lambda^N}{\lambda^N - 1} \frac{\epsilon}{\lambda^{N-i} =},$$

for all $0 \leq i \leq N$. We now show that p is the good gandidate.

For $0 \le i < j \le N-K$,

$$d(f^i(p), f^i(x)) < \frac{\epsilon}{4}, \quad d(f^j(p), f^j(x)) < \frac{\epsilon}{4}, \quad d(f^i(x), f^j(x)) \ge \epsilon$$

and we obtain $d(f^i(p), f^j(p)) > \frac{1}{2}\epsilon > \eta\epsilon$.

For $N-K \le i < j \le N$

$$d(f^i(p), f^i(x)) < \epsilon, \quad d(f^j(p), f^j(x)) < \epsilon, \quad d(f^i(x), f^j(x)) \ge d(K) > 3\epsilon$$

and we obtain in this case $d(f^i(p), f^j(p)) > \epsilon > \eta\epsilon$.

For $0 \le i < N-K < j < N$, $d(f^i(x), f^j(x)) \ge \epsilon$ and

$$d(f^i(p), f^i(x)) < \frac{\lambda^K}{\lambda^K - 1}\,\frac{\epsilon}{\lambda^K}, \quad d(f^j(p), f^j(x)) < \frac{\lambda^K}{\lambda^K - 1}\,\frac{\epsilon}{\lambda},$$

and we obtain $d(f^i(p), f^j(p)) > \epsilon(1 - \frac{\lambda^K}{\lambda^K-1}(\frac{1}{\lambda} + \frac{1}{\lambda^K})) > \eta\epsilon$. ■

Proof of Theorem 8, part (ii). Let $\mathcal{G}$ be the set of functions $A \in \mathcal{C}^\alpha$ having a unique maximizing measure supported on a periodic orbit and satisfying the property of continuously varying support. We first show that $\mathcal{G}$ is open in $\mathcal{C}^\alpha$. Let $A_0 \in \mathcal{G}$ and $\mu_0 = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k(p)}$ the unique maximizing measure supported on a periodic orbit p. By the property of continuously varying support, there exist a neighborhood $\mathcal{N}$ of $\mathrm{orb}(p)$ and a neighborhood $\mathcal{U}$ in $\mathcal{H}_\alpha$ of A_0 such that for any $A \in \mathcal{U}$ and any $\mu \in \mathcal{M}(A, f)$, the support of μ is included in $\mathcal{N}$. By the standard shadowing lemma, $\mu = \mu_0$ is the only invariant measure close to $\mathrm{orb}(p)$. Then $\mathcal{U} \cup \mathcal{H}_{\alpha+} \subset \mathcal{G}$.

We now show that $\mathcal{G}_+ = \mathcal{G} \cap \mathcal{C}^{\alpha+}$ is dense in $\mathcal{C}^{\alpha+}$. Let $A_1 \in \mathcal{C}^{\alpha+}$, then $A_1 \in \mathcal{C}^\beta$, for some $\alpha < \beta < 1$. We look for some $A = A_1 + \phi$, $\phi \in \mathcal{C}^\beta$ with small $\|.\|_\alpha$-norm such that $\Omega(A, f)$ is reduced to a periodic orbit. We actually prove the existence of a sequence (ϕ_n) of $\mathcal{C}^\beta$-functions having a β-norm ,uniformly bounded and converging to 0 in the uniform topology.

¿From Proposition 11 and 15 *(i)*, we can write

$$A_1 = m(A_1, f) + V_1 \circ f - V_1 - R_1$$

where $V_1, R_1 \in \mathcal{C}^\beta$, $R_1 \ge 0$ and $R_1 = 0$ on $\Omega(A_1, f)$.

We first construct some $\phi_1 \in \mathcal{C}^\beta$, $\phi_1 \ge 0$ with small $\|.\|_\alpha$-norm such that $A_1 + \phi_1$ admits a maximizing measure supported on a periodic orbit. If $\Omega(A, f)$ already contains a periodic orbit, we choose $\phi_1 = 0$. If not, from Lemma 19, for any $\epsilon > 0$, there exists a periodic orbit $\mathrm{orb}(p) = \{p, \ldots, f^N(p) = p\}$ satisfying $d(q, \Omega(A_1, f)) < \epsilon$ and $d(q, q') > \eta\epsilon$ for any $q, q' \in \mathrm{orb}(p)$, $q \neq q'$, where $\eta = \frac{1}{2}(1 - \frac{1}{\lambda})$. We note

$$C_1 = \mathrm{Höl}_\beta(R_1).$$

Since $R_1 = 0$ on $\Omega(A_1, f)$ and $|R_1(q)| \leq C_1 d(q, \Omega(A_1, f))^\beta$, we obtain for all $q \in \mathrm{orb}(p)$

$$\epsilon > d(q, \Omega(A_1, f)) \geq (\frac{R_1(q)}{C_1})^{\frac{1}{\beta}}.$$

We now define for each q on the orbit a localized function $\phi_{1,q}$. The functions $\{\phi_{1,q}\}_{q \in \mathrm{orb}(p)}$ have disjoint support, they satisfy $0 \leq \phi_{1,q} \leq R_1$ everywhere and $\phi_{1,q}(q) = R_1(q)$ for all $q \in \mathrm{orb}(p)$, by the following formula

$$\phi_{1,q}(x) = [R_1(q) - D_1(q) d(x, q)^\beta]^+$$

where $D_1(q) = (\frac{2}{\eta})^\beta C_1$. Clearly $D_1(q) \geq C_1$ and by the Hölder property of R_1, we have $\phi_{1,q} \leq R_1$. The support of each $\phi_{1,q}$ is included in the ball of center q and radius

$$[\frac{R_1(q)}{D_1(q)}]^{1/\beta} = [\frac{R_1(q)}{C_1}]^{1/\beta} \frac{\eta}{2} < \frac{1}{2}\epsilon\eta$$

which shows that $\{\phi_{1,q}\}_{q \in \mathrm{orb}(p)}$ have disjoint support.

On the one hand, the $\mathrm{Höl}_\beta$-seminorm of $\phi_{1,q}$ is uniformly bounded, independently of ϵ,

$$\mathrm{Höl}_\beta(\phi_{1,q}) \leq C_1 + D_1(q) \leq 2(\frac{2}{\eta})^\beta C_1$$

(we have used : $|a^+ - b^+| \leq |a - b|$, $|a^\beta - b^\beta| \leq |a - b|^\beta$ when $\beta < 1$.)

On the other hand, the $\|.\|_\infty$-norm of $\phi_{1,q}$ tends to 0 when ϵ tends to 0

$$\|\phi_{1,q}\|_0 \leq R_1(q) \leq C_1 \epsilon^\beta.$$

We thus have obtained a non-negative function

$$\phi_1 = \sum_{q \in \mathrm{orb}(p)} \phi_{1,q}$$

satisfying $\|\phi_1\|_0 \leq C_1\epsilon^\beta$, $\text{Höl}_\beta(\phi_1) \leq 4(2/\eta)^\beta C_1$, $0 \leq \phi_1 \leq R_1$ everywhere and $\phi_1(q) = R_1(q)$ for all points $q \in \text{orb}(p)$. By using Lemma 17, we obtain some $\phi_1 \in \mathcal{C}^\beta$ with small $\|.\|_\alpha$-norm. Moreover $m(A_1 + \phi_1, f) = m(A_1, f)$ and $\Omega(A_1 + \phi_1, f)$ contains both $\Omega(A_1, f)$ and $\text{orb}(p)$. Indeed

$$A_1 + \phi_1 = V_1 \circ f - V_1 - R_1 + \phi_1 + m(A_1, f).$$

If $x \in \Omega(A_1, f)$, $\epsilon > 0$ and z, $n > 0$ have been chosen such that $d(x, z) < \epsilon$, $d(x, f^n(z)) < \epsilon$ and $|\sum_{k=0}^{n-1} R \circ f^k(z)| < \epsilon$, then $|\sum_{k=0}^{n-1} \phi_1 \circ f^k(z)| < \epsilon$ and we have shown that $x \in \Omega(A_1 + \phi_1, f)$.

We now choose $\phi_2 \geq 0$, $\phi_2 \in \mathcal{C}^\beta$, with small $\|.\|_\beta$-norm such that $\phi_2 = 0$ on $\text{orb}(p)$ and $\phi_2 > 0$ else where. Then $A = A_1 + \phi_1 - \phi_2 \in \mathcal{C}^\beta$ is close to A_0 in the $\mathcal{C}^\alpha$-topology and admits a unique maximizing measure $\mu = \frac{1}{\#\,\text{orb}(p)} \sum_{q \in \text{orb}(p)} \delta_q$. ■

We finally prove Theorem 8, *part (iii)*, which is actually a direct consequence of the following proposition.

Proposition. 20 *Let f be a $\mathcal{C}^1$-expanding map, $\mathcal{G}_+$ be the set defined in Theorem 8 and $A \in \mathcal{C}^\beta$, $\beta > \alpha$. The following are equivalent.*

(i) $A \notin \mathcal{G}_+$.

(ii) $\Omega(A, f)$ *is not reduced to the orbit of a periodic point.*

(iii) *A is a limit in the $\mathcal{C}^\alpha$-topology of $\mathcal{C}^\beta$-functions (A_n), uniformly bounded in the $\mathcal{C}^\beta$-topology, having a unique maximizing measure μ_n whose support is strictly egodic and has positive topological entropy.*

The main difficulty in proving Proposition 20 is the implication: (ii) implies (iii). We first use Grillenberger's theorem [10] and [11] to construct strictly ergodic compact invariant sets which have positive entropy and are close to any compact invariant sets not equal to a finite union of periodic orbits. We then show that, when $\Omega(A, f)$ is not reduced to a single periodic orbit, there exists a sub-action V such that the V-action set contains a compact invariant set which is not equal to a finite union of periodic orbits.

Lemma. 21 *For any compact invariant set K not equal to a finite union of periodic orbits, for any neighborhood $\mathcal{U}$ of K, there exists an invariant strictly ergodic compact set L included in $\mathcal{U}$ and admitting a positive topological entropy.*

Proof. We first assume that K is a sub-shift of $\mathcal{B}(D)$ (the two-sided full shift of D symbols). Let $\mathcal{U}$ be an open set containing K and $C_1 \cup \cdots \cup C_N$ a finite union of cylinders covering $\mathcal{B}(D) \setminus \mathcal{U}$ and disjoint from K. Let $\tilde{K}$ be defined by the exclusion rules:

$$\tilde{K} = \{x \in \mathcal{B}(D) \mid \forall i = 1 \ldots N \quad \forall n \in \mathbb{Z} \sigma^n(x) \notin C_i\}$$

Then $\tilde{K}$ is a sub-shift of finite type containing K and included in $\mathcal{U}$. Since $\tilde{K}$ is not equal to a finite union of periodic orbits, the topological entropy of $\sigma \mid \tilde{K}$ is positive. By Grillenberger, there exists $L \subset \tilde{K}$ strictly ergodic with positive entropy.

To conclude the proof we notice that (S^1, f) is a factor of $(\mathcal{B}(D), \sigma)$, (i.e. there exists a continuous and surjective map $\pi : \mathcal{B}(D) \to S^1$ such thta $\pi \circ \sigma = f \circ \pi$). It is then easy to prove that, if L is strictly ergodic, then $\pi(L)$ is strictly ergodic too. moreover π is obtained as the composition of $\pi_1 : \mathcal{B}(D) \to \mathcal{B}^+(D)$, (the one-sided full shift), and of $\pi_2 : \mathcal{B}^+(D) \to S^1$. By definition of the topological entropy (by counting all words of a given length for all lengths), if L is a sub-shift of $\mathcal{B}(D)$, $\pi_1(L)$ has the same topological entropy as L. Since π_2 is almost 2-to-1, by using [4], we conclude that $\pi(L)$ has the same entropy as L. ■

Definition. 22 Let V be a sub-action and Ω_0, Ω_1 be two disjoint V-irreducible compact invariants sets of $\Omega(A, f)$. We say that (Ω_0, Ω_0) are strongly V-connected if there exists a complete orbit $\underline{x} = (x_n)_{n \in \mathbb{Z}}$ included in the V-action set such that $\alpha(\underline{x}) \subset \Omega_0$ and $\omega(\underline{x}) \subset \Omega_1$.

If Ω_0 and Ω_1 are periodic orbits of $\Omega(A, f)$ and strongly V-connected, such complete orbit $\underline{x}$ satisfying $\alpha(\underline{x}) \subset \Omega_0$ and $\omega(\underline{x}) \subset \Omega_1$ could be called an heteroclinic orbit.

Lemma. 23 *Let V be a sub-action and let us assume that $\Omega(A, f)$ is equal to a finite disjoint union of V-irreducible compact invariant sets $\{\Omega_i\}_{i=1}^N$. Let $i \neq j$. If (Ω_i, Ω_j) are V-connected, then there exist a chain $(i_0, i_1, \ldots, i_r)$ of pairewise distinct indexes in $\{1, \ldots, N\}$ such that $i_0 = i$, $i_r = j$ and each $(\Omega_{i_{k-1}}, \Omega_{i_k l})$ are strongly V-connected.*

Proof. Step one. We assume that $\Omega(A, f)$ can be written as a disjoint union of two (not necessarily V-irreducible) compact invariant sets Ω_0 and

Ω_1. We assume moreover that there exist $p_0 \in \Omega_0$ and $p_1 \in \Omega_1$ which are V-connected. We are going to show the existence of a complete orbit $\underline{x} = (x_n)$ included in the V-action set such that $x_0 \notin \Omega(A, f)$, $\alpha(\underline{x}) \subset \Omega_0$ and $\omega(\underline{x})$ is V-connected to Ω_1. We fix once for all two open sets U_0 and U_1 containing each Ω_0 and Ω_1 such that U_0 and $f(U_0)$ are disjoint from U_1. Since (p_0, p_1) are V-connected, for every $\epsilon > 0$, there exist $n_\epsilon \geq 1$ and p_ϵ such that $d(p_0, p_\epsilon) < \epsilon$, $f^{n_\epsilon}(p_\epsilon) = p_1$ and

$$V(p_1) - V(p_\epsilon) - \epsilon \leq S_n(A - m)(p_\epsilon) \leq V(p_1) - V(p_\epsilon).$$

Let us show, for all $0 \leq k < l \leq n$,

$$V \circ f^l(p_\epsilon) - V \circ f^k(p_\epsilon) - \epsilon \leq S_{l-k}(A - m) \circ f^k(p_\epsilon) \leq V \circ f^l(p_\epsilon) - V \circ f^k(p_\epsilon).$$

On the one hand, because V is a sub-action

$$S_{l-k}(A - m) \circ f^k(p_\epsilon) \leq V \circ f^l(p_\epsilon) - V \circ f^k(p_\epsilon)$$

On the other hand

$$v(p_1) - v(p_\epsilon) - \epsilon \leq S_{n-l}(A - m) \circ f^l(p_\epsilon) + S_{l-k}(A - m) \circ f^k(p_\epsilon) + S_k(A - m)(p_\epsilon),$$

and

$$S_k(A - m)(p_\epsilon) \leq V \circ f^k(p_\epsilon) - V(p_\epsilon),$$
$$S_{n-l}(A - m) \circ f^l(p_\epsilon) \leq V(p_1) - V \circ f^l(p_\epsilon).$$

Combining these inequalities we obtain

$$V \circ f^l(p_\epsilon) - V \circ f^k(p_\epsilon) - \epsilon \leq S_{l-k}(A - m) \circ f^k(p_\epsilon).$$

Let s_ϵ be the first time the iterates of p_ϵ escape U_0, that is, $f^k(p_\epsilon) \in U_0$ for $k = 0, 1, \ldots, s_\epsilon - 1$ and $f^{s_\epsilon}(p_\epsilon) \notin U_0$. Let $x_{-k}(\epsilon) = f^{s_\epsilon - k}(p_\epsilon)$. By a procedure of diagonal extraction, we can find a subsequence of ϵ's converging to zero such that each $x_{-k}(\epsilon)$ converges to some x_{-k}, (remark that $s_\epsilon \to \infty$ when $\epsilon \to 0$). By construction $x_0 \notin U_0 \cup U_1$, $x_{-k} \in \overline{U}_0$ for all $k = 1, 2, \ldots$ and the complete orbit $\underline{x} = (x_n)$ is included in the V-action set. Moreover any x_n is V-connected to p_1 and $\alpha(\underline{x}) \subset \overline{U}_0$. Since $\alpha(\underline{x})$ is a subset of $\Omega(A, f)$, necessarily $\alpha(\underline{x}) \subset \Omega_0$.

Step two. We assume for now on that $\Omega(A, f)$ is equal to a disjoint union of irreducible compact invariant sets $\{\Omega_i\}_{n=1}^N$ and that Ω_i is V-connected to Ω_j.

Let $i_0 = i$. There exists a complete orbit $\underline{x}_{i_0}$ not included in $\Omega(A, f)$ such that $\alpha(\underline{x}_{i_0}) \subset \Omega_{i_0}$ and $\omega(\underline{x}_{i_0})$ is V-connected to Ω_j. But $\omega(\underline{x}_{i_0})$ intersects some Ω_{i_1} (after Lemma 14-*iv*), and $i_1 \neq i_0$, otherwise, by irreducibility of Ω_{i_0} we would have

$$\alpha(\underline{x}_{i_0}) \stackrel{V}{\to} x_{i_0}(0) \stackrel{V}{\to} \omega(\underline{x}_{i_0} \stackrel{V}{\to} \alpha(\underline{x}_{i_0}).$$

and $\underline{x}_{i_0}(0)$ would belong to Ω_{i_0}. Either $i_1 = j$ and we are done or we repeat the previous construction. There exists $\underline{x}_{i_1}$ such that $\alpha(\underline{x}_{i_1}) \subset \Omega_{i_1}$, $\omega(\underline{x}_{i_1})$ intersects some Ω_{i_2} and $\underline{x}_{i_1}(0) \notin \Omega(A, f)$. By irreducibility of Ω_{i_0} or Ω_{i_1}, i_2 cannot be equal to i_0 or i_1. This process has to stop since the number of irreducible parts is finite. ■

We now show how to construct particuliar subactions.

Definition. 24 Let $(x, y) \in S^1$ and $A : S^1 \to \mathbb{R}$. We call action potential of A from x to y the following quantity :

$$S_A(x, y) = \lim_{\epsilon \to 0} \sup\{S_n(A - m)(x') \mid n \geq 1, \quad d(x', x) < \epsilon, \quad f^n(x') = y\}$$

(Notice that we take the supremum over all paths of length $n \geq 1$).

As in the definition of V_A in Proposition 11, $S_A(x, y)$ is uniformly bounded from above but it may happen that $S_A(x, y) = -\infty$. The interesting case takes place when $x \in \Omega(A, f)$.

Lemma. 25 *Let f be a $\mathcal{C}^1$-expanding map and A be a $\mathcal{C}^\alpha$ function. Then the action potential of A satisfies the following properties :*

(i) For any sub-action V, for any $x, y \in S^1$, $S_A(x, y) \leq V(y) - V(x)$ and $x \stackrel{V}{\to} y$ if and only if $S_A(x, y) = V(y) - V(x)$.

(ii) For any $x, y, z \in S^1$, $S_A(x, y) + S_A(y, z) \leq S_A(x, z)$, $S_A(x, x) \leq 0$ and $S_A(x, x) = 0$ if and only if $x \in \Omega(A, f)$.

(iii) For any $x \in \Omega(A, f)$, $S_A(x, .)$ is a $\mathcal{C}^\alpha$ sub-action. (In particular, $S_A(x, y) > -\infty$ for any $y \in S^1$).

Proof. (We only prove part *(iii)*). Let $x \in \Omega(A, f)$ fixed.

Step one. We first show that $S_A(x,.)$ never takes the value $-\infty$. Let us define define an approximate action potential :

$$S_A^\epsilon(x,y) = \sup\{S_N(A-m)(x') \mid d(x,x') < \epsilon, \quad f^n(x') = y, \quad n \geq 1\}$$

where the supremum is taken over all paths of lenght $n \geq 1$ sarting within ϵ of x and ending at y. Then $S_A(x,y) = \inf_\epsilon S_A^\epsilon(x,y)$. Let $\epsilon_0 < \frac{1}{2}$ and $n_0 \geq 1$, x' such that $d(x,x') < \epsilon_0$, $f^{n_0}(x') = y$ and $S_{n_0}(A-m)(x')$ is close to $S_A^\epsilon(x,y)$. Since x belongs to $\Omega(A,f)$, for every $\epsilon > 0$, tehere exist $n \geq 1$, as large as we want $\lambda^{-n}\epsilon_0 < \frac{\epsilon}{2}$, z such that $d(z,x) < \frac{\epsilon}{2}$, $f^n(z) = x$ and $|S_n(A-m)(z)| < \epsilon$. We choose z' in the same inverse branch of lenght n as z such that $f^n(z') = x'$. Then $d(z,z') \leq \lambda^{-n} d(x,x') < \frac{\epsilon}{2}$, $d(x,z') < \epsilon$ and

$$|S_n(A-m)(z) - S_n(A-m)(z')| < \frac{\text{Höl}_\alpha(A)}{\lambda^\alpha - 1}\epsilon_0,$$
$$S_{n_0+n}(A-m)(z') \geq S_{n_0}(A-m)(x') - \frac{\text{Höl}_\alpha(A)}{\lambda^\alpha - 1}\epsilon_0 - \epsilon.$$

By letting ϵ goes to zero, we obtain

$$S_A(x,y) > S_A^{\epsilon_0}(x,y) - \frac{\text{Höl}_\alpha(A)}{\lambda^\alpha - 1}\epsilon_0.$$

This section shows in addition that, in the definition of $S_A^\epsilon(x,y)$, n can be as large as we want.

Step two. Let $(y,z) \in S^1$ such that $d(y,z) < \frac{1}{2}$. Let $\epsilon > 0$ and $n \geq 1$ such that $\lambda^{-n} < \epsilon$. Then there exists x' such that $d(x,x') < \epsilon$, $f^n(x') = y$ and $S_A^\epsilon(x,y) < S_n(A-m)(x') + \epsilon$. Let x'' be the unique n-preimage of z in the same inverse branch as x'. Then $d(x'x'') < \epsilon$, $d(x,x'') < 2\epsilon$ and

$$S_n(A-m)(x') \leq s_n(A-m)(x'') + \frac{\text{Höl}_\alpha(A)}{\lambda^\alpha - 1} d(y,z).$$

Combining all these inequalities we obtain

$$\begin{aligned}
S_A(x,y) &\leq S_A^\epsilon(x,y) \\
&\leq S_n(A-m)(x') + \epsilon \\
&\leq S_A^{2\epsilon} + \epsilon + \frac{\text{Höl}_\alpha(A)}{\lambda^\alpha - 1} d(y,z) \\
&\leq S_A(x,z) + \epsilon + \frac{\text{Höl}_\alpha(A)}{\lambda^\alpha - 1}[2\epsilon + d(y,z)].
\end{aligned}$$

and $S_A(x,.)$ is $\mathcal{C}^\alpha$ with α-Hölder constant $\text{Höl}_\alpha(A)/(\lambda^\alpha - 1)$.

Step three. We show that $S_A(x,.)$ is a sub-action. For $\epsilon > 0$ we have clearly

$$S_A^\epsilon(x,y) + (A-m)(y) \leq S_A^\epsilon(x, f(y)).$$

By letting ϵ goes to zero we obtain

$$(A-m)(y) \leq S_A(x, f(y)) - S_A(x,y).$$

■

Proof of Proposition 20. We only prove the implication *(ii)⇒(iii)*.

Step one. We first show there exists a $\mathcal{C}^\beta$-sub-action V whose action set contains an invariant compact set K, $f(K) = K$, which is not equal to a finite union of periodic orbits. Since $\Omega(A,f)$ is included en any V-action set, we have the following alternative. Either $\Omega(A,f)$ is not equal to a finite union of periodic orbits and we choose $K = \Omega(A,f)$ and V is any sub-action or $\Omega(A,f)$ is equal to the union of at least two periodic orbits. We choose p, q in two distinct periodic orbits and $V(x) = S_A(p,x)$. Then V is a sub-action, by Lemma 25, and $p \overset{V}{\to} q$ for any $q \in S^1$. Lemma 23 tells us there exists a complete orbit $\underline{x}$ included in the V-action set such that $\alpha(\underline{x})$ and $\omega(\underline{x})$ do not belong to the same periodic orbit; in that case K is the closure of $\text{orb}(\underline{x})$.

Step two. As in the proof of theorem 8, we start by writting the cohomological equation for the sub-action founded in step one

$$A - m = V \circ f - V - R$$

where $m = m(A,f)$ and R is a non-negative $\mathcal{C}^\beta$-function which is equal to zero on K. We want to find a sequence of non-negative $\mathcal{C}^\beta$-functions (ϕ_n) converging to zero in the uniform topology and uniformly bounded in the $\mathcal{C}^\beta$-topology such that V is still a sub-action of $A + \phi_n$ and the V-action set of $A + \phi_n$ contains a neighborhood of K. Using a remark of T. Bousch [3], we define

$$\phi_n = \min(R, \frac{1}{n}).$$

Then $0 \leq \phi_n \leq R$, $A + \phi_n - m \leq V \circ f - V$ and ϕ_n equals R on a neighborhood $\mathcal{U}_n$ of K (notice that $m(A,f) = m(A+\phi_n, f)$). By using Lemma 21, $\mathcal{U}_n$ contains a strictly ergodic compact invariant set L_n of positive topological entropy. We finaly choose ψ_n non-negative, $\psi_n = 0$ on L_n and $\psi_n > 0$ on $S^1 \setminus L_n$ with small $\|.\|_\beta$-norm; $A_n = A + \phi_n - \psi_n$ is the desired sequence. ■

3 Lyapunov minimizing measures

The idea of the proof of Theorem 2 is first to conjugate f to a fixed map T by preserving the same geometry of the orbits and second to keep the smoothness of f into two conjugacy invariants (A, μ) where $A = -\ln f'$ has pressure zero and μ, the Lebesgue measure, is the unique equilibrium measure associated to (A, f).

We first show in the following lemma that any maps (f, g) in $\mathcal{F}_\alpha$ are conjugate by bi-hölder maps and that the Hölder exponent can be as close to 1 as we want depending on the $\mathcal{C}^{1+\alpha}$-distance of f and g. Notice that the composition of an α-Hölder map with a β-Hölder map is $\alpha\beta$-Hölder. We use the notation : $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

Lemma. 26 *For simplicity we assume all maps in $\mathcal{F}_\alpha$ fixe $1 \in S^1$.*

(i) For any $f, g \in \mathcal{F}_\alpha$, there exists a unique orientation preserving homeomorphism $\theta = \theta_{f,g}$ such that $\theta(1) = 1$ and $f \circ \theta = \theta \circ g$. The conjugating map is Hölder :

$$d(\theta(x), \theta(y)) \leq C\, d(x, y)^\gamma \quad \forall\, x, y \in S^1$$

with Hölder exponent $\gamma = \gamma(f, g)$ and γ-Hölder norm $C = C(f) \wedge C(g)$:

$$\gamma(f, g) = \inf\{\frac{\ln f'(x)}{\ln g'(y)} \mid d(x, y) \leq \frac{\|\overline{f} - \overline{g}\|_0}{\lambda(f) \vee \lambda(g) - 1}\}$$
$$C(g) = 2D \exp(\frac{H\ddot{o}l_\alpha(g')}{\lambda(g)^\alpha - 1}).$$

(ii) Let $T \in \mathcal{F}_\alpha$ fixed. Then for any $f, g \in \mathcal{F}_\alpha$, $\theta_f = \theta_{f,T}$ and $\theta_g = \theta_{g,T}$ satisfy

$$\|\overline{\theta}_f - \overline{\theta}_g\|_0 \leq \frac{\|\overline{f} - \overline{g}\|_0}{\lambda(f) \vee \lambda(g) - 1}$$
$$\|(\overline{\theta}_f)^{-1} - (\overline{\theta}_g)^{-1}\|_0 \leq C(T) \|\overline{\theta}_f - \overline{\theta}_g\|_0^{\gamma(T,f) \vee \gamma(T,g)}.$$

where $\overline{\theta}_f, \overline{\theta}_g : \mathbb{R} \to \mathbb{R}$ denote the lifts of θ_f, θ_g with $\overline{\theta}_f(0) = \overline{\theta}_g(0) = 0$.

Proof. Let $f \in \mathcal{F}_\alpha$, $z_0^f = 1$ and $z_1^f \ldots z_{D-1}^f$ the preimages of 1 by f ordered canonically on $S^1 \setminus \{1\}$. More generally, we call $Z(f)$ the set of preimages of

1 by (f^n), $n \geq 0$ and code the points in $Z(f)$ by points of the form $\{z^f_{i_1 \dots i_n}\}$ where $i_n = 0, 1, \dots, D-1$ and for each n the preimages of 1 by f^n are ordered by:

$$1 = z^f_{0\dots000},\ z^f_{0\dots001},\ z^f_{0\dots010},\ z^f_{0\dots011},\ z^f_{0\dots100},\ \dots\ , z^f_{1\dots111} < 1.$$

Moreover $Z(f)$ is dense in S^1, otherwise any non-empty connected component of $S^1 \backslash Z(f)$ would be permuted without containing the point 1 and this would contredict the uniform growth of the length.

Since the conjugating map $\theta_{f,g}$ has to preserve the order of the preimages, we have necessarily

$$\theta_{f,g}(z^g_i) = z^f_i, \quad \forall\, i = i_1 \dots i_n.$$

Then $\theta_{f,g}$ extends uniquely to a homeomorphism which conjugates f and g. Let $\overline{f}$ be the lifting of f fixing 0. We denote by $f_i^{-1} : [0,1[\to [0,1[$, $i = 0, 1 \dots D-1$, the inverse branches of $\overline{f}$:

$$\overline{f} \circ f_i^{-1}(w) = w + i \quad \forall w \in [0,1[.$$

Then the points $(z^f_{i_1 \dots i_n})$ on S^1 correspond to the points

$$\overline{z}^f_{i_1 \dots i_n} = f_{i_1}^{-1} \circ \dots \circ f_{i_n}^{-1}(0)$$

on $[0,1[$. We first prove that for any w on $[0,1[$, $n \geq 1$:

$$\| w^f_{i_1 \dots i_n} - w^g_{i_1 \dots i_n} \| \leq \frac{\|\overline{f} - \overline{g}\|_0}{\lambda(f) \vee \lambda(g) - 1}$$

where $w^f_{i_1 \dots i_n} = f_{i_1}^{-1} \circ \dots \circ f_{i_n}^{-1}(\overline{w})$. By induction

$$\begin{aligned} &|w^f_{ji_1 \dots i_n} - w^g_{ji_1 \dots i_n}| \\ &\qquad \leq |f_j^{-1}(w^f_{i_1 \dots i_n}) - f_j^{-1}(w^g_{i_1 \dots i_n})| + |f_j^{-1}(w^g_{i_1 \dots i_n}) - g_j^{-1}(w^g_{i_1 \dots i_n})| \\ &\qquad \leq \frac{1}{\lambda(f)}[|w^f_{i_1 \dots i_n} - w^g_{i_1 \dots i_n}| + \|\overline{f} - \overline{g}\|_0]. \end{aligned}$$

(The second term is obtained using the estimate

$$|\overline{f}^{-1}(u) - \overline{g}^{-1}(u)| = |\overline{f}^{-1} \circ \overline{g} \circ \overline{g}^{-1}(u) - \overline{f}^{-1} \circ \overline{f} \circ \overline{g}^{-1}(u)|$$

for any $u \in [0, D[$.) In particular, this proves

$$\|\overline{\theta}_{f,T} - \overline{\theta}_{g,T}\|_\infty \leq \sup_{i=i_1\dots i_n} |\overline{\theta}_{f,T}(\overline{z}_i^T) - \overline{\theta}_{g,T}(\overline{z}_i^T)| = \sup_{i=i_1\dots i_n} |\overline{z}_i^f - \overline{z}_i^g|$$
$$\leq \frac{\|\overline{f} - \overline{g}\|_0}{\lambda(f) \vee \lambda(g) - 1}$$

Assuming that $\theta_f = \theta_{f,T}$ is β-Hölder, we obtain

$$\|\overline{\theta}_f^{-1} - \overline{\theta}_g^{-1}\|_0 = \|\overline{\theta}_f^{-1} \circ \overline{\theta}_g \circ \overline{\theta}_g^{-1} - \overline{\theta}_f{-1} \circ \overline{\theta}_f \circ \overline{\theta}_g^{-1}\|_0$$
$$\leq \text{Höl}_\gamma(\overline{\theta}_f^{-1})\|\overline{\theta}_g - \overline{\theta}_f\|_0^\gamma.$$

We now prove that $\theta = \theta_{f,g}$ is Hölder. We begin by proving that, for all $n \geq 1$, for any consecutive preimages of order n:

$$\overline{z}_i^g = g_{i_1}^{-1} \circ \dots \circ g_{i_n}^{-1}(0), \quad \overline{z}_{i+1}^g = g_{i_1}^{-1} \circ \dots \circ g_{i_n}^{-1}(1)$$

we have $\overline{z}_i^f = \overline{\theta}(\overline{z}_i^g)$ and $|\overline{z}_i^f - \overline{z}_{i+1}^f| \leq |\overline{z}_i^g - \overline{z}_{i+1}^g|^\beta$. Indeed

$$|\overline{z}_i^f - \overline{z}_{i+1}^f| = \int_0^1 \prod_{k=1}^n [\frac{d}{dw} f_{i_k}^{-1}](w_{i_{k+1}\dots i_n}^f)\, dw.$$

Since $|w_{i_k\dots i_n}^f - w_{i_k\dots i_n}^g| \leq \|\overline{f} - \overline{g}\|_\infty / [\lambda(f) \vee \lambda(g) - 1]$, we obtain by definition of $\gamma(f,g)$:

$$|\overline{z}_i^f - \overline{z}_{i+1}^f| \leq \int_0^1 \prod_{k=1}^n [\frac{d}{dw} g_{i_k}^{-1}]^\gamma (w_{i_{k+1}\dots i_n}^g)\, dw \leq |\overline{z}_i^g - \overline{z}_{i+1}^g|^\gamma$$

(We use Jensen's inequality to extract the power *gamma* outside the integral since $\gamma(f,g) < 1$ for any $f \neq g$.) Let $\overline{z}_i^g$ and $\overline{z}_{i+1}^g$ ($i = i_1 \dots i_n$) be two consecutive preimages of order n, then $[\overline{z}_i^g, \overline{z}_{i+1}^g[$ contains D preimages of order $n+1$ that we call $\overline{z}_{ij}^g$ where $ij = i_i \dots i_n j$ and $j = 0, 1 \dots D-1$. We show that for any $j = 0, \dots D-1$:

$$|\overline{z}_i^g - \overline{z}_{i+1}^g| \leq \frac{1}{2} C(g) |\overline{z}_{ij}^g - \overline{z}_{ij+1}^g|.$$

where $C(g)$ is given in the above Lemma. It is enough to show that the distortion of $g_{i_1}^{-1} \circ \dots \circ g_{i_n}^{-1}$ is uniformly bounded by $C(g)/2D$, that is :

$$|\frac{\frac{d}{dw} g_{i_1}^{-1} \circ \dots \circ g_{i_n}^{-1}(u)}{\frac{d}{dw} g_{i_1}^{-1} \circ \dots \circ g_{i_n}^{-1}(v)}| = \prod_{k=1}^n |\frac{g'(v_{i_k\dots i_n}^g)}{g'(u_{i_k\dots i_n}^g)}| \leq \exp \sum_{k=1}^n |g'(v_{i_k\dots i_n}^g) - g'(u_{i_k\dots i_n}^g)|$$

is bounded by $C(g)/2D$ which is a consequence of

$$|v^g_{i_k\dots i_n} - u^g_{i_k\dots i_n}| \le \lambda(g)^{n-k+1}.$$

We conclude the proof. Given $x, y \in S^1$ there exist $n \ge 1$ and $z \in [x, y]$ such that each $[x, z]$ and $[z, y]$ can be covered by an interval of the form $[z^g_i, z^g_{i+1}]$ of order n and such that each $[x, z]$ and $[z, y]$ contains at least an interval of the form $[z^g_{ij}, z^g_{ij+1}]$ of order $n + 1$. Then

$$\begin{aligned} d(\theta(x), \theta(z)) &\le |\overline{z}^f_i - \overline{z}^f_{i+1}| \le |\overline{z}^g_i - \overline{z}^g_{i+1}|^\gamma \\ &\le (\frac{1}{2}C(g))^\beta |\overline{z}^g_{ij} - \overline{z}^g_{ij+1}|^\beta \le \frac{1}{2}C(g)d(x, y)^\gamma \end{aligned}$$

and an equivalent estimate for $d(\theta(z), \theta(y))$. ■

In the next lemma, we show that any $\mathcal{C}^{1+\beta}$-map of the circle can be approximated by smooth maps, in the $\mathcal{C}^{1+\alpha}$-topology, for any $\alpha < \beta$ (and not in the $\mathcal{C}^{1+\beta}$-topology). For the need of this paper we need a more precise statement.

Lemma. 27 *For any $0 < \alpha < 1$, the space $\mathcal{F}_\alpha \cap \mathcal{C}^\infty$ of smooth maps of S^1 is dense in $\mathcal{F}_{\alpha+}$ in the $\mathcal{C}^{1+\alpha}$-topology.*

Proof. Let $f \in \mathcal{F}_\beta$ for some $\beta > \alpha$, $\overline{f}$ the lift to $\mathbb{R}$ of f, (ρ_n) a smooth approximation of unity (i.e. $\int \rho_n(t)dt = 1$, the support of ρ_n is included in $]-\frac{1}{n}, \frac{1}{n}[$) and $\overline{f}_n = \overline{f} * \rho_n - \overline{f} * \rho_n(0)$, the convolution of f with ρ_n. Then $\overline{f}_n$ is a smooth map, $\overline{f}_n(x+1) = \overline{f}_n(x) + D$, $\overline{f}_n(0) = 0$ and $\lambda(f_n) \ge \lambda(f)$. The sequence of maps $(\overline{f}_n)$ converge to $\overline{f}$ in the $\mathcal{C}^1$-topology. In particular, for large n, f_n is a smooth covering of degree D of the circle, preserves orientation and satisfies the same constraints for ist derivative. Moreover the β-Hölder semi-norm of f'_n is uniformly bounded:

$$\begin{aligned} |f'_n(x) - f'_n(y)| &\le \int |f'(x-t) - f'(y-t)|\rho(t)dt \\ &\le \text{Höl}_\beta(f')|x-y|^\beta. \end{aligned}$$

By using lemma 17, (f'_n) converges to f' in the $\mathcal{C}^\alpha$-topology for any $\alpha < \beta$, and (f_n) converges to f in the $\mathcal{C}^{1+\alpha}$topology. ■

We now give a short summary of the theory of Thermodynamic Formalism for any $T \in \mathcal{F}_\alpha$, which are particular expanding Markov maps. An interested reader may find a comprehensive exposition of this theory in the book

[20]. Given a potential $A : S^1 \to \mathbb{R}$, we define the Ruelle-Peron-Frobenius opperator $\mathcal{L}_A$, acting on the set of countinuous functions, by the following formula:

$$(\mathcal{L}_A.h)(x) = \sum_{T(y)=x} h(y)e^{A(y)} \quad \forall x \in S^1$$

where $h : S^1 \to \mathbb{R}$ is a continuous function and the summation is taken over the D preimages of x. If A is α-Hölder, $\mathcal{L}_A$ also acts on α-Hölder functions and by duality, $\mathcal{L}_A$ acts on the set of probability measures by

$$\int h\, d(\mathcal{L}_A^*.\mu) = \int \mathcal{L}_A.h\, d\mu$$

for any continuous test functions h. For such Markov expanding and mixing map T and for such α-Hölder potentials A, the spectrum $\sigma(\mathcal{L}_A)$ of $\mathcal{L}_A$, acting on $\mathcal{C}^\alpha$-potentials, admits an isolated simple eigenvalue $e^{P_T(A)}$ which dominates the whole spectrum. $P_T(A)$ is called the pressure of A and the rest of the spectrum has a strictly smaller modulus:

$$\sup_{z \neq \exp P_T(A)} \{|z| \mid z \in \sigma(\mathcal{L}_A)\} < \exp P_T(A).$$

Moreover there exist a unique probability measure μ_A, called the Gibbs measure, and a unique positive α-Hölder function h_A such that

$$\mathcal{L}_A^*.\mu_A = e^{P_T(A)}\mu_A \quad \text{and} \quad \mathcal{L}_A.h_A = e^{P_T(A)}h_A.$$

We also recall that a map T possesses a Jacobian $J : S^1 \to]0, +\infty[$ with respect to μ if

$$\mu(T(U)) = \int_U J\, d\mu$$

for all Borel set U where $T : U \to S^1$ is one-to-one. Then $e^{P_T(A)-A}$ is the Jacobian of T with respect to μ_A. The measure $\mu_A^{\text{equi}} = h_A\mu_A$ is called the equilibrium measure associated to A, is T-invariant and

$$\frac{h_A}{h_A \circ T}\exp(P_T(A) - A)$$

is the Jacobian of T with respect to μ_A^{equi}.

In the following lemma we define a kind of "local" chart $(\mathcal{U}_T, \Phi_T)$ about any $T \in \mathcal{F}_{\alpha+}$. The difficulty comes from the loss of differentiability in the conjugating map θ which is not Lipschitz but γ-Hölder with γ as close to 1 as we want provided $\mathcal{U}_T$ is small enough.

Lemma. 28 *Let $0 < \alpha < \beta$, $T \in \mathcal{F}_\beta$ and $A = -\ln T'$.*

(i) There exists a $\mathcal{C}^{1+\beta}$-neighborhood $\tilde{\mathcal{U}_T}$ of T such that $-\ln f' \circ \theta_{f,T}$ is a $\mathcal{C}^\alpha$-function depending continuously on $f \in \tilde{\mathcal{U}_T}$.

(ii) Conversely, there exist a $\mathcal{C}^\beta$-neighborhood $\mathcal{U}_T$ of A and a continuous map $\Phi_T : \mathcal{U}_T \to \mathcal{F}_{\alpha+}$ such that for every $F \in \mathcal{U}_T$, $f = \Phi_T(F)$ is the solution of the equation,

$$F - P_T(F) = -\ln f' \circ \theta_{f,T}.$$

Moreover $\mathcal{U}_T$ can be any neighborhood of the form

$$\mathcal{U}_T = \{F \in \mathcal{C}^\beta \mid \|F\|_\beta < C \quad \text{and} \quad \|F - A\|_0 < \epsilon\}$$

where C is any positive constant and $\epsilon = \epsilon(C)$ is sufficiently small.

Proof. *Part (i):* Since θ_f is $\frac{\ln \lambda}{\ln D}$-Hölder, $A = \mathcal{P}(f)$ is $\frac{\ln \lambda}{\ln D}$-Hölder too for any f in $\mathcal{F}_1(\lambda, \Lambda)$. If $B = \mathcal{P}(g)$, then

$$|A(x) - B(x)| \leq |f' \circ \theta_f(x) - g' \circ \theta_g(x)|$$

(we have used that $|f'| > \lambda > 1$), and

$$|A(x) - B(x)| \leq \|f' - g'\|_\infty + \Lambda d(\theta_f(x), \theta_g(x))$$

By using lemma 26, we obtain

$$\|A - B\|_\infty \leq \|f' - g'\|_\infty + \frac{\Lambda}{\lambda - 1}\|\overline{f} - \overline{g}\|_\infty.$$

Moreover the $\frac{\ln \lambda}{\ln D}$-Hölder norm of $\mathcal{P}(f)$ is uniformly bounded:

$$|A(x) - A(y)| \leq |f' \circ \theta_f(x) - f' \circ \theta_f(y)| \leq \Lambda C(\lambda) d(x, y)^{\ln \lambda / \ln D}.$$

Continuity of $\mathcal{P}$ now follows from lemma 17.

Part (ii): For any $\mathcal{C}^\beta$-function F, $\exp(P_T(F)-F)$ is the jacobian of T with respect to the Gibbs measure μ_F. If $\theta_F : S^1 \to S^1$ is an homeomorphism, $\exp(P_T(F)-F)\circ\theta_F^{-1}$ is the jacobian of $f=\theta_F\circ T\circ\theta_F^{-1}$ with respect to the push measure $(\theta_F)_*\mu_F$. We are therefore looking for a θ_F which satisfies $(\theta_F)_*\mu_F = Leb$ and $\theta_F(1)=1$. The only possibility for θ_F is given by the formula

$$Leb(\overline{\theta}_F[0,x]) = \overline{\theta}_F(x) = \overline{\mu}_F([0,x]) \qquad \forall x\in[0,1[$$

where $\overline{\theta}_F$ is the lift of θ_F and $\overline{\mu}_F$ is the corresponding measure on $[0,1[$. Since μ_F gives positive measure to non-empty open sets and has no mass $\theta_F : S^1\to S^1$ is an homeomorphism. Let $f=\theta_F\circ T\circ\theta_F^{-1}$, then f is a covering of degree D and preserves the orientation. Moreover $J=\exp(P_T(F)-F\circ\theta_F^{-1})$ is the Jacobian of f with respect to the Lebesgue measure and

$$\overline{f}(y)-\overline{f}(x) = \int_x^y \overline{J}(t)\,dt$$

for any $0<x<y<1$. This shows in particular that f is $\mathcal{C}^1$ and $-\ln f'\circ\theta_F = F-P_T(F)$. By lemma 26, $\theta_f=\theta_F$, θ_F^{-1} is $\gamma(T,f)$-Hölder and its γ-Hölder constant is bounded by $C(T)$. Then f' is $\beta\gamma$-Hölder and

$$\mathrm{Höl}_{\beta\gamma}(f') \le \|\exp(P_T(A)-A\|_0 \mathrm{Höl}_\beta(A) C(T)^\beta.$$

Let $\gamma^*<1$, close to 1, such that $\beta\gamma^*>\alpha$. Let $\lambda^*<\lambda(T)<\Lambda^*$, $C^*>\mathrm{Höl}_\beta(A)$ and

$$\mathcal{U}_T = \{F\in\mathcal{C}^\beta \mid \mathrm{Höl}_\beta(F)<C^* \quad \text{and} \quad \|F-A\|_0<\epsilon\}.$$

We want to show that, for ϵ small enough,

$$\max_{F\in\mathcal{U}_T}\exp(P_T(F)-F)\le\Lambda^*, \qquad \min_{F\in\mathcal{U}_T}\lambda(f) = \min_{F\in\mathcal{U}_T}\exp(P_T(F)-F)\ge\lambda^*$$

$\gamma(T,f)\ge\gamma^*$, the $\beta\gamma^*$-Hölder constant of f' is uniformly bounded and f' depends continuously with respect to F. Using lemma 17, this would prove the continuity of $\Phi_T(F)$ with respect to $F\in\mathcal{U}_T$ as a $\mathcal{C}^{1+\alpha}$ maps.

We first notice the following estimates:

$$\|\overline{f}-\overline{g}\|_0 \le \|f'-g'\|_0$$

where $f' = \exp(P_T(F) - F) \circ \theta_F^{-1}$, $g' = \exp(P_T(G) - G) \circ \theta_G^{-1}$. Then we have

$$\|f' - g'\|_0 \leq \Lambda^* \left[2\|F - G\|_0 + C^* \|(\overline{\theta}_F)^{-1} - (\overline{\theta}_G)^{-1}\|_0^\beta \right]$$
$$\|(\overline{\theta}_F)^{-1} - (\overline{\theta}_G)^{-1}\|_0 \leq C(T) \|\overline{\theta}_F - \overline{\theta}_G\|_0^{\ln \lambda^* / \ln \Lambda^*}$$

To conclude, it is now enough to prove that $\overline{\theta}_F$ depends continuously with respect to $F \in \mathcal{U}_T$ in the uniform topology.

Let (F_n) be a sequence of β-Hölder functions converging to F in the $\mathcal{C}^\beta$-topology and (f_n) the corresponding maps. We show that $(\overline{\theta}_{F_n})$ converges to $\overline{\theta}_F$ uniformly. Restricted to $[0,1]$, $\overline{\theta}_{F_n}$ corresponds to the distribution of the measure $\overline{\mu}_{F_n}$. Since $\overline{\theta}_F$, $\overline{\theta}_{F_n}$ are increasing and continuous, by using Helly's theorem, it is enough to show that $(\overline{\theta}_{F_n})$ converges pointwise to $\overline{\theta}_F$ or that $(\overline{\mu}_{F_n})$ converges weakly to $\overline{\mu}_F$ (we use here the fact that $\overline{\mu}_A$ has no atom).

Let ν be a weak limit of some sub-sequence $(\mu_{F_{n'}})$. Since $\mathcal{L}_{F_n}$ converges to $\mathcal{L}_F$ in the uniform topology, for any test funnction $h : S^1 \to \mathbb{R}$:

$$\lim_{n' \to \infty} \int \mathcal{L}_{F_{n'}}.h \, d\mu_{F_{n'}} = \int \mathcal{L}_F.h \, d\nu$$
$$\int \mathcal{L}_{F_{n'}}.h \, d\mu_{F_{n'}} = e^{P(F_{n'})} \int h \, d\mu_{F_{n'}}$$
$$\lim_{n' \to \infty} e^{P(B_{n'})} \int h \, d\mu_{B_{n'}} = e^{P(F)} \int h \, d\nu.$$

We thus obtain that ν is solution of $\mathcal{L}_F.\nu = e^{P(F)}\nu$, and by uniqness of Gibbs measures, $\nu = \mu_F$. Since $\gamma(T, f)$ depends continuously with respect to f in the $\mathcal{C}^1$-topology, $\gamma(T, f)$ can be made bigger than γ^* for every $F \in \mathcal{U}_T$ as soon as ϵ is choosen small enough. ■

Remark. 29 In the previous lemma, $(\mathcal{U}_T, \Phi_T)$ is not a one-to-one parametrization. Although we do not need it, we could have been more precise. The set $\mathcal{P} = \{A \in \mathcal{C}^\alpha \mid A < 0 \text{ and } P_T(A) = 0\}$ is actually a $\mathcal{C}^1$-manifold: for each $A \in \mathcal{P}$, $\mathcal{P}$ is homeomorphic to an open set in a Banach space $\mathcal{T}_A = \{F \in \mathcal{C}^\alpha \mid F < 0 \text{ and } \int (F - A) d\mu_A^{\text{equi}} = 0\}$. Indeed if $F \in \mathcal{P}$,

$$\frac{\int A \, d\mu_A^{\text{equi}}}{\int F \, d\mu_A^{\text{equi}}} F \in \mathcal{T}_A \quad \text{and} \quad \frac{\int A \, d\mu_A^{\text{equi}}}{\int F \, d\mu_A^{\text{equi}}} \geq 1$$

where $\mu_A^{\rm equi}$ is the equilibrium measure associated to A. Conversely, if $F \in \mathcal{T}_A$, the pressure P is $\mathcal{C}^1$-differentiable on the space of $\mathcal{C}^\alpha$-potentials (see [17]) and

$$P_T'(A).H = \int H \, d\mu_A^{\rm equi}$$

The map $p(t,F) = P_T(tF)$ is therefore $\mathcal{C}^1$ with respect to t and F,

$$\frac{\partial p}{\partial t} = \int F \, d\mu_F^{\rm equi} < 0, \quad p(0,F) = \ln D \quad \text{and} \quad p(1,F) \leq 0.$$

The function $p(t,F)$ is decreasing with respect to t, there exists a unique $0 < \delta(F) \leq 1$ such that $p(\delta(F),F) = 0$. By the implicit function theorem, the function $F \mapsto \delta(F)$ is $\mathcal{C}^1$.

Proof of Theorem 2. Let $\mathcal{G}$ be the set of maps f in $\mathcal{F}_\alpha$ admiting a unique Lyapunov minimizing measure supported on a periodic orbit and satisfying the property of countinuously varying support. We first show that $\mathcal{G}$ is open in $\mathcal{F}_\alpha$ and that $\mathcal{G}_+ = \mathcal{G} \cap \mathcal{F}_{\alpha+}$ is dense in $\mathcal{F}_{\alpha+}(\lambda,\Lambda)$.

To show that $\mathcal{G}$ is open, we proceed as in the proof of theorem 8 *Part (ii)*. Let $f_0 \in \mathcal{G}$, $\mu_0 = \frac{1}{N}\sum_{k=0}^{N-1} \delta_{f_0^k(p_0)}$ the unique minimizing periodic measure, $\mathcal{N}$ a neighborhood of the orbit $\mathrm{orb}(p_0) = \{p_0, f_0(p_0), \ldots, f_0^N(p_0) = p_0\}$ and $\mathcal{U}$ a neighborhood of f_0 in the $\mathcal{C}^{1+\alpha}$-topology such that, for any $f \in \mathcal{U}$, f has a unique periodic orbit p of period N in $\mathcal{N}$ which is also the only f-invariant compact set in $\mathcal{N}$. We note $= u_f = \frac{1}{N}\sum_{k=0}^{N-1} \delta_{f^k(p)}$ the corresponding periodic measure. Thanks to the property of continuously varying support, $\mathcal{U}$ can be cosen small enough so that, for any $f \in \mathcal{U}$, any Lyapunov minimizing measure μ for f has a support included in $\mathcal{N}$ and is therefore equal to μ_f. To prove that any $f \in \mathcal{U}$ also satisfies the property of continuously varying support, we choose (f_n) converging to f in the uniform topology and notice that any accumulation set K of the sequence $(\mathrm{supp}(\mu_{f_n})$ is equal to the unique periodic orbit of f in $\mathcal{N}$.

We now show that $\mathcal{G}_+$ is dense in $\mathcal{F}_{\alpha+}$. Let $T \in \mathcal{F}_{\alpha+}$, then $T \in \mathcal{F}_\beta$ for some $\beta > \alpha$ and $A = -\ln T'$ is $\mathcal{C}^\beta$. According to Theorm 8 (and more precisely its proof) there exists a sequence (F_n) of β-Hölder functions having a β-Hölder norm uniformly bounded converging to A in the $\mathcal{C}^0$-topology and such that $F_n \in \mathcal{G}_+$ for all n. Thanks to lemma 28, $(\Phi_T(F_n))$ converges to T as $\mathcal{C}^{1+\alpha}$ maps and the set of Lyapunov minimizing measures of $\Phi_T(F_n)$ coincides with the set of maximizing measures of F_n.

The proof of the second part of Theorem 2 follows from the equivalence between *(i)* and *(ii)* in Proposition 20. ■

NOT YET FINISHED

The next proposition appeared in a slight different form in (see Prop 4 in [16]).

Proposition. 30 *Suppose μ_A, maximizing A is unique, then $\hat{\mu_t}$, when t goes to $-\infty$, converges to μ_A.*

Proof *: Suppose $\hat{\mu_t}$ does not converges to μ_A, then for some $\epsilon > 0$ and a subsequence t_n,*

$$\int A d\hat{\mu_{t_n}} < \int A d\mu_A - \epsilon.$$

Take t_n negative enough such that $t_n \epsilon + \log 2 < 0$.
Then

$$h(\mu_A) - t_n \int A d\mu_A \geq h(\mu_A) - t_n (\int A d\hat{\mu_{t_n}} + \epsilon) \geq$$

$$h(\mu_A) - t_n \epsilon - t_n \int A d\hat{\mu_{t_n}} \geq h(\hat{\mu_{t_n}}) - \log 2 - t_n \epsilon - t_n \int A d\hat{\mu_{t_n}} \geq$$

$$h(\hat{\mu_{t_n}}) - t_n \int A d\hat{\mu_{t_n}}.$$

This is a contradiction with the fact that $\hat{\mu_{t_n}}$ is equilibrium state for A.

The result above shows that when t goes to infinity, in the problem of the pressure described above, we are putting less and less stressing in the entropy of the measure and more and more emphasys on the integral of the measure.

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