Generic Properties of Geodesic Flows

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Geodesic Flow

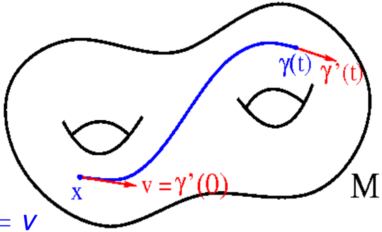
M closed C^∞ manifold [compact connected, $\partial M \neq 0$]

$g = \langle\ ,\ \rangle_x$ C^∞ riemannian metric on M.

unit tangent bundle = sphere bundle of (M, g)

$$SM = \{\, (x, v) \in TM \mid \|v\|_x = 1 \,\}$$

$$\begin{array}{rccc} \pi : & SM & \longrightarrow & M \\ & (x, v) & \longmapsto & x \end{array}$$



$(x, v) \in SM$

$\gamma : \mathbb{R} \to M$

geodesic s.t. $\gamma(0) = x$, $\dot\gamma(0) = v$

"locally length minimizing curve with $|\dot\gamma| \equiv 1$"

Geodesic Flow

$$\begin{array}{rccc} \phi_t : & SM & \longrightarrow & SM \\ & (x, v) & \longmapsto & (\gamma(t), \dot\gamma(t)) \end{array}$$

TOPOLOGICAL ENTROPY

1. Measures the "complexity" of the orbit structure of the flow.
   Measures the difficulty in predicting the position of an orbit given an approximation of its initial state.

   Dynamic ball: $\theta \in SM$, ε, $T > 0$

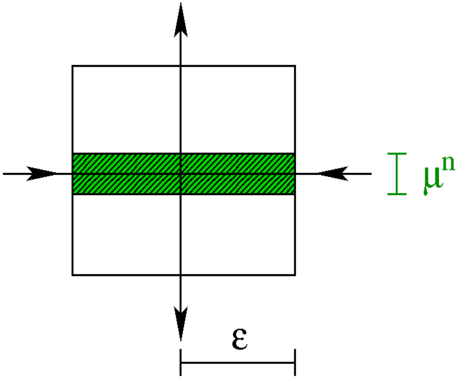
$$\mathbb{B}(\theta, \varepsilon, T) := \{\, \omega \in SM \;:\; d(\phi_t\theta, \phi_t\omega) \leq \varepsilon, \; \forall t \in [0, T] \,\}$$

Points whose orbit stay near the orbit of θ for times in $[0, T]$.

$$N_\varepsilon(T) := \min\{\#\mathcal{C} \mid \mathcal{C} = \text{cover of } SM \text{ by } (\varepsilon, T)\text{-dynamic balls}\,\}$$

$$h_{top}(g) := \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \tfrac{1}{T} \log N_\varepsilon(T).$$

$$N_\varepsilon(T) \approx e^{h_{top} \cdot T}.$$



If $h_{top} > 0$
some dynamic balls
must contract exponentially
at least in one direction.

2. For C^∞ riemannian metrics

Mañé $$h_{top}(g) = \lim_{T\to\infty} \frac{1}{T} \log \int_{M\times M} n_T(x,y)\, dx\, dy$$

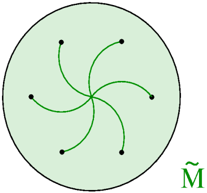
$n_T(x,y) := \#\{\,$geod. arcs $x \to y$ of length $\leq T\,\}$.

$h_{top} > 0 \implies$ positive measure of (x,y) s.t.

$n_T(x,y)$ is exponentially large.

TOPOLOGY $\Longrightarrow$ Some manifolds have always $h_{top}(g) > 0$.

- Dinaburg: $\pi_1(M)$ exponential growth
  $\Longrightarrow \quad h_{top} > 0$.
  [# of dynamic balls grows exponentially]



Also if $\lim_{R\to\infty} \frac{1}{R} \log\Big(\text{vol}(\widetilde{B}(x,R))\Big) > 0$.

"volume entropy"

- Paternain-Petean: If $H_*(\text{Loop space}(M), x)$ grows exponentially
  $\Longrightarrow$ many arcs $x \mapsto y \in \widetilde{\pi}^{-1}(x)$
  $\Longrightarrow \quad h_{top} > 0$.

GEOMETRY

sectional curvatures $K < 0 \Longrightarrow \phi_t$ Anosov $\Longrightarrow h_{top} > 0$.

$K > 0$ not clear.

If the geodesic flow ϕ_t contains a "horseshoe"
= a non-trivial hyperbolic basic set
$\Longrightarrow h_{top}(g) > 0$.

$\exists$ hyperbolic periodic orbit with transversal homoclinic point $\Longleftrightarrow$ $\exists$ horseshoe.

Symbolic dynamics $\Longrightarrow$
the number of closed orbits grows exponentially with the period.

Theorem

$\dim M \geq 2$
$\exists\, \mathcal{U} \subset \mathcal{R}^2(M)$ *open and dense* s.t.
$g \in \mathcal{U} \implies \phi_t^g$ *has a horseshoe.*

Corollary

If $g \in \mathcal{U}$ *then* $h_{top} > 0$ *and*
the number of closed geodesics grows expo. with the length.

Application:
A. Delshams, R. de la Llave, T. Seará:
Initial system that allows Arnold's diffusion by perturbations with generic periodic potentials.

Comparison with other systems:

General Hamiltonian systems.

S. Newhouse: (M^{2n}, ω) closed symplectic manifold

$$\exists\, \mathcal{H} \subset C^2(M, \mathbb{R}) \text{ residual s.t.}$$

$$h \in \mathcal{H} \Longrightarrow \begin{array}{c}\text{Hamiltonian}\\ \text{flow for } H\end{array} \begin{cases} \bullet \text{ Anosov or} \\ \bullet \text{ has a generic 1-elliptic periodic orbit} \end{cases}$$

1-elliptic = 2 (elliptic) eigenvalues of modulus 1.

1 eigenvalue $\lambda = 1$ (direction of Hamiltonian vector field).

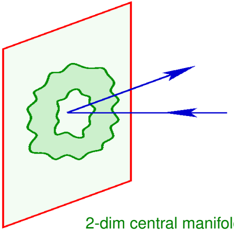
1 eigenvalue $\lambda = 1$ ($\pitchfork$ direction to energy level).

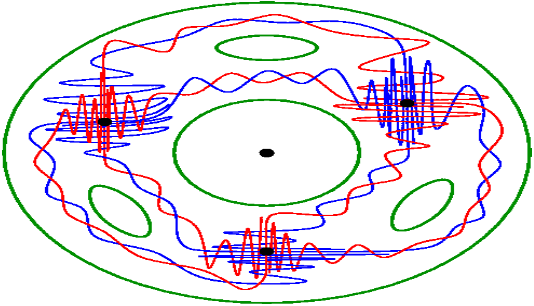
$2n - 4$ hyperbolic eigenvalues.

In the 1-elliptic case:

Poincaré map restricted to energy level is

twist map $\times$ normally hyperbolic





$\Longrightarrow$ homoclinic orbits

Newhouse theorem uses the closing lemma.

The closing lemma is not known for geodesic flows.

reason: Proof uses local perturbations.

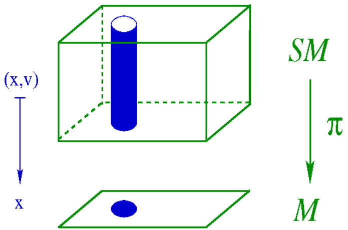
Perturbations of riemannian metrics $g_{ij}(x)$

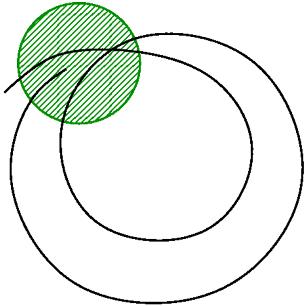
are never local in phase space $= SM$.

Newhouse theorem for geodesic flows is only known

for $M = \mathbb{S}^2$ and $M = \mathbb{RP}^2$. (Contreras - Oliveira ETDS 2004)

Non-local perturbations



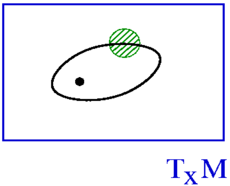


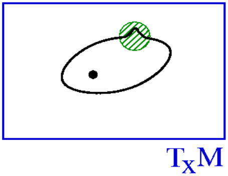
"the orbit to close could have passed through the cylinder before coming back"

General Finsler metrics:

= norm $\|\cdot\|_x$ on tangent spaces $T_x M$.

The unit sphere does not have to be symmetric or a level set of a quadratic form.





- The closing lemma holds.
- The Newhouse theorem should hold.

Bumpy metrics

M^{n+1}

$J^k_s(n) = \{\, k\text{-jets of symplectic diffeos } f : (\mathbb{R}^{2n}, 0) \hookleftarrow \}$

$Q \subset J^k_s(n)$ is *invariant* iff $\sigma\, Q \sigma^{-1} = Q \quad \forall \sigma \in J^k_s(n)$.

$\mathcal{R}^r(M) = C^\infty$-riemannian metrics on M with the C^r topology.

Theorem (Anosov, Klingenberg-Takens)

If $Q \subset J^k_s(n)$ is open, dense and invariant then the set of metrics such that the Poincaré map of every closed geodesic is in Q contains a residual set in $\mathcal{R}^r(M)$.

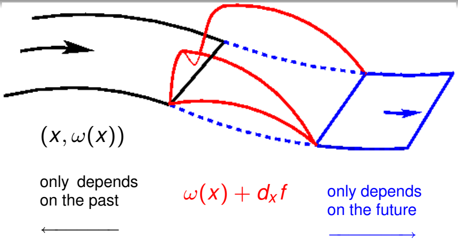
The Kupka-Smale theorem

Theorem

If $Q \subset J^k_s(n)$ is residual and invariant
$\Longrightarrow \forall r \geq k+1 \quad \exists \mathcal{G} \subset \mathcal{R}^r(M)$ *residual s.t.* $\forall g \in \mathcal{G}$:

a. *[Anosov, Klingenberg-Takens]*
Poincaré maps of all periodic orbits of all periodic orbits for ϕ^g are in Q.
b. *All heteroclinic intersections for ϕ^g are transversal.*

Simple proof of (b):



- W^s is lagangian in (T^*M, w_0).
- Choose a place where W^s is locally a lagrangian graph.
- Deform W^s to another lagrangian graph (by adding a $d_x f$).
  $w_0 = dp \wedge dx$ fixed canonical sympectic form.
- Change the metric s.t. $H(new\ W^s) \equiv 1$.
  $\Longrightarrow$ New W^s is invariant. (Hamilton-Jacobi thm)

Similar arguments can be used to perturb single orbits.

Twist maps

Elliptic fixed points:

Symplectic diffeomorphism $F : (\mathbb{R}^{2n}, 0) \hookleftarrow$

will be the Poincaré map of a closed geodesic

q-elliptic periodic point = non-degenerate
+ $2q$ eigenvalues of modulus 1.

$\Longrightarrow \exists\ 2q$-dim central manifold which is normally hyperbolic.

We choose $Q \subset J^3_s(n)$ 3-Jets of symplectic C^∞ diffeos $F : (\mathbb{R}^{2n}, 0) \hookleftarrow$

such that the map restricted to the central manifold is a “weakly monotonous” twist map, i.e.

- The map can be written in Birkhoff normal form (4-elementary condition).
- Twist condition for the Birkhoff normal form.

Conditions on $Q \subset J^3_s(n)$:

- The elliptic eigenvalues $\rho_1, \ldots, \rho_q$ and $\overline{\rho}_1, \ldots, \overline{\rho}_q$ are 4-*elementary*:

$$1 \leq \sum_{i=1}^{q} |\nu_i| \leq 4 \quad \Longrightarrow \quad \prod_{i=1}^{q} \rho_i^{\nu_i} \neq 1.$$

- The Birkhoff normal form $P(x, y) = (X, Y)$

$$Z_k = e^{2\pi i \phi_k} z_k + g_k(z),$$
$$\phi_k(z) = a_k + \sum_{\ell=1}^{q} \beta_{k\ell} |z_\ell|^2,$$

where $z_k = x_k + i\, y_k$, $Z_k = X_k + i\, Y_k$, and the 3-jet $j^3 g_k(0) = 0$, satisfies $\det[\beta_{k\ell}] \neq 0$.

Using techniques of J. Moser, M. Herman and M-C. Arnaud,

Theorem

If $F : (\mathbb{R}^{2n}, 0) \hookleftarrow$ is a germ of s sympl. diffeo. such that 0 is an elliptic fixed point and $j^3 F(0) \in Q$, then F has a 1-elliptic periodic point.

Such a 1-elliptic periodic point has a normally hyperbolic 2D central manifold where F is a twist map of the annulus.

In particular, if it is Kupka-Smale then F has a transversal homoclinic orbit.

Many closed geodesics

Theorem (Bangert, Hinston, Rademacher)

$\exists \mathcal{D} \subset \mathcal{R}^k(M)$ residual set s.t.
$g \in \mathcal{D} \Longrightarrow (M, g)$ has infinitely many closed geodesics.

In fact one can take $\mathcal{D} =$ metrics s.t. the arguments of the elliptic eigenvalues of Poincaré maps of closed orbits are algebraically independent.

Stable Hyperbolicity

$Sp(n)$:= sympletic linear isomorphisms of $\mathbb{R}^{2n}$.

$T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ linear map is *hyperbolic*

if it has no eigenvalue of modulus 1.

a sequence $\xi : \mathbb{Z} \to Sp(n)$ is *periodic* if (will be the time 1 Poincaré map)

$\exists m \quad \xi_{i+m} = \xi_i$

A periodic sequence is *hyperbolic* if $\prod_{i=1}^{m} \xi_i$ is hyperbolic.

A family of periodic sequences $\{\xi^a\}_{\alpha\in\mathcal{A}}$

is *bounded* if $\exists B > 0$ $\left\|\xi_i^a\right\| \leq B, \quad \forall i \in \mathbb{Z}, \forall \alpha \in \mathcal{A}$.

is *hyperbolic* if ξ^α is hyperbolic $\forall \alpha \in \mathcal{A}$.

Families $\xi = \{\xi^\alpha\}_{\alpha\in\mathcal{A}}$, $\eta = \{\eta^\alpha\}_{\alpha\in\mathcal{A}}$ are *periodically equivalent* iff $\forall\alpha$ ξ^α, η^α have same periods.

For period. equiv. families $\xi = \{\xi^\alpha\}_{\alpha\in\mathcal{A}}$, $\eta = \{\eta^\alpha\}_{\alpha\in\mathcal{A}}$ define

$$\|\xi - \eta\| := \sup\{\, \|\xi^\alpha - \eta^\alpha\| \;:\; \alpha \in \mathcal{A},\; n \in \mathbb{Z}\,\}$$

This determines how to pertub: up to a fixed amount in each time 1 Poincaré map.

$\Longrightarrow$ the following theorem will be useful
only in the C^1 topology for flows
= C^2 topology for riem. metrics (or Hamiltonians).

A family ξ is *stably hyperbolic* iff
$\exists \varepsilon > 0$ s.t. if η family period. equiv. to ξ &
$\|\eta - \xi\| < \varepsilon \implies \eta$ is hyperbolic.

A family ξ is *uniformy hyperbolic* iff
$\exists M > 0$ s.t.

$$\left\| \prod_{i=0}^{M} \xi_{i+j}^{\alpha}\Big|_{E^s(\xi_j^{\alpha})} \right\| < \frac{1}{2}, \qquad \left\| \left[\prod_{i=0}^{M} \xi_{i+j}^{\alpha}\Big|_{E^u(\xi_j^{\alpha})} \right]^{-1} \right\| < \frac{1}{2}$$

$$\forall \alpha \in \mathcal{A}, \qquad \forall j \in \mathbb{Z}.$$

Theorem

Let ξ be a bounded periodic family of symplectic linear maps: if ξ is stably hyperbolic $\Longrightarrow$ ξ is uniformly hyperbolic.

Remark:

- Families in $Sp(n)$: stably hyperbolic $\Longrightarrow$ uniformly hyperbolic.
- Families in $GL(n)$: stably hyperbolic $\Longrightarrow$ dominated splitting, i.e.

$$\left\| \prod_{i=0}^{M} \xi_{i+j}^{\alpha}\Big|_{E^s(\xi_j^{\alpha})} \right\| \cdot \left\| \left[\prod_{i=0}^{M} \xi_{i+j}^{\alpha}\Big|_{E^u(\xi_j^{\alpha})} \right]^{-1} \right\| < \tfrac{1}{2}.$$

Perturbation Lemma ("Franks Lemma")

Example: Statement for diffeos $f : M \to M$.

$\exists \varepsilon_0 > 0 \quad \forall \varepsilon \in]0, \varepsilon_0] \quad \exists \delta > 0$ s.t. if

$\mathcal{F} = \{ x_1, \dots, x_N \} \subset M$ any finite set

U any neighbourhood of $\mathcal{F}$

$A_i \in L(T_{x_i}M, T_{f(x_i)}M)$ "candidate for $Df(x_i)$"

$\|Df(x_i) - A_i\| < \varepsilon$

$\Longrightarrow$

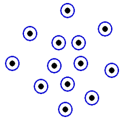
$\exists g \in Diff(M)$ s.t.

$g|_{M \setminus U} = f|_{M \setminus U}$

$g(x_i) = f(x_i) \qquad \forall x_i \in \mathcal{F}$

$Dg(x_i) = A_i$

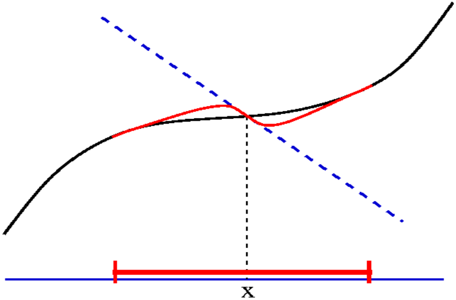
$\|f - g\|_{C^1} < \delta$



arbitrarily small support

no size problem as in closing lemma

Example: In dimension 1.



U arbitrarily small

Analogous lemma for geodesic flows:

Realize any perturbation in $Sp(n)$
in a fixed distance of the derivative of the Poincaré map
at any geodesic segment of length 1.

- fixing the geodesic.
- with support in an arbitrarily narrow strip U.
- outside a neighbourhood of a given set of finitely many transversal segments.

By a metric which is C^2 close.

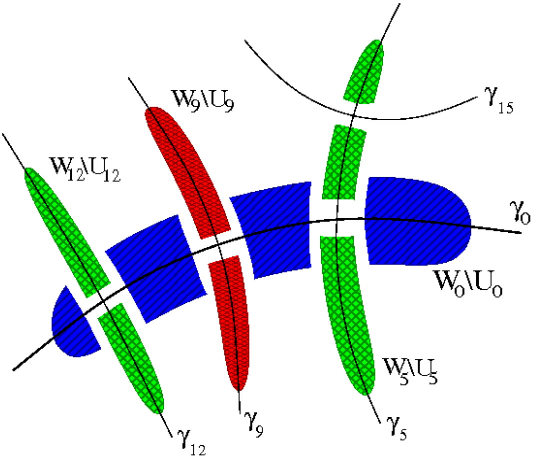


Figure: Avoiding self-intersections.

The perturbation is made in the neighbourhood of one point.

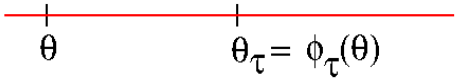
The following result allowed to pass from dim 2 to higher dimensions.

Theorem

$\exists \mathcal{G} \subset \mathcal{R}^\infty(M)$ residual s.t
$\forall g \in \mathcal{G} \quad \forall \theta \in SM \quad \exists t_0 \in [0, \frac{1}{2}]$
s.t. the sectional curvature matrix

$$K_{ij}(\theta) = \langle R(\theta, e_i)\,\theta, e_j \rangle$$

has no repeated eigenvalues.



The End

The Perturbation Lemma

Derivative of the geodesic flow

$$d\phi_t(J(0), \dot{J}(0)) = (J(t), \dot{J}(t))$$

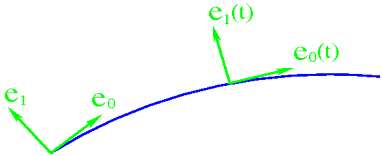
$J(t)$ = Jacobi field orthogonal to the geodesic $\gamma(t) = \pi \circ \phi_t(\theta)$.

Jacobi Equation: $J'' + K(t)\, J = 0$

$$K(u, v) = \langle R(u, \dot{\gamma})\, v, \dot{\gamma} \rangle.$$

1. Can change the Jacobi equation at will.

Use Fermi coordinates:



$e_0 = \dot\gamma$, $e_1, \dots, e_n$ = parallel transport of orthonormal basis along γ.

$$F(t = x_0, x_1, \dots, x_n) = \exp\left(\sum_{i=1}^{n} x_i\, e_i(t)\right)$$

exp for a fixed initail metric g_0.

Our general perturbation of the metric g^0 is

$$g_{00}(t,x) = [g^0(t,x)]_{00} + \sum_{i=1} \alpha_{ij}\, x_i\, x_j$$

$$g_{ij}(t,x) = [g^0(t,x)]_{ij} \qquad \text{if } (i,j) \neq (0,0).$$

This perturbation:

1. Preserves the geodesic γ.
2. Preserves the metric along γ.

   (orthogonal vector fields along γ are still orthogonal)
3. Changes the curvature along γ by

$$K(t) = K_0(t) - \alpha(t,x)$$

4. If the perturbation term is

$$x^* \alpha x = \varphi(x) \;\; x^* P(t) x$$

$\varphi(x) =$ bump function in $x_1, \ldots, x_n$.

and supp(φ) is sufficiently small

$$\Longrightarrow \qquad \|x^* \alpha x\|_{C^2} \sim \|P(t)\|_{C^0}$$

We will use

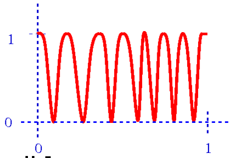
$$x^*\alpha\, x = h(t)\, \varphi(x)\, x^* P(t)\, x$$

$\varphi(x) =$ bump function in $x_1, \dots x_n$.

$h(t) =$ approximation of characteristic function of $[0,1] \setminus F^{-1}(\mathcal{F})$.

i.e. $\quad 0 \le h(t) \le 1, \qquad \operatorname{supp}(h) \cap \begin{pmatrix} \text{intersecting} \\ \text{points} \end{pmatrix} = \emptyset,$

$$\int_0^1 h \ge 1 - \varepsilon,$$



[only $\|h\|_{C^0}$ counts if supp(φ) is small]

$$P(t) = a\,\delta(t) + b\,\delta'(t) + c\,\delta''(t) + d\,\delta'''(t)$$

$$a, b, c, d \in \mathrm{Sym}(n \times n) =: \mathcal{S}(n) \qquad d_{ii} = 0 \qquad d \in \mathcal{S}^*(n)$$

$\delta(t) =$ approximation of Dirac δ at some point τ near $\frac{1}{2}$ where $K(t)$ has no repeated eigenvalues:

$$\min_{i \neq j} \left|\lambda_i - \lambda_j\right| > \eta = \eta(\mathcal{U}) > 0.$$

$\mathcal{U}$ neighbourhood of g_0.

2. Estimate the perturbation in the solution of the Jacobi equation.

$$J'' + K(t)\,J = 0$$

$$\begin{bmatrix} J \\ J' \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix}}_{\mathbb{A}} \begin{bmatrix} J \\ J' \end{bmatrix}$$

$$X' = \mathbb{A}\,X, \qquad X \in \mathbb{R}^{n\times n}$$

$$X(0) = I \qquad \Longrightarrow \qquad X(t) = d\phi_t$$

Fundamental solution

$$X' = \mathbb{A}\, X, \qquad \mathbb{A} = \begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix}$$

Remarks:

- Can only perturb K not the whole matrix $\mathbb{A}$.
- Only perturbations $K \mapsto K + \alpha$. K, α symmetric matrices
  (because the perturbation term was $x^* K\, x$)

The solutions X are symplectic linear maps.

$$Sp(n) = \{\, X \in \mathbb{R}^{n\times n} \;:\; X^* J\, X = J \,\}, \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

$$T_X Sp(n) = \{\, XY \;:\; Y^* J + JY = 0 \,\},$$

$$= \left\{\, XY \;:\; Y = \begin{bmatrix} \beta & \gamma \\ \alpha & -\beta^* \end{bmatrix} \quad \alpha, \gamma \text{ symmetric} \quad \beta \text{ arbitrary} \,\right\}.$$

Strategy:

Think on 1-parameter family of metrics $s \mapsto g_s$.

$$s \longmapsto K_s(t) = K(t) + s\,\alpha(t)$$
$$s \longmapsto X_s(t) = d\phi_t^{g_s}$$

$$\alpha(t) = \alpha(t, E), \qquad (a, b, c, d) \in \mathcal{S}(n)^3 \times \mathcal{S}^*(n)$$

$\mathcal{S}(n) = \mathrm{Sym}(n \times n)$, $\quad S^*(n) = \mathrm{Sym}(n \times n)$ & diag $\equiv 0$.
$\mathcal{S}(n)^3 \times \mathcal{S}^*(n)$ has the same dimension as $T_X Sp(n)$.

Take the derivative

$$Z_s = \frac{d\,X_s}{ds} = \frac{d}{ds}\,(d\phi_t^s)$$

Strategy:

Take the derivative

$$Z_s = \frac{d\,X_s}{ds} = \frac{d}{ds}\,(d\phi_t^s)$$

Prove that

$$\|Z_s(1)\| \geq k\ \|E\| \sim \|x^*\alpha\, x\|_{C^2}$$
$$\textit{with} \quad k = k(\mathcal{U})$$

k uniform for every geodesic segment of length 1 and $\forall g \in \mathcal{U}$

$\Longrightarrow \quad \{\, d\phi_1^g \ :\ g \in \mathcal{U}\,\}$ covers a neighbourhood of the original linearized Poincaré map $d\phi_1^{g_0}$ whose size depends only on the C^2 norm of the perturbation.

The derivative of the Jacobi equation

$$X' = \mathbb{A}_s \, X_s$$

$$Z_s = \frac{dX_s}{ds}, \quad \mathbb{A}_s = \mathbb{A} + s\,\mathbb{B}, \quad \mathbb{B} = \begin{bmatrix} 0 & 0 \\ P(t) & 0 \end{bmatrix}$$

$$P(t) = a\,\delta(t) + b\,\delta'(t) + c\,\delta''(t) + d\,\delta'''(t).$$

$$Z' = \mathbb{A}\,Z + \mathbb{B}$$

“variation of parameters”: $Z = X\,Y$

$$X\,Y' = \mathbb{B}\,X$$

$$Y(t) = \int_0^t X^{-1}\,\mathbb{B}\,X$$

$$Z(1) = X(1)\underbrace{\int_0^1 X^{-1}\mathbb{B}(t) X\,dt}$$

$$Z(1) = X(1) \underbrace{\int_0^1 X^{-1} \mathbb{B}(t) X \, dt}$$

want this to cover $\begin{bmatrix} \beta & \gamma \\ \alpha & -\beta^* \end{bmatrix}$ β arbitrary

Integrating by parts:

$$\int_0^1 X^{-1} \mathbb{B}(t) X \, dt \approx$$

$$\approx X_\tau^{-1} \left\{ \begin{bmatrix} & \\ a & \end{bmatrix} + \begin{bmatrix} b & \\ & -b \end{bmatrix} + \begin{bmatrix} & -2c \\ -(Kc + cK) & \end{bmatrix} \right.$$

b is symmetric, not arbitrary.

$$\left. + \begin{bmatrix} -Kd - 3dK & \\ & 3Kd + dK \end{bmatrix} \right\} X_\tau$$

To solve

$$b - (Kd + 3dK) = \beta$$

b symm d symm β is arbitrary

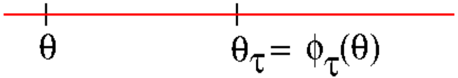
Is equivalent to solve

$$K\,e - e\,K = f$$

e sym f antisym

may not have solution unless K has no repeated eigenvalue.

A generic condition on the curvature



Theorem

$\exists \mathcal{G} \subset \mathcal{R}^\infty(M)$ residual s.t.
$\forall g \in \mathcal{G} \quad \forall \theta \in SM \quad \exists \tau \in [0, \frac{1}{2}]$ s.t. the Jacobi matrix

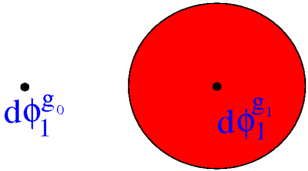
$$K_{ij}(\theta_\tau) = \langle R(\theta_\tau, e_i)\,\theta_\tau, e_j \rangle$$

has no repeated eigenvalue.

Why do we need this theorem and not just a preliminary perturbation?

Preliminary perturbation to separate the eigenvalues $\longrightarrow$ The Franks lemma depends on the amount of separation of the eigenvalues

$$g_0 \longmapsto g_1$$



Franks lemma on g_1

Strategy

Strategy: Use a transversality argument.

Know: can perturb the Jacobi matrix (curvature) at will.

$$\Sigma = \{\, A \in \mathrm{Sym}(n \times n) =: \mathcal{S}(n) \;\;:\;\; A \text{ has repeated eigenvalues } \,\}$$

it is an algebraic set with singularities.

$$A \in \Sigma \quad \Longleftrightarrow \quad \det p_A(A) = \prod (\lambda_i - \lambda_j)^2 = 0.$$

$$p_A(x) := \det \left[xI - A \right]$$

Enough to show that the geodesic vector field "crosses Σ transversally"

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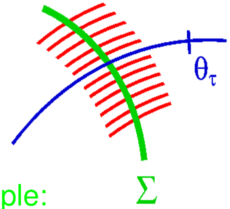

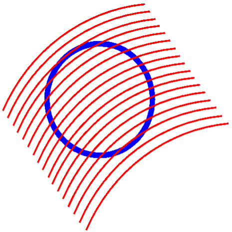

Example:

- Flow in $\mathbb{R}^2$ without singularities.
- $\Sigma = S^1$.
- Can ask that a chosen orbit segment is $\pitchfork$ S^1 but not all.
- Can ask that tangency is not of order 2.

If Σ where a smooth manifold:

$$J^k\Sigma = \text{ k-jets of curves inside } \Sigma.$$

$$\dim J^k\Sigma = (k+1)\ \dim\Sigma.$$

coefs of Taylor series in local chart $\quad t \mapsto a_0 + a_1 t + \cdots a_k t^k$
$a_i \in \mathbb{R}^\sigma, \quad \sigma = \dim\Sigma.$

$$\dim J^k\mathcal{S}(n) = (k+1)\ \dim\mathcal{S}(n)$$

$$\operatorname{codim}_{\mathcal{S}(n)}\Sigma = r \geq 1$$

$$\operatorname{codim}_{J^k\mathcal{S}(n)} J^k\Sigma = (k+1)\, r \longrightarrow \infty$$

when $k \to \infty$

$$\begin{aligned} F : \mathcal{R}^\infty(M) \times SM \times]0,1[&\longrightarrow J^k\mathcal{S}(n) \\ (g,\theta,\tau) &\longmapsto J^k_\tau K(g,\phi_\tau\theta) \end{aligned}$$

K = Jacobi matrix

J^k_τ = k-jet at t=τ

If $F \pitchfork J^k\Sigma$

$$\Longrightarrow \quad \exists \quad \text{residual } \mathcal{G} \subset \mathcal{R}^\infty(M) \text{ s.t.}$$
$$g \in \mathcal{G} \quad \Longrightarrow \quad F(g,\cdot,\cdot) \pitchfork J^k\Sigma.$$

K large $\Longrightarrow$ $\operatorname{codim} J^k\Sigma > \dim(SM\times]0,1[)$

$\pitchfork$ $\Longrightarrow$ no intersection.

$+$ compactness $\Longrightarrow$ required bounds on eigenvalues.

use

$$\min_{\theta\in SM} \max_{t\in[0,1]} \prod_{i\neq j} |\lambda_i - \lambda_j|^2 > 0. \quad \text{when } \pitchfork$$

But Σ has singularities.

Algebraic Jet space

$$\mathcal{L}_k(\Sigma) = \text{polynomies } a_0 + a_1\, t + \cdots + a_k\, t^k = p(t)$$
$$\text{s.t.} \qquad f \circ p(t) \equiv 0 \pmod{t^{k+1}}$$

Arc space

$$\mathcal{L}_\infty(\Sigma) = \text{formal power series } p(t) \quad \text{s.t.} \quad f \circ p \equiv 0.$$

$$\pi_k : \mathcal{L}_\infty(\sigma) \to \mathcal{L}_k(\Sigma) \qquad \text{truncation.}$$

$\mathcal{L}_k(\Sigma)$ is an algebraic variety.

$\pi_k(\mathcal{L}_\infty(\Sigma)) \subset \mathcal{L}_k(\sigma)$ is a finite union of algeraic subsets.

(it is "constructible")

$J^k\Sigma = k$-jets of C^∞ curves in Σ.

$\Longrightarrow \quad J^k\Sigma \subset \pi_k(\mathcal{L}_\infty(\Sigma)) \subset \mathcal{L}_k(\Sigma).$

Denef & Loeser:

$$\dim \pi_k(\mathcal{L}_\infty(\Sigma)) \leq (k+1)\ \dim \Sigma.$$

(same bound as in the smooth case).