KUPKA-SMALE thm for geodesic flows:

① Can make generic the k-Jets of Poincaré maps of closed geodesics:

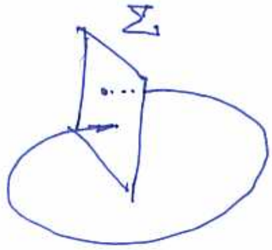
(a) Klingenberg-Takens for perturbations of each periodic orbit.

(b) Needs Anosov for the 1-Jet (to make periodic orbits of similar period isolated)

Formally: $J_s^k(n) := $ k-Jets of smooth symplectic maps $(\mathbb{R}^n, 0) \circlearrowleft$

$Q \subset J_s^k(n)$ is invariant iff

$\forall \sigma \in J_s^k(n)$, $\sigma Q \sigma^{-1} = Q$



OBS: Q invariant $\Rightarrow$ Property "Poincaré map $\in Q$" is independent of the section Σ.

Theorem:

If $Q \subset J_s^k(n)$ is open, dense & invariant.

$\Rightarrow \forall r \geq k+1 \; \exists \mathcal{G} \subset \mathcal{R}^k(M)$ residual s.t. $\forall g \in \mathcal{G}$.

① The Poincaré maps of all periodic orbits of ϕg are in Q

② All heteroclinic intersections are transversal.

OBS:
(a) Also holds for Q residual and invariant
(b) Donnay for $n=2$, Petroll $n \geq 2$ show how to perturb a single non-transverse intersection. But perhaps this is not enough.

② Transversality of invariant manifolds

(a) Hamilton-Jacobi Thm:

For a hamiltonian H if $\mathbb{L}$ is a Lagrangian submfld s.t. $H(\mathbb{L}) = \text{const.}$ $\Rightarrow$ $\mathbb{L}$ is invariant.

Γ. The ham. v.f. $X \in \mathbb{L}$ because if not it can be added to $T\mathbb{L}$ to make a bigger isotropic subspace $\Rightarrow\!\Leftarrow$

$$\ell \in T\mathbb{L} \ , \ \omega(X, \ell) = - dH(\ell) = 0$$

• W^s is Lagrangian in (T^*M, ω_0)

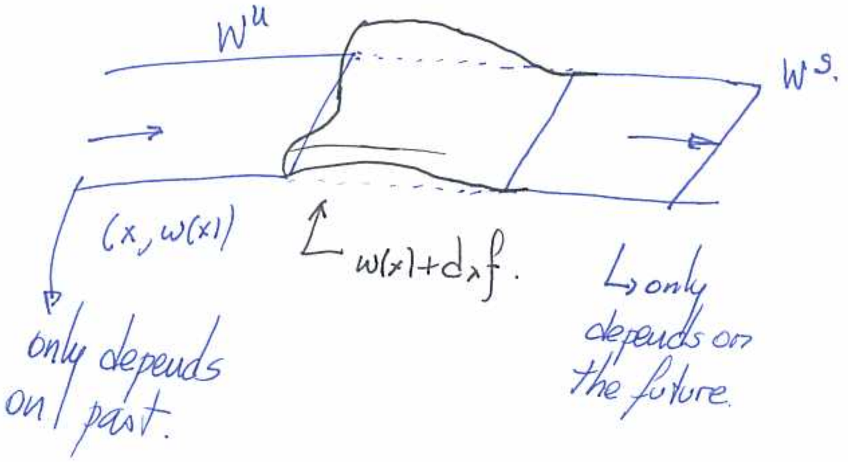
Γ $\underbrace{\omega_0(d\phi_s \cdot u \ , d\phi_s \cdot v)}_{0 \ \ s \to +\infty} = \omega_0(u, v)$ $\therefore$ isotropic + dim. $\perp$

•

- Choose a place where W^s is locally a lagrangian graph
- $\Rightarrow$ $W^s \approx \{(x,\eta(x)) \mid \eta$ closed 1-form on $U \subset M\}$.

- Deform to another lagrangian graph which is $\pitchfork W^u$ (by adding a $d_x f$).

$\omega_0 = dp \wedge dx$ fixed canonical symplectic form on T^*M.



- Change the metric s.t. $H(\text{new } W^s) = 1$

  recall $H(x;p) = \frac{1}{2}\sum p_i g^{ij}(x) p_j$

  $[g^{ij}(x)] = [g_{ij}(x)]^{-1}$ $\to$ riem. metric on TM

$\Rightarrow$ new W^s is invariant

Elliptic Closed Geodesics.

DEF:

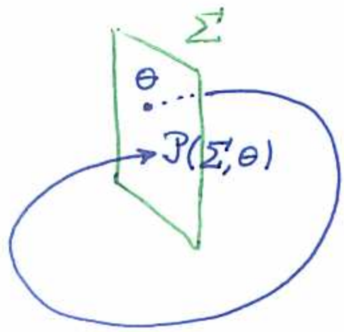
- A periodic orbit is *q-elliptic* iff its Poincaré map has $2q$ eigenvalues of modulus 1.
- ... is *elliptic* if it is q-elliptic for some $q \geq 1$

Want: $\boxed{\text{Elliptic} + \text{Kupka-Smale} \Longrightarrow \exists \text{ horseshoe.}}$

Elliptic = Normally hyperbolic twist map

Sup. θ is q-elliptic, $1 \leq q \leq n$, periodic pt.

$P = \mathcal{P}(\Sigma, \theta)$ linearized Poincaré map.



$T_\theta \Sigma = E^s \oplus E^c \oplus E^u$

P-invariant subspaces s.t.
$$\begin{cases} P|_{E^s} & \text{eigenvalues} & |\mu| < 1 \\ P|_{E^u} & \text{"} & |\lambda| > 1 \\ P|_{E^c} & \text{"} & |\rho| = 1 \end{cases}$$

$\exists$ invariant mflds W^s, W^u, W^c s.t. $T_\theta W^\alpha = E^\alpha$, $\alpha = s, u, c$.

symplectic form ω : $\omega|_{W^s} \equiv 0$, $\omega|_{W^u} \equiv 0$, $\omega|_{W^c}$ non-deg.

$\Rightarrow \mathcal{P}|_{W^c}$ is a symplectic map in nbhd of θ.

1 $\boxed{\begin{array}{l} 1\text{-elliptic} \\ + \text{ Kupka-Smale} \end{array} \Rightarrow \exists \text{ horseshoe}}$

$\mathcal{P}|_{W^c}$ is a K-S exact twist map.

$$\begin{tikzcd}
(x,y) \arrow[r] \arrow[rr, bend left, "G"] & (r,\theta) \arrow[r] & (\frac{1}{2}r^2,\theta) = (R,\theta) \\
\mathbb{D}^* \arrow[r, "\mathcal{P}"] \arrow[d, "f_0"'] & \mathbb{R}^+ \times S^1 \arrow[r] \arrow[d] & \mathbb{R}^+ \times S^1 \arrow[d, "T"] \\
\mathbb{D}^* \arrow[r] & \mathbb{R}^+ \times S^1 \arrow[r] & \mathbb{R}^+ \times S^1
\end{tikzcd}$$

$\mathbb{D}^* = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$

$G(x,y) = (\frac{1}{2}r^2, \theta) = (R,\theta)$

$G^*(R\,d\theta) = \frac{1}{2}(x\,dy - y\,dx) =: \lambda$

$x = r\cos\theta$ $\quad dx = \cos\theta\,dr - r\,\mathrm{sen}\,\theta\,d\theta$

$y = r\,\mathrm{sen}\,\theta$ $\quad dy = \mathrm{sen}\,\theta\,dr + r\cos\theta\,d\theta$

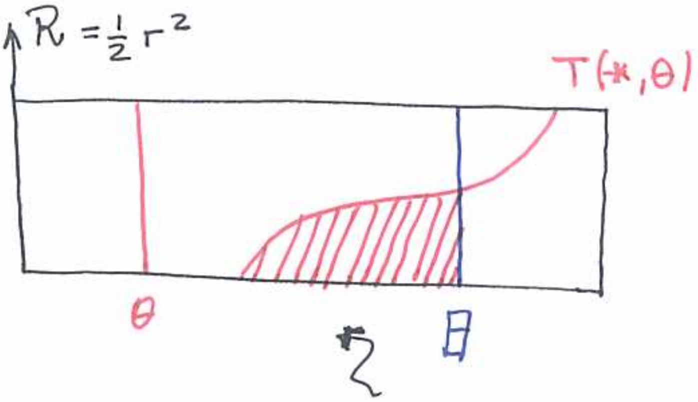
$$\begin{aligned} x\,dy - y\,dx &= r\cos\theta(\mathrm{sen}\,\theta\,dr + r\cos\theta\,d\theta) \\ &\quad - r\,\mathrm{sen}\,\theta(\cos\theta\,dr - r\,\mathrm{sen}\,\theta\,d\theta) \\ &= \dots = r^2\,d\theta \end{aligned}$$

$d\lambda = dx \wedge dy$ $\leftarrow$ area form in $\mathbb{D}$

$\mathbb{D}$ contractible $\Rightarrow f_0^*(\lambda) - \lambda$ exact.

$T^*(R\,d\theta) - R\,d\theta$ exact.

$\square$



<u>Action</u> $\quad A(\theta, \Theta) =$ area

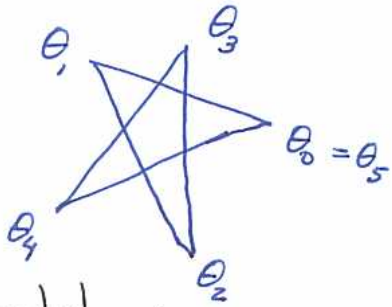
critical pts of action $\mathbb{A}(\theta_0, \dots, \theta_N) = \sum_{i=0}^{N-1} \mathbb{A}(\theta_i, \theta_{i+1})$
correspond to orbits of T by

$$(\theta_i, \theta_{i+1}) \longrightarrow T(*, \theta_i) \cap (*, \theta_{i+1})$$

For periodic sequences

$\mathbb{A}(\theta_0, \dots, \theta_N)$, $\theta_N = \theta_0$

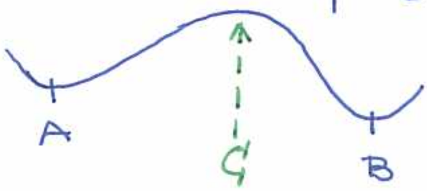
obtain:



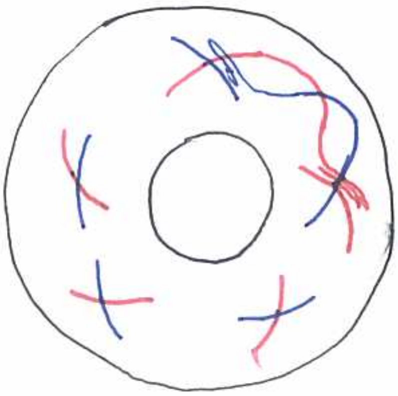
- <u>minimum</u> $\longrightarrow$ hyperbolic periodic orbit (MacKay).
- From $\underset{A}{(\theta_0, \dots, \theta_N)} \rightsquigarrow \underset{B}{(\theta_1, \dots, \theta_N, \theta_1)}$

B = 1/5 rotation of per. seq. of A

obtain mountain pass geometry



<u>minimax</u> $\longrightarrow$ elliptic periodic point



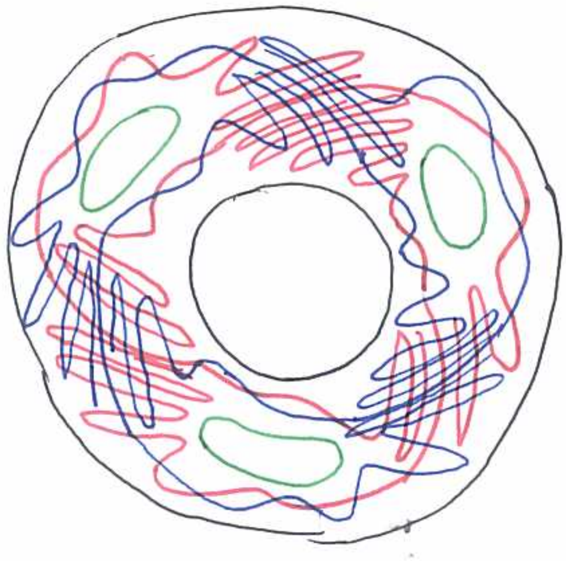
Obtain variationally a homoclinic

$\min \mathbb{A}(\varphi_{-N}, \dots, \varphi_N)$

$N \longrightarrow \infty$.

$\left.\begin{array}{l}\varphi_{-N}\\ \varphi_N\end{array}\right\}$ projections of minim. periodic orbit:

$\varphi_{-N} = \theta_0$, $\varphi_N = \theta_3$



Normally hyperbolic

$\Rightarrow$ hyperbolic orbits for the twist map are also hyperbolic on the ambient manifold Σ_i = Poincaré section

$\pitchfork$ homoclinics for the twist map are $\pitchfork$ homoclinics in ambient mfld.

(3)

q-Elliptic fixed point + Kupka-Smale	$\Rightarrow$	weakly monotonous exact twist map C^1 near totally integrable

$\mathbb{P} : (\mathbb{R}^{2n}, \theta) \circlearrowleft$ q-elliptic fixed pt. of symplectic map

$P = d_\theta \mathbb{P}$

$\rho_1, \dots, \rho_q ; \bar\rho_1, \dots, \bar\rho_q$ eigenvalues of P with modulus 1.

θ is 4-elementary if

$$1 \leq \sum_{i=1}^{q} |\nu_i| \leq 4 \Longrightarrow \prod_{i=1}^{q} \rho^{\nu_i} \neq 1$$

4-eltary $\Rightarrow$ Birkhoff normal form.

$\exists$ sympl. coords. $(x_1, \dots, x_q ; y_1, \dots, y_q)$ in W^c s.t.

$$\omega|_{W^c} = \sum dy_i \wedge dx_i$$

$\mathbb{P}|_{W^c}$ writes as $\mathbb{P}(x,y) = (X, Y)$

$$Z_k = e^{2\pi i \phi_k} z_k + g_k(z)$$

$$\phi_k(z) = a_k + \sum_{\ell=1}^{q} \beta_{k\ell} |z_\ell|^2$$

where $z = x + iy$; $Z = X + iY$; $\rho_i = e^{2\pi i a_k}$

$g(z) = g(x,y)$ has vanishing 3-Jet at θ

Say θ is weakly monotonous iff $\det \beta_{k\ell} \neq 0$

$\hookrightarrow$ (this prop. is indep. of choice of normal form).

In Birkhoff coords. we have

$$\begin{tikzcd}
(x,y) \arrow[r, Rightarrow] \arrow[rr, bend left, "Q"] & (\theta,\rho) \arrow[r] & (\theta, \rho^2/\epsilon) = (\theta, r) \\
\mathbb{D}^* \arrow[r, "\mathbb{P}"] \arrow[d, "f"'] & \mathbb{T}^q \times \mathbb{R}^q_+ \arrow[r, "R"] & \mathbb{T}^q \times \mathbb{R}^q_+ \arrow[d, "F_\epsilon"] \\
\mathbb{D}^* \arrow[r, "\mathbb{P}"] & \mathbb{T}^q \times \mathbb{R}^q_+ \arrow[r] & \mathbb{T}^q \times \mathbb{R}^q_+
\end{tikzcd}$$

To preserve usual area form in $\mathbb{T}^q \times \mathbb{R}^q_+$

$\mathbb{D}^* = \{ (x,y) \in \mathbb{R}^q \times \mathbb{R}^q \mid 0 < |x_i|^2 + |y_i|^2 < 1 \}$

$f = \mathbb{P}|_{W^u}$ in Birkhoff coords.

$\mathbb{T}^q = \mathbb{R}^q / \mathbb{Z}^q$

$\mathbb{P}^{-1}$ is $x_i = \rho_i \cos(2\pi\theta_i)$, $y_i = \rho_i \sin(2\pi\theta_i)$

f preserves $\omega = dx \wedge dy$

$Q = R \circ \mathbb{P}$ $\qquad Q(x,y) = (\theta, r) \qquad r_i = \rho_i^2/\epsilon$

$Q^*(r\,d\theta) = \frac{1}{2\pi\epsilon}(x\,dy - y\,dx) =: \lambda_\epsilon$

$\mathbb{D}$ simply connected $\Rightarrow f^*(\lambda_\epsilon) - \lambda_\epsilon$ exact

$\Rightarrow F_\epsilon^*(r\,d\theta) - r\,d\theta$ exact.

$G_\epsilon(\theta, r) = (\theta + \alpha + \epsilon\beta r, r)$

↳ 1st term in Birkhoff normal form is

- symplectic diffeo
- "totally integrable"
- weakly monotonous $(\det\beta \neq 0)$
- C^1 near F_ϵ.

Will prove

2 F (weak) Twist map + exact + C^1 near completely integrable $\Big\} \Longrightarrow$ F has a 1-elliptic periodic orbit

3 q-Elliptic fixed point + Kupka-Smale $\Big\} \Longrightarrow$ Twist map with conditions 2 $\mathbb{T}^q \times \mathbb{R}^q \circlearrowleft$ x normally hyp.

SYMPLECTIC TWIST MAPS ON $\mathbb{T}^n \times \mathbb{R}^n$

Uses techiques from M.C. Arnaud & M. Herman

$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$

Liouville 1-form $\lambda = r\,d\theta = \sum r_i\,d\theta_i$, $(\theta, r) \in \mathbb{T}^n \times \mathbb{R}^n$

$\omega := d\lambda$ symplectic form

$F: \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$ is symplectic iff $F^*\omega = \omega$

⌜ In coords. $T(\mathbb{T}^n \times \mathbb{R}^n) \simeq \mathbb{R}^n \times \mathbb{R}^n$

$\omega(x, y) = x^* J y \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

F symplectic $\iff (dF)^* J (dF) = J$ ⌟

F exact symplectic iff $F^*\lambda - \lambda$ exact form.

F weakly monotonous iff writing $F(\theta, r) = (\Theta, R)$

$$\det \frac{\partial \Theta}{\partial r} \neq 0$$

Torsion: $b := \dfrac{\partial \Theta}{\partial r}$ not neces. symmetric.

is "positive definite, neg. def., signature (p,q)" if $b + b^*$ is pos. def, neg. def., sign. (p,q)

$$\text{signature } (p,q) = \begin{cases} p \text{ negative e.vs} & - \\ q \text{ positive e.v.} & + \\ n-(p+q) \text{ zero e.v.} & 0. \end{cases}$$

$G : \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$ C^1 diffeo is completely integrable iff

$G(\theta, r) = (\theta + \beta(r), r)$ some $\beta \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, $\beta(0) = 0$.

PROPERTIES

① $\left.\begin{array}{l} G \text{ completely integrable} \\ + \text{ symplectic} \end{array}\right\} \Rightarrow$ its torsion $\dfrac{\partial \beta}{\partial r}$ is symmetric.

② ... $\Rightarrow G^*\lambda - \lambda = r\, d\beta$ is exact

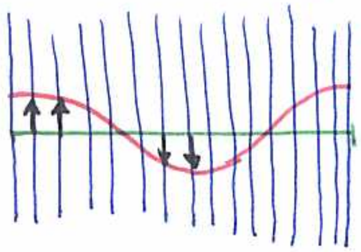
because it is closed form in $\mathbb{R}^n$.

Fixed points in a nearly integrable twist map.

$F : \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$ weakly monotonous, exact symplectic C^r diffeo, $r \geq 1$, C^1 near to a totally integrable map G.

- For tot. integ. G: zero section $= \mathbb{T}^n \times \{0\} \subset \mathrm{Fix}(G)$.
- Look for fixed pts for F

(1) Construct radially transformed torus



$\mathcal{Z} = \mathrm{Graph}(\eta)$ solving

$$F(\theta, \eta(\theta)) = (\theta, *)$$

⌈ Using implicit funct thm in

$$\Theta(\theta, \eta(\theta), F) = \theta$$

where $F(\theta, r) = (\Theta, R)$, continuing solution $\eta_G \equiv 0$ for G

by weak. monot. cond. $\det\left[\frac{\partial \Theta}{\partial r}\right] \neq 0$

$F \in C^r \Rightarrow \eta \in C^r$.

⌈ want $\mathcal{H}(\eta, F) \equiv 0$ where $\mathcal{H}(\eta, F)(\theta) = \Theta(\theta, \eta(\theta), F) - \theta$

need $\frac{\partial \mathcal{H}}{\partial \eta}$ non-singular : $\frac{\partial \mathcal{H}}{\partial \eta} = \frac{\partial \Theta}{\partial r}$ i.e.

$$\frac{\partial \mathcal{H}}{\partial \eta}(\theta) \cdot \zeta(\theta) = \frac{\partial \Theta}{\partial r}(\theta, \eta(\theta), F) \circ \zeta(\theta) = h(\theta)$$

⌋

2 F exact symplectic $\Rightarrow \exists$ generating function $S : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$

$$dS = F^*\lambda - \lambda = R\,d\Theta - r\,d\theta$$

on radially transformed torus $\mathcal{Z}$:

$$dS|_{\mathcal{Z}} = (R - r)\,d\theta$$

$$\therefore \quad \mathrm{Fix}(F|_{\mathcal{Z}}) \subset \mathrm{Crit}(S|_{\mathcal{Z}})$$

Define radial function $\varphi = \mathcal{L}(F) : \mathbb{T}^n \longrightarrow \mathbb{R}$

$$\varphi(\theta) := S(\theta, \eta(\theta))$$

- $\varphi \in C^1 \Longrightarrow \# \mathrm{Fix}(F) \geq n+1 = \text{cup length}(\mathbb{T}^n)$
- φ Morse function $\Longrightarrow \# \mathrm{Fix}(F) \geq 2^n$.

KURKA SMALE

$Q \subset J^3_s(n)$ conditions on 3-jet of ell. per. pt.

(i) $\neq$ eigenvalues

(ii) 4-eltary condition $1 \leq \sum_{i=1}^{q} |\nu_i| \leq 4 \Rightarrow \prod_{i=1}^{q} \rho_i^{\nu_i} \neq 1$

(iii) Birkhoff normal form is weakly monotonous. ρ_i, $i = 1, \dots, q$ e.v. with $|\rho_i| = 1$.

DEF: $F : \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$ is kupka-Smale iff

(i) $z \in \mathrm{Per}(F)$, $\mathrm{per}(z, F) = m \Longrightarrow DF^m(z) \in Q$.

(ii) All heteroclinic intersections are $\pitchfork$.

A. Lemma [M. Herman]

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \quad \text{symplectic matrix}$$

$a, b, c, d \in \mathbb{R}^n$, $\det(b) \neq 0$

For $\lambda \in \mathbb{C}$, let

$$M_\lambda := b^{-1}a + db^{-1} - \lambda b^{-1} - \lambda^{-1}(b^{-1})^*$$

$$\Rightarrow \quad \operatorname{rank}(\lambda I - M) = n + \operatorname{rank} M_\lambda$$

In particular

$$\lambda \text{ eigenvalue of } M \iff \det M_\lambda = 0.$$

B. Lemma: $\varphi = L(F)$ radial funct on radially transf. Torus $\mathcal{Z}$.

$(\theta, \eta(\theta)) \in \operatorname{Fix}(F) \cap \mathcal{Z}$, $M = DF(\theta, \eta(\theta))$

$$\Rightarrow \quad M_\lambda = D^2\varphi(\theta) + (1-\lambda)b^{-1} + (1-\lambda^{-1})(b^{-1})^*$$

If fixed pts of F non-deg. $\Rightarrow$ $\varphi = L(F)$ is Morse funct.

C. Lemma

$z \in \mathcal{Z} \cap \operatorname{Fix}(F)$

$a_n z^n + \dots + a_0 =: P(z).$

$\Rightarrow \exists$ Poly $P \in \mathbb{R}[x]$ of $\deg P = n$ s.t.

λ eig.val of $DF(z) \iff P(2 - \lambda - \lambda^{-1}) = 0$

- Leading coef. of $P = a_n = \det b^{-1}$,

  $b = \frac{\partial \Theta}{\partial r}$ the torsion

- Independent term of $P = a_0 = \det D^2\varphi(\theta)$.

Theorem

$F : \mathbb{T}^n \times \mathbb{R}^n \circlearrowleft$ C^4 Kupka-Smale, weakly monotonous exact symplectic diffeomorphism

$+$ C^1-near to a symplectic compl. integr. diffeo G

$\Longrightarrow$ F has a 1-elliptic periodic point near $\mathbb{T}^n \times 0$

Proof:

$n \geq 2$

claim I: F as above $\Rightarrow \exists\, z_0 \in \mathrm{Fix}(F)$ elliptic $\times$ hyperbolic

i.e. q_0-elliptic $\quad 1 \leq q_0 < n$

with this

$$F|_{\text{near } z_0} \xrightarrow[\text{normal form}]{\text{Birkhoff}} F_{q_0} : \mathbb{T}^{q_0} \times \mathbb{R}^{q_0} \circlearrowleft \text{ twist map. satisf hypot}$$

$$F = F_{q_0} \times \text{normally hyperbolic}$$

$\Longrightarrow$ F_{q_0} has z_1 : q_1-elliptic fixed pt.

$1 \leq q_1 < q_0$

$\Rightarrow \ldots \Rightarrow n > q_0 > q_1 > \ldots > q_m = 1$

$\blacksquare$ claim I

G completely integrable $\quad G \underset{C^1\text{-near}}{\frown} F$

$G(\theta, r) = (\theta + \beta(r), r)$

$\left.\begin{array}{l} G \text{ symplectic} \\ + \text{compl. int.} \end{array}\right\} \Rightarrow$ its torsion $b_0 = \frac{\partial \beta}{\partial r}$ is symmetric

$(q, n-q) := $ signature of b_0^{-1}

Claim II

torsion of F "radial function"

$\boxed{A}$ $z \in \mathcal{Z} \cap \mathrm{Fix}(F)$ & $(-1)^n (\det b) \det D^2\varphi(z) < 0$

$\Rightarrow DF(z)$ has a hyperbolic eigenvalue.

$\boxed{B}$ For any $0 \le p \le n$ $\exists z \in \mathcal{Z} \cap \mathrm{Fix}(F)$ s.t.

- $D^2\varphi(z)$ has signature $(p, n-p)$
- $DF(z)$ has at least $2|p-q|$ eigenvalues of modulus 1.

Now take $\sigma := \mathrm{sgn}\left[(-1)^n \det b\right]$

$\mathrm{sgn}\left[(-1)^n (\det b) \det D^2\varphi(z)\right] = \sigma (-1)^p$

$\boxed{1}$ If $\sigma < 0$ want p even and $|q-p| \ge 1$

$q \neq 0$ take $p = 0$

$q = 0$ take $p = 2$ $\leftarrow$ (because $n \ge 2$).

$\boxed{2}$ If $\sigma > 0$ want p odd

$q \neq 1$ take $p = 1$

$q = 1$ & $n \ge 3$ take $p = 3$

[$\sigma > 0$, $q = 1$, $n = 2$, take $p = 1$ (special case).

$\omega_i = 2 - \lambda_i - \lambda_i^{-1}$

$\sigma(-1)^p = \omega_1 \omega_2 < 0 \Rightarrow \omega_1, \omega_2 \in \mathbb{R}$ otherwise complex conjugate

$\omega_1 < 0$, $\omega_2 > 0$

hyp. $\quad \hookrightarrow \omega_2 \in [0, 4] \Rightarrow$ elliptic.

because close to compl. integ.

$\boxtimes$ Thm.

$\lambda = 1 \iff \omega = 0$

$\lambda \in S^1 \iff \omega \in [0,4]$

$\lambda \in \mathbb{R} \iff \omega \in \mathbb{R} \setminus [0,4]$

$\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup S^1) \iff \omega \in \mathbb{C} \setminus \mathbb{R}$

Proof of claim II

Write $\omega = 2 - \lambda - \lambda^{-1}$

(1) Compl. integ. G has all e.v. $\lambda = 1$, $\omega = 0$.

F C^1 near G $\Longrightarrow$ can assume all $|\omega| < 4$

(2) $z \in Z \cap \mathrm{Fix}(F)$, $\lambda_1, \lambda_1^{-1}, \dots, \lambda_n, \lambda_n^{-1}$ = e.v. of $DF(z)$

$\omega_i = 2 - \lambda_i - \lambda_i^{-1}$

$\star$ $(-1)^n (\det b) \det D^2\varphi(z) = \omega_1 \cdots \omega_n$

$\omega_i \in \mathbb{C} \setminus \mathbb{R} \Rightarrow \bar{\omega}_i$ also appears and $\omega_i \bar{\omega}_i = |\omega_i|^2 > 0$

$\therefore$ $\star < 0 \Rightarrow 2$ real hyperbolic e.v.

claim A.

(3) For G, $\varphi_G \equiv 0 \Rightarrow$ can assume $D^2\varphi_F$ near 0

Γ $D\varphi(\theta) = dS|_Z = R(\theta, \eta(\theta)) - \eta(\theta)$

$D^2\varphi(\theta) = M_{\lambda=1}$ = in terms of $DF(\theta)_\perp$.

$\therefore$ can assume

$D^2\varphi(\theta) + 2[b^{-1} + (b^{-1})^*]$ has same signature as

$[b^{-1} + (b^{-1})^*]$ $\quad b = \frac{\partial H}{\partial r}$ torsion of F.

(4) Since φ Morse function on $\mathbb{T}^n$ $\quad$ ($\Leftarrow$ (Lemma C) $DF(z)$ non-deg $\forall z \in \mathcal{Z} \cap \mathrm{Fix}(F)$)

$\forall\ 0 \leq p \leq n \quad \exists \binom{n}{p}$ crit. pts θ where $\operatorname{sgn} D^2\varphi(\theta) = (p, n-p)$.

$[0,\pi] \ni \alpha \xmapsto{N} M_{e^{i\alpha}} \qquad M_\lambda$ from Lemma B.

$N(\alpha) = M_{e^{i\alpha}} = D^2\varphi(\theta) + (1-e^{i\alpha})\, b^{-1} + (1-e^{-i\alpha})\,(b^{-1})^*$

hermitian $\Rightarrow$ real e.v.

$N(0) = M_{\lambda=1} = D^2\varphi(\theta) \longrightarrow$ signature $(p, n-p)$

$N(\pi) = M_{\lambda=-1} = D^2\varphi(\theta) + 2[b^{-1} + (b^{-1})^*] \longrightarrow$ sign. $(q, n-q)$.

$\therefore$ $|p-q|$ values of $\lambda = e^{i\alpha}$, $\alpha \in [0,\pi]$

where $\det M_\lambda = 0$ (counting multip.)

$\underset{\text{Lemma A}}{\Longrightarrow}$ $DF(z)$ has $\geq 2|p-q|$ eigenvalues in S^1

(considering cx conj. $\bar{\lambda} = e^{-i\alpha}$, $-\alpha \in [-\pi, 0]$).

1. LEMMA BOUNDED STABLE PART

$\{\xi^\alpha\}_{\alpha\in\mathcal{a}}$ Bdd stably hyperbolic

$\Rightarrow \exists\, \epsilon>0,\ \exists\, K>0$ s.t.

$\{\eta^\alpha\}_{\alpha\in\mathcal{a}}$ periodically equiv. family

$d(\eta,\xi)<\epsilon \Rightarrow \forall \alpha\in\mathcal{a}\quad \forall i\in\mathbb{Z}$

$$\Big\| \prod_{j=0}^{m-1} \eta^\alpha_{i+j}\Big|_{E^s(\eta^\alpha)} \Big\| < K, \quad m=\mathrm{Per}(\eta^\alpha).$$

2 LEMMA CONTRACTING STABLE PART AT PERIOD

$\{\xi^\alpha\}_{\alpha\in\mathcal{a}}$ Bdd stably hyperbolic

$\exists\, \epsilon>0 \quad K>0 \quad 0<\lambda<1$ s.t.

$\{\eta^\alpha\}_{\alpha\in\mathcal{a}}$ Per. equiv fam. $d(\eta,\xi)\le\epsilon$

$\Rightarrow$

$$\forall\alpha\in\mathcal{a}\quad \forall i\in\mathbb{Z}\quad \Big\| \prod_{j=0}^{m-1} \eta^\alpha_{i+j}\Big|_{E^s_i(\eta^\alpha)} \Big\| < K\lambda^m$$

$m=\mathrm{Per}(\eta^\alpha)$.

3. LEMMA (BOUNDED ANGLE)

$\{\xi^\alpha\}_{\alpha\in a}$ Bdd stably hyperbolic

$\exists\, \epsilon>0 \quad \exists\, \gamma>0 \quad \exists\, N_0\in\mathbb{Z}^+$

$\{\eta^\alpha\}_{\alpha\in a}$ per. equiv. fam. $d(\eta,\xi)\le\epsilon$

$\Rightarrow \quad \measuredangle\left(E^s_i(\eta^\alpha), E^u_i(\eta^\alpha)\right) > \gamma$

$\forall i\in\mathbb{Z} \quad \forall \alpha\in a$ with minimal period $> N_0$.

SKETCH OF PROOF OF THM

$$\boxed{\left.\begin{array}{c}\text{STABLE HYPERBOLIC}\\ \text{FAMILY}\\ +\\ \text{SYMPLECTIC.}\end{array}\right\}\Rightarrow \text{HYPERBOLICITY.}}$$

① ONCE WE KNOW THE ANGLES ARE UNIFORMLY BOUNDED BELOW FOR ANY PERTURBATION, WE CAN ASSUME THAT E^s AND E^u ARE ORTHOGONAL.

i.e. the metric adapted to splitting $E^s\oplus E^u$:

$$[\![\textstyle\sum x_i e_i + y_i f_i]\!] = \sum |x_i|^2 + \sum |y_i|^2$$

$\{e_i\}$ basis E^s , $\{f_i\}$ basis E^u

Norms of matrices in $[\![\ ]\!]$ are comparable to the norms on usual metric.

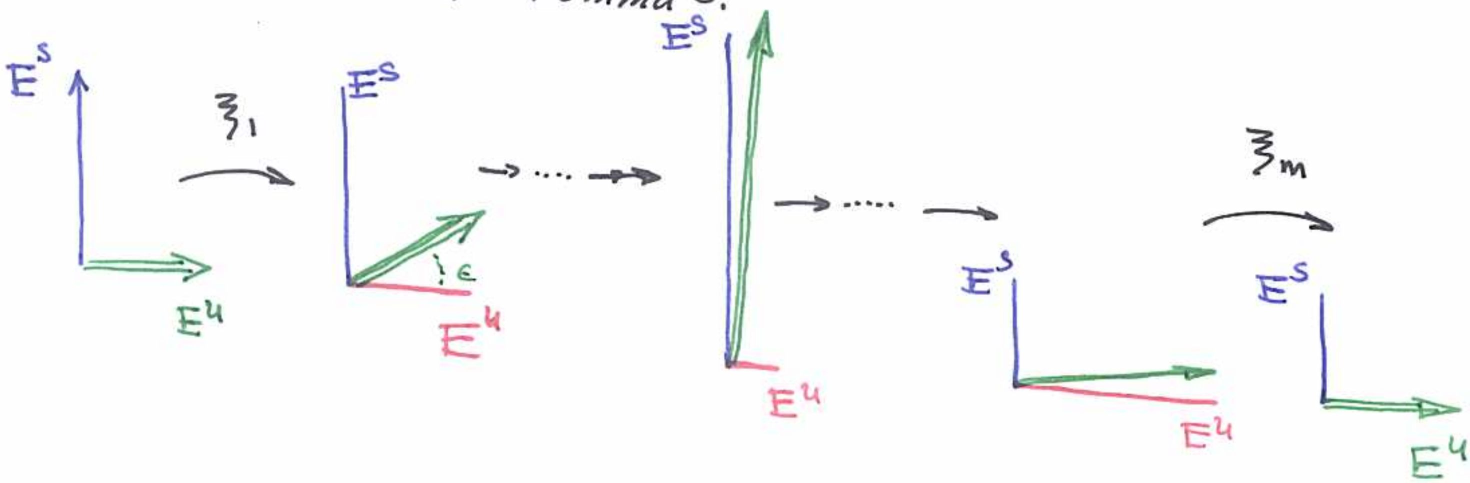
② If a seq. does not uniformly contract E^s

$$\left[\text{say} \quad \left\| \prod_{i=1}^{k} \xi_i \Big|_{E^s} \right\| \geq \frac{1}{2} \right]$$

multiply its stable component by $(1+\epsilon)^m$
unstable " " $(1+\epsilon)^{-m}$

so that at some iterate, say k,
it expands E^s & contracts E^u.

Then the following perturbation of only the maps ξ_1 and ξ_m, $m = \mathrm{Per}(\xi^\alpha)$, obtains a small angle $\measuredangle(E^s, E^u)$ at the k-th iterate.
This contradicts Lemma 3.



Arc Spaces

X = algebraic variety on $\mathbb{R}^N$ (= zeroes of polynomial eqs).

Path space on X

$$\mathcal{C}(X) := \left\{ (a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{R}^N \;\middle|\; \begin{array}{l} \exists \gamma \in C^\infty(\mathbb{R},\mathbb{R}^N) \\ \gamma(\mathbb{R}) \subset X \\ \frac{1}{n!}\gamma^{(n)}(0) = a_n \quad \forall n\in\mathbb{N} \end{array} \right\}$$

$F = (f_1, \dots, f_q)$ generators of the ideal $I(X) = \{ f \in \mathbb{R}[x_1, \dots, x_N] \mid f|_X \equiv 0 \}$

Arc space $\mathcal{L}(X)$:

$$\mathcal{L}_\bullet(X) := \left\{ (a_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}}^{\bullet} \mathbb{R}^N \;\middle|\; F\left(\sum_{k=0}^{\infty} a_k t^k\right) \equiv 0 \right\}$$

where $\equiv$ is equality as formal power series.

$\mathcal{L}_n(X)$

$\dim_{\mathbb{R}} (X \cap \mathbb{R}) \leq$

$\mathcal{L}_n(X)$ is an algebraic variety

$\pi : \mathcal{L}(X) \longrightarrow \mathcal{L}_n(X)$ the projection $(a_k)_{k\in\mathbb{N}} \longrightarrow (a_k)_{k=0}^{n}$

$\pi_n(\mathcal{L}(X))$ is a constructible set in $\mathcal{L}_n(X)$

(finite union of algebraic sets)

$\overline{\pi_n(\mathcal{C}(X))}$:= Zariski closure of $\pi_n(\mathcal{C}(X))$ = minimal alg var. containing

PROPOSITION

(a) $\dim \overline{\pi_n(\mathcal{C}(X))} \leq (n+1)\dim X$

(b) The fibers of $\pi_{n+1}(\mathcal{C}(X)) \longrightarrow \pi_n(\mathcal{C}(X))$ have dimension $\leq \dim X$.

PROOF

(1) Enough to prove for an algebraic variety X in $\mathbb{C}^N$
[Because $\dim_{\mathbb{R}}(X \cap \mathbb{R}) \leq \dim_{\mathbb{C}} X$]

(2) Obs: (b) $\Rightarrow$ (a)

(3) Fix $\overline{a} = (a_0, \ldots, a_n) \in \pi_n(\mathcal{C}(X))$

$$Z_{n+1} := \{(t,x) \in \mathbb{C} \times \mathbb{C}^N \,/\, F(a_0 + \cdots + a_n t^n + t^{n+1} x) = 0\}$$

For $t \in \mathbb{C}$ let

$$Z_{n+1}(t) := \{x \in \mathbb{C}^N \,/\, (t,x) \in Z_{n+1}\}$$

The limit W_{n+1} at $t=0$ of the 1-parameter family of varieties $Z_{n+1}(t)$ exists. ["1-dim. families are flat" alg. closed field.]

$W_{n+1} := \lim_{t\to 0} Z_{n+1}(t)$

i.e. if $Z^*_{n+1} := \{(t,x) \in Z_{n+1} \mid t \neq 0\}$
then $Z^*_{n+1} \cup W_{n+1}$ is the Zariski closure of Z^*_{n+1}.

$\mathbb{F}_{\bar{a}} := \Theta_n^{-1}(a)$ fiber of $\Theta_n : \pi_{n+1}(\mathcal{G}(X)) \longrightarrow \pi_n(\mathcal{G}(X))$ over $\bar{a}$

Claim 1: $\mathbb{F}_{\bar{a}} \subset W_{n+1}$

$\Vdash$ Let $a_{n+1} \in \mathbb{F}_{\bar{a}}$

Since $(a_0, \ldots, a_n, a_{n+1}) \in \pi_{n+1}(\mathcal{G}(X))$

$\exists \gamma \in C^\infty(\mathbb{R}, \mathbb{R}^N)$ s.t. $F \circ \gamma \equiv 0$

$\gamma(t) = a_0 + \cdots + a_n t^n + a_{n+1} t^{n+1} + \theta(t^{n+2}), \quad t \in \mathbb{R}$

Let $x_t := \frac{1}{t^n}\left[\gamma(t) - \sum_{k=0}^{n} a_k t^k\right] = a_{n+1} + \theta(t) \in Z_{n+1}(t) \subset \mathbb{C}^N$

This implies $a_{n+1} \in W_{n+1}$ //

Claim 2: $\dim W_{n+1} \leq \dim X$

$\boxed{\text{claim 2} \Rightarrow \text{PROPO}}$

$\Vdash$

① For $t \neq 0$, variety $Z_{n+1}(t) \overset{\text{iso}}{\approx} X$ by

invertible change of variables

$Z_{n+1}(t) \ni z \longleftrightarrow x \in X$

$x = a_0 + a_1 t + \cdots + a_n t^n + t^{n+1} z$

$\therefore \dim Z_{n+1}(t) = \dim X$ when $t \neq 0$.

② Need projective varieties

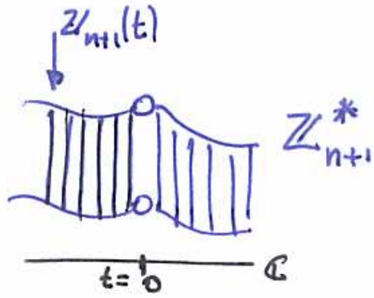
consider $\mathbb{C}^N = \mathbb{C}^N \times \{1\} \subset \mathbb{CP}^N \subset \mathbb{C}^N \cup \mathbb{CP}^{N-1}$

and corresp. varieties:

$$\mathbb{Z}_{n+1}(t), \quad \mathbb{Z}^*_{n+1} = \bigcup_{t \neq 0} \mathbb{Z}_{n+1}(t)$$

$$\mathbb{W} = \lim_{t \to 0} \mathbb{Z}_{n+1}(t)$$

Then $W_{n+1} = \mathbb{W}_{n+1} \cap \mathbb{C}^N$.



- Since for generic fiber $t \neq 0$ $\dim \mathbb{Z}_{n+1}(t) = \dim X$

we have $\dim \mathbb{Z}^*_{n+1} = \dim X + 1$

If $\dim W_{n+1} > \dim X$

$\Rightarrow \dim \mathbb{W}_{n+1} \geq \dim W_{n+1} \geq \dim X + 1$

$\Rightarrow \mathbb{W}_{n+1}$ contains an irreducible component of $\overline{\mathbb{Z}^*_{n+1}}$

this is incompatible with $\mathbb{W}_{n+1} = \lim_{t \to 0} \mathbb{Z}_{n+1}(t)$

$\square$