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On the creation of conjugate points for Hamiltonian systems

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Abstract. For a fixed Hamiltonian H on the cotangent bundle of a compact manifold M and a fixed energy level e, we prove that the set $\mathcal{A}_e$ of potentials ϕ on M such that the Hamiltonian flow of $H+\phi$ is Anosov, is the interior in the C^2 topology of the set $\mathcal{B}_e$ of potentials such that the flow has no conjugate points.

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1. Introduction

Let M be a closed connected Riemannian manifold and T^*M its cotangent bundle. By a Hamiltonian on T^*M we shall mean a C^∞ function $H : T^*M \to \mathbb{R}$ satisfying the following conditions.

(a) *Convexity:* for all $q \in M$, $p \in T_q^*M$, the Hessian matrix $(\partial^2 H/\partial p_i\, \partial p_j)(q, p)$ (calculated with respect to linear coordinates on T_q^*M) is positive definite.

(b) *Superlinearity:*

$$\lim_{|p|\to\infty} \frac{H(q, p)}{|p|} = +\infty\,.$$

The *Hamiltonian equation* for H is defined as

$$q' = H_p \qquad p' = -H_q \tag{1}$$

where H_q and H_p are the partial derivatives with respect to q and p. Observe that the Hamiltonian function H is constant along the solutions of (1). Its level sets $\Sigma_e = H^{-1}(e)$ are called *energy levels* of H. Then the compactness of M and the superlinearity hypothesis imply that the energy levels are compact. Since the Hamiltonian vector field (1) is Lipschitz, the solutions of (1) are defined on all $\mathbb{R}$. Denote by ψ_t the corresponding *Hamiltonian flow* on T^*M.

Let $\pi : T^*M \to M$ be the canonical projection and define the *vertical subspace* on $\theta \in T^*M$ by $\mathbb{V}(\theta) = \ker(\mathrm{d}\pi(\theta))$. Two points θ_1, $\theta_2 \in T^*M$ are said to be *conjugate* if $\theta_2 = \psi_\tau(\theta_1)$ for some $\tau \neq 0$ and $\mathrm{d}\psi_\tau(\mathbb{V}(\theta_1)) \cap \mathbb{V}(\theta_2) \neq \{0\}$.

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We recall that a smooth flow ψ_t generated by the vector field X on a compact manifold N, is called *Anosov* if there exist continuous invariant sub-bundles E^u, E^s of TN and constants $c > 0$, $a > 0$, such that

(a) $TN = E^s \oplus E^u \oplus \mathbb{R}X$

(b) $|\mathrm{d}\psi_t(v^s)| \leqslant ce^{-at}|v^s|$ for all $t > 0$, $v^s \in E^s$

$|\mathrm{d}\psi_t(v^u)| \leqslant ce^{at}|v^u|$ for all $t < 0$, $v^u \in E^u$.

For $e \in \mathbb{R}$, let $\mathcal{A}_e$ be the set of $\phi \in C^\infty(M)$ such that the flow of $H + \phi$ is Anosov in $(H + \phi)^{-1}(e)$ and let $\mathcal{B}_e$ be the set of $\phi \in C^\infty(M)$ such that $(H + \phi)^{-1}(e)$ contains no conjugate points. As is well known $\mathcal{A}_e$ is open in C^k topology and $\mathcal{B}_e$ is closed. On the other hand Paternain and Paternain [Pa] have shown that $\mathcal{A}_e$ is contained in $\mathcal{B}_e$. They also proved that if the flow is Anosov on the energy level Σ_e, then $e > \max\{H(q, 0) : q \in M\}$ and Σ_e is diffeomorphic to the unitary sphere bundle in T^*M. In the appendix of [An] Margulis proved that if a compact 3-manifold admits an Anosov flow, then its fundamental group has exponential growth. Therefore there is no Hamiltonian on T^*T^2 with Anosov energy levels. On the other hand the geodesic flow for the plane metric on T^2 has no conjugate points anywhere. Of course there is plenty of Hamiltonians with Anosov energy levels if M has a metric with negative curvature. The purpose of this paper is to prove the following

Theorem 1. *The interior of $\mathcal{B}_e$ in the C^2 topology is $\mathcal{A}_e$.*

This theorem is an extension to the Hamiltonian setting of a result of Ruggiero [Ru] for the geodesic flow. Our main tools are the Green bundles, theorem 2 below describing $\mathcal{A}_e$ inside $\mathcal{B}_e$, and the index form in the Lagrangian setting.

2. Green bundles and hyperbolicity

The Green bundles were constructed for the geodesic flow by Green in [Gr] using the Jacobi equation. For Hamiltonian systems they were constructed in [C-I], where proofs for the statements in this section can be found.

Suppose that the orbit of $\theta \in T^*M$ does not contain conjugate points and $H(\theta) = e$ is a regular value of H. Then there exist two $\mathrm{d}\psi_t$-invariant Lagrangian sub-bundles $\mathbb{E}$, $\mathbb{F} \subset T(T^*M)$ along the orbit of θ given by

$$\mathbb{E}(\theta) = \lim_{t\to+\infty} \mathrm{d}\psi_{-t}(\mathbb{V}(\psi_t(\theta)))$$
$$\mathbb{F}(\theta) = \lim_{t\to+\infty} \mathrm{d}\psi_{t}(\mathbb{V}(\psi_{-t}(\theta))).$$

Moreover, $\mathbb{E}(\theta) \cup \mathbb{F}(\theta) \subset T_\theta\Sigma$, $\mathbb{E}(\theta) \cap \mathbb{V}(\theta) = \mathbb{F}(\theta) \cap \mathbb{V}(\theta) = \{0\}$, $X(\theta) \in \mathbb{E}(\theta) \cap \mathbb{F}(\theta)$ and $\dim \mathbb{E}(\theta) = \dim \mathbb{F}(\theta) = \dim M$, where $X(\theta) = (H_p, -H_q)$ is the Hamiltonian vector field and $\Sigma = H^{-1}\{e\}$.

Fix a Riemannian metric on M and the corresponding induced metric on T^*M. Then $T_\theta T^*M$ splits as a direct sum of two Lagrangian subspaces: the vertical subspace $\mathbb{V}(\theta) = \ker(\mathrm{d}\pi(\theta))$ and the horizontal subspace $\mathbb{H}(\theta)$ given by the kernel of the connection map. Using the isomorphism $K : T_\theta T^*M \to T_{\pi(\theta)}M \times T^*_{\pi(\theta)}M$, $\xi \mapsto (\mathrm{d}\pi(\theta)\xi, \nabla_{\pi\xi}\theta)$, we can identify $\mathbb{H}(\theta) \approx T_{\pi(\theta)}M \times \{0\}$ and $\mathbb{V}(\theta) \approx \{0\} \times T^*_{\pi(\theta)}M \approx T_{\pi(\theta)}M$. Let $E \subset T_\theta T^*M$ be an n-dimensional subspace such that $E \cap \mathbb{V}(\theta) = \{0\}$. Then E is the graph of some linear map $S : \mathbb{H}(\theta) \to \mathbb{V}(\theta)$. It can be verified that E is Lagrangian if and only if in symplectic coordinates S is symmetric.

Take $\theta \in T^*M$ and $\xi \in T_\theta T^*M = \mathbb{H}(\theta) \oplus \mathbb{V}(\theta) \approx T_{\pi(\theta)}M \oplus T_{\pi(\theta)}M$. Writing $\mathrm{d}\psi_t(\xi) = (h(t), v(t))$, we obtain the Hamiltonian Jacobi equations

$$\dot{h} = H_{pq}h + H_{pp}v \qquad \dot{v} = -H_{qq}h - H_{qp}v \tag{2}$$

where the covariant derivatives are evaluated along $\pi(\psi_t(\theta))$, and H_{qq}, H_{qp}, H_{pp} and H_{qq} are linear operators on $T_{\pi(\theta)}M$, that in local coordinates coincide with the matrices of partial derivatives $(\partial^2 H/\partial q_i \partial q_j)$, $(\partial^2 H/\partial q_i \partial p_j)$, $(\partial^2 H/\partial p_i \partial p_j)$ and $(\partial^2 H/\partial p_i \partial q_j)$. Moreover, since the Hamiltonian H is convex, then H_{pp} is positive definite.

Let E be a Lagrangian subspace of $T_\theta T^*M$. Suppose that for t in some interval $]-\varepsilon, \varepsilon[$ we have that $\mathrm{d}\psi_t(E) \cap \mathbb{V}(\psi_t(\theta)) = \{0\}$. Then we can write $\mathrm{d}\psi_t(E) = \operatorname{graph} S(t)$, where $S(t) : \mathbb{H}(\psi_t\theta) \to \mathbb{V}(\psi_t\theta)$ is a symmetric map. That is, if $\xi \in E$ then

$$\mathrm{d}\psi_t(\xi) = (h(t), S(t)h(t)).$$

Using equation 2 we have that

$$\dot{S}h + S(H_{pq}h + H_{pp}Sh) = -H_{qq}h - H_{qp}Sh.$$

Since this holds for all $h \in \mathbb{H}(\psi_t(\theta))$ we obtain the Hamiltonian version of the Riccati equation, which is well known in the Lagrangian setting (e.g. [Ge])

$$\dot{S} + SH_{pp}S + SH_{pq} + H_{qp}S + H_{qq} = 0. \tag{3}$$

Suppose there are no conjugate points in the interval $[-T, T]$, we can construct matrix solutions $(Z_T(t), V_T(t))$, $(Z_{-T}(-t), V_{-T}(-t))$, $0 \leqslant t \leqslant T$ to the Jacobi equation with $Z_{\pm T}(0) = I$, $Z_T(T) = Z_{-T}(-T) = 0$. Let $K_{\pm T}(t) = V_{\pm T}(t)Z_{\pm T}(t)^{-1}$ be the corresponding solutions to the Riccati equation. Let $E_T(\theta) = \mathrm{d}\psi_{-T}(\mathbb{V}(\psi_T(\theta)))$ and $F_T(\theta) = \mathrm{d}\psi_T(\mathbb{V}(\psi_{-T}(\theta)))$, then $\mathrm{d}\psi_t(E_T(\theta)) = \operatorname{graph} K_T(t)$ and $\mathrm{d}\psi_t(F_T(\theta)) = \operatorname{graph} K_{-T}(t)$. If there are no conjugate points along all the orbit we have that $K_T(0) \to \boldsymbol{S}(\theta)$, and $K_{-T}(0) \to \boldsymbol{U}(\theta)$ as $T \to +\infty$; where $\boldsymbol{S}(\theta), \boldsymbol{U}(\theta)$ are the the symmetric solutions of the Riccati equation (3) corresponding to the Green bundles $\mathbb{E}(\theta) = \operatorname{graph}(\boldsymbol{S}(\theta))$ and $\mathbb{F}(\theta) = \operatorname{graph}(\boldsymbol{U}(\theta))$.

The following theorem relates the hyperbolicity of the flow ψ_t with the transversality of the Green bundles.

Theorem 2. *Let $\Sigma = H^{-1}\{e\}$ be a regular energy level without conjugate points. Then the following statements are equivalent:*

(a) $\psi_t|_\Sigma$ is Anosov,
(b) for all $\theta \in \Sigma$, $\mathbb{E}(\theta)$ and $\mathbb{F}(\theta)$ are transversal in $T_\theta\Sigma$,
(c) for all $\theta \in \Sigma$, $\mathbb{E}(\theta) \cap \mathbb{F}(\theta) = \mathbb{R}X(\theta)$,
(d) if $\theta \in \Sigma$, $v \in T_\theta\Sigma$, $v \notin \mathbb{R}X(\theta)$ then $\sup_{t\in\mathbb{R}} |\mathrm{d}\psi_t(\theta) \cdot v| = +\infty$.

3. The index form

The index form is more naturally defined in the Lagrangian setting. Since the Hamiltonian is convex in the fibres we can use the Legendre transform to obtain a Lagrangian $L : TM \to \mathbb{R}$ (see [Ar]). The Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}x = v \qquad \frac{\mathrm{d}}{\mathrm{d}t}L_v = L_x \tag{4}$$

are equivalent to the Hamilton equations (1). We derive the Jacobi equation in this Lagrangian setting. Let $\gamma(t)$ be a solution of the Euler–Lagrange equation. Considering a variation $f(s,t)$ of $\gamma(t) = f(0,t)$ made of solutions $t \mapsto f(s,t)$ of the Euler–Lagrange equation and taking the covariant derivative $\frac{\mathrm{D}}{\mathrm{d}s}$, we obtain the *Jacobi equation*

$$\frac{\mathrm{D}}{\mathrm{d}t}(L_{vx}k + L_{vv}\dot{k}) = L_{xx}k + L_{xv}\dot{k} \tag{5}$$

where $k = \frac{\partial f}{\partial s}(0,t)$, $\dot{k} = \frac{\mathrm{D}}{\mathrm{d}t}\frac{\partial f}{\partial s}$ and the derivatives of L are evaluated on $\gamma(t) = f(0,t)$. Here we have used that $\frac{\mathrm{D}}{\mathrm{d}s}\frac{\partial F}{\partial t} = \frac{\mathrm{D}}{\mathrm{d}t}\frac{\partial F}{\partial s}$ for the variation map $F(s,t) = L_v(f(s,t), \frac{\partial f}{\partial s}(s,t)) \in T^*M$, where $\frac{\mathrm{D}}{\mathrm{d}s}$ and $\frac{\mathrm{D}}{\mathrm{d}t}$ are the covariant derivatives on the Riemannian manifold T^*M. The linear operators L_{xx}, L_{xv}, L_{vv} coincide with the corresponding matrices of partial derivatives in local coordinates. The solutions of (5) satisfy

$$\mathrm{D}\varphi_t(k(0), \dot{k}(0)) = (k(t), \dot{k}(t)) \in T_{\gamma(t)}(TM)$$

where φ_t is the Lagrangian flow on TM.

Let Ω_T be the set of continuous piecewise C^2 vector fields along $\gamma_{[-T,T]}$. Define the *index form* on Ω_T by

$$I(\xi,\eta) = \int_{-T}^{T} (\dot{\xi}L_{vv}\dot{\eta} + \dot{\xi}L_{vx}\eta + \xi L_{xv}\dot{\eta} + \xi L_{xx}\eta)\,\mathrm{d}t \tag{6}$$

which is the second variation of the action functional for variations $f(s,t)$ with $\frac{\partial f}{\partial s} \in \Omega_T$.

The following expresion of the index form is taken from [C-I] and originally due to Clebsch [Cl]. Let $\theta \in T^*M$ and suppose that the orbit of θ, $\psi_t(\theta)$, $-T \leqslant t \leqslant T$ does not have conjugate points. Let $Z(t), V(t)$ be a matrix solution of the Hamiltonian Jacobi equation (2) such that $\det Z(t) \neq 0$. Let $\eta \in \Omega_T$ and define $\zeta \in \Omega_T$ by $\eta(t) = Z(t)\zeta(t)$. Then the integrand of $I(\eta,\eta)$ in (6) is $(Z\dot{\zeta})^*(H_{pp})^{-1}Z\dot{\zeta} + [(Z\zeta)^*V\zeta]'$. Since η and ζ are continuous, we have that

$$I(\eta,\eta) = \int_{-T}^{T} (Z\dot{\zeta})^*(H_{pp})^{-1}Z\dot{\zeta}\,\mathrm{d}t + (Z\zeta)^*V\zeta|_{-T}^{T}. \tag{7}$$

Corollary 1. *If $\theta \in T^*M$ and the segment $\{\psi_t(\theta) | t \in [-T,T]\}$ has no conjugate points then the index form is positive definite on*

$$\Gamma_T = \{\xi : [-T,T] \to TM | \xi(t) \in T_{\pi\psi_t\theta}M, \xi \text{ is piecewise } C^2, \xi(-T) = 0, \xi(T) = 0\}.$$

Proof. Let $\xi \in \Gamma_T$, $\xi \neq 0$. Write $\xi(t) = Z(t)\zeta(t)$. Since $\det Z(t) \neq 0$ for all $t \in [-T,T]$, $\zeta(-T) = 0$, $\zeta(T) = 0$ and $\zeta \neq 0$, then $\dot{\zeta} \neq 0$. Now use formula (7). □

One can extend formula (7) using different solutions (Z_i, V_i) of the Jacobi equation on disjoint subintervals $\{]t_i, t_{i+1}[: i = 0, \ldots, N\}$ of $[-T,T]$. Then

$$I(\eta,\eta) = \sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} (Z_i\dot{\zeta}_i)^*(H_{pp})^{-1}Z_i\dot{\zeta}_i\,\mathrm{d}t + [(Z_i\zeta_i)^*V_i\zeta_i]_{t_i^+}^{t_{i+1}^-} \tag{8}$$

where $\eta|_{]t_i,t_{i+1}[} = Z_i\zeta_i$.

4. Proof of theorem 1

The proof will proceed as follows. We take a system that has no conjugate points but is not Anosov, make a small perturbation in a neighbourhood of an orbit segment to end up with a system that has conjugate points. We will prove that the perturbed system has conjugate points by showing that the index form is not positive definite on that segment.

Let $\varphi \in \mathcal{B}_e \setminus \mathcal{A}_e$ and $\varepsilon > 0$. We can assume that $\varphi = 0$. By theorem 2 there are $\theta \in \Sigma$, $v \in T_\theta \Sigma$ such that $v \in \mathbb{E}(\theta) \cap \mathbb{F}(\theta) \setminus \mathbb{R} X(\theta)$. Let $\gamma(t) = \pi \psi_t(\theta)$. Since $X(\theta) \in \mathbb{E}(\theta) \cap \mathbb{F}(\theta)$ and $\dot\gamma(0) = \mathrm{d}\pi(X(\theta)) \neq 0$, we can assume that $|\mathrm{d}\pi(v)| = 1$ and $\langle \mathrm{d}\pi(v), \dot\gamma(0)\rangle = 0$. Let $C > 0$ be such that the segment $\gamma|_{[-C,C]}$ is injective, and let $T > C$.

Let $Z_{\pm T}(t)$, $V_{\pm T}(t)$, $K_{\pm T}(t)$, $0 \leqslant t \leqslant T$ and $E_T(\theta)$, $F_T(\theta)$ be as in section 2.

Let $\zeta_T \in E_T(\theta)$, $\eta_T \in F_T(\theta)$ be such that $\mathrm{d}\pi \zeta_T = \mathrm{d}\pi \eta_T = \mathrm{d}\pi v$.

Define $\xi_T : [-T, T] \to TM$ by

$$\xi_T(t) = \begin{cases} \mathrm{d}\pi \, \mathrm{d}\psi_t(\eta_T) = Z_{-T}(t)\mathrm{d}\pi(v) & \text{for } t \in [-T, 0] \\ \mathrm{d}\pi \, \mathrm{d}\psi_t(\zeta_T) = Z_T(t)\mathrm{d}\pi(v) & \text{for } t \in [0, T]. \end{cases} \tag{9}$$

By (8)

$$\begin{aligned} I(\xi_T, \xi_T) &= \xi_T(T)^* K_T(T) \xi_T(T) - \xi_T(-T)^* K_{-T}(-T) \xi_T(-T) \\ &\quad + \xi_T(0)^* (K_{-T}(0) - K_T(0)) \xi_T(0) \\ &= \mathrm{d}\pi(v)^* (K_{-T}(0) - K_T(0)) \, \mathrm{d}\pi(v). \end{aligned} \tag{10}$$

There exist $\lambda \in (0, C/2)$ and $T_0 > 0$ such that $|\bar\xi_T(t)|^2 > \frac{1}{2}$ for $|t| \leqslant \lambda$ and $T > T_0$, where $\bar\xi_T(t)$ is the component of $\xi(t)$ orthogonal to $\dot\gamma(t)$.

Since $K_T(0) \to \boldsymbol{S}(\theta)$, $K_{-T}(0) \to \boldsymbol{U}(\theta)$ as $T \to \infty$ and $v = (\mathrm{d}\pi(v), \boldsymbol{S}(\theta)\mathrm{d}\pi(v)) = (\mathrm{d}\pi(v), \boldsymbol{U}(\theta)\mathrm{d}\pi(v))$, we have that $|(K_{-T}(0) - K_T(0)) d\pi(v)| \to 0$ as $T \to \infty$. Fix T sufficiently large to have $I(\xi_T, \xi_T) < \varepsilon\lambda/4$. Henceforth we omit the subscript T in ξ.

Choose coordinates $\bar x = (x_1, \ldots, x_n) : U \to [-C, C] \times \mathbb{R}^{n-1}$ on a tubular neighbourhood of $\gamma([-C, C])$ such that $\bar x \circ \gamma([-C, C]) = [-C, C] \times \{\mathbf{0}\}$ and $\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}$ is an orthogonal frame over the points of $\gamma([-C, C])$.

Define $\tilde H = H + \phi$ with $\phi(x_1, \ldots, x_n) = \frac{1}{2} f(x_1/C) f(|\boldsymbol{x}|/\delta)\varepsilon|\boldsymbol{x}|^2$, where $\boldsymbol{x} = (x_2, x_3, \ldots, x_n)$ and $f : \mathbb{R} \to \mathbb{R}$ is a C^∞ bump function with support in $[-1, 1]$ which is 1 on $[-\frac{1}{2}, \frac{1}{2}]$. Thus $\tilde H_p = H_p$, $\tilde L = L - \phi$, $\tilde L_v = L_v$, $\tilde L_{vv} = L_{vv}$, $\tilde L_{vx} = L_{vx}$. Since

$$\phi_x = \tfrac{1}{2}\varepsilon(C^{-1} f'(x_1/C) f(|\boldsymbol{x}|/\delta)|\boldsymbol{x}|^2, \, f(x_1/C)[f'(|\boldsymbol{x}|/\delta)|\boldsymbol{x}|/\delta + 2 f(|\boldsymbol{x}|/\delta)]\boldsymbol{x})$$

and $\phi_{xx} = \dfrac{1}{2}\varepsilon \begin{bmatrix} a(\bar x) & \boldsymbol{b}(\bar x) \\ \boldsymbol{b}(\bar x)^t & \boldsymbol{A}(\bar x) \end{bmatrix}$ where

$$a = C^{-2} f''(x_1/C) f(|\boldsymbol{x}|/\delta)|\boldsymbol{x}|^2 \qquad \boldsymbol{b} = C^{-1} f'(x_1/C)[f'(|\boldsymbol{x}|/\delta)|\boldsymbol{x}|/\delta + 2 f(|\boldsymbol{x}|/\delta)]\boldsymbol{x}$$

$$\boldsymbol{A} = f\left(\frac{x_1}{C}\right)([f''(|\boldsymbol{x}|/\delta)/\delta^2 + (3/\delta|\boldsymbol{x}|) f'(|\boldsymbol{x}|/\delta)]\boldsymbol{x} \otimes \boldsymbol{x} + [f'(|\boldsymbol{x}|/\delta)|\boldsymbol{x}|/\delta + 2 f(|\boldsymbol{x}|/\delta)]I)$$

we have $|\phi_x| \leqslant c\delta\varepsilon$, $|\phi_{xx}| \leqslant c\varepsilon$ and then

$$|\tilde X(\theta) - X(\theta)| \leqslant c\delta\varepsilon \qquad |D\tilde X(\theta) - DX(\theta)| \leqslant c\varepsilon$$

where c is a constant. Using (6), we have

$$\begin{aligned} \tilde I(\xi, \xi) = I(\xi, \xi) &+ \int_{-T}^{T} (\dot\xi(L_{vv}(\tilde\psi_t(\theta)) - L_{vv}(\psi_t(\theta)))\dot\xi + 2\dot\xi(L_{vx}(\tilde\psi_t(\theta)) - L_{vx}(\psi_t(\theta)))\xi) \, \mathrm{d}t \\ &+ \int_{-T}^{T} \xi(\tilde L_{xx}(\tilde\psi_t(\theta)) - \tilde L_{xx}(\psi_t(\theta)) + \tilde L_{xx}(\psi_t(\theta)) - L_{xx}(\psi_t(\theta)))\xi \, \mathrm{d}t. \end{aligned} \tag{11}$$

Since $\lim_{\delta \to 0} \tilde\psi_t(\theta) = \psi_t(\theta)$ uniformly on $t \in [-T, T]$, for $\delta > 0$ sufficiently small we have

$$\tilde I(\xi, \xi) \leqslant I(\xi, \xi) + \int_{-T}^{T} \xi(\tilde L_{xx}(\psi_t(\theta)) - L_{xx}(\psi_t(\theta)))\xi \, \mathrm{d}t + \varepsilon\lambda/4.$$

Let $\Delta(t) = \xi(t)(\tilde{L}_{xx}(\psi_t(\theta)) - L_{xx}(\psi_t(\theta)))\xi(t)$. If $\gamma(t) \in \bar{x}^{-1}([-C, C] \times \{\mathbf{0}\})$,

$$\Delta(t) = -\xi(t)\phi_{xx}(\gamma(t))\xi(t) = -\varepsilon\xi(t)\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & f(x_1 \circ \gamma(t)/C) \end{bmatrix}\xi(t). \tag{12}$$

Then

$$\Delta(t) = \begin{cases} -\varepsilon f(x_1 \circ \gamma(t)/C)|\bar{\xi}|^2 & \text{if } \gamma(t) \in \bar{x}^{-1}([-C, C] \times \{\mathbf{0}\}) \\ 0 & \text{if } \gamma(t) \notin B(C, \delta) \end{cases} \tag{13}$$

where $B(C, \delta) = \{q \in M : \bar{x}(q) \in [-C, C] \times B_\delta(\mathbf{0})\}$. Therefore

$$\int_{-T}^{T} \Delta(t)\mathrm{d}t < -\varepsilon\lambda + \int_{D(T,C,\delta)} \Delta(t)\,\mathrm{d}t$$

where $D(T, C, \delta) = ([-T, T] \setminus [-C, C]) \cap \gamma^{-1}(B(C, \delta))$.

Lemma 1. *If the orbit $\gamma(t)$ is not periodic*

$$\lim_{\delta\to 0} \int_{D(T,C,\delta)} \Delta(t)\,\mathrm{d}t = 0.$$

Proof. If the segment $\gamma|_{[-T,T]}$ is injective, then we can choose $\delta > 0$ sufficiently small to have $D(T, C, \delta) = \emptyset$ and hence the integral vanishes.

Suppose that $([-T, T] \setminus [-C, C]) \cap \gamma^{-1}(\gamma([-C, C])) = \{s_1, \ldots, s_N\}$. Let $t_1, \ldots, t_N \in [-C, C]$ with $\gamma(s_i) = \gamma(t_i)$. Let $\bar{\gamma}_i'$ be the component of $(\bar{x} \circ \gamma)'(s_i)$ orthogonal to $(\bar{x} \circ \gamma)'(t_i)$. Let $r > 0$ be such that $|\bar{\gamma}_i'| > 2r$ for $i = 1, \ldots, N$ and define $h_i(t) = \langle \bar{x} \circ \gamma(t), \bar{\gamma}_i' \rangle / |\bar{\gamma}_i'|$ for $|t - s_i|$ small. Then $\exists\alpha > 0$ such that $h_i'(t) > r$ and $|h_i(t)| > |t - s_i|r$ for $|t - s_i| < \alpha$. If δ is sufficiently small, then $\gamma(t) \notin B(C, \delta)$ if $\delta/r < |t - s_i|$ for any $i = 1, \ldots, N$ and so

$$\int_{D(T,C,\delta)} |\Delta(t)|\mathrm{d}t \leqslant \sum_{i=1}^{N} \int_{s_i-(\delta/r)}^{s_i+(\delta/r)} |\Delta(t)|\,\mathrm{d}t. \tag{14}$$

The lemma follows. □

Lemma 2. *If the orbit $\gamma(t)$ is periodic, then*

$$\lim_{\delta\to 0} \int_{D(T,C,\delta)} \Delta(t)\,\mathrm{d}t \leqslant 0.$$

Proof. Let τ be the period of γ. Defining

$$D'(T, C, \delta) = ([-T, T] \setminus \bigcup_{m=0}^{\infty} [m\tau - C, m\tau + C]) \cap \gamma^{-1}(B(C, \delta))$$

for $k = [T + C/\tau]$ we have

$$\int_{D(T,C,\delta)} \Delta(t)\mathrm{d}t = \int_{D'(T,C,\delta)} \Delta(t)\mathrm{d}t + \sum_{m=1}^{k} \int_{m\tau-C}^{m\tau+C} \Delta(t)\mathrm{d}t.$$

As in lemma 1 one proves that $\lim_{\delta\to 0} \int_{D'(T,C,\delta)} \Delta(t)\,\mathrm{d}t = 0$.

By (13) we have $\sum_{m=1}^{k} \int_{m\tau-C}^{m\tau+C} \Delta(t)\,\mathrm{d}t \leqslant 0$. □

Thus, for $\delta > 0$ sufficiently small we have $\tilde{I}(\xi, \xi) \leqslant I(\xi, \xi) - \varepsilon\lambda + \varepsilon\lambda/4 < 0$ and corollary 1 implies that $H + \phi$ has conjugate points in $(H + \phi)^{-1}(e)$.

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