C^2 DENSELY THE 2-SPHERE HAS AN ELLIPTIC CLOSED GEODESIC

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Abstract. We prove that a riemannian metric on the 2-sphere or the projective plane can be C^2 approximated by a C^∞ metric whose geodesic flow has an elliptic closed geodesic.

In this paper we show how to overcome a difficulty presented by Henri Poincaré in the C^2 generic case. In 1905, H. Poincaré [34, p. 259] claimed that any convex surface in $\mathbb{R}^3$ should have an elliptic or degenerate non self-intersecting closed geodesic. This is, the linearized Poincaré map of the geodesic flow at the closed geodesic has an eigenvalue of modulus 1. In 1980, A. Grjuntal [13] showed a counterexample to Poincaré's claim. Victor Donnay [9], [10] constructs an example of a C^∞ riemannian metric on the 2-sphere S^2 which has positive metric entropy and all whose closed geodesics but a finite number (which are degenerate) are hyperbolic. Donnay's theorem is not known in positive curvature. It is also not known if there exists a C^∞ riemannian metric on S^2 all of whose closed geodesics are hyperbolic.

Here we prove

Theorem A.
A riemannian metric on the 2-sphere or the projective plane can be C^2 approximated by a C^∞ metric with an elliptic closed geodesic.

In August 2000 at Rio de Janeiro's dynamical systems conference, Michel Herman [15] conjectured the above result and announced a proof of it in the case of positive curvature. His proof used, as we shall also do, Ricardo Mañé's theory on dominated splittings adapted to the geodesic flow and also an equivariant version of Brouwer's translation theorem. We shall see that in the positive curvature case one can replace the use of Brouwer's translation theorem by the intermediate value theorem on the interval. For the non convex case we use Hofer, Wysocki, Zehnder theory on Reeb flows for generic tight contact forms on the 3-sphere S^3.

Theorem A is a version for geodesic flows of a theorem by Sheldon Newhouse [31]. In 1977, Newhouse proved that if H is a smooth hamiltonian on a symplectic manifold, 0 is a regular value for H and the energy level $H^{-1}\{0\}$ is compact, then there is a C^2 perturbation H_1 of H such that the hamiltonian flow on $H_1^{-1}\{0\}$ is either Anosov or it has an elliptic closed orbit. But Newhouse's arguments heavily rely on a C^2 closing lemma for hamiltonian systems, which is not known for geodesic flows. Newhouse's theorem applied to the hamiltonian which corresponds to the geodesic flow would give a C^2 approximation to the metric g by a *Finsler* metric and not a riemannian metric. This is because all the known proofs for the closing lemma rely on local perturbations of a vector field in its phase space. A perturbation

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of a riemannian metric is never a local perturbation of the geodesic flow on the unit tangent bundle because the support of the perturbation is a union of fibers.

By a theorem of Wilhelm Klingenberg and Floris Takens [24] (see also the Kupka-Samle theorem in page 21) for any $r > 0$ one can perturb in the C^∞ topology a degenerate or elliptic periodic geodesic to make the r-jet of its Poincaré map generic, in particular, with torsion.

By Jurgen Moser's invariant circle theorem [29, 2.11] and Birkhoff's normal form, a C^4 generic elliptic closed geodesic in a C^∞ surface has an invariant torus separating the phase space. Theorem A then implies that there is a C^2 dense set of riemannian metrics on S^2 or $\mathbb{P}^2$ whose geodesic flow is not ergodic for the Liouville's measure. The eigenvalues of the linearized Poincaré map of a generic C^4 elliptic geodesic is invariant under topological equivalences, because they can be seen as limiting rotation numbers on invariant torii converging to the geodesic. Then theorem A implies that there are no structurally stable geodesic flows on S^2 or $\mathbb{P}^2$.

Theorem A also allows to partially generalize a result of Vladimir Lazutkin [27] which says that the billiard map in the interior of a C^∞ embedded curve in $\mathbb{R}^2$ with positive curvature is not ergodic. In our case we obtain that a strictly convex domain in $\mathbb{R}^3$ with C^∞ boundary in a residual set in the C^2 topology, the billiard map in its interior has a set of positive measure of invariant quasi-periodic tori. Indeed, using the KAM theorem, Svanidze [36] announced in $\mathbb{R}^3$, and Valery Kovachev and Georgi Popov [26, 25] proved this result in $\mathbb{R}^n$, $n \geq 3$, provided that the geodesic flow on the boundary has an elliptic periodic geodesic which is k-elementary, $k \geq 5$.

We sketch the proof of theorem A . Since S^2 is a double cover of $\mathbb{P}^2$, the result on $\mathbb{P}^2$ can be inferred from the result on S^2. Let $\mathcal{H}(S^2)$ be the set of C^∞ riemannian metrics on S^2 all of whose closed geodesics are hyperbolic. Let $\mathcal{F}^1(S^2)$ be the interior of $\mathcal{H}(S^2)$ in the C^2 topology. One has to prove that $\mathcal{F}^1(S^2)$ is empty. In [8] it is proven that if $g \in \mathcal{F}^1(S^2)$ then the closure $\overline{\mathrm{Per}(g)}$ of the set of periodic orbits for g is a uniformly hyperbolic set. Moreover, it contains a non-trivial basic set Λ. One can also assume that the geodesic flow for g is Kupka-Smale because in [8] it is proven that the Kupka-Smale metrics are C^2 dense.

If the metric g has positive curvature, George Birkhoff [2] shows that there exists a simple closed geodesic γ such that the geodesic flow on $T^1S^2\backslash T\gamma$ admits a global transversal section homeomorphic to an open annulus. The Poincaré's return map f to the section preserves a finite area form and the return time is uniformly bounded away form zero and infinity. In section §3 we show how to use that $\overline{\mathrm{Per}(g)}$ is infinite, uniformly hyperbolic and that f is Kupka-Smale and area preserving to overcome the closing lemma problem and arrive to a contradiction.

When the curvature of g is not strictly positive, the return map to the Birkhoff's section may not be defined everywhere. Instead of using the Birkhoff section we chose to lift the geodesic flow to a Reeb flow of a (tight) contact form on S^3. If the metric is Kupka-Smale, Hofer, Wysocki, Zehnder [21] theory implies that either

- there is a periodic orbit γ such that the Reeb flow on $S^3\backslash\gamma$ admits a global transversal section homeomorphic to a disc with Poincaré's return map preserving a finite area and uniformly bounded return times. This is called the dynamically convex case and the proof in this case also holds for the Birkhoff's section in the positive curvature case.

or

- there is a finite set of periodic orbits $\gamma_1, \dots, \gamma_N$ and a global system of transversal sections on $S^3 \setminus \cup_{i=1}^N \gamma_i$. The return map preserves an area form of finite total area but it is not defined everywhere. The return time is bounded away from zero but it becomes infinite only at the intersection of the stable manifolds of a fixed finite set of periodic orbits. This case is called non dynamically convex.

In section §1 we show how to lift the geodesic flow to the Reeb flow of a tight contact form on S^3. In section §2 we summarize the Hofer, Wysocki, Zehnder theory on global sections for generic Reeb flows of tight contact forms on S^3 and prove the lemmas that we need for the non dynamically convex case. In section §3 we prove theorem A .

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1. Lifting of the geodesic flow.

A contact manifold is a pair (N, λ), where N is an odd dimensional smooth manifold and λ is a contact 1-form, i.e. $\lambda \wedge d\lambda^n$ is a volume form on N, where $\dim N = 2n+1$. The Reeb vector field Y of (N, λ) is the unique vector field determined by

$$i_Y\, d\lambda \equiv 0 \qquad \text{and} \qquad \lambda(Y) \equiv 1.$$

Write $S^3 = \{ z \in \mathbb{C}^2 \mid |z| = 1 \}$, $z = (z_1, z_2) = (q_1 + ip_1, q_2 + ip_2)$. The standard contact form on S^3 is

$$\lambda_0 := \tfrac{1}{2} \sum_{\alpha=1}^{2} \big[p_\alpha\, dq_\alpha - q_\alpha\, dp_\alpha \big] \Big|_{S^3} = \tfrac{1}{2} \big[p\, dq - q\, dp \big] |_{S^3}.$$

1.1. **Proposition.** *For any riemannian metric g on S^2, the double cover of the geodesic flow of (S^2, g) is conjugate to the Reeb flow of a positive multiple $f(z)\, \lambda_0$, $f(z) > 0$ of the standard contact form λ_0 on S^3.*

Proof: We need to see the geodesic flow on S^2 as a Reeb vector field. For background material see [23] or [32]. Let $\pi : TS^2 \to S^2$ be the projection $\pi(x, v) = x$. The tangent space to TS^2 can be decomposed $T_\theta TS^2 = H(\theta) \oplus V(\theta)$ where $V(\theta) = \ker d_\theta \pi$ and $H(\theta)$ is the kernel of the connection map. The maps $d_\theta \pi : H(\theta) \to T_{\pi(\theta)} S^2$, $\nabla_\theta : V(\theta) \to T_{\pi(\theta)} S^2$ are linear isomorphisms and they induce the Sasaki's riemannian metric on TS^2 by

$$\langle\!\langle \zeta_1, \zeta_2 \rangle\!\rangle_\theta := \langle h_1, h_2 \rangle_{\pi(\theta)} + \langle v_1, v_2 \rangle_{\pi(\theta)},$$

where $\zeta_i = (h_i, v_i) \in H \oplus V$. The Liouville's 1-form Θ on TS^2 is defined $\Theta_{(x,v)} \zeta := \langle v, d\pi(\zeta) \rangle_x$, where $\pi : T^1 S^2 \to S^2$ is the projection $\pi(x, v) = x$. Its differential $\omega = d\Theta$ is a symplectic form on TS^2 and is computed as

$$\omega(\zeta_1, \zeta_2) = \langle v_1, h_2 \rangle_x - \langle v_2, h_1 \rangle_x,$$

where $\zeta_i = (h_i, v_i) \in H \oplus V$. In particular $\omega \wedge \omega$ is a volume form in TS^2. The vector field $Z(x, v) = (0, v)$ is the unit normal vector to the unit tangent bundle $T^1 S^2$ under the Sasaki metric. Its contraction $i_Z \omega$ is the Liouville's form Θ. Then $\Theta \wedge d\Theta = \frac{1}{2}\, i_Z(\omega \wedge \omega)$ is a volume form on $T^1 S^2$ and $(T^1 S^2, \Theta)$ is a contact manifold.

In the decomposition $T(TS^2) = H \oplus V$, the geodesic vector field is written as $X(x,v) = (v,0)$. Then on the unit tangent bundle T^1S^2 we have that

$$\Theta(X(x,v)) = \langle v,v\rangle_x \equiv 1,$$
$$(i_X\omega)_\theta(\zeta) = \langle \theta, d\pi(\zeta)\rangle_{\pi(\theta)} = \langle\!\langle Z(\theta), \zeta\rangle\!\rangle_\theta = 0.$$

Then the geodesic vector field is the Reeb vector field of the Liouville's 1-form.

The unit sphere bundle T^1S^2 of S^2 is diffeomorphic to the special orthogonal group $SO(3)$ by identifying the orthogonal matrix with columns $[x, v, x\times v]$ with the unit tangent vector (x,v). The map $f: SO(3)\to S^2$, $f([x,v,x\times v]) = x$ corresponds to the projection $\pi: T^1S^2 \to S^2$. The fibers of the map $g([x,v,x\times v]) = x\times v$ are the unit tangent vectors to the oriented great circles of S^2 (with axis $x\times v$).

We show that the double cover of $SO(3)$ is the 3-sphere S^3. We identify S^3 as the unit norm quaternions and S^2 as the unit norm quaternions $q\in\mathbb{Q}$ with zero real part.

$$\mathbb{Q} = \{x_0 + x_1 i + x_2 j + x_3 k \mid x_\alpha\in\mathbb{R},\ i^2=j^2=k^2=-1,\ ij=k,\ jk=i,\ ki=j\}.$$

The covering map is $R: S^3\to SO(3)$, $R_q(x) := q^{-1}\,x\,q = \overline{q}\,x\,q$. Indeed, if $|x|=1$, then $|R_q(x)| = 1$ so that R_q is orthogonal. Since $R_q(1)=1$, $\rho(q)$ preserves the above embedding $S^2\subset S^3$. Since $R_1 = id$ and $q\mapsto R_q$ is continuous, we have that R_q preserves orientation. The group $SO(3)$ is the set of orientation preserving rotations. It is generated by the sets U_i, U_j, U_k of rotations around the three coordinate axis. In order to prove that the map R_q is surjective it is enough to prove that U_i, U_j, U_k are in the image of R. We only prove that $U_i\subset R(S^3)$. The rotation axis of R_q, $q = a+y$, $a\in\mathbb{R}$, $\Re(y)=0$ is the direction of y in $\mathbb{R}^3 = \Re^{-1}\{0\}\subset\mathbb{Q}$, because in this case $q\,y = y\,q$ and then $R_q(y) = \overline{q}\,y\,q = q$. If $q = a+ib = e^{i\theta}$ and $x = zj$, with $z\in\mathbb{C}$, then $R_q(zj) = e^{-i\theta}zj\,e^{i\theta} = e^{-2i\theta}zj$. So that R_q is the rotation of angle 2θ with axis i.

Define the map $F: S^3\to SO(3) = T^1S^2$ by $F(q) = [R_q(j), R_q(k), R_q(i)] = [x, v, x\times v]$. The lift under F of the tangents to the oriented great circles of S^2 are those $q\in S^3$ for which $R_q(i)$ takes a fixed value. We show that $F^*\Theta_0 = 2\,\lambda_0$, where Θ_0 is the Liouville's form for the metric g_0 with curvature $+1$.

Let $\langle\cdot,\cdot\rangle$ be the euclidean inner product on $\mathbb{R}^4\approx\mathbb{Q}$. Observe that for q_1, $q_2\in\mathbb{Q}$ we have that

$$\langle q_1, q_2\rangle = \Re(q_1\,\overline{q_2}) = \Re(\overline{q_1}\,q_2) = \Re(q_2\,\overline{q_1}).$$

For $\zeta\in T_qS^2$, $q = q_1 + p_1\,i + q_2\,j + p_2\,k$, $\zeta = x + y\,i + z\,j + w\,k$, the standard contact form λ_0 is written as

$$\begin{aligned}\lambda_0(q)\cdot\zeta &= p_1\,x - q_1\,y + p_2\,z - q_2\,w\\ &= \langle p_1 - q_1 i + p_2 j - q_2 k, \zeta\rangle = \langle iq, \zeta\rangle\\ &= \Re[-\overline{\zeta}\,i\,q].\end{aligned}$$

Since $\pi\circ F(q) = \big(R_q(j), R_q(k)\big) = (x,v)$, the first component of $d\pi\circ d_qF(\zeta)$ is

$$d_qF(\zeta) = \big(d_qR(j)\cdot\zeta, *\big) = (\overline{\zeta}\,j\,q + \overline{q}\,j\,\zeta, *).$$

The pull-back $F^*\Theta_0$ is given by

$$\begin{aligned}(F^*\Theta_0)_q\cdot\zeta &= \Theta_0(\overline{q})\cdot d_qF(\zeta) = \langle R_q(k),\, \overline{\zeta}\, j\, q + \overline{q}\, j\, \zeta\rangle \\ &= \langle \overline{q}\, k\, q,\, \overline{\zeta}\, j\, q + \overline{q}\, j\, \zeta\rangle = \Re\left[(\overline{\zeta}\, j\, q)\,\overline{\overline{q}\, k\, q} + \overline{\overline{q}\, k\, q}\,(\overline{q}\, j\, \zeta)\right] \\ &= \Re\left[\overline{\zeta}\, j\, q\,\overline{q}\,(-k)\, q - \overline{q}\, k\, q\,\overline{q}\, j\, \zeta\right] \\ &= \Re\big[-\overline{\zeta}\, i\, q\big] + \Re\big[\overline{q}\, i\, \zeta\big].\end{aligned}$$

Since $\Re[\,\overline{q}\, i\, \zeta\,] = \Re[\,\overline{\overline{q}\, i\, \zeta}\,] = \Re[-\overline{\zeta}\, i\, q]$,

$$\big(F^*\Theta_0\big)_q\cdot\zeta = 2\,\Re\big[-\overline{\zeta}\, i\, q\big] = 2\,\lambda_0.$$

If g is another riemannian metric on S^2, using isothermal coordinates (e.g. [7]) one can show that there exists a diffeomorphism $h : S^2 \to S^2$ and a smooth positive function $f : S^2 \to \mathbb{R}^+$ such that $h^*g = f g_0$. Since h is an isometry between $(S^2, f g_0)$ and (S^2, g), the map $dh : TS^2 \to TS^2$ conjugates their geodesic flows. Using h we can assume that $g = f\, g_0$. Let $H : (T^1S^2, g_0) \to (T^1S^2, g)$ be the map $(x, v) \mapsto \big(x, v/\sqrt{f(x)}\big)$ and let Θ be the Liouville's form for the metric $g = f\, g_0$. Then

$$\begin{aligned}H^*\Theta_{(x,v)}\cdot\zeta &= \Theta_{H(x,v)}\cdot d_{(x,v)}H(\zeta) \\ &= \Theta\big(\tfrac{v}{\sqrt{f}}\big)\cdot d\pi(\zeta) \\ &= g_x\big(\tfrac{v}{\sqrt{f}},\, d\pi\cdot\zeta\big) = f(x)\; g_0\big(\tfrac{v}{\sqrt{f}},\, d\pi\cdot\zeta\big) \\ &= \sqrt{f(x)}\, g_0(v, d\pi\cdot\zeta) = \sqrt{f(x)}\;\Theta_0(x, v)\cdot\zeta.\end{aligned}$$

So that $H^*\Theta = \sqrt{f\circ\pi}\;\Theta_0$. Then $F^*H^*\Theta = 2\sqrt{f\circ\pi\circ F}\;\lambda_0$. If X is the vector field of the geodesic flow for $(S^2, g = f\, g_0)$, since X is the Reeb flow of the Liouville's form for g, then $(F\circ H)^{-1}(X)$ is the Reeb vector field for $2\sqrt{f\circ\pi\circ F}\;\lambda_0$. □

2. Generic tight contact flows on S^3

In this section we summarize the theory of contact flows on S^3 by Hofer, Wysocki and Zehnder that we shall need. A description of the theory is given in [20] and the proofs can be found in [21].

Write $S^3 := \{\, z \in \mathbb{C}^2 \,|\, |z| = 1\,\}$, $z = (z_1, z_2) = (q_1 + ip_1, q_2 + ip_2)$ with $z_j \in \mathbb{C}$ and $p_j,\, q_j \in \mathbb{R}$. Let

$$\lambda_0 := \tfrac{1}{2}\sum_{j=1}^{2}\big[q_j\, dp_j - p_j\, dq_j\big]\Big|_{S^3}$$

be the standard contact form in S^3. In the following we shall only consider contact forms

$$\lambda = f\cdot\lambda_0,$$

where $f : S^3 \to]0, +\infty[$ that we shall call tight contact forms. Any such λ is indeed a contact form in S^3 because

$$\lambda\wedge d\lambda = f^2\cdot(\lambda_0\wedge d\lambda_0)$$

is a volume form if $f^{-1}\{0\} \neq \emptyset$. The *Reeb vector field* of λ is the vector field X on S^3 defined by

$$i_X\, d\lambda \equiv 0 \quad\text{and}\quad \lambda(X) \equiv 1.$$

Its flow φ_t is called the *Reeb flow* of λ. The *contact structure* of λ is the distribution of linear 2-dimensional subspaces

$$\xi = \ker \lambda.$$

The contact form λ and its contact structure ξ are invariant under the derivative of the Reeb flow because

$$L_X\lambda = d(i_X\lambda) + i_X d\lambda \equiv 0.$$

The derivative $d\lambda$ is a symplectic (i.e area-) form on any 2-dimensional subspace which is transversal to the vector field X, in particular on the contact structure ξ.

2.1. **Closed orbits of the Reeb flow.**

Let $x : [0, T] \to S^3$ be a periodic orbit for φ_t with period T. Then T is a multiple of the minimal period of x. We shall say that (x, T) is *non-degenerate* if the number $+1$ is not an eigenvalue of its linearized Poincaré map $d\varphi_T|_\xi$, restricted to $\xi = \ker\lambda$. Since $d\varphi_T|_\xi$ preserves the area form $d\lambda|_\xi$, its eigenvalues have the form $\{\mu_1, \mu_2\} = \{\mu, \overline{\mu}, \mu^{-1}, (\overline{\mu})^{-1}\}$. Hence, if (x, T) is non-degenerate, either $|\mu| = 1$ or $\mu \in \mathbb{R}\setminus\{0\}$. We say that (x, T) is *hyperbolic* if $\mu \in \mathbb{R}$, $\mu \neq \pm 1$ and that (x, T) is *elliptic* if $\mu \in \mathbb{C}\setminus\mathbb{R}$, $|\mu| = 1$. We shall also distinguish between (+)-hyperbolic orbits, when $\mu, \mu^{-1} > 0$; and $(-)$-hyperbolic orbits, when $\mu, \mu^{-1} < 0$.

We describe now one characterization of the Conley-Zehnder index of a non-degenerate periodic orbit (x, T). Assume that (x, T) is non-degenerate. Since $\pi_1(S^3) = 0$, $x : \mathbb{R}/T\mathbb{Z} \to S^3$ is contractible. Choose a map $v_D : D \to S^3$, $D := \{\, z \in \mathbb{C} \,|\, |z| \le 1 \,\}$ such that $v_D(e^{2\pi i s}) = x(Ts)$. Then the contact bundle v_D^* is trivial. Choose a trivialization $v_D^* \to D \times \mathbb{R}^2 \approx D \times \mathbb{C}$. Write the derivative of the Reeb vector field on this trivialization

$$\Phi(s) := d\varphi_{sT}|_\xi \in \mathrm{Sp}(1) = \{\, A \in \mathbb{R}^{2\times 2} \mid \det A = +1 \,\}\,, \qquad 0 \le s \le 1.$$

This arc of symplectic matrices starts at the identity $\Phi(0) = I$ and ends at the linearized Poincaré map $\Phi(1) = d\varphi_t|_\xi$. Let $z \in \mathbb{C}\setminus\{0\}$ and $z(s) := \Phi(s)\, z$. Choose a continuous argument $\theta(s)$ for

$$e^{2\pi i\theta(s)} = \frac{z(s)}{|z(s)|}, \qquad 0 \le s \le 1.$$

Define the winding number of $\Phi(s)z$ by

$$\Delta(z) := \theta(1) - \theta(0) \in \mathbb{R},$$

and the winding interval of the arc ϕ by

$$I(\Phi) := \big\{\, \Delta(z) \;\big|\; z \in \mathbb{C}\setminus\{0\} \,\big\}.$$

2.1. **Lemma.** length $\big[I(\Phi)\big] \le \frac{1}{2}$.

Proof: Let $z(s) := \Phi(s)z$, $w(s) := \Phi(s)\, w$ and $u(s) := z(s)\,\overline{w(s)} \in \mathbb{C}$. Observe that $\Delta(u) = \Delta(z) - \Delta(w)$. Assume that $|\Delta(u)| \ge \frac{1}{2}$. Then there exists $0 < s_0 < 1$ such that $z(s_0) = \tau\, w(s_0)$ for some $\tau \in \mathbb{R}\setminus\{0\}$. Then $z(s) = \tau\, w(s)$ for all $s \in [0, 1]$. Hence $\Delta(z) = \Delta(w)$, which contradicts $|\Delta(z) - \Delta(w)| \ge \frac{1}{2}$. $\square$

Then the winding interval either lies between two consecutive integers or contains an integer. Define the *Conley-Zehnder index* of the non-degenerate periodic orbit (x, T) by

$$\mu(x, T) := \mu(\Phi) := \begin{Bmatrix} 2k+1 & \text{if } \; I(\Phi) \in]k, k+1[\\ 2k & \text{if } \; k \in I(\Phi) \end{Bmatrix}, \quad k \in \mathbb{Z}.$$

The integer $\mu(\Phi)$ depends only on the homotopy type of the chosen disc map $v_D : D \to S^3$. Since $\pi_2(S^3) = 0$, the index[1] $\mu(x,T)$ is well defined.

Observe that the winding number $\Delta(z_0)$ is an integer if and only if $\Phi(1)z_0 = \mu z_0$ for some $\mu > 0$. Hence

$$\begin{aligned} \mu(x,T) \text{ is even} &\iff (x,T) \text{ is } (+)\text{-hyperbolic.}\\ \mu(x,T) \text{ is odd} &\iff (x,T) \text{ is elliptic or } (-)\text{-hyperbolic.}\end{aligned}$$

We also define the self-linking of a closed orbit (x,T).Take a disc map $v_D : D \to S^3$ such that $v_D(e^{2\pi i s}) = x(sT)$, $0 \le s \le 1$. Choose a nowhere vanishing section Z of the pull-back bundle $v_D^*\xi$. Then Z is nowhere tangent to x. Pushing x slightly in the direction of Z we obtain a loop x' which is transversal to ξ and disjoint from x. The two loops x, x' have a natural orientation induced by the orientation of ∂D. Define the *self-linking number of* (x,T)

$$sl(x,T) := I(x', v_D) \in \mathbb{Z},$$

as the oriented intersection number of x' with v_D. This number does not depend on the choices of v_D or Z. Indeed, using a trivialization of $v_D^*\xi$, the section Z is a function $Z : D \to \mathbb{C}\setminus\{0\}$. Since $\pi_2(\mathbb{C}\setminus\{0\}) = 0$, Z is homotopic to a constant function $Z : D \to \{1\}$. Then the loop x' is isotopic to the loop $\{\overline{x}\}$ obtained from Z and thus it has the same intersection number as x. Similarly, $sl(x,T)$ only depends on the homotopy type of v_D, but since $\pi_2(S^3) = 0$, there is only one homotopy type.

2.2. **Finite energy surfaces.**

An *almost complex structure compatible with* λ on S^3 is a linear bundle map $J : \xi \to \xi$ such that $J^2 = -Id$ and such that the quadratic form

$$\xi \times \xi \ni (h,k) \longmapsto d\lambda(h, Jk)$$

is positive definite on each fiber of the product bundle $\xi \times \xi$. The set of smooth almost complex structures compatible with λ is always non-empty and contractible. We extend J to an almost complex structure $\tilde{J}$ on the product $\mathbb{R} \times S^3$ by setting

$$\tilde{J}(a, bX + h) = \big(-b\ ,\ aX + Jh\big), \qquad \text{for} \quad a,b \in \mathbb{R},\ h \in \xi.$$

Let $\Sigma := \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and let $\Gamma \subset \Sigma$ be a finite set (of "punctures"). Let $j : T\Sigma \to T\Sigma$, $j(z) = iz$, be the complex structure on Σ.

2.2. **Definition** (cf. [20, appendix]).
A (spherical) **finite energy surface** *is a map* $\tilde{u} = (a,u) : \Sigma \setminus \Gamma \to \mathbb{R} \times S^3$ *such that*

(1) $$\tilde{u} : \Sigma \setminus \Gamma \to \mathbb{R} \times S^3 \textit{ is proper and non-constant,}$$

(2) $$d\tilde{u} \circ j = \tilde{J} \circ d\tilde{u},$$

(3) $$\int_{\Sigma\setminus\Gamma} u^* d\lambda < +\infty.$$

A **finite energy plane** *is a spherical finite energy surface with only one puncture* $\Gamma = \{\infty\}$.

[1] i.e. two such discs $v_i : D_i \to S^3$, $i = 0,1$, can be joined to form a sphere $u : S^2 = D_0 \cup D_1 \to S^3$. Since $\pi_2(S^3) = 0$, $u(S^2)$ is the boundary of a 3-ball B^3 in S^3. One can use the ball B^3 to construct a homotopy v_t, between v_0 and v_1 with $v_t(e^{2\pi i s}) = x(sT)$ for all $s,t \in [0,1]$.

The integrand in (3) is always non-negative, and it is positive at the points where the projected map $u : \Sigma \setminus \Gamma \to S^3$ is transversal to the Reeb vector field.

The set of punctures Γ must be non-empty because if $\Gamma = \emptyset$ then $\tilde{u}$ is constant [20, lemma 3.4]. The behaviour of a finite energy surface near a puncture is classified by the following

2.3. Lemma ([20])**.** *Let $\tilde{u} : \Sigma \setminus \Gamma \to \mathbb{R} \times S^3$ be a finite energy surface and $\gamma \in \Gamma$ a puncture. Then one of the following cases hold, where $\tilde{u} = (a, u)$:*

- *positive puncture:* $\lim\limits_{z\to\gamma} a(z) = +\infty$;
- *negative puncture:* $\lim\limits_{z\to\gamma} a(z) = -\infty$;
- *removable puncture:* $\lim\limits_{z\to\gamma} a(z)$ exists in $\mathbb{R}$.

In the case of a removable puncture γ there exists a neighbourhood $\mathcal{U}(\gamma)$ of γ such that $\tilde{u}$ is bounded on $\mathcal{U}(\Gamma) \setminus \{\gamma\}$. By Gromov's removable singularity theorem [14], $\tilde{u}$ can be extended smoothly over the puncture γ. Hence we only consider positive and negative punctures:

$$\Gamma = \Gamma^+ \cup \Gamma^-.$$

A special example of a finite energy surface is the following. Let (x, T) be a periodic orbit of the Reeb vector field. Then the map $\tilde{u} : \mathbb{C} \setminus \{0\} \to \mathbb{R} \times S^3$,

$$\tilde{u}\big(e^{2\pi(s+it)}\big) = \big(Ts, x(Tt)\big) \in \mathbb{R} \times S^3.$$

is an embedded finite energy surface in $\mathbb{R} \times S^3$. Its $d\lambda$-energy vanishes:

$$\int_{\mathbb{C}\setminus\{0\}} u^* d\lambda = 0.$$

Its image $F = \tilde{u}(\mathbb{C} \setminus \{0\})$ is fixed under the $\mathbb{R}$-action on $\mathbb{R} \times S^3$.

At a puncture γ, a finite energy surface converges to a periodic orbit (x_γ, T_γ) of the Reeb vector field which is called the *asymptotic limit* of the surface at the puncture. Its period T_γ can be a multiple of its minimal period τ_γ. When $T_\gamma = \tau_\gamma$ we say that (x_γ, T_γ) is simply covered by $\tilde{u}$. Now we describe better the local form of $\tilde{u} = (a, u) : \Sigma \setminus \Gamma \to \mathbb{R} \times S^3$ near a puncture.

Assume that the asymptotic limit (x, T) associated with the puncture is a non-degenerate periodic orbit. Fix holomorphic polar coordinates $\sigma(s, t)$ in a punctured neighbourhood of the puncture $\gamma \in \Gamma$, we have the asymptotic behaviour

$$\lim_{s\to\infty} u \circ \sigma(s, t) = x(mt) \quad \text{ in } C^\infty(S^1),$$

where $\gamma = \lim_{s\to\infty} \sigma(s, t)$ and (x, T) is a periodic orbit of the Reeb vector field with (perhaps non-minimal) period $T = |m|$. When $\gamma \in \Gamma^+$, $m = T$ and when $\gamma \in \Gamma^-$, $m = -T$.

One can give coordinates to a neighbourhood of the asymptotic limiting periodic orbit P of the form $S^1 \times \mathbb{R}^2$, $S^1 = \mathbb{R}/\mathbb{Z}$, with $P = S^1 \times \{0\}$ and $\{0\} \times \mathbb{R}^2$ tangent to the constant structure ξ. In such coordinates

either

- there exists $c \in \mathbb{R}$ such that

$$\tilde{u} \circ \sigma(s, t) = (ms + c, mt), \qquad \text{for } (s, t) \in S^1 \times \mathbb{R}^2,$$

and its whole projection $u(\Sigma \setminus \Gamma) = P$ is only the asymptotic limit, as in the example above;

or,

- the projected map u has the form
$$u \circ \sigma(s,t) \approx \big(mt, e^{\mu s} e(t)\big), \qquad \text{as } s \to \infty;$$
modulo lower order terms in s. In a positive puncture $s \to +\infty$ and $\mu < 0$, in a negative puncture $s \to -\infty$ and $\mu > 0$. Therefore near such a puncture there is a precise directional convergence of u to the periodic orbit given by the periodic non-vanishing vector field
$$e(t) = \lim_{s\to\infty} \frac{\partial_s u}{|\partial_s u|} \in \xi_{x(mt)} \setminus \{0\}, \qquad t \in S^1 = \mathbb{R}/\mathbb{Z} \tag{4}$$
in the contact structure ξ along P.

In the later case above, by Stokes theorem
$$\sum_{\gamma\in\Gamma^+} T_\gamma - \sum_{\gamma\in\Gamma^+} T_\gamma = \int_{\Sigma\setminus\Gamma} u^* d\lambda > 0,$$
where (x_γ, T_γ) is the asymptotic limit of $\tilde{u}$ at γ. It follows that such $\tilde{u}$ must have at least one positive puncture.

A neighbourhood of the puncture looks like $]0,+\infty[\times S^1$. To this we add $\{+\infty\} \times S^1$. In this way one obtains a circle compactification $\overline{\Sigma}$ of $\Sigma \setminus \{\gamma\}$ and $u : \Sigma \setminus \Gamma \to S^3$ can be extended to a smooth map $\overline{u} : \overline{\Sigma} \to S^3$ such that the boundary circles parametrize the periodic orbits which are the asymptotic limits of the punctures. On a positive puncture, $\gamma \in \Gamma^+$, the orientation of the limiting periodic orbit coincides with the orientation of the boundary circle of $\overline{\Sigma}$ and on a negative puncture the orientations are reversed.

Define the *Conley-Zehnder index of* $\tilde{u} : \Sigma \setminus \Gamma \to \mathbb{R} \times S^3$ by
$$\mu(\tilde{u}) = \sum_{\gamma\in\Gamma^+} \mu(x_\gamma, T_\gamma) - \sum_{\gamma\in\Gamma^-} \mu(x_\gamma, T_\gamma),$$
where (x_γ, T_γ) is the asymptotic limit of u at γ and $\mu(x, T)$ is the Conley-Zehnder index of a periodic orbit (x, T) defined above.

Note that given a finite energy surface $\tilde{u} = (a, u) : \Sigma \setminus \Gamma \to \mathbb{R} \times S^3$, and $c \in \mathbb{R}$, the translated map
$$\tilde{u}_c(z) := \big(a(z) + c, u(z)\big), z \in \Sigma \setminus \Gamma,$$
is also a finite energy surface because $\tilde{J}$ is invariant under translations in the first factor $\mathbb{R}$. Hence a finite energy surface always belong to a 1-parameter family of finite energy surfaces, invariant under the $\mathbb{R}$-action on $\mathbb{R} \times S^3$.

We define the *index* of an *embedded* finite energy surface $F = \tilde{u}(\Sigma \setminus \Gamma)$ by
$$\mathrm{Ind}(F) := \mu(F) - \chi(S^2) + \sharp F,$$
where $\mu(F) = \mu(\tilde{u})$, $\chi(S^2) = 2$ the Euler characteristic of S^2 and $\sharp F := \sharp\Gamma$. Observe that if $\tilde{u}$ is an embedding $\mu(F)$ only depends on the image F and not on $\tilde{u}$. This index describes the dimension of the moduli space consisting of nearby embedded finite energy surfaces having the same topological type.

2.3. Zoology of embedded finite energy surfaces with low indices.

In the following we shall only be interested in finite energy surfaces

$$\tilde{u} = (a, u) : \Sigma \setminus \Gamma \to \mathbb{R} \times S^3$$

such that

- its projection $u : \Sigma \setminus \Gamma \to S^3$ is an embedding,
- have exactly one positive puncture,
- its image $F = u(\Sigma \setminus \Gamma)$ has indices $\mu(F) \in \{1, 2, 3\}$ and $\mathrm{Ind}(F) \in \{1, 2\}$.

Assume that all the periodic orbits are non-degenerate. Let (x, T) be an asymptotic limit of such $\tilde{u}$. Take a disc map $v_D : D \to S^3$such that $v_D(e^{2\pi t}) = x(tT)$ and choose a nowhere vanishing section Z of the pull-back $v_D^* \xi$ of the contact bundle ξ. Then there exists a nowhere vanishing function $f(t) \in \xi_{x(t)}$ along the orbit such that the linearized Reeb flow along the orbit has the representation

$$d_{x(0)} \varphi_t \cdot v = f(t) \cdot Z(x(t)),$$

where the dot at the right denotes the complex multiplication with respect to the complex structure J on ξ. By the definition of the Conley-Zehnder index we have the following table

$\mu(x,T)$	change of argument of f along a full period		eigenvalues of $d_{x(0)}\varphi_T$
1	$0 < \Delta \arg f < 2\pi$		elliptic or $(-)$-hyperbolic
2	$\Delta \arg f = 2\pi$	if v is an eigenvector of the linearized Poincare map	$(+)$-hyperbolic
2	$\pi < \Delta \arg f < 3\pi$	otherwise	$(+)$-hyperbolic
3	$0 < \Delta \arg f < 2\pi$		elliptic or $(-)$-hyperbolic

If the projection u of $\tilde{u}$ is an embedding and $\gamma \in \Gamma$ is a puncture there is a precise directional convergence in ξ towards the asymptotic periodic orbit, described by the vector field $e(t) \in \xi_{x(t)}$ from (4).

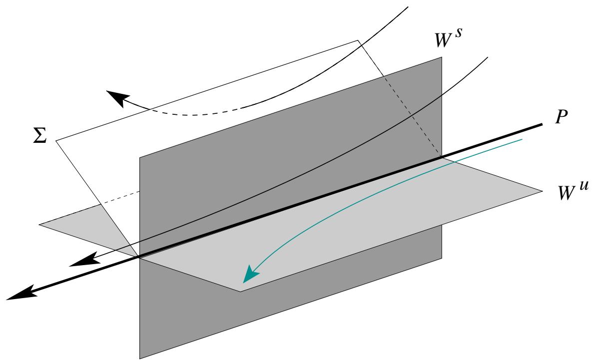


FIGURE 1. This figure shows the stable and unstable manifolds of a binding periodic orbit P of Conley-Zehnder index 2 and the approach of a finite energy surface Σ having P as asymptotic limit.

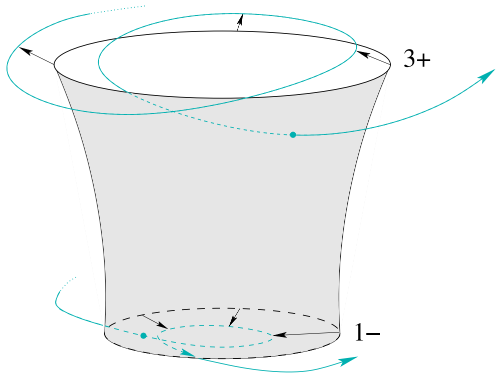


FIGURE 2. The figure shows the winding of the flow near periodic orbits of Conley-Zehnder indices 1 and 3, relative to a finite energy cylinder. The Reeb vector field is transversal to the finite energy surface. Two orbits intersecting the surface are shown.

Consider for example $\gamma \in \Gamma^+$ and assume that the asymptotic limit P_γ has Conley-Zehnder index 2 or 3. Then it can be proved (cf. [19, 21]) that the vector e describing the approach towards the periodic orbit has winding number at most 1 with respect to the nowhere vanishing section Z. This means that the flow turns faster (more than 2π) around the periodic orbit than the approaching surface if the Conley-Zehnder index is 3.

If the Conley-Zehnder index is 2, the situation is a bit more subtle. Assume that the vector $e(t)$ describing the directional convergence has winding number 1. Since $\mu(P_\gamma) = 2$, the stable and unstable subspaces of the periodic orbit have winding number 1 (they are generated by nowhere vanishing sections of ξ whose winding can be measured with respect to Z) and intersect a transversal section in a pair of transversal lines creating four quadrants. The trace of e on this transversal section will lie on one of this quadrants. See figures 1, 3, 5 for more detail.

If $\gamma \in \Gamma^-$ and if the asymptotic limit has Conley-Zehnder index 1 or 2, then the winding number of the vector $e(t)$ describing the asymptotic approach is at least 1. This time the surface turns faster than the flow around the orbit if the Conley-Zehnder index is 1. If the index is equal to 2, the periodic orbits is $(+)$-hyperbolic and the surface turns as fast as the flow, similar to the behaviour near the positive puncture of index 2. See figures 1, 2, 3, 4.

Now assume that $\tilde{u}$ satisfies the three items above. Let μ^+ be the Conley-Zehnder index of its (unique) positive puncture. The negative punctures in Γ^- all have μ-indices ≥ 1 if $\Gamma^- \neq \emptyset$. Denote by N_j the number of negative punctures having μ-index equal to j, where $j \geq 1$. We have that $\mathrm{Ind}(F) \geq 1$. Hence

$$\mathrm{Ind}(F) = \mu^+ - \sum_{j=1}^{\ell} j\, N_j - 2 + (1 + \sharp\Gamma^-) \geq 1.$$

This is equivalent to

$$\sum_{j=1}^{\ell}(j-1)\,N_j \le \mu^+ - 2. \tag{5}$$

We conclude for the index of the positive puncture that

$$2 \le \mu^+.$$

If $\mu^+ = 2$, it follows from (5) that $N_j = 0$ for all $j \ge 2$. If $\mu^+ = 3$, we conclude that $N_2 \le 1$ and $N_j = 0$ for all $j \ge 3$. In both cases there is no restriction on the number of negative punctures with μ-index equal to 1. In order to list the types of such surfaces F we introduce the vectors

$$(\mu^+, \mu_1^-, \dots, \mu_N^-),$$

where $N = \sharp\Gamma^-$ is the number of negative punctures of F, μ^+ is the Conley-Zehnder index of the unique positive puncture and μ_j^- are the indices of the negative punctures ordered so that $\mu_j^- \ge \mu_{j+1}^-$. Then F must have one of the following types:

$$\begin{aligned} &(3, 1_1, \dots, 1_N), && \mathrm{Ind}(F) = 2;\\ &(3, 2, 1_1, \dots, 1_{N-1}), && \mathrm{Ind}(F) = 1;\\ &(2, 1_1, \dots, 1_N), && \mathrm{Ind}(F) = 1. \end{aligned} \tag{6}$$

The number N of negative punctures can be zero. If this happens, the first and third cases represent finite energy planes. The second case, for $N = 1$, represents a finite energy cylinder connecting a periodic orbit of index 3 [elliptic or $(-)$-hyperbolic] with a periodic orbit of index 2 [$(+)$-hyperbolic].

We point out that the only index that can occur at a positive and a negative puncture, although not simultaneously, is the index equal to 2, which belongs to a $(+)$-hyperbolic orbit, where the asymptotic approach of F lies in a quadrant between the stable an unstable manifolds. This will be important later on in the decomposition of families of leaves at their ends which always takes place along a hyperbolic orbit of index 2.

2.4. **Global system of transversal sections.**

2.4. **Definition.** *We say that a contact form $\lambda = f\lambda_0$ is* **non-degenerate** *if all the periodic orbits (x, T) of its Reeb vector field are non-degenerate, i.e. the eigenvalues of the linearized Poincare map for their minimal periods, restricted to $\xi = \ker\lambda$, are not roots of unity.*

We say that

2.5. **Theorem. [Hofer, Wysocki, Zehnder** [21, 20]**]**

If all the periodic orbits of the Reeb vector field X of the contact form $f\lambda_0$ are non-degenerate, then there exists a non-empty set $\mathcal{P}$ of finitely many periodic orbits distinguished periodic orbits of X which have self linking number -1 and Conley-Zehnder indices in the set $\{1, 2, 3\}$ so that the complement

$$S^3 \setminus \mathcal{P}$$

is smoothly foliated into leaves which are embedded punctured Riemann spheres, transversal to the Reeb vector field X and converging at the punctures to periodic orbits from $\mathcal{P}$.

The leaves $F = u(\Sigma \setminus \Gamma)$ are projections of embedded finite energy surfaces with indices $\mathrm{Ind}(F) \in \{1,2\}$ *whose asymptotic limits (in $\mathcal{P}$) are simply covered. Each leaf has precisely one positive puncture, but an arbitrary number of negative punctures.*

Moreover, there is at least one leaf which is the projection of a finite energy plane and whose asymptotic limit P has Conley-Zehnder index $\mu(P) \in \{2,3\}$, (c.f. [21, prop. 7.1]).

Since the leaves $F = u(\Sigma \setminus \Gamma)$ are transversal to the Reeb vector field, the 2-form $d\lambda|_F$ is a positive area form on F and its total area $\int_F d\lambda$ is finite.

The periodic orbits in $\mathcal{P}$ are called *binding periodic orbits* and the leaves F with index $\mathrm{Ind}(F) = 1$ are called *rigid leaves.*

The decomposition of S^3 in theorem 2.5 comes from a foliation on $\mathbb{R} \times S^3$ by embedded finite energy surfaces. The binding periodic orbits are projections of surfaces $\tilde{F} = \tilde{u}(\Sigma \setminus \Gamma) \subset \mathbb{R} \times P$, $P \in \mathcal{P}$, which are fixed under the $\mathbb{R}$-action on $\mathbb{R} \times S^3$.

The rigid leaves are projections of surfaces $\tilde{F} = \tilde{u}(\Sigma \setminus \Gamma)$, with $\mathrm{Ind}(\tilde{F}) = 1$, which belong to a 1-parameter family of surfaces having the same asymptotic limits, namely, the orbit of $\tilde{F}$ under the $\mathbb{R}$ action on S^3. The projection of the family is an isolated embedded punctured sphere. There are finitely many binding orbits.

The leaves $F = u(\Sigma \setminus \Gamma)$ with $\mathrm{Ind}(F) = 2$ belong to a 2-parameter family of finite energy surfaces all with the same asymptotic limits. One parameter is given by the orbit under the $\mathbb{R}$-action on $\mathbb{R} \times S^3$. After projecting by $\mathbb{R} \times S^3 \to S^3$, it remains a 1-parameter family

The foliation gives a decomposition

$$S^3 = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2,$$

where $\mathcal{S}_0$ is the set of points in the binding periodic orbits, $\mathcal{S}_1$ is the set of points in the finitely many rigid leaves and $\mathcal{S}_2$ is the set of points in the leaves with index $\mathrm{Ind}(F) = 2$ which occur in 1-dimensional families parametrized either by S^1 or by an open interval $I =]0,1[$. In the case of an I-family F_τ, the surfaces decompose at each of the ends where $\tau \to 0$ and $\tau \to 1$ into two rigid surfaces $\{C^+, C^-\}$ along a hyperbolic binding orbit P whose Conley-Zehnder index is equal to $\mu(P) = 2$. On C^+ the orbit P is the asymptotic limit at its positive puncture and on C^-, P is the asymptotic limit at a negative puncture.

2.6. **Definition.** *We say that the contact form $f\lambda$ is* **dynamically convex** *if it admits a foliation as in theorem 2.5 which has an S^1-family of leaves F_τ with index* $\mathrm{Ind}(F_\tau) = 1$.

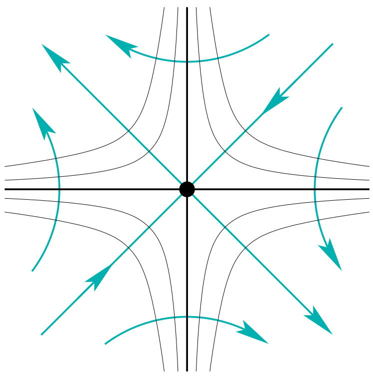


FIGURE 3.

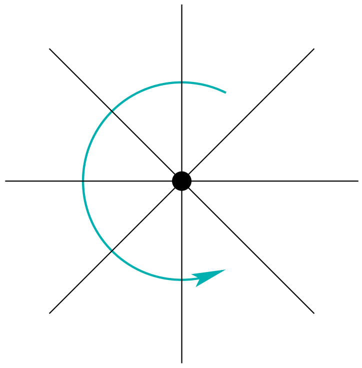


FIGURE 4.

Figures 3 and 4 show the behaviour of the foliation near a binding orbit P. The periodic orbit is shown perpendicular to the page and we only draw the trace of the foliation on the page as black lines. The big dot is the trace of the periodic orbit.

In figure 3 the binding orbit has Conley-Zehnder index 2, and then it is hyperbolic with positive eigenvalues. The inward pointing arrows are the trace of the local stable manifold and the outward point arrows are the trace of the local unstable manifold. The black lines are the traces of 1-parameter families of leaves which split into two rigid surfaces when they approach the periodic orbit. Two rigid surfaces are horizontal and the other two are vertical. The grey curves indicate the flow.

In figure 4 the binding orbit has Conley-Zehnder index 1 or 3. It is either elliptic or hyperbolic with negative eigenvalues. The foliation looks the same in both cases. When the orbit is hyperbolic the stable and unstable manifolds are Möbius bands which intersect transversally the leaves of the foliation. The grey curve represents the flow.

Consider the Reeb flow in a neighbourhood of a binding periodic orbit $P = (x, T)$ of index $\mu(P) \in \{1, 3\}$. The P is either elliptic or $(-)$-hyperbolic. Fix a disk map $v_D : D \to S^3$ with $v_D(e^{2\pi t}) = x(tT)$ and a nowhere vanishing section Z of the pull back $v_D^*\xi$ of the contact bundle ξ. The Reeb flow turns around the periodic orbit at a speed approaching the argument of the eigenvalues of the linearized Poincaré map. When $\mu(P) = 1$ and P is elliptic, the angular speed may be slow but bounded away from zero. The foliation gives an open book decomposition of a tubular neighbourhood of P and the flow is transversal to the foliation. All the transverse asymptotic vectors $e(t)$ of the leaves with asymptotic limit P must have the same the same (finite) winding number with respect to Z. Then the orbits of the flow in a neighbourhood of P return (in the future and in the past) to each leaf with asymptotic limit P at a uniformly finite time (c.f. [18, lemma 5.2]). Since there are only a finite number of rigid surfaces, there must be at least one 1-dimensional family of leaves F_τ with index $\mathrm{Ind}(F) = 2$. If $f\lambda_0$ is not dynamically convex, then the family decomposes on two rigid surfaces, at least one of them having P as an asymptotic limit. Thus if $f\lambda_0$ is not dynamically convex, then on a neighbourhood of P, the orbits have uniformly finite return times to a rigid surface.

In the case of a non dynamically convex contact form, the points which lie in a stable manifold of a binding orbit P of μ-index equal to 2 eventually do not return in the future to a rigid surface. Similarly, the points in the unstable manifold of P eventually do not return in the past to a rigid surface. Any other point x which is not in a binding periodic orbit returns to a rigid surface in a finite (but not uniformly bounded) time. Because flowing it a bit if necessary, it lies in an I-family F_τ of leaves of index $\mathrm{Ind}(F_\tau) = 2$. The parameter $\tau(x, t)$ defined by $\varphi_t(x) \in F_{\tau(x,t)}$ is strictly monotonous. Its derivative does not approach to zero if it is not in the local stable manifold of a binding orbit of μ-index equal to 2. Then it reaches one end of the interval where the leaves decompose into rigid leaves.

Hence in the non dynamically convex case:

- The set of rigid surfaces $\mathbf{\Sigma} := \mathcal{S}_1$ is composed of finitely many connected components, each with finite $d\lambda$-area and the form $d\lambda$ is non-degenerate on each of them.
- The points which are not in the stable manifold of a binding periodic orbit of Conley-Zehnder index 2 return to $\mathbf{\Sigma}$ infinitely many times in the future.

- The first return map f is an area preserving local diffeomorphism. It is defined in all $\mathbf{\Sigma}$ but the first intersection of the unstable manifolds of binding periodic orbits of index 2 with $\mathbf{\Sigma}$.

In the dynamically convex case, the foliation contains an S^1-family of leaves with index $\mathrm{Ind}(F) = 2$. Then S^3 decomposes as the leaves of the family plus the binding orbits: $S^3 = \mathcal{S}_0 \cup S_2$. Since the foliation contains a finite energy plane (with only one puncture), the leaves of the family are topological open discs and there is only one binding periodic orbit P with index $\mu(P) = 3$ by table (6). This gives an open book decomposition of S^3. In such case it has no rigid leaves but any leaf of the family is a (disc-like) global transversal section for the Reeb flow restricted to $S^3 \setminus \{P\}$. Thus in the dynamically convex case we have that

- every orbit of the flow intersects infinitely many times the (disc-like) section $\mathbf{\Sigma}$,
- $d\lambda|_{\mathbf{\Sigma}}$ is an area form on $\mathbf{\Sigma}$,
- the first return time is finite and uniformly bounded,
- the first return map $f : \mathbf{\Sigma} \to \mathbf{\Sigma}$ is an area preserving diffeomorphism, with area form $d\lambda|_{\mathbf{\Sigma}}$.

Define the diameter of a set $A \subset S^3$ by

$$\mathrm{diam}(A) := \sup_{x,y \in A} d_{S^3}(x,y)$$

We shall need the following

2.7. Proposition.

Suppose that the Reeb vector field of the contact form $\lambda = f\lambda_0$ is Kupka-Smale, i.e. all its periodic orbits non-degenerate and the stable and unstable manifolds of its hyperbolic orbits, when they intersect do it transversally. Furthermore, suppose that (S^3, λ) is not dynamically convex.

Then there exists $A > 0$ such that if P is a binding orbit of Conley-Zehnder index $\mu(P) = 2$ and $\mathcal{W}$ is a connected component of $W^s(P) \cap \mathbf{\Sigma}$ or $W^u(P) \cap \mathbf{\Sigma}$, then $\mathrm{diam}\,\mathcal{W} \geq A$.

Proof:

Let $\mathbb{P}$ be a binding periodic orbit of Conley-Zehnder index $\mu(\mathbb{P}) = 2$. We will prove the proposition for $W^s(\mathbb{P}) \cap \mathbf{\Sigma}$. The proof for the unstable manifold is similar, flowing the Reeb flow forwards.

Since the Conley-Zehnder index of $\mathbb{P}$ is 2, $\mathbb{P}$ is (+)-hyperbolic, i.e. its eigenvalues are positive. Then the stable manifold $W^s(\mathbb{P})$ has two connected components which are immersed open cylinders. Observe that $d\lambda|_{W^s(\mathbb{P})} \equiv 0$, because $W^s(\mathbb{P})$ is invariant under the Reeb flow and hence its tangent space contains the Reeb vector field X, but $i_X d\lambda \equiv 0$ on S^3.

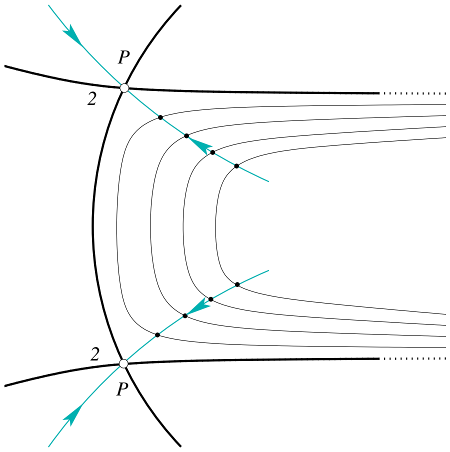


FIGURE 5. This figure shows how the stable manifold of a binding orbit of Conley-Zehnder index 2 cuts embedded circles on nearby leaves of the foliation belonging to a 1-parameter family.

Choose one connected component of $W^s(\mathbb{P})$. By lemma 7.6 in [21] there is a 1-dimensional family of leaves F_τ parametrized by $\tau \in [0,1]$ which at $\tau \to 0$ decomposes into two rigid surfaces (C^+, C^-) having P as a common asymptotic limit such that the chosen component of the local stable manifold $W^s_{loc}(\mathbb{P})$ intersects F_τ in a smooth embedded circle S_τ for all τ close to 0. See figure 5. Flowing $W^s_{loc}(\mathbb{P})$ backwards, since the flow is transversal to the foliation, the stable manifold keeps intersecting the leaves F_τ in smooth embedded circles for all $\tau \in]0,1[$. When $\tau \to 1$, the family decomposes into two rigid orbits (C_1^+, C_1^-) with a common binding orbit P_1 of index 2. Recall that (lemma 5.2 in [18]) in a neighbourhood of a binding orbit of odd μ-index the solutions make a full turn around the binding orbit in a short interval of time. When the family of loops $\mathbf{\Sigma}_\tau$ arrives at $\tau \to 1$ to the rigid surfaces either the connected component of intersection $W^s(\mathbb{P}) \cap (C_1^+ \cup C_1^-)$ is inside one of the rigid surfaces, and in that case the connected component $\mathbf{\Sigma}_1$ is also an embedded circle, or it intersects both surfaces.

2.8. **Lemma.** *There exist a positive constant*

$$0 < A < \min\{\, \mathrm{diam}(P) \mid P \ \text{ binding orbit}\,\},$$

such that if

- *R is a rigid surface of the foliation given by theorem 2.5.*
- *P is a binding orbit of Conley-Zehnder index $\mu(P) = 2$.*
- *S is a connected component of $W^s(P) \cap R$ or of $W^u(P) \cap R$ which is an embedded circle in R.*

Then $\quad \mathrm{diam}(S) > A$.

Proof: We only prove the lemma for the stable manifold, the same prove applies to the unstable manifold. Observe that the binding periodic orbit P must be $(+)$-hyperbolic. Then the connected components of $W^s(P)$ are two topological cylinders. Let $\mathcal{W}$ be the connected component of $W^s(P)$ which contains S. Then S is also an embedded circle in $\mathcal{W}$. Hence its homotopy class is either 0 or ± 1 in $\pi_1(\mathcal{W}) = \mathbb{Z}$.

If S is homotopically trivial then it bounds a disk in $\mathcal{W}$. Since the Reeb vector field X is tangent to $\mathcal{W}$ and has no singularities, there is a point $s \in S$ where $X(s)$ is tangent to S. Since S is also included in the leaf R and the vector field X is transversal to F, $X(s)$ can not be tangent to S. Therefore S is not homotopically trivial.

Then S is either homologous to P or to $-P$ inside $\mathcal{W}$. Choose the orientation in S such that it is homologous to P. Since S is a simple closed curve in $\mathcal{W}$, there is an embedded surface $Q \subset \mathcal{W}$ such that as a 2-chain $\partial Q = S - P$. Since $d\lambda|_{W^s(P)} \equiv 0$, by Stokes theorem

$$\oint_S \lambda = \int_Q d\lambda + \oint_P \lambda = \oint_P \lambda = T(P),$$

where $T(P)$ is the minimal period of P (recall that the binding orbits are simply covered).

Let $\mathbb{Y}$ be the set of closed continuous curves $y : S^1 \to \mathbf{\Sigma}$ inside a rigid surface which are not homotopically trivial. Since the set of rigid surfaces is finite and their ends are a finite set of closed orbits,

$$a := \tfrac{1}{2} \inf\{\, \operatorname{diam}(y) \mid y \in \mathbb{Y} \,\} > 0$$

is positive. Then if S is not homotopically trivial on the rigid surface R then

$$\operatorname{diam}(S) > a.$$

If S is homotopically trivial on R then it bounds an embedded disk $D \subset R$. Then

$$\int_D d\lambda = \oint_S \lambda > T(P).$$

Let

$$b := \min\{\, T(P) \mid P \text{ binding orbit } \} > 0.$$

Since the rigid surfaces are embedded, transversal to the non-singular vector field X, with compact closure and finitely many,

$$c := \inf\{\, \operatorname{diam} D \mid D \hookrightarrow \mathbf{\Sigma} \text{ embedded disk, } d\lambda\text{-area}(D) > b \,\} > 0$$

is positive. Then

$$\operatorname{diam}(S) = \operatorname{diam}(D) > c.$$

Now let

$$A := \min\{\, a,\ b,\ c \,\} > 0.$$

□

In the case the family $\mathbf{\Sigma}_\tau$ intersects at an embedded circle $\mathbf{\Sigma}_1$ only one of the rigid surfaces, say C_1, we use lemma 2.8 to show that $\operatorname{diam}(\mathbf{\Sigma}_1) > A$ and continue flowing backwards along a family of leaves F_τ, $\tau \in]1, 2[$. The intersection $\mathbf{\Sigma}_1$ continues as embedded circles $\mathbf{\Sigma}_\tau$ for all $\tau \in]1.2[$ and we repeat the argument.

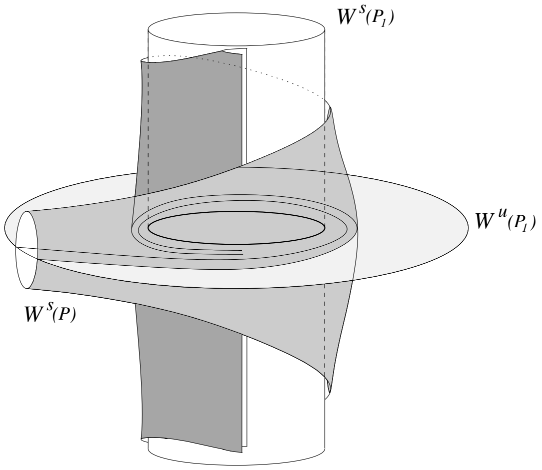


FIGURE 6. This figure shows how the stable manifold of an hyperbolic periodic orbit P accumulates on the whole stable manifold of another hyperbolic periodic orbit when there is an heteroclinic transversal intersection.

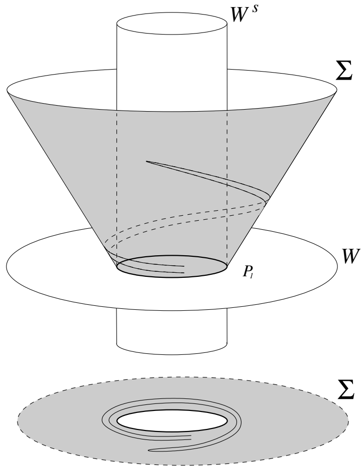


FIGURE 7. This figure shows the approach of rigid surface Σ to an asymptotic limit P_1 of index $\mu(P_1) = 2$. If the unstable manifold of P_1 intersects transversally the stable manifold $W^s(P)$ of another binding orbit P, as in figure 6, then figure 7 also shows how the intersection of $W^s(P)$ with the rigid surface Σ accumulates on the whole asymptotic limit P_1. The lower part of the figure is a representation of a neighbourhood of the puncture corresponding to P_1 in the rigid surface Σ and the intersection $W^s(P) \cap \Sigma$.

In the case the family $\mathbf{\Sigma}_\tau$ hits both rigid surfaces C_1^+ and C_1^- then in the limit $\tau \to 1$ the stable manifold must intersect the unstable manifold of binding orbit P_1 of index 2 which is a common asymptotic limit to C_1^+ and C_1^-. The family S_τ decomposes into three sets: a subset in $W^s(\mathbb{P}) \cap W^u(P_1)$, which for negative time has no return to the union of rigid surfaces $\mathbf{\Sigma}$; and two connected components of $W^s(\mathbb{P}) \cap \mathbf{\Sigma}$, one component $\mathbf{\Sigma}_1$ in C_1^+ and another $\mathbf{\Sigma}_1^-$in C_1^-. Both components are immersions of the real line $\mathbb{R}$ in C_1 or C_1^- and both contain the periodic orbit P_1 in their closure. Hence

$$\operatorname{diam}(\mathbf{\Sigma}_1^+) \geq \operatorname{diam}(P_1) > A,$$

and similarly $\operatorname{diam}(\mathbf{\Sigma}_1^-) > A$.

Choose one of the components, say $\mathbf{\Sigma}_1^+$. When we continue flowing backwards $\mathbf{\Sigma}_1^+$, it accumulates on a branch of the stable manifold $W^s(P_1)$ by the λ-lemma. The argument repeats, the branch of $W^s(P_1)$ cuts a family of surfaces F_τ parametrized by $\tau \in]1,2[$ on embedded circles S_τ^1, and the backward flow of the component $\mathbf{\Sigma}_1^+$ intersects the family on connected subsets which accumulate on *the whole* $S_\tau^1 \subset W^s(P_1) \cap F_\tau$, (see figures 6, 7).

When $\tau \to 2$ the family F_τ decomposes into two rigid surfaces C_2^+, C_1^- and the family of connected components $\mathbf{\Sigma}_\tau^+ \subset W^s(\mathbb{P}) \cap F_\tau$ either intersects only one of them or both. If it intersects only one, say C_2^+, the intersection $\mathbf{\Sigma}_2$ is in the interior of C_2^+ and its closure contains the intersection $S_{\tau=2}^1 \subset W^s(P_1) \cap C_2^+$, which is an embedded circle. Then

$$\operatorname{diam}(\mathbf{\Sigma}_{\tau=2}^+) \geq \operatorname{diam}(S_{\tau=2}^1) > A.$$

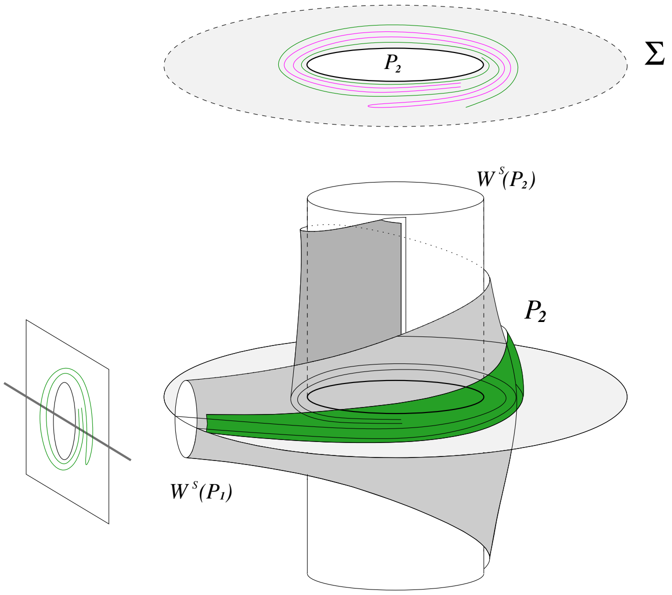


FIGURE 8. This figure shows how the backward flow of a small interval J of a stable manifold $W^s(P)$ which intersects transversally the unstable manifold of a binding orbit P_2 accumulates on the unstable manifold of P_2 and its intersection on a rigid surface with asymptotic limit P_2 contains the orbit P_2 in its closure. In this case, $W^s(P)$ first accumulates on $W^s(P_1)$ which, in turn, accumulates on $W^s(P_2)$.

If the family $\mathbf{\Sigma}_\tau^+$ as $\tau \to 2$ intersects both surfaces C_2^+ and C_1^+, since $\mathbf{\Sigma}_\tau^+$ is connected, it intersects (transversally) $W^u(P_2)$. Choose for example C_2^+. Let $\mathbb{G}$ be a connected component of the intersection of $\mathbf{\Sigma}_\tau^+$ as $\tau \to 2$ with C_2^+. The $\mathbb{G}$ is a 1-dimensional submanifold whose forward flow contains an open segment $\mathbb{J} \subset W^s(\mathbb{P})$ with an endpoint in $W^u(P_2)$. which is transversal to $W^u(P_2)$. Flowing backwards $\mathbb{G}$ as in figure 8, we have that the connected component $\mathbb{G} \subset C_2^+$ must accumulate on the whole periodic orbit P_2. Hence

$$\operatorname{diam}(\mathbb{G}) \geq \operatorname{diam}(P_2) > A.$$

Flowing backwards, we have that $\mathbb{G}$ accumulates on the unstable manifold $W^u(P_2)$, and the argument goes on.

Alternatively, we could say that the connected component $\mathbb{G}$ accumulates on a whole component B of $W^s(P_1) \cap C_2^+$, which in turn accumulates on the whole periodic orbit P_2. Thus

$$\operatorname{diam}(\mathbb{G}) \geq \operatorname{diam}(B) \geq \operatorname{diam}(P_2) > A;$$

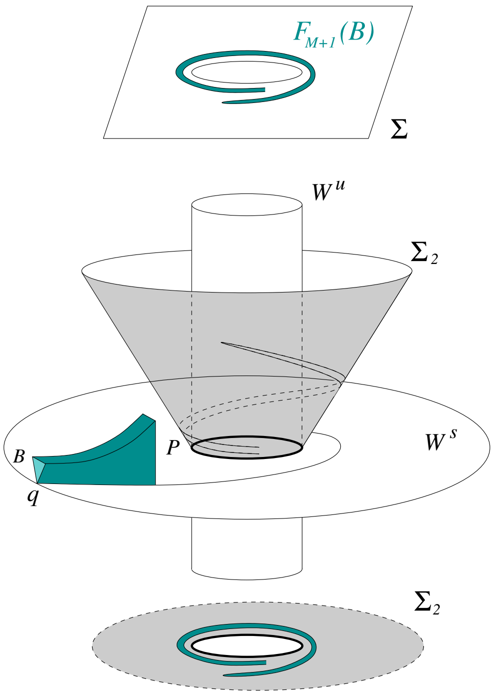
and the argument goes on. $\square$

We shall be interested in returns to a rigid surface. Let Σ be a rigid surface. Define the return times $\tau_n : \Sigma \to \mathbb{R} \cup \{+\infty\}$ and return maps $F_n : [\tau_n < +\infty] \to \Sigma$ to Σ by

$$\begin{aligned} \tau_0 &:\equiv 0, \\ F_n(x) &:= \varphi_{\tau_n(x)}(x), \qquad \text{if} \quad \tau_n(x) < +\infty\,; \\ \tau_{n+1}(x) &:= \inf\{\, t > \tau_n(x) \mid \varphi_t(x) \in \Sigma\,\}. \end{aligned}$$

2.9. **Proposition.** *Suppose that the pair (S^3, λ) satisfies the hypothesis of proposition 2.7. Let $A > 0$ be the constant given by proposition 2.7. Let B be an open subset of a rigid surface Σ such that $\tau_N|_B$ is finite but not bounded for some $N > 0$. Then* $\operatorname{diam}(F_N(B)) > A$.

Proof: Since $\tau_N|_B$ is not bounded, there is a point q in the boundary ∂B such that $\tau_N(q) = +\infty$. Let $M := \max\{\, k \geq 0 \mid \tau_k(q) < +\infty \,\}$. Since $\tau_0 \equiv 0$, we have that $0 \leq M \leq N-1$.



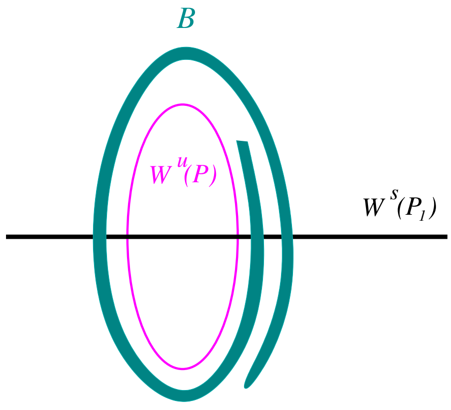


FIGURE 10. This figure shows how if $W^u(P)$ intersects another stable manifold $W^s(P_1)$ before returning to Σ, then $W^s(P_1)$ must intersect the interior of the forward iterate of B before it returns to Σ.

FIGURE 9. The figure shows how the forward flow of an open set B in a rigid surface Σ with a boundary point q in the stable manifold $W^s(P)$ intersects a rigid surface with asymptotic limit P in a subset which accumulates on the *whole* orbit P. If its forward flow does not intersect other stable manifolds of binding perioidc orbits of index 2, then its returns to Σ are sets which contain in their closure an intersection of the unstable manifold $W^u(P)$ of P which is a topological circle.

Since $M < N$, $\tau_M|B < +\infty$. By the continuity of the flow the set $[\tau_M < +\infty]$ is open. The map F_M is a local diffeomorphism on the open set $[\tau_M < +\infty]$. We have that $F_M(B) \subseteq \Sigma$, $F_M(q) \in \partial F_M(B)$. Moreover, $F_M(q) \in W^s(P) \cap \Sigma$ for a binding periodic orbit P with index $\mu(P) = 2$. Flowing forward $F_M(B)$ it reaches a neighbourhood of P, then intersects a rigid surface accumulating on the whole orbit P as in figure 9, and then it follows accumulating on $W^u(P)$. The forward orbit of $F_M(q)$ accumulates on the periodic orbit P and does not return to Σ.

Suppose that $W^u(P)$ intersects another stable manifold $W^s(P_1)$ of a binding periodic orbit P_1 with $\mu(P_1) = 2$, with point accumulating on P_1 before it intersects Σ. Then by hypothesis $W^u(P)$ intersects $W^s(P_1)$ transversally. Since the forward flowing of $F_M(B)$ accumulates on the whole $W^u(P)$, it intersects $W^s(P_1)$ before it returns to Σ (see figure 10). This implies that there are points $x \in B$ for which $\tau_{M+1}(x) = +\infty$. Since $M+1 \le N$, this contradicts the hypothesis $\tau_N|_B < +\infty$.

Then the unstable manifold $W^u(P)$ intersects Σ in a circle S_{M+1} and $F_{M+1}(B)$ accumulates on the whole circle S_{M+1}. By lemma 2.8,

$$\operatorname{diam}(F_{M+1}(B)) = \operatorname{diam}(\overline{F_{M+1}(B)}) \ge \operatorname{diam} S_{M+1} > A.$$

If $M+1 < N$, for each of the following iterates F_k, $M+1 < k \le N$, the argument above show that the forward flowing of S_{M+1} does not intersect a stable manifold of a binding orbit of index 2 before returning to Σ. Then $S_{M+1} \subset [\tau_{N-M-1} < +\infty]$, $S_N := F_{N-M-1}(S_{M-1})$ is a circle and $F_B(B)$ accumulates on S_N. Then $\operatorname{diam} F_N(B) \ge \operatorname{diam} S_N > A$.

□

3. Proof of theorem A .

The Kupka-Smale theorem for geodesic flows is proven in [8], we recall it here. Let $J^r_s(n-1)$ be the set of r-jets of symplectic automorphisms of $\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$ that fix the origin. Clearly one can identify $J^1_s(n-1)$ with $Sp(n-1)$. A set $Q \subset J^r_s(n-1)$ is said to be *invariant* if for all $\sigma \in J^r_s(n-1)$, $\sigma Q \sigma^{-1} = Q$.

Kupka-Smale theorem for geodesic flows. *Let $Q \subset J^{r-1}_s(n-1)$ be open, dense and invariant. Then there exists a residual subset $\mathcal{O} \subset \mathcal{G}^r$ such that for all $g \in \mathcal{O}$:*

- *The $(r-1)$-jet of the Poincaré map of every closed geodesic of g belongs to Q.*
- *All heteroclinic points of hyperbolic closed geodesics of g are transversal.*

Let M be a closed 2-dimensional smooth manifold and let $\mathcal{H}^1(M)$ be the set of C^r riemannian metrics, $r \ge 4$ on M all of whose closed geodesics are hyperbolic, endowed with the C^2 topology and let $\mathcal{F}^1(M) = \operatorname{int}\big(\mathcal{H}^1(M)\big)$ be the interior of $\mathcal{H}^1(M)$ in the C^2 topology. If $g \in \mathcal{F}^1(M)$, let $\overline{\operatorname{Per}(g)}$ be the closure of the periodic orbits of the geodesic flow of g.

A hyperbolic set Λ is said *locally maximal* if there exists an open neighbourhood U of Λ, such that Λ is the maximal invariant subset of U, i.e. $\Lambda = \bigcap_{t\in\mathbb{R}} d\phi^g_t(U)$. A *basic set* is a locally maximal hyperbolic set with a dense orbit. It is *non-trivial* if it is not a single closed orbit. It follows that in a basic set the periodic orbits are dense in its relative non-wandering set [22, cor. 6.4.19], [4, 5].

3.1. Theorem ([8, prop. 5.5, cor. 5.9]).

If $g \in \mathcal{F}^1(M)$ then the closure $\overline{\operatorname{Per}(g)}$ is hyperbolic. Moreover, $\overline{\operatorname{Per}(g)}$ decomposes into a finite number of hyperbolic basic sets and at least one of them is non-trivial.

The non-triviality of the basic set in this theorem (c.f. [8, cor. 5.9]) is obtained from the fact that the geodesic flow for M has infinitely many closed orbits. If M is non simply connected, then $\pi_1(M)$ is infinite, and any riemannian metric has a minimizing closed geodesic in each free homotopy class. If $M = S^2$, then Bangert [1] and Franks [11] proved that M has infinitely many geometrically distinct geodesics. For the projective space $\mathbb{P}^2$, lifting the riemannian metric on $\mathbb{P}^2$ to its double cover $S^2 \to \mathbb{P}^2$ one obtains that also $\mathbb{P}^2$ has infinitely many closed geodesics.

The arguments in the above theorem need C^2 perturbations. For example, it is not known if a riemannian metric whose geodesic flow is topologically equivalent to an Anosov flow can be C^r approximated, $r \geq 3$, by a metric whose geodesic flow is Anosov.

Define an equivalence relation on $\text{Per}(g)$ by saying that $x \sim y$ if and only if $W^s(x) \cap W^y(y) \neq \emptyset$ and $W^u(x) \cap W^s(y) \neq \emptyset$. Its equivalence classes are called *homoclinic classes.* The spectral decomposition theorem [35], [22, ex. 18.3.7], used to obtain the basic set in theorem 3.1, states that the basic set is a *whole homoclinic class.*

Proof of theorem A :

Assume by contradiction that (S^2, g_0) is a C^∞ riemannian metric which can not be C^2 approximated by one having an elliptic closed geodesic. Then $g \in \mathcal{F}^1(S^2)$. We can assume that g is a Kupka-Smale metric. By theorem 3.1, there is a hyperbolic non-trivial basic set Λ for the geodesic flow ϕ_t of (S^2, g). Let m be the normalized Liouville's measure for (T^1S^2, ϕ_t).

Since g is C^r, $r \geq 3$, we have that f is C^2. Bowen and Ruelle [6, cor. 5.7.(b)] proved that since f is C^2, if Λ has positive measure, then Λ must be open. Since Λ is closed, either $m(\Lambda) = 0$ or $\Lambda = \Sigma$. If $\Lambda = \Sigma$, then the geodesic flow must be Anosov. But there are no Anosov[2] geodesic flows for S^2. Thus $m(\Lambda) = 0$.

Let $Q = W^s(\Lambda) \cup W^u(\Lambda) = \{\, x \in \Sigma \mid \alpha\text{-lim}(x) \subseteq \Lambda \text{ or } \omega\text{-lim}(x) \subseteq \Lambda \,\}$. Then Hirsch, Palis, Pugh and Shub [16], proved that

$$Q = \bigcup_{t\in\mathbb{R}} \bigcup_{x\in\Lambda} \phi_t\big(W^{ss}_\varepsilon(x) \cup W^{uu}_\varepsilon(x)\big); \tag{7}$$

where for $x \in \Lambda$, $W^{ss}_\varepsilon(x) := \cap_{t\geq 0}\, \phi_{-t}\big(B_\varepsilon(\phi_t(x))\big)$ and $W^u_\varepsilon(x) := \cap_{t\geq 0}\, \phi_t\big(B_\varepsilon(\phi_{-t}(x))\big)$ are the local strong stable and unstable manifolds. Since the homoclinic class Λ for the *flow* ϕ_t is transitive, Q is connected[3]. The points in $Q \setminus \Lambda$ are wandering. By Poincaré's recurrence theorem $m(Q \setminus \Lambda) = 0$. Hence $m(Q) = 0$.

We say that a point $x \in \Lambda$ is an *interior point of* Λ, if x is an accumulation point of Λ on both connected components of $W^{ss}_\varepsilon(x)$ and also of $W^{uu}_\varepsilon(x)$. Since Λ is non-trivial, it has infinitely many interior points.

We shall need the following two lemmas:

[2] because $\pi_1(T^1S^2)$ would have exponential growth (Margulis [28], Plante & Thurston [33] or Klingenberg [23]) or also because (S^2, g) would not have conjugate points (Klingenberg [23]).

[3] but its intersections with a transversal "surface of section" may be not connected.

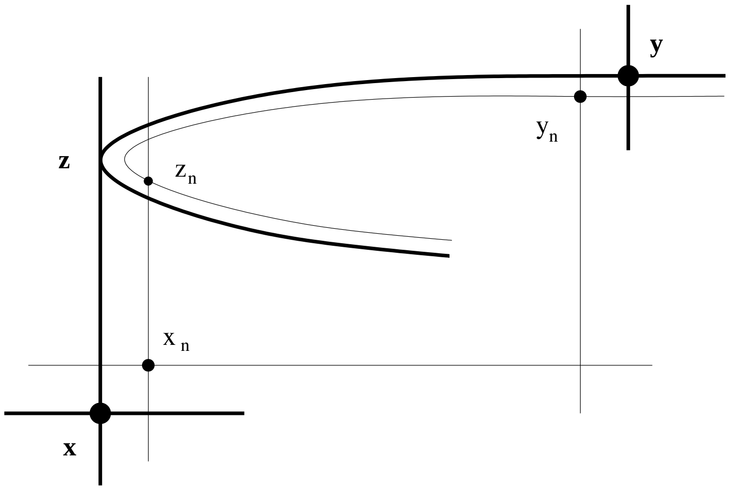


FIGURE 11. This figure shows how if one of x, $y \in \Lambda$ is an interior point of the homoclinic class Λ, then any $z \in W^s(x) \cap W^u(y)$ is accumulated by homoclinic points z_n of periodic points x_n, y_n in the same homoclinic class Λ.

3.2. **Lemma.** $\Lambda = W^s(\Lambda) \cap W^u(\Lambda)$.

Proof: Clearly $\Lambda \subseteq W^s(\Lambda) \cap W^u(\Lambda)$. Suppose that $z \in W^s(\Lambda) \cap W^u(\Lambda)$. By (7) there are $x, y \in \Lambda$ such that $z \in W^s(x) \cap W^u(y)$ or $z \in W^u(x) \cap W^s(y)$. Suppose it is the first case, the second is similar. We have to prove that $z \in \Lambda$.

The point z may be in a tangential intersection of $W^s(x)$ with $W^u(y)$ or in a topologically transversal intersection. The topologically transversal case is easier and uses the same methods, so we prove only the tangential case.

Suppose that x is an interior point of Λ. The case when y is an interior point of Λ is similar. Then x is accumulated by points of Λ in its four quadrants made by $W^s_{loc}(x)$ and $W^u_{loc}(x)$. Since the periodic points are dense in Λ, there exist sequences $x_n \to x$, $y_n \to y$ in $\Lambda \cap \mathrm{Per}(g)$ and points $z_n \in W^s(x_n) \cap W^u(y_n)$ such that $\lim_n z_n = z$, as in figure 11. Since g is Kupka-Smale and x_n, y_n are periodic points, the intersections $W^s(x_n) \pitchfork W^u(y_n)$ are transversal. Since x_n, y_n are in the homoclinic class Λ, we also have that $W^u(x_n) \cap W^s(y_n) \neq \emptyset$. Then there are hyperbolic sets H inside the homoclinic class Λ which accumulate on z_n, see figure 12. Therefore $z_n \in \Lambda$. Since Λ is closed, $z = \lim_n z_n \in \Lambda$.

Now suppose that x and y are not interior points of Λ. Then (cf. [3, prop. 2.1.1] or [30]) they are in the invariant manifolds of periodic points. Then the intersection of $W^s(x)$ and $W^u(y)$ at z is transversal. The same argument of last paragraph then shows that $z \in \Lambda$.

□

3.3. **Lemma.**

Let x be an interior point of the homoclinic class Λ. Then there is $\varepsilon = \varepsilon(x) > 0$ such that the set $Q \cap \overline{B_\varepsilon(x)}$ is closed, where $B_\varepsilon(x)$ is the ε-ball centered at x.

Proof: Observe that for $\delta > 0$ small enough, the set

$$Q_\delta := \bigcup_{x \in \Lambda} \overline{W^{ss}_\delta(x)} \cup \overline{W^{uu}_\delta(x)}$$

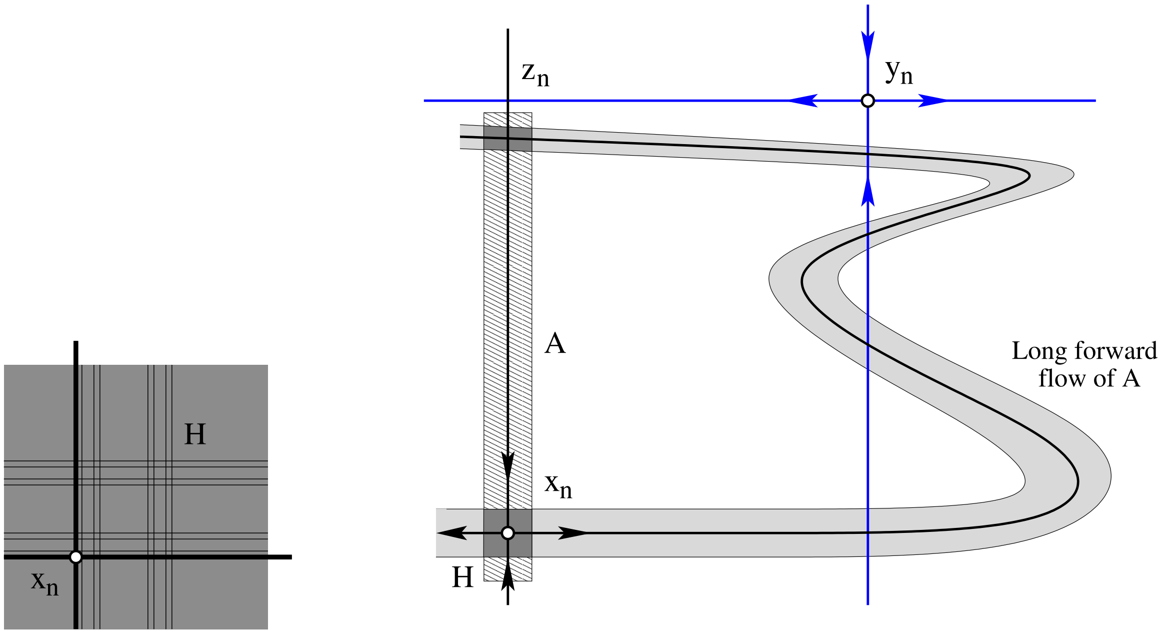


FIGURE 12. This figure shows why the point $z_n \in W^s(x_n) \cap W^u(y_n)$ is in the homoclinic class Λ. Since also $W^u(x_n) \cap W^s(y_n) \neq \emptyset$, the point z_n is accumulated by small (transitive) horseshoes $H = H_m$, each of them also accumulating on x_n. By the local product structure on the uniformly hiperbolic set $\overline{\text{Per}(g)}$ the homoclinic classes are (relatively) open subsets of $\overline{\text{Per}(g)}$. Thus, when this construction is made with a sufficiently thin rectangle A and a very large forward iterate by the flow, the periodic points in the horseshoe H are in the same homoclinic class Λ as x_n. In fact the whole (transitive) horseshoe is inside the homoclinic class Λ, because, by the shadowing lemma, for any $\varepsilon > 0$, there are ε-dense periodic orbits on Λ.

is closed. It is enough to prove that for $\varepsilon = \varepsilon(x) > 0$ sufficiently small, $B_\varepsilon(x) \cap Q = B_\varepsilon(x) \cap Q_\delta$.

Let $\mathcal{R}$ be a small surface transversal to the vector field of ϕ_t containing x in its interior, which is a topological rectangle whose sides are in two weak stable manifolds and two weak unstable manifolds:

$$\partial\mathcal{R} \subset \bigcup_{i=1,2} W^s_\varepsilon(x_i) \cup W^u_\varepsilon(x_i), \qquad \text{with } x_1,\ x_2 \in \Lambda.$$

Since x is an interior point of Λ, the diameter of $\mathcal{R}$ can be chosen arbitrarily small. Let $\tau > 0$ be such that $\phi_t(\mathcal{R}) \cap \phi_s(\mathcal{R}) = \emptyset$ for all $-\tau \leq t < s \leq \tau$. For simplicity assume that $\tau = 2$.

Recall that (cf. [16, §6] or [17]) since Λ is uniformly hyperbolic, its local weak invariant manifolds $W^s(x)$, $W^u(x)$, $x \in \Lambda$ are graphs of functions $A(x) : E^s(x) \oplus \langle \frac{d}{dt}\phi_t \rangle \to E^u(x)$ whose domain contains a ball of some fixed radius $2\,\rho_1 > 0$. In particular their diameters are larger than ρ_1. Similarly for the strong invariant manifolds.

We take the diameter of $\mathcal{R}$ small enough so that if $z \in \phi_s(\mathcal{R}) \cap \Lambda$, $s \in [-1,1]$, then $\big[\cup_{t\in[-\delta,\delta]} W^{ss}_\delta(\phi_t(z))\big] \cap \phi_s(\mathcal{R})$ and $\big[\cup_{t\in[-\delta,\delta]} W^{uu}_\delta(\phi_t(z))\big] \cap \phi_s(\mathcal{R})$ are differentiable curves which cross the rectangle $\phi_s(\mathcal{R})$. Now take $\varepsilon = \varepsilon(x) > 0$ such that $\overline{B_\varepsilon(x)} \subset \phi_{[-1,1]}(\mathcal{R})$.

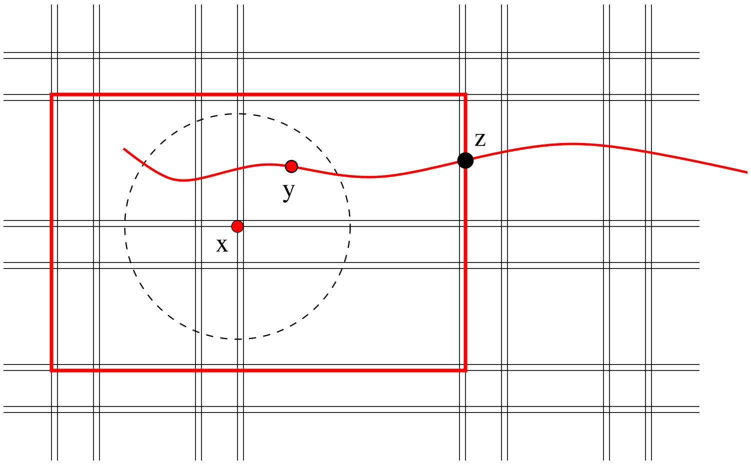


FIGURE 13

Let $y \in Q \cap \phi_s(\mathcal{R})$, $s \in [-1,1]$, and suppose that $y \in W^s(w)$, with $w \in \Lambda$. The case $y \in W^u(w)$, $w \in \Lambda$ is similar. Consider the connected component Γ of the intersection of the weak stable manifold $W^s(w)$ with the rectangle $\phi_s(\mathcal{R})$. Suppose first that Γ intersects the boundary of $\phi_s(\mathcal{R})$, necessarily at one of its endpoints z, as shown in figure 13. Then z is in the intersection of $W^s(w)$ with $W^u(x_1)$ or $W^u(x_2)$. Therefore, by lemma 3.2, z is in the homoclinic class Λ. By our choice of the diameter of $\mathcal{R}$, we have that $y \in \cup_{t\in[-\delta,\delta]} W^{ss}_\delta(\phi_t(z))$. Hence $y \in Q_\delta$.

Now suppose that Γ does not intersect the boundary of $\phi_s(\mathcal{R})$. The curve $\Gamma = W^s(w) \cap \phi_s(\mathcal{R}) \subset \text{int}(\mathcal{R})$ must have at least one well defined endpoint z, which is in the orbit of some $w \in \Lambda$. By our choice of the size of $\mathcal{R}$, the stable manifold $W^s(z) = W^s(w)$ of $z \in \Lambda \cap \phi_s(\mathcal{R})$ crosses $\phi_s(\mathcal{R})$. Therefore $\Gamma \cap \partial\,[\phi_s(\mathcal{R})] \neq \emptyset$ and hence this case can not occur.

We have proved that $Q \cap \overline{B_\varepsilon(x)} \subseteq Q \cap \phi_{[-1,1]}(\mathcal{R}) \subseteq Q_\delta$.

$\square$

Since g is a Kupka-Smale metric on S^2, by proposition 1.1 we can lift the geodesic flow to a reparametrization of the Reeb flow of a non-degenerate tight contact form on S^3.

The dynamically convex case.

In this case there is a closed orbit γ for the lifted geodesic flow ψ_t to S^3 and a smooth 2-dimensional open disk $\mathbb{D} \subset S^3$ such that $\partial\mathbb{D} = \gamma$, $\mathbb{D}$ is transversal to the lifted geodesic flow ψ_t and intersects every orbit in $S^3 \setminus \gamma$ infinitely many times. Moreover, the differential $d\lambda$ of the lifted Liouville's form λ is non-degenerate on $\mathbb{D}$, the total area $\int_{\mathbb{D}} d\lambda = \text{period}(\gamma)$ is finite and the Poincaré's return map $f : \mathbb{D} \to \mathbb{D}$ is C^2 and preserves λ. Also the first return time $\tau : \mathbb{D} \to]0, +\infty[$ is uniformly bounded above and below.

Let $\widehat{\Lambda}$ be the lift of the basic set Λ to S^3 and let $\Gamma := \widehat{\Lambda} \cap \mathbb{D}$. Let μ be the normalized $d\lambda$-measure on $\mathbb{D}$. The invariant measure for ψ_t induced by μ is the lift of the Liouville's measure m. Then $\mu(\Gamma) = 0$. Since there are $0 < a < b$ such that the return time τ is uniformly bounded: $a < \tau(x) < b$, $\forall x \in \mathbb{D}$, the set Γ is a uniformly hyperbolic set for the Poincaré map $f(x) = \psi_{\tau(x)}(x)$.

The hyperbolic set Γ may contain the boundary closed orbit $\gamma = \partial\mathbb{D}$. In that case Γ accumulates on γ and in particular is not closed. But nevertheless it is locally compact *inside* $\mathbb{D}$.

Since Γ is non-trivial it has an interior point x. Let $\epsilon := \min\{\varepsilon(x), \frac{1}{2}\, d(x, \partial\mathbb{D})\} > 0$, where $\varepsilon(x)$ is from lemma 3.3. Let $B_\epsilon(x)$ be the ball of radius ϵ in $\mathbb{D}$. Let $\mathcal{Q}$ be the lift of Q intersected with $\mathbb{D}$. Then $\mathcal{Q} = W^s(\Gamma) \cap W^u(\Gamma)$. Since the points in $\mathcal{Q} \setminus \Gamma$ are wandering, $\mu(\mathcal{Q}) = \mu(\Gamma) + \mu(\mathcal{Q} \setminus \Gamma) = 0$. Let R be a connected component of $\mathbb{D} \setminus Q$ which is contained in $B_\varepsilon(x)$. Such R always exist because x is an interiror point of Γ, $\overline{B_\varepsilon(x)} \cap \mathcal{Q}$ is closed by lemma 3.3 and $\mu(\mathcal{Q}) = 0$. Moreover R is a topological rectangle whose sides are pieces of stable and unstable manifolds of points in Γ:

$$\partial R \subset W^s_\varepsilon(x_1) \cup W^s_\varepsilon(x_2) \cup W^u_\varepsilon(y_1) \cup W^u_\varepsilon(y_2).$$

Since R has positive $d\lambda$-measure, by Poincaré's recurrence there exists $N > 0$ such that $f^N(R) \cap R \neq \emptyset$. Since Q is invariant and R is a connected component of $\mathbb{D} \setminus Q$, we have that $f^N(R) = R$. Since[4] f^N is continuous, $f^N(\partial R) = \partial R$. Since local stable manifolds are sent to local stable manifolds, using twice the period if necessary, we have that $f^{2N}(\overline{I_1}) \subset \overline{I_1}$, where $I_1 := W^s_\varepsilon(x_1) \cap R$. By the intermediate value theorem there is a fixed point $y \in \overline{I_1}$, $f^{2N}(y) = y$. Since $g \in \mathcal{F}^1(S^2)$, y is a hyperbolic point. Since I_1 is an invariant curve under f^{2N} and $y \in \overline{I_1}$, we have that $I_1 \subset W^s(y)$. Parallel sides of the rectangle R are disjoint $W^u_\varepsilon(x_1) \cap W^u_\varepsilon(x_2) = \emptyset$. We can assume that $y \notin W^u_\varepsilon(x_1)$. Since g is Kupka-Smale, $W^u_\varepsilon(x_1)$ intersects transversally I_1 at one of its endpoints w. Since $w \in I_1 \subset W^s(y)$, we have that $f^{2Nk}(W^u_\varepsilon(x_1))$ is transversal to I_1 and contains the point $f^{2Nk}(w)$ which approaches y as $k \to +\infty$. Then $\emptyset \neq f^{2Nk}(W^u_\varepsilon(x_1)) \cap \operatorname{int}(R) \subset Q \cap \operatorname{int}(R)$ for some $k \in \mathbb{N}$. But this contradicts the choice of R.

$\square$

The non-dynamically convex case.

Let ψ_t be the lift of the geodesic flow to S^3. Then ψ_t is the Reeb flow of a tight contact form and it is Kupka-Smale. The lift $\overline{\operatorname{Per}(\psi)}$ of $\overline{\operatorname{Per}(g)}$ is also a hyperbolic set. Let $\widehat{\Lambda}$ be the lift of Λ and $\widehat{Q}$ the lift of Q.

For $\epsilon > 0$ and $x \in \overline{\operatorname{Per}(\psi)}$, let $W^s_\epsilon(x)$, $W^u_\epsilon(x)$ be the ϵ-balls in $W^s(x)$, $W^u(x)$. Since the set $\overline{\operatorname{Per}(\psi)}$ is hyperbolic (cf. [22, p. 6.4.13], [4]), for all $\epsilon > 0$ there exists $\delta = \delta[\epsilon] > 0$ such that

$$\text{if} \quad y, z \in \overline{\operatorname{Per}(\psi)} \quad \text{and} \quad d(y,z) < \delta[\epsilon] \quad \text{then} \quad W^u_\epsilon(y) \cap W^s_\epsilon(z) \neq \emptyset. \tag{8}$$

Let Σ be a rigid surface such that $\Sigma \cap \widehat{\Lambda} \neq \emptyset$. Let $x \in \widehat{\Lambda} \cap \Sigma$ be an interior point of $\widehat{\Lambda} \cap \Sigma$. Let

$$\varepsilon = \min\left\{ \delta\left[\tfrac{1}{2}\, d(x, \partial\Sigma)\right],\ A,\ \varepsilon(x) \right\} > 0,$$

[4]Another proof from this point is the following: since R is homeomorphic to a disc, and f^N preserves a finite measure, by Brouwer's translation theorem there is a fixed point y for f^N inside R. The point y belongs to a periodic orbit for ψ. Since $\overline{\operatorname{Per}(\psi)}$ is uniformly hyperbolic, if $\epsilon > 0$ is small enough then $\emptyset \neq W^s(y) \cap \partial R \subset W^s(y) \cap W^u(\Lambda)$. Then y is in the homoclinic class Λ. Since $y \in R$, this contradicts the choice of R. We shall use this argument in the non-dynamically convex case. We present another argument here to show that Brouwer's translation theorem is not needed for the dynamically convex case or for the positively curvature case.

where $A > 0$ is from proposition 2.9, $\varepsilon(x) > 0$ is from lemma 3.3 and $\delta[\cdot]$ is a function satisfying (8).

Let $B_\varepsilon(x)$ be the ball of radius ε in Σ centered at x. By lemma 3.3, $\overline{B_\varepsilon(x)} \cap \widehat{Q} \cap \Sigma$ is closed. The contact 1-form λ is the lift of the Liouville's form on T^1S^2. The form $d\lambda$ on Σ is finite and is preserved by the return map to Σ it induces a smooth invariant probability μ on Σ . Since the flow ψ_t is not Anosov, $\mu(\widehat{\Lambda} \cap \Sigma) = 0$. Since the points in $(\widehat{Q} \cap \Sigma) \setminus \widehat{\Lambda}$ are wandering, by Poincaré's recurrence, $\mu[(\widehat{Q} \cap \Sigma) \setminus \widehat{\Lambda}] = 0$. Thus $\mu(\widehat{Q} \cap \Sigma) = 0$. Therefore $\overline{B_\varepsilon(x)} \setminus \widehat{Q}$ is an open non-empty set.

Let R be a connected component of $B_\varepsilon(x) \setminus \widehat{Q}$ whose closure is in $\mathrm{int}(B_\varepsilon(x))$. Such component exists because x is not isolated in $\widehat{\Lambda}$. Actually, R is an open rectangle with two sides in each of $W^s(\widehat{\Lambda})$ and $W^u(\widehat{\Lambda})$. Let $\tau_n : \Sigma \to [0, +\infty[\cup \{+\infty\}$ and $F_n : \Sigma \to \Sigma$ be the return times and return maps to Σ, i.e.

$$\tau_0 \equiv 0, \qquad F_n(x) = \psi_{\tau_n(x)}(x), \qquad \tau_{n+1}(x) = \inf \{\, t > \tau_n(x) \mid \psi_t(x) \in \Sigma \,\}.$$

By Poincaré's recurrence, there is some $N > 0$ such that $F_N(R \cap [\tau_N < +\infty]) \cap R \neq \emptyset$.

Suppose that[5] $\mathrm{int}(R) = R \subset [\tau_N < +\infty]$. Then $F_N|_R$ is a local diffeomorphism and then $F_N(R)$ is connected. If $F_N(R) \cap \partial R \neq \emptyset$ then there is a point $q \in \partial R \cap F_N(R) \subset \widehat{Q} \cap F_N(R)$. Since the set $\widehat{Q}$ is invariant, there is $q' \in R \cap \widehat{Q}$, which contradicts the choice of R. Hence $F_N(R) \subseteq R$. Since the map $F_N : R \to R$ is area preserving and the $d\lambda$-area of R is finite, it can not have a translation domain. By Brouwer's translation theorem [12], F_N has a fixed point in R. Then the flow ψ_t has a periodic point $p \in R \cap \overline{\mathrm{Per}(\psi)}$. By the definition of $\delta = \delta[\epsilon]$ above, we have that $W^s(p) \cap W^u(x) \neq 0$ and $W^u(p) \cap W^s(x) \neq 0$. Then p is in the same homoclinic class as x, thus $p \in \widehat{\Lambda}$. But this contradicts the choice of R.

Then $R \cap [\tau_N = +\infty] \neq \emptyset$. By the choice of N there exists $q \in R \cap [\tau_N < +\infty]$ such that $F_N(q) \in R$. Let B be the connected component of $R \cap [\tau_N < +\infty]$ containing q. Since $[\tau_N < +\infty]$ is open, we have that B is open. Since $R \cap [\tau_N = +\infty] \neq \emptyset$, τ_N is unbounded in B. By proposition 2.9, $\mathrm{diam}\, F_N(B) > A$. Since $\mathrm{diam}(R) < \varepsilon \leq A$, we have that $F_N(B) \not\subset R$. But since $F_N(B)$ is connected, the same argument as above shows that $F_N(B) \subseteq R$ because $F_N(B) \cap \partial R \subseteq F_N(B) \cap \widehat{Q} = \emptyset$. This is a contradiction.

□

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