GEODESIC FLOWS WITH POSITIVE TOPOLOGICAL ENTROPY, TWIST MAPS AND HYPERBOLICITY

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Abstract. We prove a perturbation lemma for the derivative of geodesic flows in high dimension. This implies that a C^2 generic riemannian metric has a non-trivial hyperbolic basic set in its geodesic flow.

1. Introduction

Let M^{n+1} be a closed (compact without boundary) manifold of dimension $n+1$, $n \geq 1$, endowed with a C^∞ riemannian metric g and let $\phi_t = \phi_t^g$ be the geodesic flow of g on the unit tangent bundle S^gM. The simplest invariant which measures the complexity of the flow ϕ_t^g is its *topological entropy* which we denote by $h_{top}(g)$. The topological entropy measures the difficulty in predicting the position of an orbit given an approximation of its initial state. Namely, if $\theta \in S^gM$ is a unit vector and $T, \delta > 0$ define the (δ, T)-*dynamic ball* about θ as

$$B(\theta, \delta, T) = \{\, \vartheta \in S^gM \,:\, d(\phi_t^g(\vartheta), \phi_t^g(\theta)) < \delta \,\},$$

where d is the distance function in S^gM. Let $N_\delta(T)$ be the minimal quantity of (δ, T)-dynamic balls needed to cover S^gM. The topological entropy is the limit on δ of the exponential growth rate of $N_\delta(T)$:

$$h_{top}(g) := \lim_{\delta \to 0} \limsup_{T \to +\infty} \frac{1}{T} \log N_\delta(T). \tag{1}$$

Thus, if $h_{top}(g) > 0$, some dynamic balls must contract exponentially at least in one direction. R. Mañé [23] showed that

$$h_{top}(g) = \lim_{T \to +\infty} \frac{1}{T} \log \int_{M \times M} n_T(x, y)\, dx\, dy, \tag{2}$$

where $n_T(x, y)$ is the number of geodesic arcs of length $\leq T$ joining $x \in M$ to $y \in M$ and the integral is with respect to the volume on $M \times M$.

Some manifolds have all their riemannian metrics with positive entropy. For example, if the fundamental group of M has exponential growth (see Dinaburg [9], Manning [24]), using the definition (1) of the topological entropy, or when the homology of the loop space of M growths exponentially (see Paternain and Petean [28]), using (2).

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A way of obtaining positive topological entropy is by showing that the flow has a non-trivial *hyperbolic basic set*. A *locally maximal* invariant set is a compact subset $\Lambda \subset S^gM$ such that $\phi^g_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$ and there is a neighbourhood U of Λ such that

$$\Lambda = \bigcap_{t\in\mathbb{R}} \phi^g_t(U).$$

A *hyperbolic set* is a compact ϕ^g_t-invariant subset $\Lambda \subset S^gM$ such that the restriction of the tangent bundle of S^gM to Λ has a splitting

$$T_\Lambda S^gM = E^s \oplus \langle X\rangle \oplus E^u,$$

where $\langle X\rangle$ is the subspace generated by the vector field X of ϕ^g_t, E^s and E^u are $d\phi^g_t$ invariant sub-bundles and there are constants C, $\lambda > 0$ such that

(i) $|d\phi^g_t(\xi)| \le C\, e^{-\lambda t}\, |\xi|$ for all $t > 0$, $\xi \in E^s$;
(ii) $\big|d\phi^g_{-t}(\xi)\big| \le C\, e^{-\lambda t}\, |\xi|$ for all $t > 0$, $\xi \in E^u$.

A *non-trivial hyperbolic basic set* is a locally maximal compact invariant subset $\Lambda \subset S^gM$ which is hyperbolic, has a dense orbit and which is not a single periodic orbit.

Using symbolic dynamics one shows that if a flow contains a non-trivial hyperbolic basic set then it has positive topological entropy. It also has infinitely many periodic orbits and their number growths exponentially with their period, namely

$$h_{top}(g) \ge h_{top}(\phi^g|_\Lambda) = \lim_{T\to+\infty} \frac{1}{T}\,\log P(T) > 0,$$

where $P(T)$ is the number of periodic orbits in Λ with period $\le T$.

If a manifold has negative sectional curvature, its geodesic flow is Anosov and hence it contains a non-trivial hyperbolic basic set. On manifolds with positive curvature it is not so clear that one can perturb the metric to obtain positive topological entropy. In this work we prove

Theorem A.

On any closed manifold M with $\dim M \ge 2$ the set of C^∞ riemannian metrics whose geodesic flow contains a non-trivial hyperbolic basic set is open and dense in the C^2 topology.

Corollary B.

Let M be a closed manifold with $\dim M \ne 1$. There is a set $\mathcal{G}$ of C^∞ riemannian metrics on M such that $\mathcal{G}$ is open and dense in the C^2 topology and if $g \in \mathcal{G}$, $h_{top}(g) > 0$ and

$$\lim_{T\to+\infty} \frac{1}{T}\,\log P(T) > 0,$$

where $P(T)$ is the number of closed geodesics of length $\le T$.

G. Knieper and H. Weiss [20] prove Theorem A for surfaces in the C^∞ topology. Their methods are restricted to dimension 2. G. Paternain and the author proved Theorem A for surfaces in [6]. This paper generalize their methods. For general hamiltonian flows S. Newhouse [27] proves a stronger result: C^2-generically the hamiltonian flow is either Anosov or it has a generic 1-elliptic periodic orbit. In both cases the flow contains a hyperbolic basic set. The Newhouse Theorem was proved for riemannian metrics on $\mathbb{S}^2$ or $\mathbb{RP}^2$ in Contreras & Oliveira [5]. The techniques of this paper are not enough to prove it for general manifolds because of the lack of closing lemma for geodesic flows.

If instead of riemannian metrics we were considering Finsler metrics then the same techniques as in [27] would prove the Newhouse Theorem and in particular Theorem A. However, perturbation results within the set of riemannian metrics are harder, due to the fact that when we change the metric in a neighbourhood of a point of the manifold we affect all the geodesics leaving from those points; in other words, even if the size of the neighbourhood in the manifold is small, the effect of the perturbation in the unit sphere bundle is necessarily large. This is the main reason why the closing lemma is not known for geodesic flows (see Pugh & Robinson [30]), even though there is a closing lemma for Finsler metrics.

An application of this paper is that the metrics obtained in Theorem A satisfy the conditions H1, H2, (a periodic orbit with a transversal homoclinic point) required in a recent paper by A. Delshams, R. de la Llave and T. Seara [7] to obtain orbits with unbounded energy (Arnold's diffusion type phenomenon) for perturbation of geodesic flows by quasi-periodic potentials. See also section 2 in [7] for a discussion on the abundance of this situation.

We show how to obtain Theorem A from the results proved in the following sections. A closed geodesic is said *degenerate* if its linearized Poincaré map has an eigenvalue which is a root of unity. A riemannian metric is said *bumpy* if all its closed geodesics are non-degenerate. A closed geodesic is *hyperbolic* if it has no eigenvalue of modulus 1 and it is *elliptic* if it is non-degenerate and non-hyperbolic. An elliptic geodesic is q-elliptic if it has precisely $2q$ eigenvalues of modulus 1.

If γ and η are hyperbolic periodic orbits for ϕ_t^g a *heteroclinic orbit* from η to γ is an orbit $\phi_{\mathbb{R}}^g(\theta)$ such that

$$\lim_{t\to-\infty} d(\phi_t^g(\theta),\eta)=0 \qquad \text{and} \qquad \lim_{t\to+\infty} d(\phi_t^g(\theta),\gamma)=0.$$

The orbit $\phi_{\mathbb{R}}^g(\theta)$ is said to be *homoclinic* if $\eta=\gamma$. The *weak stable* and *weak unstable* manifolds of a hyperbolic periodic orbit γ are

$$W^s(\gamma):=\big\{\,\theta\in S^gM\ :\ \lim_{t\to+\infty} d(\phi_t^g(\theta),\gamma)=0\,\big\},$$
$$W^u(\gamma):=\big\{\,\theta\in S^gM\ :\ \lim_{t\to-\infty} d(\phi_t^g(\theta),\gamma)=0\,\big\}.$$

The sets $W^s(\gamma)$ and $W^u(\gamma)$ are $(n+1)$-dimensional invariant immersed submanifolds of S^gM. Then a heteroclinic orbit is an orbit in the intersection $W^s(\gamma)\cap W^u(\eta)$. If $W^s(\gamma)$ and $W^u(\eta)$ are transversal at $\phi_{\mathbb{R}}^g(\theta)$ we say that the heteroclinic orbit is transverse. A standard argument in dynamical systems (see [16, §6.5.d] for diffeomorphisms[1]) shows that

[1]Note that for geodesic flows the closed orbits never reduce to fixed points.

if a flow contains a transversal homoclinic orbit then it contains a non-trivial hyperbolic basic set. Therefore for Theorem A it is enough to look for a homoclinic orbit.

Denote by $\mathcal{R}^k(M)$ the set of C^k riemannian metrics in M provided with the C^k topology. In section 2 we recall the Kupka-Smale theorem 2.1 for geodesic flows, which was proven in [6] using results of Anosov [2] and Klingenberg and Takens [19]. In particular, it says that for a generic riemannian metric in $\mathcal{R}^k(M)$, $k \geq 2$, all heteroclinic orbits are transverse.

In section 3 we prove the following

Theorem C. *There is a subset $\mathcal{G}_0 \subset \mathcal{R}^2(M)$ such that*

(i) *For all $4 \leq k \leq \infty$, $\mathcal{G}_0$ contains a residual set in $\mathcal{R}^k(M)$.*
(ii) *If the geodesic flow of a metric $g \in \mathcal{G}_0$ contains a non-hyperbolic orbit, then it contains a non-trivial hyperbolic basic set.*

This is obtained by showing that such a metric g contains a generic elliptic geodesic. Using the Birkhoff normal form one obtains a region nearby the elliptic periodic orbit where the Poincaré map is conjugate to a Kupka-Smale twist map on $\mathbb{T}^n \times \mathbb{R}^n$, $n = \dim M - 1$. In Theorem 4.1, using arguments of M-C. Arnaud and M. Herman we prove that such twist maps have a generic 1-elliptic periodic orbit.

The restriction of the Poincaré map of this 1-elliptic orbit to its central manifold is a twist map of the annulus $S^1 \times \mathbb{R}$. Such Kupka-Smale twist maps have homoclinic orbits. Since the central manifold is normally hyperbolic, the homoclinic orbit for the twist map is a homoclinic orbit for the whole Poincaré map, and it is transverse by the Kupka-Smale condition on the Poincaré map.

Theorem C can be used to obtain density of hyperbolicity in the C^∞ topology when a non-hyperbolic geodesic is known to exist. Interesting cases are obtained in W. Ballman, G. Thorgbersson and W. Ziller [4], where they give conditions under which the existence of a closed non-hyperbolic geodesic is guaranted (see specially Theorem B). Combining this result with Theorem C one obtains that any 1/4-pinched metric in S^n may be approximated in the C^∞ topology by a metric with a non-trivial hyperbolic basic set.

Having a non-trivial basic set for ϕ^g is an open condition on the C^2 topology on g (this is, the C^1 topology on ϕ^g), because basic sets can be analytically continued (cf. [29, Th. 5.1]). Therefore Theorem C covers the case in Theorem A when a metric can be C^2 approximated by one with an elliptic periodic orbit.

The remaining case is covered by the following Theorem D. Let

$$\mathcal{P}(g) := \{\, \gamma \ : \ \gamma \text{ periodic orbit for } g \,\},$$

$$\mathrm{Per}(g) := \bigcup_{\gamma \in \mathcal{P}(g)} \gamma(\mathbb{R}),$$

$$\mathcal{H}(M) := \{\, g \in \mathcal{R}^\infty(M) \mid \forall\, \gamma \in \mathcal{P}(g) \ : \ \gamma \text{ is hyperbolic} \,\},$$

$$\mathcal{F}^2(M) := \mathrm{int}_{C^2}\, \mathcal{H}(M).$$

Theorem D.
There is a set $\mathcal{D} \subset \mathcal{R}^2(M)$ such that

(i) *For all $2 \le k \le \infty$, $\mathcal{D} \cap \mathcal{R}^k(M)$ is residual in $\mathcal{R}^k(M)$.*
(ii) *If $g \in \mathcal{D} \cap \mathcal{F}^2(M)$, then $\Lambda = \overline{\mathrm{Per}(g)}$ contains a non-trivial hyperbolic basic set.*

This finishes the proof of Theorem A because $\mathcal{F}^2(M)$ is the open set in the C^2 topology of C^∞ metrics which can not be C^2-approximated by a metric with an elliptic periodic orbit and the set $\mathcal{D}$ is C^2-dense in $\mathcal{F}^2(M)$.

When $\dim M = 2$ Theorem D was proven in Contreras & Paternain [6]. The proof of Theorem D appears in section 9 and follows from Rademacher's theorem [31] (which says that a generic riemannian metric has infinitely many closed geodesics), Smale's spectral decomposition theorem for hyperbolic sets and the following Theorem E, also proved in section 9:

Given a set $A \subset SM$, define

$$\mathcal{P}(g, A) := \{ \gamma \in \mathcal{P}(g) \ : \ \gamma(\mathbb{R}) \subset A \},$$

$$\mathrm{Per}(g, A) := \bigcup_{\gamma \in \mathcal{P}(g,A)} \gamma(\mathbb{R}),$$

$$\mathcal{H}(A) := \{ g \in \mathcal{R}^\infty(M) \mid \forall \gamma \in \mathcal{P}(g, A) \ : \ \gamma \text{ is hyperbolic} \},$$

$$\mathcal{F}^2(A) := \mathrm{int}_{C^2} \mathcal{H}(A).$$

Theorem E.
There is a set $\mathcal{G}_1 \subset \mathcal{R}^2(M)$ such that

(i) *$\mathcal{G}_1$ is open in $\mathcal{R}^2(M)$ and $\mathcal{G}_1 \cap \mathcal{R}^\infty(M)$ is dense in $\mathcal{R}^\infty(M)$.*
(ii) *If $g \in \mathcal{G}_1 \cap \mathcal{F}^2(A)$, then $\Lambda = \overline{\mathrm{Per}(g, A)}$ is a hyperbolic set.*

Theorem E is proved in section 9 by adapting R. Mañé's theory of stable hyperbolicity, developed for the stability conjecture in [22], to the case of geodesic flows. One first considers the linearized Poincaré maps of small segments of the closed geodesics in the set A. These are periodic sequences of symplectic matrices in $\mathbb{R}^{2n}$. Denote by $Sp(n)$ the set of symplectic linear maps in $\mathbb{R}^{2n}$. In Theorem 8.1 we prove that if these sequences are stably hyperbolic under uniform perturbations in $Sp(n)$, then they are uniformly hyperbolic. Such uniform hyperbolicity is inherited by the closure $\overline{\mathrm{Per}(g, A)}$.

In order to reduce the problem to sequences of symplectic matrices we need a perturbation lemma, proved in Theorem 7.1, which is the main technical difficulty in the paper. One has to perturb the linearized Poincaré map on any orbit segment, in an arbitrary direction in $Sp(n)$, on an arbitrarily small neighbourhood of the segment, without moving neither the orbit segment nor the possible self-intersections with the remaining of the periodic orbit, without changing the metric above the segment and covering a perturbation size on $Sp(n)$ which is uniform for all orbit segments of a given length, say 1, but possibly depending on the riemannian metric g.

Such a perturbation had been done by Klingenberg and Takens [19] and Anosov [2] but not with the uniform estimate. We prove the perturbation lemma only for a special set of metrics $\mathcal{G}_1 \subset \mathcal{R}^\infty(M)$: those such that every geodesic segment of length $\frac{1}{2}$ has a point whose curvature matrix has all its eigenvalues distinct and separated by a uniform bound.

In Theorem 6.1 we prove that such set $\mathcal{G}_1$ is open and dense in $\mathcal{R}^k(M)$ for all $k \geq 2$. The use of the set $\mathcal{G}_1$ is the main difference with the perturbation lemma in dimension 2, proved in [6], which only needs the riemannian metric to be C^4. We only prove the density of $\mathcal{G}_1$ for C^∞ metrics.

The lengths 1 and $\frac{1}{2}$ above are chosen for simplicity of the exposition and they can be any number smaller than the injectivity radius ℓ of the metric. In their application in the proof of Theorem E in section 9.1, we use $\frac{1}{2} \leq 1 = 2 \cdot \frac{1}{2} < \frac{1}{4}\,\ell$. Multiplying the riemannian metrics by a constant, without loss of generality we can assume that all the metrics in this work have injectivity radius larger than 4.

Finally, in section 5 we introduce the Fermi coordinate system and the kind of perturbations of the metrics that are used in Theorem 6.1 and Theorem 7.1

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2. The Kupka-Smale theorem.

Let M^{n+1} be a closed manifold of dimension $n+1$. Let ϕ_t^g be the geodesic flow of a riemannian metric g acting on SM, the unit sphere bundle of M. Let $\pi : SM \to M$ be the canonical projection. Non-trivial closed geodesics on M are in one-to-one correspondence with the periodic orbits of ϕ_t^g. Given a closed orbit $\gamma = \{\phi_t^g(\theta) \mid t \in [0,a]\}$ of period $a > 0$, define the *Poincaré map* $\mathcal{P}_g(\Sigma,\theta)$ as follows: Choose a local hypersurface Σ in SM containing θ and transversal to γ. Then there are open neighbourhoods Σ_0 and Σ_a of θ and a differentiable function $\delta : \Sigma_0 \to \mathbb{R}$ such that the map $\mathcal{P}_g(\Sigma,\theta) : \Sigma_0 \to \Sigma_a$ given by $\vartheta \mapsto \phi^g_{\delta(\vartheta)}(\vartheta)$ is a diffeomorphism.

Recall (c.f. Klingenberg [18]) that there is a canonical splitting of the tangent bundle $T(TM) = H \oplus V$, where the *vertical subspace* $V = \ker d\pi$ is tangent to the fibers of π and the *horizontal subspace* H is the kernel of the connection map $K : T(TM) \to TM$. There is a natural identification $T_\theta TM = H(\theta) \oplus V(\theta) \leftrightarrow T_\theta M \oplus T_\theta M$ given by $\zeta = (h,v) \leftrightarrow (d_\theta\pi(\zeta), K(\zeta))$. Under this identification the tangent space to the unit tangent bundle is $T_\theta SM = H(\theta) \oplus N(\theta)$, where $N(\theta) = \{\,\vartheta \in TM \,|\, \langle\vartheta,\theta\rangle_{\pi(\theta)} = 0\,\}$. The geodesic flow preserves the canonical contact form $\lambda(\zeta) = \langle\theta, h\rangle_{\pi(\theta)} = \langle\theta, d\pi(\zeta)\rangle_{\pi(\theta)}$ and hence its kernel

$$\mathcal{N}(\theta) := \ker\lambda \cap T_\theta SM = N(\theta) \oplus N(\theta) \subset H(\theta) \oplus V(\theta)$$

defines an invariant codimension 1 subspace in $T_\theta SM$, transversal to the geodesic flow. The canonical symplectic form $\omega := d\lambda$ is invariant under the geodesic flow and non-degenerate on $\mathcal{N}(\theta)$. We choose the local hypersurface Σ above such that $T_\theta\Sigma = \mathcal{N}(\theta)$. The *linearized*

Poincaré map $P_g(\theta) := d_\theta \mathcal{P}_g(\Sigma,\theta)$ is an ω-symplectic linear map on $\mathcal{N}(\theta)$ and

$$P_g(J(0),\dot{J}(0)) = (J(a),\dot{J}(a)),$$

where J is a normal Jacobi field along the geodesic $\pi\circ\gamma$ and $\dot{J}$ denotes the covariant derivative along the geodesic. After choosing a symplectic linear basis for $\mathcal{N}$ we can identify the group of ω-symplectic linear maps on $\mathcal{N}$ with the symplectic linear group $Sp(n)$ on $\mathbb{R}^n\times\mathbb{R}^n$. Although the distribution $\mathcal{N}$ is not integrable, the symplectic form ω is still non-degenerate in $T_\vartheta\Sigma$ for ϑ in a neighbourhood of θ and the Poincaré map $\mathcal{P}_g(\Sigma,\vartheta)$ preserves $\omega|_\Sigma$.

Let $J^k_s(n)$ be the set of k-jets of C^k symplectic automorphisms of $\mathbb{R}^n\times\mathbb{R}^n$ which fix the origin. One can identify $J^1_s(n)$ with $Sp(n)$. A set $Q\subset J^k_s(n)$ is said to be *invariant* if for all $\sigma\in J^k_s(n)$, $\sigma Q\sigma^{-1}=Q$. In this case, the property that says that the Poincaré map $\mathcal{P}_g(\Sigma,\theta)$ belongs to Q is independent of the section Σ.

A closed orbit is said to be *hyperbolic* if its linearized Poincaré map has no eigenvalues of modulus 1. If γ is a hyperbolic closed orbit and $\theta=\gamma(0)$, define the strong stable and strong unstable manifolds of γ at θ by

$$W^{ss}(\theta) = \{\,\vartheta\in SM \mid \lim_{t\to+\infty} d\big(\phi^g_t(\vartheta),\phi^g_t(\theta)\big)=0\,\},$$
$$W^{uu}(\theta) = \{\,\vartheta\in SM \mid \lim_{t\to-\infty} d\big(\phi^g_t(\vartheta),\phi^g_t(\theta)\big)=0\,\}.$$

Define the weak stable and weak unstable manifolds by

$$W^s(\gamma) := \bigcup_{t\in\mathbb{R}} \phi^g_t\big(W^{ss}(\theta)\big), \qquad W^u(\gamma) := \bigcup_{t\in\mathbb{R}} \phi^g_t\big(W^{uu}(\theta)\big).$$

It turns out that they are immersed submanifolds of dimension

$$\dim W^s(\gamma) = \dim W^u(\gamma) = \dim M = n+1.$$

A heteroclinic point is a point in the intersection $W^s(\gamma)\cap W^u(\eta)$ for two hyperbolic closed orbits γ and η. We say that $\theta\in SM$ is a transversal heteroclinic point if $\theta\in W^s(\gamma)\cap W^u(\eta)$, and $T_\theta W^s(\gamma)+T_\theta W^u(\eta)=T_\theta SM$.

Let $\mathcal{R}^k(M)$ be the Banach manifold of C^k riemannian metrics on M endowed with the C^k topology. Using results from Anosov [2] and Klingenberg & Takens [19], in [6, Theorem 2.5] we proved the following analogous to the Kupka-Smale theorem for geodesic flows:

2.1. **Theorem.**

Let $Q\subset J^k_s(n)$ be open, dense and invariant. Then there exists a residual subset $\mathcal{O}\subset \mathcal{R}^{k+1}(M)$ such that for all $g\in\mathcal{O}$:

- *The k-jet of the Poincaré map of every closed geodesic of g belongs to Q.*
- *All heteroclinic points of hyperbolic closed geodesics of g are transversal.*

Since countable intersections of residual subsets are residual, in theorem 2.1 we can replace Q by a residual invariant subset in $J^k_s(n)$. Also, using the natural projection $\pi : J^{k+1}_s(n) \to J^k_s(n)$ by truncation, in Theorem 2.1 one obtains a residual subset $\mathcal{O} \subset \mathcal{R}^r(M)$ for any $r \geq k+1$.

3. Elliptic closed geodesics.

We say that a periodic orbit is *q-elliptic* if its linearized Poincaré map has $2q$ eigenvalues of modulus 1 and that it is *elliptic* if it is q-elliptic for some $q > 0$.

Suppose that θ is a q-elliptic periodic point, $q \leq n$. Let $P = d_\theta \mathcal{P}(\Sigma, \theta)$ be its linearized Poincaré map. Let $T_\theta \Sigma = E^s \oplus E^u \oplus E^c$ be the decomposition into the stable, unstable and center subspaces for P. This is, E^s, E^u and E^c are invariant under P and $P|_{E^s}$ has only eigenvalues ρ of modulus $|\rho| < 1$, $P|_{E^u}$ has only eigenvalues ρ of modulus $|\rho| > 1$ and $P|_{E^c}$ has only eigenvalues ρ of modulus $|\rho| = 1$. Then there are local embeddings $W^s : (\mathbb{R}^p, 0) \to (\Sigma, \theta)$, $W^u : (\mathbb{R}^p, 0) \to (\Sigma, \theta)$, $p = n - q$ and $W^c : (\mathbb{R}^{2q}, 0) \to (\Sigma, \theta)$, such that $T_\theta W^s = E^s$, $T_\theta W^u = E^u$, $T_\theta W^c = E^c$ which are locally invariant under $\mathcal{P} = \mathcal{P}(\Sigma, \theta)$, i.e. $\mathcal{P}W^s$, $\mathcal{P}W^u$, $\mathcal{P}W^c$ are locally equal to W^s, W^u, W^c respectively, see Hirsch, Pugh, Shub [17]. They are called stable, unstable and center manifolds for (Σ, θ). The stable and unstable manifolds are unique, but the center manifold may not be unique. If $\mathcal{P}$ is of class C^k (resp. C^∞) then W^s, W^u, are C^k (resp. C^∞). If $\mathcal{P}$ is of class C^k (resp. C^∞) then W^c can be chosen C^k (resp. C^r, with r arbitrarily large) on a sufficiently small neighbourhood of θ. The submanifolds W^s, W^u are isotropic with respect to the canonical symplectic form ω (i.e. $\omega|_{W^s} \equiv 0$ and $\omega|_{W^u} \equiv 0$) because $\mathcal{P}$ preserves ω and $d\mathcal{P}$ (resp. $d\mathcal{P}^{-1}$) asymptotically contracts tangent vectors in W^s (resp. W^u). The restriction $\omega|_{E^c}$ is non-degenerate (see Robinson [32]) and hence $\mathcal{P}|_{W^c}$ is a symplectic map on a sufficiently small neighbourhood of θ.

Let $\rho_1, \ldots \rho_q; \overline{\rho_1}, \ldots, \overline{\rho_q}$ be the eigenvalues of P with modulus 1. We say that θ is *4-elementary* if

$$\prod_{i=1}^{q} \rho_i^{\nu_i} \neq 1 \qquad \text{whenever} \qquad 1 \leq \sum_{i=1}^{q} |\nu_i| \leq 4. \tag{3}$$

In this case there are symplectic coordinates $(x_1, \ldots, x_q; y_1, \ldots, y_q)$ in W^c such that $\omega|_{W^c} = \sum_{i=1}^q dy_i \wedge dx_i$ and $\mathcal{P}|_{W^c}$ is written in *Birkhoff normal form* $\mathcal{P}(x, y) = (X, Y)$, where

$$Z_k = e^{2\pi i\, \phi_k}\; z_k + g_k(z), \tag{4}$$

$$\phi_k(z) = a_k + \sum_{\ell=1}^{q} \beta_{k\ell}\, |z_\ell|^2$$

$z = x + iy$, $Z = X + iY$, $\rho_i = e^{2\pi\, i\, a_k}$ and $g(z) = g(x, y)$ has vanishing derivatives up to order 3 at the origin. We say that θ is *weakly monotonous* if the matrix $\beta_{k\ell}$ is non-singular. The property $\det \beta_{k\ell} \neq 0$ is independent of the particular choice of normal form. In these coordinates, the matrix $\beta_{k\ell}$ can be detected from the 3-jet of $\mathcal{P}$ at $\theta=(0,0)$ and it can be seen that the property $\{\,(3)$ and $\det\beta_{k\ell}\neq 0\,\}$ is open and dense in the jet space $J^3_s(q)$.

Consider the following maps

$$\begin{array}{ccccc}
(x,y) & \longrightarrow & (\theta,\rho) & \longrightarrow & (\theta,\rho^2/\varepsilon)=(\theta,r)\\
\mathbb{D}^* & \xrightarrow{P} & \mathbb{T}^q\times\mathbb{R}^q_+ & \xrightarrow{R} & \mathbb{T}^q\times\mathbb{R}^q_+\\
f\big\downarrow & & & & \big\downarrow F_\varepsilon\\
\mathbb{D}^* & \xrightarrow{P} & \mathbb{T}^q\times\mathbb{R}^q_+ & \xrightarrow{R} & \mathbb{T}^q\times\mathbb{R}^q_+
\end{array}$$

where $\mathbb{D}=\{\,(x,y)\in\mathbb{R}^q\times\mathbb{R}^q\;:\;|x|^2+|y|^2<1\,\}$, $\mathbb{D}^*=\mathbb{D}\setminus\{(0,0)\}$, $f=\mathcal{P}|_{W^c}$ in the above coordinates, $\mathbb{T}^q=\mathbb{R}^q/\mathbb{Z}^q$ and P^{-1} is given by $x_i=\rho_i\cos(2\pi\theta_i)$, $y_i=\rho_i\sin(2\pi\theta_i)$. Since the coordinates in Birkhoff normal form are symplectic, the map f preserves the form $\omega:=\sum_i dx_i\wedge dy_i=dx\wedge dy$. Let $Q=R\circ P:\mathbb{D}^*\to\mathbb{T}^q\times\mathbb{R}^q_+$ be given by $Q(x,y)=(\theta,r)$, $r_i=\rho_i^2/\varepsilon$. Then $Q^*(r\,d\theta)=\frac{1}{2\pi\varepsilon}(x\,dy-y\,dx)=:\lambda_\varepsilon$. Since $\mathbb{D}$ is simply connected, $f^*(\lambda_\varepsilon)-\lambda_\varepsilon$ is exact. Therefore $F_\varepsilon^*(r\,d\theta)-r\,d\theta$ is exact.

Let $G_\varepsilon(\theta,r):=(\theta+a+\varepsilon\,\beta\,r,r)$ be the symplectic diffeomorphism given by the first term in (4) in the coordinates (θ,r). Its N-th iterate is given by $G_\varepsilon^N(\theta,r):=(\theta+Na+\varepsilon\,N\beta\,r,r)$. This is a totally integrable (c.f. Arnaud [3, p. 11]) weakly monotonous (i.e. $\det(\varepsilon N\beta)\neq 0$) twist map of $\mathbb{T}^q\times\mathbb{R}^q_+$. Let $\mathbb{B}_\delta:=\{\,r\in\mathbb{R}^q_+\;:\;\sum_i(r_i-\frac{1}{2q})^2<\delta^2\,\}$. In [26] (see also Moser's appendix 3.3 in [18] or Arnaud [3, chap. 8]) J. Moser proves that given $\eta>0$ there exist $\delta>0$, $N\in\mathbb{N}$ and $\varepsilon>0$ such that

(i) $\left\|F_\varepsilon^N-G_\varepsilon^N\right\|_{C^1}<\eta$ in $\mathbb{T}^q\times\mathbb{B}_\delta$.

(ii) There exists a torus $\mathcal{T}^q$ radially transformed by F_ε^N in $\mathbb{T}^q\times\mathbb{B}_\delta$, i.e. $\mathcal{T}^q=\{\,(\theta,r(\theta))\;:\;\theta\in\mathbb{T}^q\,\}\subset\mathbb{T}^q\times\mathbb{B}_\delta$ such that $F_\varepsilon^N(\theta,r(\theta))=(\theta,R(\theta))$ for some $R:\mathbb{T}^q\to\mathbb{R}^q_+$.

Let S_N be a generating function for F_ε^N, i.e. a function $S_N:\mathbb{T}^q\times\mathbb{B}_\delta\to\mathbb{R}$ such that $dS_N=\left(F_\varepsilon^N\right)^*(r\,d\theta)-r\,d\theta$. On the radially transformed torus $\mathcal{T}^q$ we have that

$$dS_N(\theta,r(\theta))=\big(R(\theta)-r(\theta)\big)\,d\theta.$$

Then critical points of $dS_N|_{\mathcal{T}^q}$ correspond to fixed points of F_ε^N in $\mathcal{T}^q$. Therefore F_ε^N has at least $q-1=$ cup length$(\mathcal{T}^q)$ fixed points on $\mathcal{T}^q$. If S_N is a Morse function then F_ε^N has at least 2^q fixed points.

Let $Q\subset J^3_s(n)$ be the set of 3-jets of C^3 symplectic automorphisms T of $\mathbb{R}^n\times\mathbb{R}^n$ which fix the origin and such that

(i) The eigenvalues of d_0T are all different.
(ii) The eigenvalues of modulus 1 satisfy the 4-elementary condition (3).
(iii) The coefficients of the Birkhoff normal form (4) satisfy the weakly monotonous condition $\det\beta_{k\ell}\neq 0$.

Theorem C. *Let $\mathcal{G}_0$ be the set of C^4 riemannian metrics on M such that*

- *The k-jet of the Poincaré map of every closed geodesic of g (and its multiples) belongs to Q.*
- *All heteroclinic points of hyperbolic closed geodesics of g are transversal.*

Then

(i) *$\mathcal{G}_0$ contains a residual set in $\mathcal{R}^k(M)$ for all $k \geq 4$.*
(ii) *If the geodesic flow of a metric $g \in \mathcal{G}_0$ contains a non-hyperbolic periodic orbit then it contains a non-trivial hyperbolic set, in particular $h_{top}(g) > 0$.*

Proof: Since Q is residual and invariant in all $J_s^\ell(n)$, $\ell \geq 3$, by theorem 2.1 the set $\mathcal{G}_0$ contains a residual subset in $\mathcal{R}^k(M)$, $k \geq 4$. Now suppose that $g \in \mathcal{G}_0$ contains a non-hyperbolic periodic point $\theta \in S^g M$. We will prove that arbitrarily near to θ there is a hyperbolic periodic orbit with a transversal homoclinic point. Then (see e.g. [16, pg. 276]) there is a hyperbolic horseshoe containing the homoclinic point.

Observe that it is enough to find a 1-elliptic periodic point. For in that case the Poincaré map restricted to the 2-dimensional central manifold W^c will be a Kupka-Smale twist map which has hyperbolic orbits with homoclinic points[2] (see Le Calvez [21, Remarques p. 34]). This hyperbolic periodic orbit will be hyperbolic in the Poincaré section (c.f. Arnaud [3, lemme 8.6]). A homoclinic point in the central manifold is also a homoclinic point in the Poincaré section, and it must be transversal by the Kupka-Smale condition on $\mathcal{G}_0$.

Now suppose that there is a q-elliptic periodic point θ with $q > 1$. As stated above, Moser proves that there is a subset $\mathbb{T}^q \times \mathbb{B}_\delta$ near θ and an iterate $N \in \mathbb{N}$ such that the N-th iterate F_ε^N of the Poincaré map $F = \mathcal{P}|_{W^c}$ is a weakly monotonous twist map with fixed points which is C^1-near to a totally integrable twist map G_ε^N. In this case Theorem 4.1 below says that F has a 1-elliptic periodic point θ. Since the central manifold is normally hyperbolic, by lemma 8.6 in Arnaud [3], the periodic point θ will also be 1-elliptic for the whole Poincaré map $\mathcal{P} : \Sigma \to \Sigma$.

□

4. Symplectic twist maps on $\mathbb{T}^n \times \mathbb{R}^n$.

In this section we use the techniques developed by Arnaud and Herman in [3]. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ with its inherited addition. On $\mathbb{T}^n \times \mathbb{R}^n$ we use the coordinates $(\theta, r) \in \mathbb{T}^n \times \mathbb{R}^n$. Let $\lambda = r\,d\theta = \sum_i r_i\,d\theta_i$ be the *Liouville 1-form* on $\mathbb{T}^n \times \mathbb{R}^n = T^*\mathbb{T}^n$. The *symplectic form* on $\mathbb{T}^n \times \mathbb{R}^n$ is $\omega = d\lambda = dr \wedge d\theta$. Under the natural identification $T_{(\theta,r)}\mathbb{T}^n \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, the symplectic form is written as $\omega(x, y) = x^* J y$, where $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$. A C^1 diffeomorphism $F : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ is *symplectic* if $F^*\omega = \omega$. This is equivalent to $(dF)^* J (dF) = J$. It is *exact symplectic* if $F^*\lambda - \lambda$ is an exact form. It is *weakly monotonous* if when writing $F(\theta, r) = (\Theta, R)$, we have that $\det \frac{\partial \Theta}{\partial r} \neq 0$.

[2] In fact for such a twist map *all* hyperbolic periodic orbits have homoclinic points, see Mather [25] or Franks & Le Calvez [12].

The *torsion* of F is $b := \frac{\partial\Theta}{\partial r}$. The torsion is not necessarily symmetric and its symmetrization $b + b^*$ may be singular. We say that the *torsion is positive definite, negative definite, of signature* (p, q) if $b + b^*$ is positive definite, negative definite, of signature (p, q). Here signature (p, q) means p negative eigenvalues, q positive eigenvalues and $n - (p + q)$ zero eigenvalues.

A C^1 diffeomorphism $G : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ is *completely integrable* if it has the form $G(\theta, r) = (\theta + \beta(r), r)$ for some $\beta \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $\beta(0) = 0$. If furthermore G is symplectic then its torsion $\frac{\partial\beta}{\partial r}$ is symmetric. In this case $G^*\lambda - \lambda = r\, d\beta$ is exact because it is a closed form in $\mathbb{R}^n$.

Through this section F will denote a weakly monotonous exact symplectic C^r diffeomorphism, $r \geq 1$ which is C^1 near to a totally integrable symplectic map G.

Observe that for the totally integrable map G, the zero section $\mathbb{T}^n \times \{0\}$ consist of fixed points. We look for fixed points of F near $\mathbb{T}^n \times \{0\}$:

1. First we construct a *radially transformed torus* $\mathfrak{T} = \mathrm{Graph}(\eta)$ by solving
$$F(\theta, \eta(\theta)) = (\theta, *).$$
This can be done using the implicit function theorem applied to the equation
$$\Theta(\theta, \eta(\theta), F) = \theta,$$
where $F(\theta, r) = (\Theta, R)$, continuing the solution $\eta_G \equiv 0$ for G because by the weakly monotonous condition $\det\left[\frac{\partial\Theta}{\partial r}\right] \neq 0$. The function η is C^r if F is C^r.
2. Since F is exact symplectic, there is a *generating function* $S : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ such that
$$dS = F^*\lambda - \lambda = R\, d\Theta - r\, d\theta.$$
On the radially transformed torus $\mathfrak{T}$ we have
$$dS|_{\mathfrak{T}} = (R - r)\, d\theta.$$
Therefore a fixed point of F is a critical point for S in $\mathfrak{T}$. We define the *radial function* $\varphi = L(F) : \mathbb{T}^n \to \mathbb{R}$ as
$$\varphi(\theta) = S(\theta, \eta(\theta)). \tag{5}$$
Since φ is C^1, F has at least $n + 1 = \text{cup length}(\mathbb{T}^n)$ fixed points. If φ is a Morse function then F has at least 2^n fixed points.

Let $Q \subset J^3_s(n)$ be the subset defined by conditions (i), (ii), (iii) in section 3. We say that the diffeomorphism $F : \mathbb{T}^n \times \mathbb{R}^n \hookleftarrow$ is *Kupka-Smale* if

(i) If z is a periodic point of F with period[3] m then $DF^m(z) \in Q$.
(ii) All the heteroclinic intersections of hyperbolic periodic points are transversal.

[3] The integer m is not necessarily the minimal period of z.

4.1. **Theorem.**

If $F : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ is a C^4 Kupka-Smale weakly monotonous exact symplectic diffeomorphism which is C^1 near to a symplectic completely integrable diffeomorphism G, then F has a 1-elliptic periodic point near $\mathbb{T}^n \times \{0\}$.

In particular, there is a non-trivial hyperbolic set for F near $\mathbb{T}^n \times \{0\}$ and $h_{top}(F) \neq 0$.

4.2. **Lemma.** [M. Herman]

Let $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbb{R}^{2n\times 2n}$ be a symplectic matrix with a, b, c, $d \in \mathbb{R}^{n\times n}$ and $\det(b) \neq 0$.

For $\lambda \in \mathbb{C}$, let

$$M_\lambda := b^{-1}\,a + d\,b^{-1} - \lambda\, b^{-1} - \lambda^{-1}(b^{-1})^*.$$

Then

$$\operatorname{rank}(\lambda\, I - M) = n + \operatorname{rank} M_\lambda.$$

In particular λ is an eigenvalue of M iff $\det M_\lambda = 0$.

Proof:

Since M is symplectic, $M^* J M = J$. Therefore $a^*c = c^*a$, $b^*d = d^*b$ and $a^*d - c^*b = I$. This implies that

$$-(b^{-1})^* = c - d\,b^{-1}a. \tag{6}$$

Let $P = \begin{bmatrix} I & 0 \\ d\,b^{-1} & I \end{bmatrix}$, then

$$N := P^{-1}\,M\,P = \begin{bmatrix} a + b\,d\,b^{-1} & b \\ -(b^{-1})^* & 0 \end{bmatrix}.$$

If (v_1, v_2) is an eigenvector of N with eigenvalue λ, then $v_2 = -\lambda^{-1}\,(b^{-1})^* v_1$ and

$$(b^{-1}a + d\,b^{-1} - \lambda^{-1}\,(b^{-1})^* - \lambda\, b^{-1})\; v_1 = 0.$$

□

A periodic point z for F of period p is said *non-degenerate* if 1 is not an eigenvalue of $DF^p(z)$. Observe that if F is Kupka-Smale then all its periodic points are non-degenerate.

4.3. **Lemma.** *Let $\varphi = L(F)$ be the radial function* (5) *on the radially transformed torus $\mathfrak{T}$. At a fixed point $(\theta, \eta(\theta))$ for F on $\mathfrak{T}$, writing $M = DF(\theta, \eta(\theta))$, we have that*

$$M_\lambda = D^2\varphi(\theta) + (1-\lambda)\, b^{-1} + (1 - \lambda^{-1})\,(b^{-1})^*.$$

If the fixed points of F are non-degenerate then $\varphi = L(F)$ is a Morse function.

Proof: From the equation $\Theta(\theta, \eta(\theta)) = \theta$ for $\mathfrak{T}$ we get that $D\eta(\theta) = b^{-1}\,(I - a)$.

We have that $D\varphi(\theta) = dS|_{\mathfrak{T}} = R(\theta, \eta(\theta)) - \eta(\theta)$. Therefore, using (6),

$$D^2\varphi(\theta) = c + d\,b^{-1}(I-a) - b^{-1}(I-a) = b^{-1}a + d\,b^{-1} - b^{-1} - (b^{-1})^*. \tag{7}$$

This implies the formula.

If $\lambda = 1$ is not an eigenvalue of M, by lemma 4.2, $M_{\lambda=1} = D^2\varphi(\theta)$ is non-singular. □

4.4. **Lemma.**

If $z \in \mathfrak{T}$ is a fixed point of F then there is a polynomial $P \in \mathbb{R}[x]$ of degree n such that λ is an eigenvalue of $DF(z)$ iff $P(2-\lambda-\lambda^{-1})=0$.

The leading coefficient of P is $a_n = \det b^{-1}$, where $b = \frac{\partial \Theta}{\partial r}$ is the torsion at z and the independent term of P is $a_0 = \det D^2\varphi(\theta)$.

Proof: From lemma 4.3, $\det(M_\lambda)$ is a polynomial on $x=(1-\lambda)$ and $y=(1-\lambda^{-1})$ with maximal exponent n. Since $M_{1/\lambda} = M_\lambda^*$, this polynomial is symmetric on x and y. Therefore it can be written as a degree n polynomial on the variables $x+y=xy=2-\lambda-\lambda^{-1}$.

Write $w = 2-\lambda-\lambda^{-1}$. Then $w=0$ iff $\lambda=1$. Since $P(2-\lambda-\lambda^{-1}) = \det(M_\lambda)$, from lemma 4.3, $a_0 = P(w=0) = \det D^2\varphi(\theta)$.

Since $w = (1-\lambda)(1-\lambda^{-1})$, we have that

$$\frac{M_\lambda}{w} = \frac{D^2\varphi(\theta)}{w} + \frac{b^{-1}}{1-\lambda^{-1}} + \frac{(b^{-1})^*}{1-\lambda}.$$

The leading coefficient of P is

$$a_n = \lim_{w\to+\infty} \frac{P(w)}{w^n} = \lim_{\lambda\to-\infty} \det\left(\frac{M_\lambda}{w}\right) = \det b^{-1}.$$

□

Proof of Theorem 4.1.

If $n=1$ then F is a twist map of the annulus $S^1\times\mathbb{R}$. which is Kupka-Smale. Those maps have 1-elliptic periodic orbits (which are minimax critical points of the generating function) and also hyperbolic points with transversal homoclinic intersections (see Le Calvez [21, Remarques p. 34]).

Assume that $n \geq 2$. We shall prove that F contains a fixed point z_0 of elliptic × hyperbolic type, i.e. a q_0-elliptic point with $1 \leq q_0 < n$. Using the Birkhoff normal form about that point and Moser's theorem as in section 3 we obtain a new map $F_{q_0} : \mathbb{T}^{q_0} \times \mathbb{R}^{q_0} \hookleftarrow$ satisfying the hypothesis of theorem 4.1. Then F_{q_0} has a fixed point z_1 which is q_1-elliptic with $1 \leq q_1 < q_0$. The map F_{q_0} is conjugate to an iterate of F on a piece of the central manifold of z_0 which is normally hyperbolic (see Arnaud [3, lemme 8.6]). Therefore z_1 is a q_1-elliptic periodic point for F. Inductively obtain a sequence $z_0,\dots,z_m$ of periodic points for F, where z_i is q_i-elliptic and $n > q_0 > q_1 > \cdots > q_m = 1$. The point z_m is a 1-elliptic periodic point for F. Applying the case $n=1$ to its central manifold which is normally hyperbolic one obtains a totally hyperbolic periodic orbit for F with a homoclinic orbit. The homoclinic intersection is transversal by the Kupka-Smale hypothesis on F.

Write $w = 2 - \lambda - \lambda^{-1}$. Observe that

$$\begin{aligned} \lambda = 1 \quad &\text{iff} \quad w = 0, \\ \lambda \in \mathbb{S}^1 \quad &\text{iff} \quad w \in [0,4], \\ \lambda \in \mathbb{R} \quad &\text{iff} \quad w \in \mathbb{R}\backslash]0,4[, \\ \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \mathbb{S}^1) \quad &\text{iff} \quad w \in \mathbb{C} \setminus \mathbb{R}, \end{aligned}$$

where $\mathbb{S}^1 = \{\, w \in \mathbb{C} : |w| = 1 \,\}$. The completely integrable map G has all its fixed points degenerate, with $\lambda = 1$ and $w = 0$. Since we are assuming that F is C^1 near to G the eigenvalues λ of DF at the fixed points in $\mathfrak{T}$ are near to 1 and w is near to 0. From now on we can assume that $|w| < 4$.

Let $z \in \mathfrak{T}$ be a fixed point of F. Let $\lambda_1, \lambda_1^{-1}, \ldots, \lambda_n, \lambda_n^{-1}$ be the eigenvalues of $DF(z)$. Let $w_i = 2 - \lambda_i - \lambda_i^{-1}$, $1 \le i \le n$. By lemma 4.4

$$(-1)^n \, (\det b) \, \det D^2\varphi(\theta) = w_1 \cdots w_n. \tag{8}$$

If some $w_i \in \mathbb{C} \setminus \mathbb{R}$ then the complex conjugate $\overline{w_i} = w_j$ for some $j \neq i$. Since $w_i \, \overline{w_i} = |w_i|^2 > 0$, if the product in (8) is negative then there are at least two (real) hyperbolic eigenvalues for $DF(z)$.

Since the completely integrable map G is symplectic, its torsion $b_0 := \frac{\partial \beta}{\partial r}$ is symmetric. Therefore $b_0^{-1} + (b_0^{-1})^* = 2\, b_0^{-1}$ is non-singular. Since F is C^1 near G, we can assume that $b^{-1} + (b^{-1})^*$ is non-singular.

For the completely integrable map G we have that $\eta_G \equiv 0$, $\varphi_G \equiv 0$, $c_G = \frac{\partial R}{\partial \theta} = 0$, $a_G = \frac{\partial \Theta}{\partial \theta} = I$. Since F is C^1 near G, from (7) we have that $D^2\varphi(\theta)$ is near 0. Therefore we can assume that $\|D^2\varphi(\theta)\|$ is so small that

$$D^2\varphi(\theta) + 2\,[\, b^{-1} + (b^{-1})^*] \tag{9}$$

has the same signature as $[\, b^{-1} + (b^{-1})^* \,]$, where $b = \frac{\partial \Theta}{\partial r}$ is the torsion for F.

Since φ is a Morse function on $\mathbb{T}^n$, for any $0 \le p \le n$ there are $\binom{n}{p}$ critical points θ of φ where $D^2\varphi(\theta)$ has signature $(p, n-p)$. Suppose that the signature of $b^{-1} + (b^{-1})^*$ is $(q, n-q)$ and the signature of $D^2\varphi(\theta)$ is $(p, n-p)$. Consider the map $[0,\pi] \ni \alpha \overset{N}{\longmapsto} M_{e^{i\alpha}}$ corresponding to $\lambda = e^{i\alpha} \in \mathbb{S}^1$. Observe that

$$N(\alpha) = M_{e^{i\alpha}} = D^2\varphi(\theta) + (1 - e^{i\alpha})\, b^{-1} + (1 - e^{-i\alpha})\, (b^{-1})^*$$

is an hermitian matrix and then its has real eigenvalues. By lemma 4.3,

$$N(0) = M_{\lambda=1} = D^2\varphi(\theta) \quad \text{has signature} \quad (p, n-p),$$

By the hypothesis in (9),

$$N(\pi) = M_{\lambda=-1} = D^2\varphi(\theta) + 2\,[\, b^{-1} + (b^{-1})^*] \quad \text{has signature} \quad (q, n-q).$$

Therefore there are at least $|p-q|$ values of $\lambda = e^{i\alpha}$, $\alpha \in [0,\pi]$ where $\det M_\lambda = 0$, counting multiplicities (by $\dim \ker M_\lambda$). Thus $DF(z)$ has at least $2\,|p-q|$ eigenvalues of modulus 1, considering the complex conjugates $\overline{\lambda} = e^{-i\alpha}$, $-\alpha \in [-\pi, 0]$.

Let $\sigma := \operatorname{sign}\big[\,(-1)^n\,\det b\,\big]$. If

$$\operatorname{sign}\big[(-1)^n\,(\det b)\,\det D^2\varphi(\theta)\big] = \sigma\,(-1)^p < 0, \tag{10}$$

by (8) there are at least two (real) hyperbolic eigenvalues for $DF(z)$.

Therefore if (10) holds and $|p-q|\ge 1$, the fixed point z is of elliptic $\times$ hyperbolic type. These conditions are satisfied in the following cases:

(a) If $\sigma<0$, for (10) we want p even:
  If $q\neq 0$, take $p=0$;
  If $q=0$, since $n\ge 2$, take $p=2$.

(b) If $\sigma>0$, for (10) we want p odd:
  If $q\neq 1$, take $p=1$;
  If $q=1$ and $n\ge 3$, take $p=3$.

In the case $\sigma>0$, $q=1$ and $n=2$ take $p=1$. Then from (8) and (10) we have that $w_1\,w_2<0$. Then $w_1,\,w_2\in\mathbb{R}$ because otherwise they would be complex conjugates. Say $w_1<0$, which gives two hyperbolic eigenvalues, and $w_2>0$. But then $0<w_2<4$ because we are assuming that F is C^1 close to G. This gives two elliptic eigenvalues and hence z is of elliptic $\times$ hyperbolic type.

□

5. Coordinates and General Perturbations.

Let M^{n+1} be a closed manifold of dimension $n+1$. Given a riemannian metric g for M, denote by $\pi: S^gM\to M$ its unit tangent bundle, by $\phi^g_t: S^gM\to S^gM$ its geodesic flow and by X_g the vector field of ϕ^g. Fix a C^∞ riemannian metric $\mathfrak{g}$ and denote by SM its unit tangent bundle, which we call the *sphere bundle*. For any riemannian metric g, the map $SM\to S^gM$, $\theta\mapsto\theta/|\theta|_g$ is a diffeomorphism. Without loss of generality we shall assume that all the riemannian metrics in the paper have injectivity radius larger than 4.

Denote by $\mathcal{R}^k(M)$, $k\in\mathbb{N}\cup\{+\infty\}$ the Banach manifold of C^k riemannian metrics with the C^k topology. Let $\mathfrak{X}^k(SM)$ be the set of C^k vector fields on the sphere bundle SM with the C^k topology and $\mathfrak{F}^k(SM)$ the set of C^k flows on SM with the C^k topology.

In a local coordinate chart, the geodesic equations read

$$\ddot{x}_k = \sum_{ij}\Gamma^k_{ij}\,x_i\,x_j,$$

where the Christoffel symbols

$$\Gamma^k_{ij}(x) = \tfrac{1}{2}\sum_\ell g^{k\ell}\left(\frac{\partial g_{\ell j}}{\partial x_i}+\frac{\partial g_{i\ell}}{\partial x_j}-\frac{\partial g_{ij}}{\partial x_\ell}\right), \qquad [g^{k\ell}]=[g_{k\ell}]^{-1}$$

depend only on the 1-jet of the riemannian metric g. Thus the map $\mathcal{R}^2(M)\to\mathfrak{X}^1(SM)$, $g\mapsto X_g$ is continuous. This implies that the map $\mathcal{R}^2(M)\ni g\mapsto\phi^g\in\mathfrak{F}^1(SM)$ is continuous. In particular, the derivative of the geodesic flow $d_\theta\phi^g_t$ depends continuously on $g\in\mathcal{R}^2(M)$.

Fix a riemannian metric g_0 on M and assume that the injectivity radius of g_0 is larger than 4. We now introduce Fermi coordinates along a geodesic arc $c(t)$, $t \in [-1, 1]$ with unit speed. All the facts that we will use about Fermi coordinates can be found in [13, 18]. Take an orthonormal frame $\{ c'(0), e_1, \dots, e_n\}$ for $T_{c(0)}M$. Let $e_i(t)$ denote the parallel translation of e_i along c. Consider the differentiable map $\Phi : [-1, 1] \times \mathbb{R}^n \to M$ given by

$$\Phi(t, x) = \exp_{c(t)} \Big[\sum_{i=1}^{n} x_i \, e_i(t) \Big].$$

This map has maximal rank at $(t, 0)$, $t \in [-1, 1]$. Since $c(t)$ has no self intersections on $t \in [-1, 1]$, there exists a neighbourhood V of $[-1, 1] \times \{0\}$ in which $\Phi|_V$ is a diffeomorphism.

Let $[g_0(t, x)]_{ij}$ denote the components of the metric g_0 in the chart (Φ, V). Let $\mathcal{S}(n) \subset \mathbb{R}^{n\times n}$ be the manifold of symmetric matrices. Let $\alpha : [-1, 1] \times \mathbb{R}^n \to \mathcal{S}(n)$ be a C^∞ function with support in a neighbourhood of $[-1, 1] \times \{0\}$. We can define a new riemannian metric g by setting

$$\begin{aligned} g_{00}(t, x) &= [g_0(t, x)]_{00} + \sum_{i,j=1}^{n} \alpha_{ij}(t, x) \, x_i \, x_j \,; \\ g_{0j}(t, x) &= [g_0(t, x)]_{0j} \,, \quad 1 \le j \le n; \\ g_{ij}(t, x) &= [g_0(t, x)]_{ij} \,, \quad 1 \le i, j \le n; \end{aligned} \tag{11}$$

where we index the coordinates by $x_0 = t$ and $(x_1, \dots, x_n) = x$.

For any such metric g we have that (cf. [13, 18]):

$$\begin{aligned} g^{ij}(t, 0) = \quad g_{ij}(t, 0) \quad &= \delta_{ij}, \qquad 0 \le i, j \le n; \\ \partial_k \, g^{ij}(t, 0) = \partial_k \, g_{ij}(t, 0) &= 0, \qquad 0 \le i, j, k \le n; \end{aligned}$$

where $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$.

We need the differential equations for the geodesic flow ϕ_t in hamiltonian form. It is well known that the geodesic flow is conjugate to the hamiltonian flow of the function

$$H(x, y) = \frac{1}{2} \sum_{i,j} g^{ij}(x) \, y_i \, y_j.$$

Hamilton's equations are

$$\begin{aligned} \tfrac{d}{dt} \, x_i &= \quad H_{y_i} \;\; = \qquad \sum_j g^{ij}(x) \, y_j, \\ \tfrac{d}{dt} \, y_k &= -H_{x_k} = -\tfrac{1}{2} \sum_{i,j} \tfrac{\partial}{\partial x_k} g^{ij}(x) \, y_i \, y_j. \end{aligned}$$

Observe that for all such metrics g the curve $c(t)$ is a geodesic and the orbit $\gamma(t) = (c(t), \dot{c}(t))$ is given by the coordinates $x_0 = t$, $x = 0$, $y_0 = 1$, $y = 0$.

Using the identity $\frac{d}{dt} \, (d\phi_t) = (dX \circ \phi_t) \cdot d\phi_t$, with $X = \frac{d}{dt} \phi_t \big|_{t=0}$, we obtain the differential equation for the linearized hamiltonian flow, on the orbit $\gamma(t)$, which we call the Jacobi equation:

$$\frac{d}{dt}\bigg|_{(t,x=0)} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & I \\ \begin{smallmatrix} 0 & 0 \\ 0 & -K \end{smallmatrix} & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{12}$$

where $K(t) \in \mathbb{R}^{n\times n}$ is a symmetric matrix given by

$$K(t)_{ij} = \tfrac{1}{2} \tfrac{\partial^2}{\partial x^i \partial x^j}\, g^{00}(t,0) = -\tfrac{1}{2} \tfrac{\partial^2}{\partial x^i \partial x^j}\, g_{00}(t,0). \tag{13}$$

Let

$$K_0(t) := \tfrac{1}{2} \tfrac{\partial^2}{\partial x^2}\, g_0^{00}(t,0) \in \mathcal{S}(n).$$

It is easy to check that

$$K(t) = K_0(t) - \alpha(t,0). \tag{14}$$

By comparison with the usual Jacobi equation we get that

$$K(t)_{ij} = \langle R_g\big(\dot c(t), e_i(t)\big)\, \dot c(t),\, e_j(t)\rangle_g, \tag{15}$$

where R_g is the curvature tensor for the metric g. We call $K(0)$ the *Jacobi matrix* or the matrix of sectional curvatures of the orthonormal frame $\{\dot c(0), e_1, \dots, e_n\}$.

If we change the frame $\{\dot c(0), e_1, \dots, e_n\}$ to another orthonormal frame $\{\dot c(0), u_1, \dots, u_n\}$ with $u_i = \sum_j q_{ij}\, e_j$, the matrix $Q = [q_{ij}]_{n\times n}$ is orthogonal and the matrix $K(t)$ changes to $Q\, K(t)\, Q^*$. Therefore we have a well defined map $K_g : S^gM \to \mathcal{S}(n)/_{O(n)}$, $K_g(\dot c(0)) = [K(0)]$, from the unit tangent bundle for g to the conjugacy classes of $\mathcal{S}(n)$ by the orthogonal group. In particular, the set of eigenvalues of $K_g(\dot c(0))$ is well defined.

6. A generic condition on the curvature.

In order to make the perturbation lemma in section 7 we need to choose a metric in which every geodesic segment of length $\frac{1}{2}$ has a point in which the Jacobi matrix (15) has no repeated eigenvalues. In this section we prove that such condition is generic.

Recall that $\mathcal{R}^2(M)$ is the manifold of C^2 riemannian metrics on M endowed with the C^2 topology. Given $g \in \mathcal{R}^2(M)$, define as in section 5 the map $K_g : S^gM \to \mathcal{S}(n)/_{O(n)}$ by $K_g(\theta) = [K]$ where

$$K_{ij} = \langle R_g(\theta, e_i)\, \theta, e_j\rangle_{\pi(\theta)},$$

where $\{\theta, e_1, \dots, e_n\}$ is any orthonormal basis for $T_{\pi(\theta)}M$. Let $h : \mathcal{S}(n)/_{O(n)} \to [0, +\infty[$ be the function

$$h([K]) := \prod_{1\le i<j\le n} (\lambda_i - \lambda_j)^2, \tag{16}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of K. Let $H : \mathcal{R}^2(M) \to [0, +\infty[$ be

$$H(g) := \min_{\theta\in S^gM}\ \max_{t\in[0,\frac{1}{2}]} h(K_g(\phi_t^g(\theta))). \tag{17}$$

In this section we prove the following

6.1. **Theorem.** *The function* $H:\mathcal{R}^2(M)\to[0,+\infty[$ *is continuous and the set*

$$\mathcal{G}_1 := \big\{\, g\in\mathcal{R}^2(M)\mid H(g)>0\,\big\}$$

is open in $\mathcal{R}^2(M)$ *and* $\mathcal{G}_1\cap\mathcal{R}^\infty(M)$ *is dense in* $\mathcal{R}^\infty(M)$.

Proof:

Define the function $h:\mathcal{S}(n)/_{O(n)}\to\mathbb{R}$ by $h([A]):=(-1)^m\det[Dp_A(A)]$, where $p_A(x)=\det(xI-A)$ is the characteristic polynomial of a representative $A\in\mathcal{S}(n)$, Dp_A is its derivative and $m=\binom{n}{2}=\frac{n(n-1)}{2}$. It is easy to see that h is well defined and, by calculating its value on a diagonal representative of $[A]$ in $\mathcal{S}(n)/_{O(n)}$, that

$$h([A]) := (-1)^m\det[Dp_A(A)] = \prod_{1\le i<j\le n}(\lambda_i-\lambda_j)^2,$$

where the λ_i's are the eigenvalues of the class $[A]$. Moreover, the function h is continuous.

In a coordinate chart, the curvature tensor

$$R(\tfrac{\partial}{\partial x_i},\tfrac{\partial}{\partial x_j})\,\tfrac{\partial}{\partial x_k}=\sum_{ijk}R^\ell_{ijk}\,\tfrac{\partial}{\partial x_\ell}\,,$$

$$R^s_{ijk}=\sum_\ell\Gamma^\ell_{ik}\,\Gamma^s_{j\ell}-\sum_\ell\Gamma^\ell_{jk}\,\Gamma^s_{i\ell}+\frac{\partial\Gamma^s_{ik}}{\partial x_j}-\frac{\partial\Gamma^s_{jk}}{\partial x_i}\,,$$

depends only on the 2-jet of the riemannian metric. Thus the Jacobi matrix $K_g(\theta)$ depends continuously on $g\in\mathcal{R}^2(M)$.

Define the map $\mathbb{K}:\mathcal{R}^2(M)\times SM\times[0,\frac12]\to\mathcal{S}(n)/_{O(n)}$, by

$$\mathbb{K}(g,\theta,t):=K_g\big[\phi^g_t(\tfrac{\theta}{|\theta|_g})\big]. \tag{18}$$

Since both h and $\mathbb{K}$ are continuous and SM is compact, the function $H:\mathcal{R}^2(M)\to[0,+\infty[$ defined in (17) is continuous. Hence $\mathcal{G}_1=H^{-1}(\mathbb{R}^+)$ is open in $\mathcal{R}^2(M)$.

Let $FM\to M$ be the frame bundle over M:

$$FM=\{\,\Theta=(\theta_0,\theta_1,\dots,\theta_n)\in(T_xM)^{n+1}\,|\,x\in M,\ \Theta\text{ is a }\mathfrak{g}\text{-orthonormal basis}\,\}.$$

Let $J^k\mathcal{S}(n)$ be the k-jet bundle of curves in $\mathcal{S}(n)$, i.e. $J^k\mathcal{S}(n)$ is the set of equivalence classes of smooth curves $a:]-\varepsilon,\varepsilon[\to\mathcal{S}(n)$ under the relation $a_1\sim a_2$ iff there is a smooth chart $\psi:U\to\mathbb{R}^d$ for $\mathcal{S}(n)$ about $a_1(0)$ such that $D^j(\psi\circ a_1)(0)=D^j(\psi\circ a_2)(0)$ for all $j=0,1,\dots,k$, where $d:=\dim\mathcal{S}(n)=\frac{n(n+1)}{2}$. Then $J^k\mathcal{S}(n)$ is a smooth bundle over $\mathcal{S}(n)$ whose fiber is the set $P_{k,d}$ of polynomials $p:\mathbb{R}\to\mathbb{R}^d$ of degree $\le k$ with $p(0)=0$. Therefore

$$\dim J^k\mathcal{S}(n)=\dim\mathcal{S}(n)+\dim P_{k,d}=d+k\,d=(k+1)\,d. \tag{19}$$

Consider the map $K:\mathcal{R}^\infty(M)\times FM\times\mathbb{R}\to\mathcal{S}(n)$ defined by

$$K(g,\Theta,t)_{ij}:=\big\langle R_g(\theta^g_0(t),\theta^g_i(t))\,\theta^g_0(t),\theta^g_j(t)\big\rangle_g,$$

where $\Theta^g = (\theta^g_0, \theta^g_1, \dots, \theta^g_n)$ is the g-orthonormal frame obtained from Θ by the Gram-Schmidt process and $\Theta^g(t)$ is its g-parallel transport along the g-geodesic $c(t) = \pi(\phi^g_t(\theta^g_0))$. Let $\mathcal{K} : \mathcal{R}^\infty(M) \times FM \to J^k\mathcal{S}(n)$ be the jet extension of K, i.e. $\mathcal{K}(g, \Theta) = J^k a(0)$ is the k-jet of the curve $a(t) := K(g, \Theta, t)$ at $t = 0$.

The perturbation given in section 5 and formula (14) show that any smooth path $a(t)$ on $\mathcal{S}(n)$ or $J^k\mathcal{S}(n)$ with $a(0) = K(g, \Theta, 0)$ can be realized by a smooth perturbation of the metric g which preserves the geodesic at θ^g_0. Therefore the map $\mathcal{K}$ is a submersion for any $k \geq 0$.

Now consider the set $\Sigma \subset \mathcal{S}(n)$ of symmetric matrices with a repeated eigenvalue. It is an algebraic subset of $\mathcal{S}(n) \approx \mathbb{R}^d$ because it is the set of zeroes of the polynomial map $h : \mathcal{S}(n) \to \mathbb{R}$, $h(A) = (-1)^m \det[Dp_A(A)]$. Since the polynomial h is non-constant, Σ has positive codimension $r > 0$ in $\mathcal{S}(n)$. This is, since Σ is an algebraic set, it has a Whitney stratification by submanifolds of $\mathcal{S}(n)$, whose maximal dimension is $d - r$. Let $J^k\Sigma \subset J^k\mathcal{S}(n)$ be the set of k-jets of C^∞ curves in $\mathcal{S}(n)$ whose image is in Σ.

Define the *arc space* $\mathcal{L}(\Sigma)$ of Σ as the set of formal power series $\ell(t) = \sum_{i=0}^\infty a_i\, t^i$, with $a_i \in \mathcal{S}(n)$ and one parameter t, such that $h(\ell(t)) \equiv 0$. For $k \in \mathbb{N}$, let $\mathcal{L}_k(\Sigma)$ be the set of polynomials $p(t) = \sum_{i=0}^k a_i\, t^i$ of degree $\leq k$ in $\mathcal{S}(n)$ such that $h(p(t)) = 0 \mod t^{k+1}$. We have a natural projection $\pi_k : \mathcal{L}(\Sigma) \to \mathcal{L}_k(\Sigma)$ given by truncation. We also have a natural injection $J^k\Sigma \hookrightarrow \pi_k(\mathcal{L}(\Sigma))$ given by the Taylor expansion of the curves up to order k. Thus we have the inclusions $J^k\Sigma \subset \pi_k(\mathcal{L}(\Sigma)) \subset \mathcal{L}_k(\Sigma) \subset J^k\mathcal{S}(n)$.

The set $\mathcal{L}_k(\Sigma)$ is algebraic because it is the set of zeroes of finitely many polynomials. The set $\pi_k(\mathcal{L}(\Sigma))$ is constructible (cf. Denef & Loeser [8, p. 202]), i.e. it is obtained by unions and subtractions of finitely many algebraic sets. Each of those algebraic sets has a Whitney stratification, therefore $\pi_k(\mathcal{L}(\Sigma))$ is a union of countably many submanifolds of $J^k\mathcal{S}(n)$. The dimension of $\pi_k(\mathcal{L}(\Sigma))$ is the maximal dimension of those submanifolds. By lemma[4] 4.3 in Denef & Loeser [8], $\dim \pi_k(\mathcal{L}(\Sigma)) \leq (k+1)\dim\Sigma \leq (k+1)(d-r)$. Also in proposition A.1 in the appendix, we prove that $\dim J^k\Sigma \leq (k+1)\dim\Sigma$, which is enough for our argument. Therefore, from (19), the codimension of $\pi_k(\mathcal{L}(\Sigma))$ in $J^k\mathcal{S}(n)$ satisfies

$$\lim_{k\to+\infty} \operatorname{codim}_{J^k\mathcal{S}(n)} \pi_k(\mathcal{L}(\Sigma)) = +\infty.$$

Since the function $\mathcal{K}$ is a submersion, it is transversal to each stratum T of $\pi_k(\mathcal{L}(\Sigma))$. By theorem 19.1 in [1] there is a residual set $\mathcal{D}_T \subset \mathcal{R}^\infty(M)$ such that for all $g \in \mathcal{D}_T$, the maps $\mathcal{K}(g, \cdot) : FM \to J^k\mathcal{S}(n)$ are transversal to T. Since $\operatorname{codim}_{J^k\mathcal{S}(n)} \pi_k(\mathcal{L}(\Sigma)) \geq k+1$, if $k + 1 > \dim FM$ and $g \in \mathcal{D}_T$, then the image of $\mathcal{K}(g, \cdot)$ does not intersect T. Since there is a countable number of strata, intersecting all those residual subsets we get a residual set $\mathcal{D}_0 \subset \mathcal{R}^\infty(M)$ such that for $g \in \mathcal{D}_0$, the image of $\mathcal{K}(g, \cdot)$ is disjoint from $\pi_k(\mathcal{L}(\Sigma))$ and also from $J^k\Sigma$.

[4]This estimate on the dimension may not be satisfied for $\mathcal{L}_k(\Sigma)$, at least for small k, see examples in Veys [34].

Since $\mathcal{R}^\infty(M)$ is a complete metric space, the residual set $\mathcal{D}_0$ is dense in $\mathcal{R}^\infty(M)$. We now prove that $H > 0$ on $\mathcal{D}_0$. Then $\mathcal{G}_1$ contains a dense set in $\mathcal{R}^\infty(M)$. Let $g \in \mathcal{D}_0$. Suppose that $H(g) = 0$. Observe that both, the maximum and minimum in (17) are attained. Since the function h in (16) is non-negative, there exists $\theta \in S^gM$ such that $h(K_g(\phi_t^g(\theta))) \equiv 0$ for all $t \in [0, \frac{1}{2}]$. Let $\theta_0 \in SM$ be such that $\theta_0/|\theta_0|_g = \theta$, and let $\Theta \in FM$ be a frame whose first vector is θ_0. Then the C^∞ curve $c(t) := K(g, \Theta, t) \in \Sigma$ for all $t \in [0, \frac{1}{2}]$. Hence $\mathcal{K}(g, \Theta) = J^k c(0) \in J^k\Sigma$. This contradicts the choice of $g \in \mathcal{D}_0$.

□

7. Franks' lemma for geodesic flows.

Let $\gamma = \{\, \phi_t^g(v) \,|\, t \in [0, 1]\}$ be a piece of an orbit of length 1 of the geodesic flow ϕ_t^g of the metric $g \in \mathcal{R}^\infty(M)$. Let Σ_0 and Σ_t be transverse sections to ϕ^g at v and $\phi_t^g(v)$ respectively. We have a Poincaré map $\mathcal{P}_g(\Sigma_0, \Sigma_t, \gamma)$ going from Σ_0 to Σ_t. One can choose Σ_0 and Σ_t such that the *linearized Poincaré map*

$$P_g(\gamma)(t) \overset{\text{def}}{=} d_v\mathcal{P}_g(\Sigma_0, \Sigma_t, \gamma)$$

is a linear symplectic map from $\mathcal{N}_0 := N(v) \oplus N(v)$ to $\mathcal{N}_t := N(\phi_t^g(v)) \oplus N(\phi_t^g(v))$ and

$$P_g(\gamma)(t)(J(0), \dot{J}(0)) = (J(t), \dot{J}(t)),$$

where J is an orthogonal Jacobi field along the geodesic $\pi \circ \gamma$ and $\dot{J}$ denotes the covariant derivative along the geodesic. Fix a set of Fermi coordinates along $\pi \circ \gamma$. Then we can identify the set of all linear symplectic maps from $\mathcal{N}_0$ to $\mathcal{N}_t$ with the symplectic group

$$Sp(n) := \{\, X \in \mathbb{R}^{n\times n} \,|\, X^*\mathbb{J}X = \mathbb{J}\,\},$$

where $\mathbb{J} = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$.

Suppose that the geodesic arc $\pi \circ \gamma(t)$, $t \in [0, 1]$, does not have any self intersection and let W be a tubular neighbourhood of it. We denote by $\mathcal{R}^\infty(\gamma, g, W)$ the set of metrics $\bar{g} \in \mathcal{R}^\infty(M)$ for which γ is a piece of orbit of length 1, $\bar{g} = g$ on $\gamma([0, 1])$ and such that the support of $\bar{g} - g$ lies in W.

When we apply the following theorem 7.1 to a piece of a closed geodesic we may have self intersections of the whole geodesic. Given any finite set of non-self intersecting geodesic segments $\mathfrak{F} = \{\eta_1, \dots, \eta_m\}$, defined on $[0, 1]$, with the following properties:

1. The endpoints of η_i are not contained in W;
2. The segment $\pi \circ \gamma|_{[0,1]}$ intersects each η_i transversally;

denote by $\mathcal{R}^\infty(\gamma, g, W, \mathfrak{F})$ the set of metrics $\overline{g} \in \mathcal{R}^\infty(\gamma, g, W)$ such that $\overline{g} = g$ in a small neighbourhood of $W \cap \cup_{i=1}^m \eta_i([0, 1])$.

Consider the map $S : \mathcal{R}^\infty(\gamma, g, W) \to Sp(n)$ given by $S(\bar{g}) = P_{\bar{g}}(\gamma)(1)$. The following result is the analogue for geodesic flows of the infinitesimal part of Franks' lemma [11, lem.

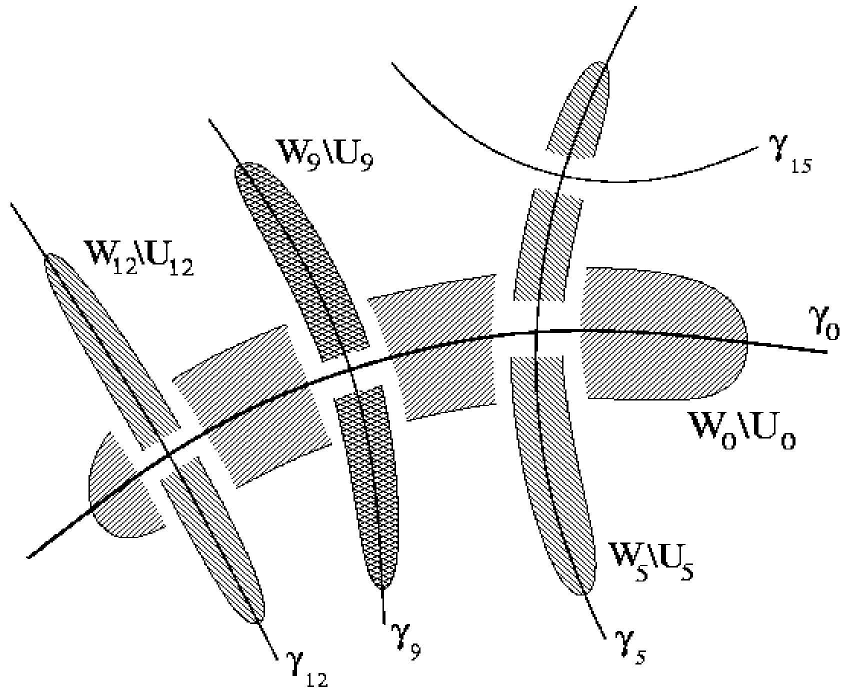


FIGURE 1. Avoiding self-intersections.

1.1] (whose proof for general diffeomorphisms is quite simple). A difference with the case of surfaces in [6] is that here we ask the original metric g_0 to be in the residual set $\mathcal{G}_1$ obtained in theorem 6.1.

7.1. **Theorem.** *Let $g_0 \in \mathcal{G}_1 \cap \mathcal{R}^r(M)$, $4 \le r \le \infty$. Given $\mathcal{U} \subset \mathcal{R}^2(M)$ a neighbourhood of g_0, there exists $\delta = \delta(g_0, \mathcal{U}) > 0$ such that given $g \in \mathcal{U}$, γ, W and $\mathfrak{F}$ as above, the image of $\mathcal{U} \cap \mathcal{G}_1 \cap \mathcal{R}^r(\gamma, g, W, \mathfrak{F})$ under the map S contains the ball of radius δ centered at $S(g_0)$.*

The time 1 in the preceding statement was chosen to simplify the exposition and the same result holds for any time τ chosen in a closed interval $[a,b] \subset]0,+\infty[$; now with $\delta = \delta(g_0, \mathcal{U}, a, b) > 0$. In order to fix the setting, take $[a,b] = [\frac{1}{2}, 1]$ and assume that the injectivity radius of M is larger than 4. This implies that there are no periodic orbits with period smaller than 8 and that any periodic orbit can be cut into non self-intersecting geodesic segments of length τ with $\tau \in [\frac{1}{2}, 1]$. We shall apply theorem 7.1 to such segments of a periodic orbit choosing the supporting neighbourhoods carefully as we now describe.

A closed geodesic is *prime* if it is not the iterate of a shorter closed geodesic. Given $g \in \mathcal{R}^r(M)$ and γ a prime periodic orbit of g let $\tau \in [\frac{1}{2}, 1]$ be such that $m\tau = \text{period}(\gamma)$ with $m \in \mathbb{N}$. For $0 \le k < m$, let $\gamma_k(t) := \gamma(t + k\tau)$ with $t \in [0, \tau]$. Given a tubular neighbourhood W of $\pi \circ \gamma$ and $0 \le k < m$ let $S_k : \mathcal{R}^r(\gamma, g, W) \to Sp(n)$ be the map $S_k(\overline{g}) = P_{\overline{g}}(\gamma_k)(\tau)$.

Let W_0 be a small tubular neighbourhood of γ_0 contained in W. Let $\mathcal{F}_0 = \{\eta_1^0, \dots, \eta_{m_0}^0\}$ be the set of geodesic segments η given by those subsegments of γ of length τ whose endpoints are outside W_0 and which intersect γ_0 transversally at $\eta(\tau/2)$ (see Figure 1). We now apply Theorem 7.1 to γ_0, W_0 and $\mathcal{F}_0$. The proof of this theorem also selects a neighbourhood U_0 of $W_0 \cap \cup_{i=1}^{m_0} \eta_i^0([0,\tau])$. We now consider γ_1 and we choose a tubular neighbourhood W_1 of γ_1 small enough so that if γ_1 intersects γ_0 transversally, then W_1 intersected with W_0 is contained in U_0 (see Figure 1). By continuing in this fashion we select recursively tubular neighbourhoods $W_0, \dots, W_{m-1}$, all contained in W, to which we successively apply Theorem 7.1. This choice of neighbourhoods ensures that there is no interference between one perturbation and the next. In the end we obtain the following:

7.2. **Corollary.**

Let $g_0 \in \mathcal{G}_1 \cap \mathcal{R}^r(M)$, $4 \le r \le \infty$. Given a neighbourhood $\mathcal{U}$ of g_0 in $\mathcal{R}^2(M)$, there exists $\delta = \delta(g_0, \mathcal{U}) > 0$ such that if $g \in \mathcal{U}$, γ is a prime closed orbit of ϕ^g and W is a tubular neighbourhood of $c = \pi \circ \gamma$, then the image of $\mathcal{U} \cap \mathcal{G}_1 \cap \mathcal{R}^r(\gamma, g_0, W) \to \Pi_{k=0}^{m-1} Sp(n)$, under the map $(S_0, \dots, S_{m-1})$, contains the product of balls of radius δ centered at $S_k(g_0)$ for $0 \le k < m$.

The arguments below can be used to show that $\overline{g} - g$ can be supported not only outside a finite number of intersecting segments but outside any given compact set[5] of measure zero in γ. This is done by adjusting the choice of the function h in (30).

The nature of these results (i.e. the independence on the size of the neighbourhood W) forces us to use the C^1 topology on the perturbation of the geodesic flow, thus the C^2 topology on the metric. The size $\delta(g_0, \mathcal{U}) > 0$ in theorem 7.1 and corollary 7.2 depends on the C^4-norm of g_0 and the value of $H(g_0)$ from Theorem 6.1.

The remaining of the section is devoted to the

Proof of Theorem 7.1:

We first describe the strategy used in the proof. At the beginning we fix most of the constants and bump functions that are needed. We show that the map S is a submersion when restricted to a suitable submanifold of the set of perturbations. To obtain a size δ that depends only on g_0 and $\mathcal{U}$ and that works for all $g \in \mathcal{U}$, γ and W we find a uniform lower bound for the norm of the derivative of S using the constants and the bump functions that we fixed before. This uniform estimate can only be obtained in the C^2 topology.

The technicalities of the proof can be summarized as follows. To obtain a C^2 perturbation of the metric preserving the geodesic segment $c = \pi \circ \gamma$ one needs a perturbation of the form (32), with $\alpha(t,x) = \varphi_\varepsilon(x)\, p(t)$, where $\varphi_\varepsilon(x)$ is a bump function supported in an ε-neighbourhood in the transversal direction to c and $p(t)$ is given by formula (33). The second factor in (33) is used to make the derivative of S surjective. The function $\delta(t)$ is an approximation to a Dirac delta at a point $t = \tau$ where $h(\mathbb{K}(g_0, \theta, \tau)) > \frac{1}{2} H(g_0)$, where H is from Theorem 6.1. This is done in order to solve equation (38), which is trivial when $\dim M = 2$ (and $\beta \in \mathbb{R}$). The first factor $h(t)$ is an approximation of a characteristic function

[5]But to use this argument to support $\overline{g} - g$ outside a given infinite set of geodesic segments of length $\ge \frac{1}{2}$ one needs to bound from below their angle of intersection with c.

used to support the perturbation outside a neighbourhood of the intersecting segments in $\mathfrak{F} = \{\eta_1, \ldots, \eta_m\}$. Then inequality (27) shows that if the neighbourhood W of c is taken small enough, then the C^2 norm of the perturbation is essentially bounded by only the C^0 norm of $p(t)$. In order to bound the C^2 norm of $p(t)$ from (33) in equation (27), we use the C^4 norm of g_0 to have a bound for the second derivative of the sectional curvature $K_0(t,0)$ of g_0 along the geodesic c.

Since $\mathcal{G}_1$ is open in the C^2 topology, we can assume that $\mathcal{U}$ is small enough so that

$$\mathcal{U} \cap \mathcal{R}^r(\gamma, g, W, \mathfrak{F}) \subset \mathcal{U} \cap \mathcal{G}_1 \cap \mathcal{R}^r(\gamma, g, W, \mathfrak{F}).$$

By shrinking $\mathcal{U}$ if necessary, we can assume that there is $k_0 = k_0(\mathcal{U}) > 0$ such that the Jacobi matrices, given in (18), satisfy

$$\|\mathbb{K}(g,\theta,t)\| \le k_0 \qquad \text{for all} \quad (g,\theta,t) \in \mathcal{U} \times SM \times [0,1]. \tag{20}$$

Let $k_1 = k_1(\mathcal{U}) > 1$ be such that if $g \in \mathcal{U}$ and ϕ_t is the geodesic flow of g, then

$$\|d_\theta \phi_t\| \le k_1 \quad \text{and} \quad \|d_\theta \phi_t^{-1}\| \le k_1 \quad \text{for all } t \in [0,1] \tag{21}$$

and all $\theta \in S^1_g M$. Given $0 < \lambda \ll \frac{1}{8}$ let $k_2 = k_2(\mathcal{U}, \lambda) > 0$ be such that $\lim_{\lambda \to 0} k_2(\lambda) = 0$ and

$$\|d_\theta \phi_s - d_\theta \phi_t\| \le k_2 \quad \text{and} \quad \|d_\theta \phi_s^{-1} - d_\theta \phi_t^{-1}\| \le k_2 \quad \text{for all } |s-t| < \lambda, \tag{22}$$

$s, t \in [0,1]$, all $g \in \mathcal{U}$ and all $\theta \in S^1_g M$. Choose $\lambda = \lambda(\mathcal{U}) > 0$ small enough such that

$$k_1^{-2} - 2\, k_1\, k_2 > 0. \tag{23}$$

Since $g_0 \in \mathcal{G}_1$, there is $a_0 > 0$ such that $H(g_0) > 2\, a_0^2$, where H is from (17). Consider the map $H_2 : \mathcal{R}^2(M) \times SM \times [0,1] \to \mathbb{R}$ given by $H_2(g,\theta,t) = h(\mathbb{K}(g,\theta,t))$, where h is from (16) and $\mathbb{K}(g,\theta,t)$ is from (18). Then H_2 is continuous. Let

$$A_0 := \big\{ (\theta,t) \in SM \times [0,1] \;\big|\; H_2(g_0,\theta,t) \ge 2\, a_0^2 \big\}.$$

Then $A_0 \subset SM \times [0,1]$ is compact and since $H(g_0) > 2\, a_0^2$,

$$A_0 \cap \big(\{\theta\} \times [\tfrac{1}{4}, \tfrac{3}{4}]\big) \ne \emptyset \quad \text{for all } \theta \in SM.$$

Since H_2 is continuous, there is a neighbourhood $\mathcal{U}_0 \subset \mathcal{U}$ of g_0 in $\mathcal{R}^2(M)$ such that

$$H_2(g,\theta,t) > a_0^2 \qquad \text{for all} \quad (g,\theta,t) \in \mathcal{U}_0 \times A_0.$$

Let $v := \gamma(0)$ and fix $\tau = \tau(v, \mathcal{U}_0) \in [\frac{1}{4}, \frac{3}{4}]$ such that $(v,\tau) \in A_0$. Then, if $i \ne j$,

$$(2\, k_0)^{2(m-1)}\, |\lambda_i - \lambda_j|^2 \ge \big(2\, \|\mathbb{K}(g,v,\tau)\|\,\big)^{2(m-1)}\, |\lambda_i - \lambda_j|^2 \ge \prod_{i \ne j} (\lambda_i - \lambda_j)^2 > a_0^2,$$

for all $g \in \mathcal{U}_0$, where $m = \binom{n}{2} = \frac{n(n-1)}{2}$ and $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of $\mathbb{K}(g,\theta,t)$. Therefore

$$\min_{i \ne j} |\lambda_i - \lambda_j| > \frac{a_0}{(2k_0)^{m-1}} =: k_3. \tag{24}$$

Let

$$k_4 := \max\left\{ k_3^{-1},\ 1 + 4\, k_0\, k_3^{-1},\ 1,\ k_0 \right\}. \tag{25}$$

Let $\delta : [0,1] \to [0,+\infty[$ be a C^∞ function such that $\delta(s) = 0$ if $|s-\tau| \ge \lambda$ and $\int_0^1 \delta(s)\ ds = 1$, where $\lambda = \lambda(\mathcal{U})$ is from (23). The C^5-norm of δ depends only on $\mathcal{U}$ and does not depend on $\tau = \tau(v)$.

By (23) there exists $\rho = \rho(\mathcal{U}) > 0$ such that

$$k_5 := \frac{k_1^{-2} - 2\, k_1 k_2 - \rho\, k_1^2\ \|\delta\|_{C^0}}{k_1\, k_4} > 0. \tag{26}$$

Given $\varepsilon > 0$, let $\varphi_\varepsilon : \mathbb{R}^n \to [0,1]$ be a C^∞ function such that $\varphi_\varepsilon(x) = 1$ if $x \in [-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]^n$ and $\varphi_\varepsilon(x) = 0$ if $x \notin [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^n$. In lemma 7.6 we prove that $\varphi_\varepsilon(x)$ can be chosen such that

$$\|\varphi_\varepsilon(x)\ x^* p(t)\, x\|_{C^2} \le k_6\, \|p\|_{C^0} + \varepsilon\, k_6\, \|p\|_{C^1} + \varepsilon^2\, k_6\, \|p\|_{C^2} \tag{27}$$

for some fixed $k_6 > 0$ (independent of ε) and any $p : [0,1] \to \mathbb{R}^{n\times n}$ of class C^2.

Let $\eta = \eta(g_0, \mathcal{U}_0) > 0$ be such that

$$4\, \eta\, k_5^{-1}\, k_6\, \|\delta\|_{C^3} < \tfrac{1}{2}\, \varepsilon_0. \tag{28}$$

Let $\varepsilon_0 = \varepsilon_0(g_0, \mathcal{U}_0) > 0$ be such that

$$\|g - g_0\|_{C^2} < \varepsilon_0 \quad \Longrightarrow \quad g \in \mathcal{U}_0. \tag{29}$$

So far, the constants chosen above, excepting τ, do not depend on γ or $\mathfrak{F}$. We shall prove that the image of $\mathcal{U}_0$ by S contains the ball in $Sp(n)$ of center $S(g_0)$ and radius $\eta = \eta(g_0, \mathcal{U})$.

Let $h : [0,1] \to [0,1]$ be a C^∞ function with support outside the intersecting points

$$\operatorname{supp}(h) \subset \,]0,1[\, \setminus (\pi\circ\gamma)^{-1}\left[\, \cup_{i=1}^m \eta_i \right]$$

and such that

$$\int_0^1 \big(1 - h(s)\big)\ ds < \rho. \tag{30}$$

From (28), there is $\varepsilon_1 = \varepsilon_1(g_0, \mathcal{U}_0, \gamma, \mathfrak{F}) > 0$ such that

$$k_5^{-1}\, \eta\, \big(\, 4\, k_6\, \|\delta\|_{C^3} + 8\, k_6\, \varepsilon_1\, \|h\|_{C^1}\, \|\delta\|_{C^4} + 16\, k_6\, \varepsilon_1^2\, \|h\|_{C^2}\, \|\delta\|_{C^5} \,\big) < \varepsilon_0. \tag{31}$$

Fix a Fermi coordinate chart (Φ, V) along the geodesic segment $c := \pi\circ\gamma$ for the metric g_0 as in section 5. Choose

$$\varepsilon_1 > \varepsilon_2 = \varepsilon_2(g_0, \mathcal{U}_0, \gamma, \mathfrak{F}, W) > 0$$

such that the segments η_i do not intersect the points with coordinates (t,x) with $|x| < \varepsilon_2$ and $t \in \operatorname{supp}(h)$ and such that $[0,1] \times [-\varepsilon_2, \varepsilon_2]^n \subset V$ and $\Phi([0,1] \times [-\varepsilon_2, \varepsilon_2]^n) \subset W$.

Let $\mathcal{S}(n) \approx \mathbb{R}^{\frac{n(n+1)}{2}}$ be the set of real $n\times n$ symmetric matrices. Let $\alpha(t,x)$ denote a C^∞ function $\alpha : [0,1] \times [-\varepsilon_2, \varepsilon_2]^n \to \mathcal{S}(n)$ with support contained in $V \setminus \Phi^{-1}(\cup_{i=1}^m \eta_i)$. Let $\mathcal{F}$ be the set of C^r riemannian metrics given by (11) endowed with the C^2 topology. One easily checks that $\mathcal{F} \subset \mathcal{R}^r(\gamma, g_0, W, \mathfrak{F})$. Let

$$\mathcal{V}_0 := \mathcal{F} \cap \mathcal{U}_0.$$

The Jacobi equation for the linearized geodesic flow on γ for the metrics on $\mathcal{F}$ is given by (12), where $K(t)$ is given by (13). Its solutions $\big(a(t), b(t)\big)$ satisfy $\dot{b}_0(t) = 0$ and $a_0(t) = a_0(0) + t\, b_0(t)$. Observe from (11) that the conditions

$$\begin{aligned} a_0(t) &= \sum_{i=1}^{n} g^{0i}(t,0)\, a_i(t) \equiv 0, \\ b_0(t) &= \dot{a}_0(t) \equiv 0, \end{aligned}$$

are invariant among the metrics $g \in \mathcal{F}$ and satisfy (12). These solutions correspond to Jacobi fields which are orthogonal to $\dot{c}(t)$. In particular, the subspaces

$$\mathcal{N}_t = \big\{ (a,b) \in T_{c(t)}TM \mid a_0 = b_0 = 0 \,\big\} \approx \mathbb{R}^n \times \mathbb{R}^n$$

are invariant under (12) for all $g \in \mathcal{F}$. From now on we reduce the Jacobi equation (12) to the subspaces $\mathcal{N}_t$.

We need uniform estimates for all $g \in \mathcal{V}_0$. Fix $g \in \mathcal{V}_0$ and write

$$\mathbb{A}_t = \mathbb{A}_t^g = \begin{bmatrix} 0 & I \\ -K(t,0) & 0 \end{bmatrix}_{2n \times 2n},$$

where $K(t,0)$ is from (13). Let $X_t = X_t^g = d\phi_t^g|_{\mathcal{N}_0} : \mathcal{N}_0 \to \mathcal{N}_t$ be the fundamental solution of the Jacobi equation (12) for g:

$$\dot{X}_t = \mathbb{A}_t\, X_t.$$

The time 1 map X_1 is a symplectic linear isomorphism: $X_1^* \,\mathbb{J}\, X_1 = \mathbb{J}$, where $\mathbb{J} = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$. Differentiating this equation we get the tangent space of the symplectic isomorphisms $Sp(n)$ at X_1: $\mathcal{T}_{X_1} = \big\{\, Y \in \mathbb{R}^{2n\times 2n} \mid X_1^* \,\mathbb{J}\, Y \text{ is symmetric } \big\}$. Observe that, since X_1 is symplectic,

$$\mathcal{T}_{X_1} = X_1 \cdot \mathcal{T}_I,$$

and that $\mathcal{T}_I$ is the space of $2n \times 2n$ matrices of the form $Y = \left[\begin{smallmatrix} \beta & \gamma \\ \alpha & -\beta^* \end{smallmatrix}\right]$, where $\alpha,\, \gamma \in \mathcal{S}(n)$ are symmetric $n \times n$ matrices and $\beta \in \mathbb{R}^{n\times n}$ is an arbitrary $n \times n$ matrix. Since $X_\tau \in Sp(n)$ is symplectic, the map $W \mapsto X_\tau^{-1}\, W X_\tau$ is a linear automorphism of $\mathcal{T}_I$.

Write

$$\begin{aligned} \mathcal{S}(n) &:= \big\{\, a \in \mathbb{R}^{n\times n} \mid a^* = a \,\big\}, \\ \mathcal{S}^*(n) &:= \big\{\, d \in \mathcal{S}(n) \mid d_{ii} = 0,\ \forall i = 1, \dots, n \,\big\}, \\ \mathcal{AS}(n) &:= \big\{\, e \in \mathbb{R}^{n\times n} \mid e^* = -e \,\big\}. \end{aligned}$$

7.3. **Proposition.** *Let $F:\mathcal{S}(n)^3\times\mathcal{S}^*(n)\to Sp(n)$ be the map $F(\omega):=X_1^g=d_v\phi_1^g|_{\mathcal{N}_0}$, where $\omega=(a,b,c;d)\in\mathcal{S}(n)^3\times\mathcal{S}^*(n)$,*

$$g=g_\omega=g_0+\sum_{i,j=1}^n \alpha_{ij}\,x_i\,x_j\;dx_0\otimes dx_0, \tag{32}$$

$$\alpha(t,x)=p(t)\,\varphi_\varepsilon(x),$$

$$p(t)=h(t)\left[\,a\,\delta(t)+b\,\delta'(t)+c\,\delta''(t)+d\,\delta'''(t)\right]. \tag{33}$$

Then if $g_\omega\in\mathcal{V}_0$,

$$\|d_\omega F\cdot\zeta\|\ge k_5\;\|\zeta\|\qquad \textit{for all}\quad \zeta\in\mathcal{S}(n)^3\times\mathcal{S}^*(n)\approx\mathbb{R}^{2n^2+n}.$$

Proof: Observe that the map $\omega\mapsto g_\omega$ is affine. Write $g:=g_\omega$ and $g^r:=g_{\omega+r\zeta}$, $\zeta=(a,b,c;d)$, $r\in\mathbb{R}$. The Jacobi equation for g^r along γ is

$$\dot X_r=\mathbb{A}^r\,X^r, \tag{34}$$

where $\mathbb{A}^r=\left[\begin{smallmatrix}0 & I\\ -K^r & 0\end{smallmatrix}\right]$, $K^r=K+r\,p(t)$ and $p(t)$ is from (33). Differentiating this equation with respect to r, we get the differential equation for $Z_t:=\frac{dX^r(t)}{dr}\Big|_{r=0}$:

$$\dot Z=\mathbb{A}\,Z+\mathbb{B}\,X, \tag{35}$$

where $\mathbb{A}=\left[\begin{smallmatrix}0 & I\\ -K & 0\end{smallmatrix}\right]$ and $\mathbb{B}=\left[\begin{smallmatrix}0 & 0\\ p(t) & 0\end{smallmatrix}\right]$. Here $Z_1=d_\omega F\cdot\zeta$.

Write $Z_t=X_t\,Y_t$, then from (34) and (35) we get that

$$X\,\dot Y=\mathbb{B}\,X.$$

Since $X^r(0)\equiv I$, we have that $Z(0)=0$ and $Y(0)=0$. Therefore

$$Y(t)=\int_0^t X_s^{-1}\,\mathbb{B}_s\,X_s\;ds.$$

Write

$$B=\begin{bmatrix}0 & 0\\ b & 0\end{bmatrix},\quad C=\begin{bmatrix}0 & 0\\ c & 0\end{bmatrix},\quad D=\begin{bmatrix}0 & 0\\ d & 0\end{bmatrix}.$$

Integrating by parts and using (34), we have that

$$\begin{aligned}\int_0^1 X_s^{-1}\,\delta'(s)\,BX_s\;ds&=\int_0^1\delta(s)\left[\,X_s^{-1}\dot X_s\,X_s^{-1}\,B\,X_s-X_s^{-1}B\,\mathbb{A}\,X_s\,\right]\,ds\\ &=\int_0^1\delta(s)\,X_s^{-1}\big[\mathbb{A}B-B\mathbb{A}\big]X_s\;ds\\ &=\int_0^1\delta(s)\,X_s^{-1}\left[\begin{smallmatrix}b & 0\\ 0 & -b\end{smallmatrix}\right]X_s\;ds.\end{aligned}$$

$$\int_0^1 X_s^{-1}\,\delta''(s)\,C X_s\,ds = \int_0^1 \delta'(s)\,X_s^{-1}\left[\begin{smallmatrix} c & 0\\ 0 & -c\end{smallmatrix}\right] X_s\,ds$$
$$= \int_0^1 \delta(s)\,X_s^{-1}\left(\mathbb{A}\left[\begin{smallmatrix} c & 0\\ 0 & -c\end{smallmatrix}\right] - \left[\begin{smallmatrix} c & 0\\ 0 & -c\end{smallmatrix}\right]\mathbb{A}\right) X_s\,ds$$
$$= \int_0^1 \delta(s)\,X_s^{-1}\left[\begin{smallmatrix} 0 & -2c\\ -(Kc+cK) & 0\end{smallmatrix}\right] X_s\,ds.$$
$$\int_0^1 X_s^{-1}\,\delta'''(s)\,D X_s\,ds = \int_0^1 \delta'(s)\,X_s^{-1}\left[\begin{smallmatrix} 0 & -2d\\ -(Kd+dK) & 0\end{smallmatrix}\right] X_s\,ds$$
$$= \int_0^1 \delta(s)\,X_s^{-1}\left(\mathbb{A}\left[\begin{smallmatrix} 0 & -2d\\ -(Kd+dK) & 0\end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 & -2d\\ -(Kd+dK) & 0\end{smallmatrix}\right]\mathbb{A}\right) X_s\,ds$$
$$= \int_0^1 \delta(s)\,X_s^{-1}\left[\begin{smallmatrix} -Kd-3dK & 0\\ 0 & 3Kd+dK\end{smallmatrix}\right] X_s\,ds.$$

Write

$$W_1 := \int_0^1 X_s^{-1}\,\frac{\mathbb{B}_s}{h(s)}\,X_s\,ds = \int_0^1 \delta(s)\,X_s^{-1}\begin{bmatrix}\beta & \gamma\\ \alpha & -\beta^*\end{bmatrix} X_s\,ds. \tag{36}$$

Then we have that

$$\begin{aligned} \alpha &= a - (Kc + cK),\\ \gamma &= -2c\,,\\ \beta &= b - Kd - 3dK. \end{aligned} \tag{37}$$

We want to solve this system at $s = \tau$ for $a, b, c \in \mathcal{S}(n)$ and $d \in \mathcal{S}^*(n)$, where $\alpha, \gamma \in \mathcal{S}(n)$ and $\beta \in \mathbb{R}^{n\times n}$ is arbitrary. We start by separating β into a sum of a symmetric and an antisymmetric matrix. Thus

$$Kd - dK = \frac{\beta - \beta^*}{2}. \tag{38}$$

Since $k_3 > 0$ in (24), the next lemma 7.4 shows that equation (38) has a solution $d \in \mathcal{S}^*(n)$.

7.4. **Lemma.** *Let K be a symmetric matrix and let $L_K : \mathcal{S}^*(n) \to \mathcal{AS}(n)$ be given by $L_K(d) := K\,d - d\,K$. Suppose that the eigenvalues λ_i of K are all distinct. For all $e \in \mathcal{AS}(n)$ there exists $d \in \mathcal{S}^*(n)$ such that $L_K\,d = e$ and*

$$\|d\| \le \frac{\|e\|}{\min_{i\ne j}|\lambda_i - \lambda_j|}.$$

Proof: Let Q be an orthogonal matrix such that $K = Q\,D\,Q^*$, where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is a diagonal matrix. Define $F_Q : \mathbb{R}^{n\times n} \hookleftarrow$ by $F_Q(a) := Q^*\,a\,Q$. Observe that F_Q preserves both $\mathcal{S}(n)$ and $\mathcal{AS}(n)$.

Moreover, we have that

$$L_K\,d = e \quad\Longleftrightarrow\quad L_D(F_Q\,d) = F_Q\,e.$$

Thus

$$L_K = F_{Q^*}\, L_D\, F_Q. \tag{39}$$

Since Q is orthogonal, F_Q is an isometry. Hence, from (39), it is enough to prove that L_D restricted to $\mathcal{S}^*(n)$ is a linear isomorphism and that

$$\left\| L_D|_{\mathcal{S}^*(n)}{}^{-1} \right\| \le \frac{1}{\min_{i\neq j}|\lambda_i - \lambda_j|}.$$

But writing the equation $L_D\, w = h$ in coordinates, we have that

$$\lambda_i\, w_{ij} - w_{ij}\, \lambda_j = h_{ij}, \quad \forall i,j = 1,\ldots,n;$$

which has the symmetric solution

$$w_{ij} = \frac{1}{\lambda_i - \lambda_j}\, h_{ij}\,, \qquad w_{ii} = 0,$$

for any antisymmetric h.

□

The rest of the solution to the system (37) is given by

$$b = \tfrac{1}{2}\,(\beta + \beta^*) + 2\,(K d + dK), \tag{40}$$

$$c = -\tfrac{1}{2}\,\gamma, \tag{41}$$

$$a = \alpha - \tfrac{1}{2}\,(K\gamma + \gamma K). \tag{42}$$

Consider the map $T : \mathcal{S}(n)^3 \times \mathcal{S}^*(n) \to \mathcal{T}_I$,

$$T(a,b,c;d) = \begin{bmatrix} \beta & \gamma \\ \alpha & -\beta^* \end{bmatrix}$$

given by the system (37). We want to estimate $\left\|T^{-1}\right\|$. Observe that

$$\|\beta\| = \sup_{|u|=|v|=1} \langle \beta\, u, v\rangle = \sup_{|u|=|v|=1} \langle u, \beta^*\, v\rangle = \|\beta^*\|\,.$$

From (38), lemma 7.4, (24) and (25),

$$\|d\| \le \frac{\left\|\frac{\beta-\beta^*}{2}\right\|}{\min\limits_{i\neq j}|\lambda_i - \lambda_j|} \le \frac{\|\beta\|}{k_3} \le k_4\; \|\beta\|\,. \tag{43}$$

From (40), (20), (43) and (25),

$$\|b\| \le \|\beta\| + 4\,k_0\; \|d\| \le \left(1 + 4\,k_0\,k_3^{-1}\right)\; \|\beta\| \le k_4\; \|\beta\|\,.$$

Also, from (41), (25) and (42),

$$\|c\| \le \|\gamma\| \le k_4\; \|\gamma\|\,,$$

$$\|a\| \le \|\alpha\| + k_0\; \|\gamma\| \le k_4\; \max\{\|\alpha\|\,, \|\gamma\|\}.$$

Write

$$\mathbb{D} := \begin{bmatrix} \beta & \gamma \\ \alpha & -\beta^* \end{bmatrix} = T(\zeta).$$

Since

$$\|\mathbb{D}\| \geq \max\{\|\alpha\|, \|\beta\|, \|\gamma\|\},$$

we get that

$$\|\zeta\| := \max\{\|a\|, \|b\|, \|c\|, \|d\|\} \leq k_4 \, \|T(\zeta)\|.$$

Thus

$$\|\mathbb{D}\| = \|T(\zeta)\| \geq \tfrac{1}{k_4} \|\zeta\|. \tag{44}$$

Write

$$W_1 := \int_0^1 \delta(s) \, X_s^{-1} \, \mathbb{D} \, X_s \, ds,$$

$$Q(s) := X_s^{-1} \, \mathbb{D} \, X_s \qquad \text{and} \qquad P(s) := \delta(s) \; X_s^{-1} \, \mathbb{D} \, X_s.$$

Given a continuous map $f : [0,1] \to \mathbb{R}^{2n\times 2n}$, define

$$\mathcal{O}_\lambda(f,\tau) := \sup_{|s-\tau|\leq\lambda} |f(s) - f(\tau)|.$$

Observe that

$$\mathcal{O}_\lambda(fg,\tau) \leq \|f\|_0 \; \mathcal{O}_\lambda(g,\tau) + \mathcal{O}_\lambda(f,\tau)\, |g(\tau)|,$$

where $\|f\|_0 := \sup_{s\in[0,1]} |f(s)|$. We have that

$$\begin{aligned}
\mathcal{O}_\lambda(Q,\tau) = \mathcal{O}_\lambda(X_s^{-1}\mathbb{D}X_s,\tau) &\leq \left\|X_s^{-1}\right\|_0 \mathcal{O}_\lambda(\mathbb{D}X_s,\tau) + \mathcal{O}_\lambda(X_s^{-1},\tau) \; \|\mathbb{D}\| \; \|X_\tau\| \\
&\leq \left\|X_s^{-1}\right\|_0 \|\mathbb{D}\| \; \mathcal{O}_\lambda(X_s,\tau) + \mathcal{O}_\lambda(X_s^{-1},\tau) \; \|\mathbb{D}\| \; \|X_\tau\| \\
&\leq 2\, k_1 k_2 \; \|\mathbb{D}\|.
\end{aligned}$$

$$\|W_1 - Q(\tau)\| = \left\| \int_0^1 \delta(s) \left[Q(s) - Q(\tau)\right] \, ds \right\| \leq \mathcal{O}_\lambda(Q,\tau) \leq 2\, k_1 k_2 \; \|\mathbb{D}\|.$$

$$\begin{aligned}
\|Y_1 - W_1\| &\leq \left\| \int_0^1 \left[h(s) - 1\right] P(s) \, ds \right\| \leq \|P\|_0 \int_0^1 |1 - h(s)| \, ds \leq \rho \; \|P\|_0 \\
&\leq \rho \, k_1^2 \; \|\delta\|_0 \; \|\mathbb{D}\|.
\end{aligned}$$

$$\|Q(\tau)\| = \left\|X_\tau^{-1} \, \mathbb{D} \, X_\tau\right\| \geq \tfrac{1}{k_1^2} \, \|\mathbb{D}\|.$$

Therefore

$$\begin{aligned}
\|Y_1\| &\geq \|Q(\tau)\| - \|W_1 - Q(\tau)\| - \|Y_1 - W_1\| \\
&\geq \left(\tfrac{1}{k_1^2} - 2k_1\, k_2 - \rho\, k_1^2 \, \|\delta\|_0\right) \|\mathbb{D}\|.
\end{aligned}$$

Using (44),

$$\|Z_1\| = \|X_1 \, Y_1\| \geq k_1^{-1} \; \|Y_1\| \geq \frac{k_1^{-2} - 2k_1\, k_2 - \rho\, k_1^2 \; \|\delta\|_0}{k_1 \, k_4} \; \|\zeta\| = k_5 \; \|\zeta\|.$$

□

7.5. **Lemma.**

Let $\mathcal{N}$ be a smooth connected riemannian m-manifold and let $F:\mathbb{R}^m\to\mathcal{N}$ be a smooth map such that

$$|d_xF(v)|\geq a>0 \qquad \textit{for all } (x,v)\in T\mathbb{R}^m \textit{ with } |v|=1 \textit{ and } |x|\leq r. \tag{45}$$

Then for all $0<b<a\,r$,

$$\big\{w\in\mathcal{N}\,|\,d\big(w,F(0)\big)<b\,\big\}\subseteq F\big\{x\in\mathbb{R}^m\,|\,|x|<\tfrac{b}{a}\,\big\}.$$

Proof: Let $w\in\mathcal{N}$ with $d\big(w,F(0)\big)<b$. Let $\beta:[0,1]\to\mathcal{N}$ be a differentiable curve with $\beta(0)=F(0)$, $\beta(1)=w$ and $|\dot\beta|<b$. Let $\tau=\sup(A)$, where $A\subset[0,1]$ is the set of $t\in[0,1]$ such that there exist a unique C^1 curve $\alpha:[0,t]\to\mathbb{R}^m$ such that $\alpha(0)=0$, $|\alpha(s)|<r$ and $F(\alpha(s))=\beta(s)$ for all $s\in[0,t]$. By the inverse function theorem $\tau>0$, A is open in $[0,1]$ and there exist a unique $\alpha:[0,\tau[\to\mathbb{R}^m$ such that $F\circ\alpha=\beta$. By (45),

$$\big|\dot\beta(s)\big|=\big\|d_{\alpha(s)}F\big\|\cdot|\dot\alpha(s)|\geq a\;|\dot\alpha(s)|\,,\qquad \text{for all } s\in[0,\tau[. \tag{46}$$

Thus, $|\dot\alpha|\leq\frac{1}{a}\max_{0\leq t\leq1}\big|\dot\beta(t)\big|<\frac{b}{a}$. This implies that α is Lipschitz and hence it can be extended continuously to $[0,\tau]$. Observe that $|\alpha(\tau)|<r$, for if $|\alpha(\tau)|\geq r$, then

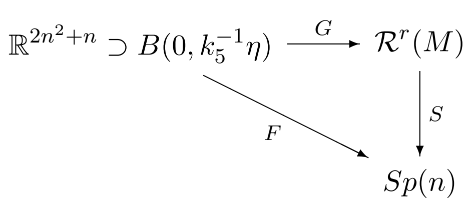
$$b\geq b\,\tau\geq\int_0^\tau\big|\dot\beta(s)\big|\,ds\geq a\int_0^\tau|\dot\alpha(s)|\;ds\geq a\,r,$$

contradicting the hypothesis $b<ar$. This implies that the set A is also closed in $[0,1]$. Thus $A=[0,1]$ and $\tau=1$. From (46), writing $x=\alpha(1)\in F^{-1}\{w\}$,

$$|x|\leq\text{length}(\alpha)=\int_0^1|\dot\alpha(t)|\;dt\leq\frac{1}{a}\int_0^1|\dot\beta(t)|\,dt<\frac{b}{a}.$$

□

Let $G:\mathbb{R}^{2n^2+n}\to\mathcal{R}^r(M)$ be the map $G(\omega)=g_\omega$, where g_ω is from (32). The following diagram commutes



By proposition 7.3 and lemma 7.5, in $Sp(n)$ the ball $B(S(g_0),\eta)\subset F\big(B(0,k_5^{-1}\eta)$. It is enough to prove that $G\big(B(0,k_5^{-1}\eta)\big)\subset\mathcal{U}_0$, for then $S(\mathcal{V}_0)\supset B(S(g_0),\eta)$.

If $f:[0,1]\to\mathbb{R}$, write

$$\|f\|_{C^r}:=\sum_{s=0}^{r}\sup_{x\in[0,1]}|D^sf(x)|.$$

Observe that

$$\|fg\|_{C^r} \leq 2^r \, \|f\|_{C^r} \, \|g\|_{C^r} .$$

If $\omega < k_5^{-1}\eta$ and $p(t)$ is from (33) and in $\varphi_\varepsilon(x)$, $\varepsilon < \varepsilon_1$, then, by lemma 7.6,

$$\begin{aligned}
\|g_\omega - g_0\|_{C^2} &= \|\varphi_\varepsilon(x)\, x^* p(t)\, x\|_{C^2} \\
&\leq k_6 \, \|p\|_{C^0} + k_6 \, \varepsilon \, \|p\|_{C^1} + k_6 \, \varepsilon^2 \, \|p\|_{C^2} \\
&\leq k_6 \, 4 \, k_5^{-1} \, \eta \, \|\delta\|_{C^3} + k_6 \, \varepsilon_1 \, 4 \, k_5^{-1} \, \eta \, (2 \, \|h\|_{C^1} \, \|\delta\|_{C^4}) + k_6 \, \varepsilon_1^2 \, 4 \, k_5^{-1} \, \eta \, (2^2 \, \|h\|_{C^2} \, \|\delta\|_{C^5}) \\
&< \varepsilon_0,
\end{aligned}$$

where the last inequality is from (31). Then, by (29), $g_\omega \in \mathcal{U}_0 \cap \mathcal{F} = \mathcal{V}_0$.

Bump functions

7.6. **Lemma.** *There exist $k_6 > 0$ and a family of C^∞ functions $\varphi_\varepsilon : [-\varepsilon, \varepsilon]^n \to [0,1]$ such that $\varphi_\varepsilon(x) \equiv 1$ if $x \in [-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]^n$, $\varphi_\varepsilon(x) \equiv 0$ if $x \notin [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^n$ and for any C^2 map $B : [0,1] \to \mathbb{R}^{n\times n}$ the function $\alpha(t,x) := \varphi_\varepsilon(x)\, x^* B(t)\, x$ satisfies,*

$$\|\alpha\|_{C^2} \leq k_6 \, \|B\|_{C^0} + \varepsilon \, k_6 \, \|B\|_{C^1} + \varepsilon^2 \, \|B\|_{C^2},$$

with k_6 independent of $0 < \varepsilon < 1$.

Proof: Let $\psi : [-1,1] \to [0,1]$ be a C^∞ function such that $\psi(x) \equiv 1$ for $|x| \leq \frac{1}{4}$ and $\psi(x) \equiv 0$ for $|x| \geq \frac{1}{2}$. Given $\varepsilon > 0$ let $\varphi = \varphi_\varepsilon : [-\varepsilon, \varepsilon]^n \to [0,1]$ be defined by $\varphi(x) = \prod_{i=1}^n \psi\left(\frac{x_i}{\varepsilon}\right)$. Let $B \in \mathbb{R}^{n\times n}$ and let $\beta(x) = \varphi(x)\, x^* B x$. Then

$$\|\beta\|_0 \leq \varepsilon^2 \, \|B\| \tag{47}$$

$$d_x\beta = (d_x\varphi)\; x^* B x + \varphi(x)\; x^*(B + B^*)$$

$$\frac{\partial\varphi}{\partial x_i} = \frac{1}{\varepsilon}\; \psi'\left(\frac{x_i}{\varepsilon}\right) \prod_{k\neq i}^{n} \psi\left(\frac{x_k}{\varepsilon}\right)$$

$$\|d_x\varphi\| \leq \frac{1}{\varepsilon} \; \|d\psi\|_0 \tag{48}$$

$$\|d_x\beta\| \leq 3\,\varepsilon \; \|B\| \; \|\psi\|_{C^1} \tag{49}$$

$$d_x^2\beta = (d_x^2\varphi)\; x^* B x + 2\,(d_x\varphi)\; x^*\,(B + B^*) + \varphi(x)\,(B + B^*)$$

$$\frac{\partial^2\varphi}{\partial x_i\,\partial x_j} = \frac{1}{\varepsilon^2}\; \psi''\left(\frac{x_i}{\varepsilon}\right) \prod_{k\neq i} \psi\left(\frac{x_k}{\varepsilon}\right) \delta_{ij} + \frac{1}{\varepsilon^2}\; \psi'\left(\frac{x_i}{\varepsilon}\right)\, \psi'\left(\frac{x_j}{\varepsilon}\right) \prod_{k\neq i,j} \psi\left(\frac{x_k}{\varepsilon}\right)\,(1 - \delta_{ij}).$$

$$\left\|d_x^2\varphi\right\| \leq \frac{1}{\varepsilon^2}\; \max\left\{\; \left\|d^2\psi\right\|_0 \;,\; \|d\psi\|_0^2 \;\right\} \leq \frac{1}{\varepsilon^2}\; \|\psi\|_{C^2}^2 .$$

$$\begin{aligned}
\left\|d_x^2\beta\right\| &\leq \|\psi\|_{C^2}^2 \; \|B\|\,(1 + 4 + 2) \\
&\leq 7\; \|\psi\|_{C^2}^2 \; \|B\| .
\end{aligned} \tag{50}$$

Let $k_6 := 4 + 3\ \|\psi\|_{C^1} + 7\ \|\psi\|^2_{C^2}$. Then from (47), (49) and (50), we have that

$$\|\beta\|_{C^2} \le k_6\ \|B\|\,. \tag{51}$$

Now let $\alpha(t,x) := \varphi(x)\, x^*\, B(t)\, x$. Observe that

$$\begin{aligned}\|\alpha\|_{C^2} &\le \sup_t \|\alpha(t,\cdot)\|_{C^2} + \sup_x \|\alpha(\cdot,x)\|_{C^2} + 2\ \left\|\tfrac{\partial^2\alpha}{\partial x\,\partial t}\right\|_0 .\\ &\le \|\beta\|_{C^2} + \varepsilon^2\ \|B\|_{C^2} + 2\ \left\|\tfrac{\partial^2\alpha}{\partial x\,\partial t}\right\|_0 .\end{aligned}$$

But, using (48),

$$\begin{aligned}\tfrac{\partial^2\alpha}{\partial x\,\partial t} &= d_x\varphi\cdot x^* B'(t)\, x + \varphi(x)\ x^*\big[\,B'(t) + B'(t)^*\,\big]\\ \left\|\tfrac{\partial^2\alpha}{\partial x\,\partial t}\right\| &\le \varepsilon\ \|\psi\|_{C^1}\ \|B'\|_0 + 2\,\varepsilon\,\|B'\|_0\\ &\le \tfrac12\, k_6\,\varepsilon\,\|B\|_{C^1}.\end{aligned}$$

Hence, using (51),

$$\|\alpha\|_{C^2} \le k_6\ \|B\|_{C^0} + k_6\,\varepsilon\ \|B\|_{C^1} + \varepsilon^2\ \|B\|_{C^2}\,.$$

□

8. Stable hyperbolicity.

In this section we prove, in Theorem 8.1, a symplectic version of R. Mañé's Lemma II.3 in [22]. In contrast to the general case in $GL(n,\mathbb{R})$, where one obtains uniform domination; in the symplectic case the result is uniform hyperbolicity.

We say that a linear map $T:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is *hyperbolic* if it has no eigenvalue of modulus 1. Equivalently, T is hyperbolic if there is a splitting $\mathbb{R}^{2n} = E^s\oplus E^u$ and an iterate $M\in\mathbb{Z}^+$ such that $T(E^s)=E^s$, $T(E^u)=E^u$ and

$$\left\|T^M|_{E^s}\right\| < \tfrac12 \quad\text{and}\quad \left\|(T|_{E^u})^{-M}\right\| < \tfrac12.$$

The subspaces E^s and E^u are called the *stable subspace* and *unstable subspace* of T.

Let $Sp(n)$ be the group of symplectic linear isomorphisms of $\mathbb{R}^{2n}$. We say that a sequence $\xi:\mathbb{Z}\to Sp(n)$ is *periodic* if there exists $m\ge 1$ such that $\xi_{i+m}=\xi_i$ for all $i\in\mathbb{Z}$. We say that a periodic sequence ξ is *hyperbolic* if the linear map $\prod_{i=1}^m \xi_i$ is hyperbolic. In this case the stable and unstable subspaces of $\prod_{i=0}^{m-1}\xi_{i+j}$ are denoted by $E^s_j(\xi)$ and $E^u_j(\xi)$ respectively.

We say that a family $\xi = \{\,\xi^\alpha\,\}_{\alpha\in\mathcal{A}}$ of sequences in $Sp(n)$ is *bounded* if there exists $Q>0$ such that $\|\xi^\alpha_i\| < Q$ for all $\alpha\in\mathcal{A}$ and $i\in\mathbb{Z}$. Given two families of periodic sequences in $Sp(n)$, $\xi = \{\,\xi^\alpha\,\}_{\alpha\in\mathcal{A}}$ and $\eta = \{\,\eta^\alpha\,\}_{\alpha\in\mathcal{A}}$, we say that they are *periodically equivalent* if they have the same indexing set $\mathcal{A}$ and for all $\alpha\in\mathcal{A}$ the periods of ξ^α and η^α coincide.

Given two periodically equivalent families of periodic sequences in $Sp(n)$, $\xi = \{\,\xi^\alpha\,\}_{\alpha\in\mathcal{A}}$ and $\eta = \{\,\eta^\alpha\,\}_{\alpha\in\mathcal{A}}$, define

$$d(\xi,\eta) = \sup\big\{\|\xi^\alpha_n - \eta^\alpha_n\| \ :\ \alpha\in\mathcal{A},\ n\in\mathbb{Z}\,\big\}.$$

We say that a family ξ is *hyperbolic* if for all $\alpha\in\mathcal{A}$, the periodic sequence ξ^α is hyperbolic. We say that a hyperbolic periodic family ξ is *stably hyperbolic* if there exists $\varepsilon>0$ such that any periodically equivalent family η satisfying $d(\eta,\xi)<\varepsilon$ is also hyperbolic.

Finally, we say that a family of periodic sequences ξ is *uniformly hyperbolic* if there exists a constant iterate $M\in\mathbb{Z}^+$ and subspaces $E^s_i(\xi^\alpha)$, $E^u_i(\xi^\alpha)$, $\alpha\in\mathcal{A}$, $i\in\mathbb{Z}$, such that

$$\xi_j(E^\tau_j(\xi^\alpha)) = E^\tau_{j+1}(\xi^\alpha), \qquad \text{for all } \alpha\in\mathcal{A},\ j\in\mathbb{Z},\ \tau\in\{s,u\}$$

and

$$\Big\|\prod_{i=0}^{M}\xi^\alpha_{i+j}\Big|_{E^s_j(\xi^\alpha)}\Big\| < \tfrac12 \quad\text{and}\quad \Big\|\Big(\prod_{i=0}^{M}\xi^\alpha_{i+j}\Big|_{E^u_j(\xi^\alpha)}\Big)^{-1}\Big\| < \tfrac12,\ \text{for all } \alpha\in\mathcal{A},\ j\in\mathbb{Z}.$$

Equivalently, if there exist $K>0$, $0<\lambda<1$ and invariant subspaces $E^s_i(\xi^\alpha)$, $E^u_i(\xi^a)$, $\alpha\in\mathcal{A}$, $i\in\mathbb{Z}$, such that

$$\Big\|\prod_{i=0}^{m-1}\xi^\alpha_{i+j}\Big|_{E^s_j(\xi^\alpha)}\Big\| < K\,\lambda^m \quad\text{and}\quad \Big\|\Big(\prod_{i=0}^{m-1}\xi^\alpha_{i+j}\Big|_{E^u_j(\xi^\alpha)}\Big)^{-1}\Big\| < K\,\lambda^m,$$

for all $\alpha\in\mathcal{A}$, $j\in\mathbb{Z}$, $m\in\mathbb{N}$. Observe that in this case the sequence ξ is hyperbolic and the subspaces $E^s_i(\xi^\alpha)$, $E^u_i(\xi^\alpha)$ necessarily coincide with the stable and unstable subspaces of the map $\prod_{j=0}^{m-1}\xi^\alpha_{i+j}$.

The remaining of the section is devoted to the proof of the following

8.1. **Theorem.**

If ξ^α is a stably hyperbolic family of periodic sequences of bounded symplectic linear maps then it is uniformly hyperbolic.

Let $\Omega = \sum_{i=1}^n dx_i\wedge dx_{i+n}$ be the canonical symplectic form on $\mathbb{R}^{2n}$ and $J\in Sp(n)$ be $J(x,y) := (-y,x)$ for $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$. The matrix of J in the canonical basis is

$$J = [J_{ij}] = \begin{bmatrix} 0 & I\\ -I & 0\end{bmatrix}.$$

Then $\Omega(x,y) = \langle x, Jy\rangle = x^*Jy$, where $\langle x,y\rangle = \sum_{i=1}^n x_i\,y_i$. Observe that $A\in Sp(n)$ iff

$$A^*JA = J. \tag{52}$$

We say that a basis $\mathcal{B} = (v_1,\dots,v_{2n})$ is *symplectic* if $\Omega(v_i,v_j) = J_{ij}$. If $T:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is a linear map with matrix A in a symplectic basis $\mathcal{B}$, then $T\in Sp(n)$ iff (52) holds.

We say that a linear subspace $E\subseteq\mathbb{R}^{2n}$ is *lagrangian* if $\Omega|_E\equiv 0$ and $\dim E = n$.

8.2. **Lemma.**

(i) *A subspace* $E \subseteq \mathbb{R}^{2n}$ *is lagrangian if and only if* $JE = E^\perp$

(ii) *If* $T \in Sp(n)$ *is a hyperbolic symplectic linear map, then its stable and unstable subspaces* $E^s(T)$, $E^u(T)$ *are lagrangian.*

(iii) *If* $T \in GL(\mathbb{R}^{2n})$ *has matrix* $\mathbb{D}$ *in a symplectic basis* $\mathcal{B} = (v_1, \dots, v_{2n})$ *and the lagrangian subspace* $E = \operatorname{span}\{v_1, \dots v_n\}$ *satisfies* $T(E) \subset E$*, then* $T \in Sp(n)$ *iff*

$$\mathbb{D} = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix},$$

where $C = (A^*)^{-1}$ *and* $A^{-1}B$ *is symmetric.*

(iv) *If* $E \subset \mathbb{R}^{2n}$ *is a lagrangian subspace and* $\mathcal{B} = (v_1, \dots, v_n)$ *is an orthonormal basis for* E*, then* $(\mathcal{B}, J\mathcal{B}) = (v_1, \dots, v_n, Jv_1, \dots Jv_n)$ *is a symplectic basis for* $\mathbb{R}^{2n} = E \oplus JE$*.*

Proof:

i. Observe that $JE = E^\perp$ if and only if $\dim E = 2n - \dim E$ and $\Omega(x, y) = \langle x, Jy \rangle = 0$ for all $x, y \in E$.

ii. Let $u, v \in E^s(\xi)$. Since T preserves the symplectic form Ω, we have that

$$\Omega(u, v) = \lim_{m \to +\infty} \Omega(T^m u, T^m v) = 0.$$

Therefore $JE^s \subset (E^s)^\perp$ and hence $\dim E^s(T) \leq n$. Similarly, $\Omega(u, v) = 0$ if u, $v \in E^u(T)$. Therefore $\dim E^s(T) = \dim E^u(T) = n$.

iii.iv. Item (iii) follows from formula (52). Item (iv) is a direct calculation.

□

8.3. **Lemma.** *If* $\{\xi^\alpha\}_{\alpha \in \mathcal{A}}$ *is a bounded stably hyperbolic family of maps in* $Sp(n)$*, then there exist* $\varepsilon > 0$ *and* $K > 0$ *such that if* $\{\eta^\alpha\}_{\alpha \in \mathcal{A}}$ *is a periodically equivalent family of maps in* $Sp(n)$ *with* $d(\xi, \eta) \leq \varepsilon$*, then the family* η *is hyperbolic and*

$$\forall\, \alpha \in \mathcal{A},\ \forall\, i \in \mathbb{Z}, \qquad \left\| \prod_{j=0}^{m-1} \eta^\alpha_{i+j} \Big|_{E^s_i(\eta^\alpha)} \right\| < K, \qquad m = \operatorname{Per}(\eta^\alpha),$$

where m *is the minimal period of* η^α*.*

Proof: Suppose the lemma is false. Then for all $\varepsilon > 0$ and $K > 0$ there exist a periodically equivalent family $\{\eta^\alpha\}_{\alpha \in \mathcal{A}}$ with $d(\eta, \xi) \leq \varepsilon$, $\alpha_0 \in \mathcal{A}$, $i_0 \in \mathbb{Z}$ and an orthonormal basis $\mathcal{B}$ for $E^s_{i_0}(\eta^{\alpha_0})$ such that in that basis $\prod_{j=0}^{m-1} \eta^{\alpha_0}_{i_0+j}$ has an entry $b = b_{k\ell}$ with $|b| \geq K$, where $m = \operatorname{Per}(\eta^{\alpha_0})$.

For simplicity assume that $i_0 = 1$. Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be the matrix given by $a_{ij} = 0$ if $(i, j) \neq (\ell, k)$ and $a_{\ell k} = \delta$, where

$$\delta = \frac{3n}{K}.$$

In the basis $(\mathcal{B}, J\mathcal{B})$ for $E^s_1(\eta^{\alpha_0}) \oplus JE^s_1(\eta^{\alpha_0})$ write

$$\prod_{i=1}^{m} \eta_i^{\alpha_0} = \begin{bmatrix} B & C \\ 0 & (B^*)^{-1} \end{bmatrix}, \qquad \mathbb{D}_s = \begin{bmatrix} I + sA & 0 \\ 0 & (I + sA^*)^{-1} \end{bmatrix}.$$

Observe that $\mathbb{D}_s \in Sp(n)$ and $|\operatorname{tr} B| < n$. We claim that

$$\|I - \mathbb{D}_s\| \leq |\, 2\,\delta s| \,. \tag{53}$$

Indeed, if $k \neq \ell$ then $(I + sA^*)^{-1} = I - sA^*$ and (53) holds. If $k = \ell$ then $I - \mathbb{D}_s$ has only two non-zero entries, which are $s\delta$ and $1 - \frac{1}{1+s\delta} \leq s\delta$.

Let $\{\zeta^\alpha(s)\}_{\alpha\in\mathcal{A}}$, $s \in [0, 1]$ be the families given by $\zeta_i(s) = \eta_i^\alpha$ if $\alpha \neq \alpha_0$ or $i \neq 1$, and $\zeta_1^{\alpha_0}(s) = \eta_1^{\alpha_0}\,\mathbb{D}_s$. Then $E^s_1(\eta^{\alpha_0})$ is an invariant subspace under $\prod_{i=1}^m \zeta_i^{\alpha_0}(s)$ for all $s \in [0, 1]$. But

$$\begin{aligned} \operatorname{tr}\left[\prod_{i=1}^{m} \zeta_i^{\alpha_0}\Big|_{E^s_1(\eta^{\alpha_0})}\right] &= \operatorname{tr} B(I + A) = \operatorname{tr} B + b\,\delta \\ &\geq b\,\delta - n \geq n. \end{aligned}$$

Therefore there is $s \in [0, 1]$ such that $\prod_{i=1}^m \zeta_i^{\alpha_0}(s)$ has an eigenvalue of modulus 1.

We have that

$$\|\zeta_1^{\alpha_0} - \eta_1^{\alpha_0}\| \leq \|\eta_1^{\alpha_0}\| \;\; |2\,\delta| \leq \|\eta_1^{\alpha_0}\| \;\; \tfrac{6n}{K}.$$

$$\begin{aligned} d(\xi, \zeta) &\leq d(\xi, \eta) + d(\eta, \zeta) \\ &\leq \varepsilon + \big(\, \|\xi_1^{\alpha_0}\| + \varepsilon\big)\, \tfrac{6n}{K}. \end{aligned}$$

Since $d(\xi, \zeta) \to 0$ when $\varepsilon \to 0$ and $K \to +\infty$, this contradicts the stable hyperbolicity of ξ.

□

8.4. **Lemma.** *If $\{\xi^\alpha\}_{\alpha\in\mathcal{A}}$ is a bounded stably hyperbolic family of maps in $Sp(n)$, then there exist $\varepsilon > 0$, $K > 0$ and $0 < \lambda < 1$ such that if $\{\eta^\alpha\}_{\alpha\in\mathcal{A}}$ is a periodically equivalent family of maps in $Sp(n)$ with $d(\xi, \eta) \leq \varepsilon$, then*

$$\forall\, \alpha \in \mathcal{A},\ \forall\, i \in \mathbb{Z}, \qquad \left\| \prod_{j=0}^{m-1} \eta_{i+j}^\alpha\Big|_{E^s_i(\eta^\alpha)} \right\| < K\,\lambda^m, \qquad m = \operatorname{Per}(\eta^\alpha),$$

where m is the minimal period of η^α.

Proof: By Lemma 8.3 there exist $\varepsilon_1 > 0$, $K_1 > 0$ such that if η is a family in $Sp(n)$, periodically equivalent to ξ with $d(\eta,\xi) \le \varepsilon_1$, then η is hyperbolic and

$$\forall\, \alpha \in \mathcal{A},\ \forall\, i \in \mathbb{Z}, \qquad \left\| \prod_{j=0}^{m-1} \eta^\alpha_{i+j}\Big|_{E^s_i(\eta^\alpha)} \right\| < K_1, \qquad m = \mathrm{Per}(\eta^\alpha).$$

Let $\varepsilon := \frac{\varepsilon_1}{2}$. Suppose that η is a periodically equivalent family with $d(\eta,\xi) \le \varepsilon = \frac{\varepsilon_1}{2}$. On the splitting $E^s_i(\eta^\alpha) \oplus JE^s_i(\eta^\alpha)$ write

$$\eta^\alpha_i = \begin{bmatrix} A_{\alpha,i} & C_{\alpha,i} \\ 0 & (A^*_{\alpha,i})^{-1} \end{bmatrix}, \qquad \mathbb{D}_{\alpha,i}(\delta) = \begin{bmatrix} (1+\delta)\, I & 0 \\ 0 & (1+\delta)^{-1} I \end{bmatrix}.$$

For all $i \in \mathbb{Z}$ let $\zeta^\alpha_i = \zeta^\alpha_i(\delta) := \eta^\alpha_i \cdot \mathbb{D}_{\alpha,i}(\delta)$ and let $\delta > 0$ be such that

$$\max\{\, \delta,\, 1 - (1+\delta)^{-1} \,\} \cdot \left[\sup_{\alpha,i} \|\xi^\alpha_i\| + \frac{\varepsilon_1}{2} \right] < \frac{\varepsilon_1}{2}.$$

Then

$$d(\zeta,\xi) < \varepsilon_1. \tag{54}$$

Therefore the family ζ is hyperbolic and we claim that

$$E^s_i(\zeta^\alpha) = E^s_i(\eta^\alpha) \qquad \text{for all } \alpha \in \mathcal{A},\ i \in \mathbb{Z}.$$

For, observe that $E^s_i(\eta^\alpha)$ is invariant under $\prod_{j=0}^{m-1} \zeta^\alpha_{i+j}$, where $m = \mathrm{Per}(\eta^\alpha)$. If for some $\alpha \in \mathcal{A}$ and $i \in \mathbb{Z}$, $E^s_i(\zeta^\alpha) \ne E^s_i(\eta^\alpha)$, then there exists $0 < \delta_1 \le \delta$ such that $\zeta^\alpha(\delta_1)$ has an eigenvalue of modulus 1. This contradicts (54).

We have that

$$(1+\delta)^m \left\| \prod_{j=0}^{m-1} \eta^\alpha_{i+j}\Big|_{E^s_i(\eta^\alpha)} \right\| = \left\| \prod_{j=0}^{m-1} \zeta^\alpha_{i+j}\Big|_{E^s_i(\zeta^\alpha)} \right\| \le K_1.$$

This gives the lemma for $\lambda = (1+\delta)^{-1}$ and $K = K_1$.

□

We shall need the following definition of angle between linear subspaces. Given a linear decomposition $\mathbb{R}^d = E \oplus F$ define

$$\measuredangle(E,F) = \|L\|^{-1},$$

where $L : E^\perp \to E$ is the linear map such that $F = \{\, x + Lx \mid x \in E^\perp \,\}$, and $E^\perp := \{\, y \in \mathbb{R}^d \mid \langle y,x\rangle = 0,\ \forall x \in E \,\}$ is the orthogonal complement of E in $\mathbb{R}^d$.

8.5. **Lemma.** *If $\{\xi^\alpha\}_{\alpha\in\mathcal{A}}$ is a bounded stably hyperbolic family of maps in $Sp(n)$ then there exist $\varepsilon > 0$, $\gamma > 0$ and $N_0 \in \mathbb{Z}^+$ such that if $\{\eta^\alpha\}_{\alpha\in\mathcal{A}}$ is a periodically equivalent family of maps in $Sp(n)$ with $d(\eta,\xi) \le \varepsilon$, then η is hyperbolic and*

$$\measuredangle\big(E^s_i(\eta^\alpha), E^u_i(\eta^\alpha)\big) > \gamma$$

for all $\alpha \in \mathcal{A}$ with minimal period $> N_0$ and all $i \in \mathbb{Z}$.

Proof: Suppose it is false. Then there exists a periodic sequence $\eta : \mathbb{Z} \to Sp(n)$ with period m arbitrarily large, periodically equivalent to a sequence ξ^α of the family ξ, with $\sup_{j\in\mathbb{Z}} \|\eta_j - \xi^\alpha_j\|$ arbitrarily small and some $i \in \mathbb{Z}$ with $\measuredangle(E^s_i(\eta^\alpha), E^u_i(\eta^\alpha))$ arbitrarily small. Shifting the sequence we can assume that $i = 1$.

By lemma 8.2, $JE^s_1(\eta) = E^s_1(\eta)^\perp$. Consider the matrix of $\prod_{i=1}^m \eta_i$ in the decomposition $\mathbb{R}^{2n} = JE^s_1(\eta) \oplus E^s_1(\eta)$:

$$\prod_{i=1}^m \eta_i = \begin{bmatrix} A & 0 \\ P & B \end{bmatrix} = \begin{bmatrix} (B^*)^{-1} & 0 \\ P & B \end{bmatrix}.$$

Since it is symplectic, choosing an orthonormal basis adapted to the decomposition, we have that $A = (B^*)^{-1}$ and that $B^{-1}P$ is symmetric. By Lemma 8.4,

$$\|B\| = \|A^{-1}\| < K\,\lambda^m. \tag{55}$$

Let $L : JE^s_1(\eta) \to E^s_1(\eta)$ be such that $E^u_1(\eta) = \{\, v \oplus Lv \mid v \in JE^s_1(\eta)\,\}$. Since $E^u_1(\eta)$ is invariant,

$$L\,A = P + B\,L.$$

Thus $L = PA^{-1} + BLA^{-1}$ and

$$\|L\| \le \|PA^{-1}\| + \|L\|\; K^2\,\lambda^{2m}.$$

If the period m is large enough, then $K^2\,\lambda^{2m} \le \frac{1}{2}$ and thus

$$\tfrac{1}{2}\,\|PA^{-1}\|^{-1} \le \|L\|^{-1} = \measuredangle\big(E^s_1(\eta), E^u_1(\eta)\big).$$

The number $\|PA^{-1}\|^{-1}$ is arbitrarily small because the angle $\measuredangle(E^s_1(\eta), E^u_1(\eta))$ is arbitrarily small.

Define the sequence $\zeta : \mathbb{Z} \to Sp(n)$ by $\zeta_i := \eta_i$ for $1 < i \le m$ and

$$\zeta_1 := \eta_1 \begin{bmatrix} I & C \\ 0 & I \end{bmatrix}$$

in the splitting $\mathbb{R}^{2n} = JE^s_1(\eta) \oplus E^s_1(\eta)$. This map ζ_1 is symplectic if the matrix C is symmetric. Then

$$\prod_{i=1}^m \zeta_i = \begin{bmatrix} A & 0 \\ P & B \end{bmatrix} \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & AC \\ P & PC + B \end{bmatrix}.$$

If we find a symmetric matrix C with arbitrarily small norm $\|C\|$ such that the last matrix has an eigenvalue 1, then we shall obtain a contradiction with the stable hyperbolicity of ξ.

For, consider the system

$$\begin{aligned} Ax + \quad A\,C\,y \quad &= x, \\ Px + (PC+B)\,y \;&= y. \end{aligned}$$

Then $x = (I-A)^{-1}A\,Cy$, and thus

$$y = (I-B)^{-1}P\left[I + A\,(I-A)^{-1}\right]C\,y.$$

Since $I + A\,(I-A)^{-1} = -A^{-1}\,(I-A^{-1})^{-1}$, we have that

$$-(I-B)^{-1}P\,A^{-1}(I-A^{-1})^{-1}\,C\,y = y.$$

Take $v \in \mathbb{R}^n$ such that $|v| = \left\|PA^{-1}\right\|^{-1}$ and $|PA^{-1}v| = 1$. Let $y = -(I-B)^{-1}PA^{-1}v$. From (55) we can assume that $\|I-B\| \le 2$. Hence $|y|^{-1} \le 2$. Now take w such that $(I-A^{-1})^{-1}w = v$. From (55), $\left\|I-A^{-1}\right\| \le 2$, so that $|w| \le 2\,|v|$. Take a symmetric matrix C such that

$$Cy = w \qquad \text{and} \qquad \|C\| = \frac{|w|}{|y|}.$$

Then $\|C\| \le 4\,|v| = 4\,\left\|PA^{-1}\right\|^{-1}$, which is arbitrarily small.

□

8.6. **Lemma.**

Let $\mathbb{R}^{2n} = E \oplus F$, where E, F are lagrangian subspaces such that $\measuredangle(E,F) > \gamma$. Then there exists $K = K(\gamma) > 0$ and a symplectic basis $\{e_1,\dots,e_n;\ f_1,\dots,f_n\}$, $e_i \in E$, $f_j \in F$, such that the norm

$$\Big\|\sum_{i=1}^{n} x_i e_i + y_i f_i\Big\|^2 := \sum_{i=1}^{n} x_i^2 + y_i^2$$

satisfies

$$\tfrac{1}{K}\,|z| \le \|z\| \le K\,|z|,$$

where $|\cdot|$ is the euclidean norm in $\mathbb{R}^{2n}$.

Proof: Define the following inner product in $\mathbb{R}^{2n}$:

$$[x_1 \oplus y_1, x_2 \oplus y_2] := \langle x_1, x_2\rangle + \langle y_1, y_2\rangle,$$

where $x_i \oplus y_i \in E \oplus F$ and $\langle\cdot,\cdot\rangle$ is the euclidean inner product in $\mathbb{R}^{2n}$. We first show that the norm $[\![\,\cdot\,]\!]$ associated to $[\cdot,\cdot]$ is equivalent to the euclidean norm.

If $x \oplus y \in E \oplus F$, then

$$\begin{aligned} |x+y|^2 &= |x|^2 + |y|^2 + 2\,\langle x, y\rangle \\ &\le |x|^2 + |y|^2 + \big(|x|^2 + |y|^2\big) \\ &\le 2\,[\![x \oplus y]\!]^2. \end{aligned}$$

Let $L : E^\perp \to E$ be a linear map such that $F = \{\, z \oplus Lz \,|\, z \in E^\perp \,\}$. Then $\|L\| < \gamma^{-1}$, in the euclidean norm. Let $z \in E^\perp$ be such that $y = z \oplus Lz$. Then

$$|y|^2 = |z|^2 + |Lz|^2 \le (1 + \gamma^{-2})\,|z|^2.$$

Hence

$$|z|^2 \ge \frac{1}{1+\gamma^{-2}}\;|y|^2.$$

The last two equations imply that

$$|Lz|^2 \le |y|^2 - |z|^2 \le \left(1 - \frac{1}{1+\gamma^{-2}}\right)\;|y|^2 = \frac{\gamma^{-2}}{1+\gamma^{-2}}\;|y|^2.$$

Since $\langle x, y\rangle = \langle x, z \oplus Lz\rangle = \langle x, Lz\rangle$, have that

$$\begin{aligned} |x+y|^2 &= |x|^2 + |y|^2 + 2\langle x, y\rangle \\ &\ge |x|^2 + |y|^2 - 2\,|x|\frac{\gamma^{-1}}{\sqrt{1+\gamma^{-2}}}\,|y| \\ &\ge \left(1 - \frac{\gamma^{-1}}{\sqrt{1+\gamma^{-2}}}\right)(|x|^2 + |y|^2), \qquad \forall\, x \oplus y \in E \oplus F. \end{aligned}$$

Writing $A(\gamma) := \max\big\{\sqrt{2}, \big(1 - \frac{\gamma^{-1}}{\sqrt{1+\gamma^{-2}}}\big)^{-\frac{1}{2}}\big\}$, we have that

$$\tfrac{1}{A(\gamma)}\;\; |x+y| \le [\![x \oplus y]\!] \le A(\gamma)\;\; |x+y|\,, \qquad \forall\, x \oplus y \in E \oplus F.$$

Now, let $K : \mathbb{R}^{2n} \hookleftarrow$ be the linear isomorphism defined by

$$[x, Ky] = \Omega(x, y), \qquad x, y \in \mathbb{R}^{2n};$$

where Ω is the canonical symplectic form in $\mathbb{R}^{2n}$. Observe that F is the orthogonal complement of E with respect to $[\cdot,\cdot]$. Since E is lagrangian, we have that $[x, Ky] = 0$ if $x, y \in E$. Thus $K(E) = F$ and similarly $K(F) = E$.

Let $e_1, \ldots, e_n$ be an orthonormal basis for E and let $f_i := K^{-1} e_i$, $i = 1, \ldots, n$. Then

$$\Omega(e_i, f_j) = [e_i, Kf_j] = [e_i, e_j] = \delta_{ij}.$$

This implies that the basis $\{e_1, \ldots, f_n\}$ is symplectic.

Observe that if $y \in \mathbb{R}^{2n}$, then

$$[\![Ky]\!]^2 = [Ky, Ky] = \Omega(Ky, y) \le |Ky|\;|y| \le A(\gamma)^2\,[\![Ky]\!]\;[\![y]\!].$$

So that

$$[\![Ky]\!] \le A(\gamma)^2 \, [\![y]\!] \qquad \text{for all } y \in \mathbb{R}^{2n}.$$

Let $x \in \mathbb{R}^{2n}$ and let $y := Jx \in \mathbb{R}^{2n}$. Then $|Jx| = |x|$ and $\Omega(Jx, x) = |x|^2$. Therefore

$$[\![Jx]\!] \, [\![Kx]\!] \ge [Jx, Kx] = \Omega(Jx, x) = |x|^2 = |Jx| \, |x| \ge \tfrac{1}{A(\gamma)^2} \, [\![Jx]\!] \, [\![x]\!].$$

Thus

$$[\![Kx]\!] \ge \tfrac{1}{A(\gamma)^2} \, [\![x]\!] \qquad \text{for all } x \in \mathbb{R}^{2n}.$$

Finally, we have that

$$\begin{aligned}
\Big[\!\!\Big[\sum_{i=1}^n x_i e_i + y_i f_i \Big]\!\!\Big]^2 &= \sum_{i=1}^n x_i^2 + \Big[\!\!\Big[\sum_{i=1}^n y_i f_i \Big]\!\!\Big]^2 \\
&= \sum_{i=1}^n x_i^2 + \Big[\!\!\Big[K^{-1} \Big(\textstyle\sum_{i=1}^n y_i e_i \Big) \Big]\!\!\Big]^2 \\
&\le \sum_{i=1}^n x_i^2 + A(\gamma)^4 \Big[\!\!\Big[\textstyle\sum_{i=1}^n y_i e_i \Big]\!\!\Big]^2 \\
&\le A(\gamma)^4 \, \sum_{i=1}^n (x_i^2 + y_i^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Big[\!\!\Big[\sum_{i=1}^n x_i e_i + y_i f_i \Big]\!\!\Big]^2 &\ge \sum_{i=1}^n x_i^2 + \frac{1}{A(\gamma)^4} \Big[\!\!\Big[\textstyle\sum_{i=1}^n y_i e_i \Big]\!\!\Big]^2 \\
&\ge \frac{1}{A(\gamma)^4} \, \sum_{i=1}^n (x_i^2 + y_i^2).
\end{aligned}$$

Hence the lemma holds for $\kappa(\gamma) := A(\gamma)^2$. $\square$

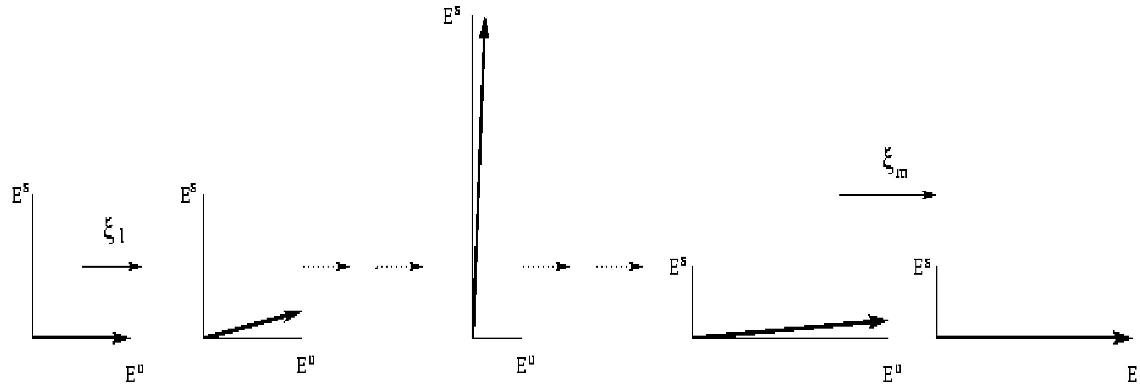


FIGURE 2. SKETCH OF THEOREM 8.1: Once we know the angles are uniformly bounded below for any perturbation, we can assume E^s and E^u are orthogonal. If a sequence does not uniformly contract E^s ($\left\|\Pi_1^k \xi_i|_{E^s}\right\| \geq \frac{1}{2}$) multiply its stable component by $(1+\varepsilon)^m$ and its unstable component by $(1+\varepsilon)^{-m}$ so that at some iterate, say k, it expands E^s and contracts E^u. Then the perturbation of only ξ_1 and ξ_m shown in the figure obtains a small angle $\sphericalangle(E^s, E^u)$ at the k-th iterate, which is a contradiction.

Proof of Theorem 8.1:

We first prove that there is $M_1 > 0$ such that

$$\left\| \prod_{j=0}^{M_1-1} \xi_{i+j}^{\alpha}\Big|_{E_i^s(\xi^\alpha)} \right\| < \frac{1}{2}, \qquad \forall\, \alpha \in \mathcal{A}, \quad \forall\, i \in \mathbb{Z}. \tag{56}$$

Since the family ξ is bounded, it is enough to prove that

$$\exists\, N > 0 \;:\; \quad \forall\, \alpha \in \mathcal{A}\,, \quad \forall\, i \in \mathbb{Z}\,, \quad \exists\, 0 < n \leq N : \qquad \left\| \prod_{j=0}^{n-1} \xi_{i+j}^{\alpha}\Big|_{E_i^s(\xi^\alpha)} \right\| < \frac{1}{2}. \tag{57}$$

For take $m > 0$ such that

$$\frac{1}{2^m}\Big(\sup_{\alpha,i} \|\xi_i^\alpha\| \Big)^N < \frac{1}{2}, \tag{58}$$

and let $M_1 := (m+1)\,N$. Writing $M_1 = n_1 + n_2 + \cdots + n_k + r$, where the $n_\ell \leq N$ are such that (57) holds for $i = n_1 + \cdots + n_{\ell-1}$ and $0 \leq r < N$ we have that $k \geq m$ and by (58), we obtain that (56) holds.

If (57) were not true, then

$$\forall\, N > 0\,, \quad \exists\, \alpha_N \in \mathcal{A}\,, \quad \exists\, i_N \in \mathbb{Z}\,, \quad \forall\, 0 < n \leq N : \qquad \left\| \prod_{j=0}^{n-1} \xi_{i_N+j}^{\alpha_N}\Big|_{E_{i_N}^s(\xi^{\alpha_N})} \right\| \geq \frac{1}{2}. \tag{59}$$

CASE I: *Suppose that the periods of the sequences ξ^{α_N} are bounded.*

Taking subsequences of α_N we can assume that

- $i_N \equiv i_0$ is constant.
- $\mathrm{Per}(\xi^{\alpha_N}) = m$ is constant.
- $\forall\, j \in \mathbb{Z}, \quad \exists\, \eta_j = \lim_N \xi^{\alpha_N}_{i_0+j}$.
- $\forall\, j \in \mathbb{Z}, \quad \exists\, E^+_j = \lim_N E^s_{i_0+j}(\xi^{\alpha_N})$.
- $\forall\, j \in \mathbb{Z}, \quad \exists\, E^-_j = \lim_N E^u_{i_0+j}(\xi^{\alpha_N})$.

Observe that the subspaces E^+_j, E^-_j are m-periodic and invariant under $\prod_{i=0}^{m-1} \eta_{j+i}$.

From (59) we have that

$$\left\| \prod_{j=0}^{n-1} \eta_j \Big|_{E^+_0} \right\| \geq \frac{1}{2}, \qquad \text{for all } n \in \mathbb{N}^+. \tag{60}$$

The stable hyperbolicity of the family ξ implies that the sequence η is hyperbolic. Then $\prod_{i=0}^{m-1} \eta_i$ is a hyperbolic matrix which is limit of the sequence of hyperbolic matrices $\prod_{j=0}^{m-1} \xi^{\alpha_N}_{i_0+j}$. This implies that $E^+_0 = \lim_N E^s_{i_0}(\xi^{\alpha_N})$, $E^-_0 = \lim_N E^u_{i_0}(\xi^{\alpha_N})$ are the stable and unstable subspaces of $\prod_{j=0}^{m-1} \eta_j$. But this contradicts (60).

CASE II: *Suppose that the periods of the sequences ξ^{α_N} are unbounded.*

Let ε, K, λ be from Lemma 8.4. Let $N_1 > 0$ and $\varepsilon_0 > 0$ be such that

$$K\, \lambda^{n_1}\, (1+\varepsilon_0)^{n_1} < \tfrac{1}{2}, \qquad \forall\, n_1 \geq N_1. \tag{61}$$

Let N_0 and γ be from Lemma 8.5. Taking a subsequence of α_N we can assume that all the periods satisfy

$$\mathrm{Per}(\xi^{\alpha_N}) > \max\{N_0, N_1\}. \tag{62}$$

If we extend the family ξ to the family of all the shifted sequences $j \mapsto \xi^{\alpha}_{i+j}$ for all $\alpha \in \mathcal{A}$, $i \in \mathbb{Z}$, then the new family is also stably hyperbolic. Using this extended family if necessary, we can assume that $i_N = 1$ in inequality (59).

We shall perturb the symplectic linear maps ξ^{α}_i so that the angle $\sphericalangle(E^s_{N+1}(\xi^{\alpha_N}), E^u_{N+1}(\xi^{\alpha_N}))$ becomes arbitrarily small, contradicting Lemma 8.5.

In the decomposition $E^s(\xi^{\alpha_N}) \oplus E^u(\xi^{\alpha_N})$, for $m = \mathrm{Per}(\xi^{\alpha_N})$, write

$$\prod_{i=1}^{N} \xi^{\alpha_N}_i = \begin{bmatrix} B & 0 \\ 0 & (B^*)^{-1} \end{bmatrix}, \qquad \prod_{i=1}^{m} \xi^{\alpha_N}_i = \begin{bmatrix} \mathbb{A} & 0 \\ 0 & (\mathbb{A}^*)^{-1} \end{bmatrix}.$$

By Lemma 8.4, (62) and (61),

$$\|\mathbb{A}\| \le K\,\lambda^m < \tfrac{1}{2}. \tag{63}$$

Then from (59) we have that $m > N$.

By Lemma 8.6 and (62), it is equivalent to measure the norms of linear maps in the decompositions $E^s(\xi^{\alpha_N}) \oplus E^u(\xi^{\alpha_N})$. Without loss of generality we may assume that $\kappa(\gamma) = 1$ in Lemma 8.6.

Define a perturbation η of ξ^{α_N} by

$$\eta_1 := \begin{bmatrix} (1+\varepsilon)\, I & 0 \\ 0 & (1+\varepsilon)^{-1} I \end{bmatrix} \xi_1^{\alpha_N} \begin{bmatrix} I & C \\ 0 & I \end{bmatrix},$$

$$\eta_i := \begin{bmatrix} (1+\varepsilon)\, I & 0 \\ 0 & (1+\varepsilon)^{-1} I \end{bmatrix} \xi_i^{\alpha_N}\,, \qquad 1 < i < m,$$

$$\eta_m := \begin{bmatrix} I & D \\ 0 & I \end{bmatrix} \xi_m^{\alpha_N} \begin{bmatrix} (1+\varepsilon)\, I & 0 \\ 0 & (1+\varepsilon)^{-1} I \end{bmatrix};$$

where C and D are small symmetric matrices defined as follows.

Observe that by (59), $\|B^*\| = \|B\| \ge \frac{1}{2}$. Let $u,\, v \in \mathbb{R}^n$ be such that $|B^* u| = 1$, $|v| = 1$,

$$|B^*\, u| \ge \tfrac{1}{2}\, |u| \qquad \text{and} \qquad |B\, v| \ge \tfrac{1}{2}\, |v|.$$

Let C be a symmetric matrix such that

$$C\,(B^* u) = \varepsilon\, |B^* u|\, v \qquad \text{and} \qquad \|C\| = \varepsilon.$$

Let D be the symmetric matrix

$$D = -(1+\varepsilon)^{2m}\, \mathbb{A}\, C\, \mathbb{A}^*.$$

From (63), (62) and (61), if $0 < \varepsilon < \varepsilon_0$ then

$$\|D\| \le K^2\, \lambda^{2m}\, (1+\varepsilon)^{2m}\ \|C\| < \|C\| = \varepsilon.$$

Therefore, since the family ξ is bounded,

$$\lim_{\varepsilon \to 0} d(\eta, \xi^{\alpha_N}) = 0 \qquad \text{uniformly on } N.$$

Observe that with this definition of D, we have that

$$\prod_{i=1}^{m} \eta_i = \begin{bmatrix} (1+\varepsilon)^m\, \mathbb{A} & 0 \\ 0 & (1+\varepsilon)^{-m}\, (\mathbb{A}^*)^{-1} \end{bmatrix}.$$

In particular,

$$\left\| \Big(\prod_{i=1}^{m} \eta_i \Big|_{E_1^u(\xi^{\alpha_N})} \Big)^{-1} \right\| = \left\| \prod_{i=1}^{m} \eta_i \Big|_{E_1^s(\xi^{\alpha_N})} \right\| \le K\, \lambda^m\, (1+\varepsilon)^m < 1.$$

Thus the sequence η is hyperbolic and has the same subspaces E_1^s, E_1^u as the sequence ξ^{α_N}.

Observe that

$$\prod_{i=1}^{N}\eta_i = \begin{bmatrix}(1+\varepsilon)^N B & 0\\ 0 & (1+\varepsilon)^{-N}(B^*)^{-1}\end{bmatrix}\ \begin{bmatrix} I & C\\ 0 & I\end{bmatrix}$$
$$= \begin{bmatrix}(1+\varepsilon)^N B & (1+\varepsilon)^N\, B\, C\\ & \\ 0 & (1+\varepsilon)^{-N}(B^*)^{-1}\end{bmatrix}.$$

The unstable subspace E^u_{N+1} at time N, is $E^u_{N+1}(\eta) = (\prod_{i=1}^N \eta_i)(E^u_1(\xi^{\alpha_N}))$. Therefore

$$E^u_{N+1}(\eta) = \{\, z\oplus Lz \in E^u_{N+1}(\xi^{\alpha_N})\oplus E^s_{N+1}(\xi^{\alpha_N}) \mid z\in E^u_{N+1}(\xi^{\alpha_N})\,\},$$

where $L : E^u_{N+1}(\xi^{\alpha_N}) \to E^s_{N+1}(\xi^{\alpha_N})$ is given by

$$L = (1+\varepsilon)^{2N}\, B\, C\, B^*.$$

The stable subspace is $E^s_{N+1}(\eta) = E^s_{N+1}(\xi^{\alpha_N})$.

We have that

$$|Lu| = (1+\varepsilon)^{2N}\,|B\,CB^*u| = (1+\varepsilon)^{2N}\,|Bv|\,\varepsilon\,|B^*u|$$
$$\ge \tfrac14\,\varepsilon\,(1+\varepsilon)^{2N}\,|u|.$$

Under the inner product $[\cdot,\cdot]$ of Lemma 8.6, $E^u_{N+1}(\xi^{\alpha_N}) = \big(E^s_{N+1}(\xi^{\alpha_N})\big)^\perp$. Thus

$$\measuredangle\big(E^s_{N+1}(\eta), E^u_{N+1}(\eta)\big) = \|L\|^{-1} \le \frac{4}{\varepsilon\,(1+\varepsilon)^{2N}},$$

which is arbitrarily small if N is large enough. This finishes the proof of (56).

It remains to prove that there is $M_2>0$ such that

$$\left\|\Big(\prod_{j=0}^{M_2-1}\xi^\alpha_{i+j}\Big|_{E^u_i(\xi^\alpha)}\Big)^{-1}\right\| < \frac12, \qquad \forall\,\alpha\in\mathcal{A},\quad \forall\, i\in\mathbb{Z}. \tag{64}$$

Let N_0 and γ be from Lemma 8.5 for ξ. Let

$$\mathcal{A}_0 := \{\,\alpha\in\mathcal{A} \mid \operatorname{Per}(\xi^\alpha) > N_0\,\}.$$

In the splitting $E^s_i(\xi^\alpha)\oplus E^u_i(\xi^\alpha)$ we have that

$$\prod_{j=0}^{M_1-1}\xi^\alpha_{i+j} = \begin{bmatrix} F & 0\\ 0 & (F^*)^{-1}\end{bmatrix},$$

with $\|F\| < \frac12$ by (56). Using the equivalent norm from Lemma 8.6, we have that

$$\left\|\Big(\prod_{j=0}^{M_1-1}\xi^\alpha_{i+j}\Big|_{E^u_i(\xi^\alpha)}\Big)^{-1}\right\| = \left\|\prod_{j=0}^{M_1-1}\xi^\alpha_{i+j}\Big|_{E^s_i(\xi^\alpha)}\right\| < \frac12, \qquad \forall\,\alpha\in\mathcal{A}_0,\quad \forall\, i\in\mathbb{Z}.$$

This finishes the proof if $\mathcal{A}_0 = \mathcal{A}$. If not, by repeating sequences in $\mathcal{A}_1 := \mathcal{A} \setminus \mathcal{A}_0$ we can assume that $\mathcal{A}_1$ is infinite. Since the periods of the sequences in $\mathcal{A}_1$ are bounded by N_0, the same argument as in CASE I above gives $M_3 > 0$ such that

$$\left\| \Big(\prod_{j=0}^{M_3-1} \xi^{\alpha}_{i+j}\Big|_{E^u_i(\xi^\alpha)} \Big)^{-1} \right\| < \frac{1}{2} \qquad \forall\, \alpha \in \mathcal{A}_1, \quad \forall\, i \in \mathbb{Z}.$$

Then for (64) take $M_2 = M_1 \cdot M_3$. In order to get (56) and (64) with the same M, take $M = M_1 \cdot M_2$.

□

9. Hyperbolicity.

Given a subset $A \subset SM$ and $g \in \mathcal{R}^\infty(M)$ let $\mathcal{P}(g, A)$ be the set of closed orbits γ for ϕ^g such that $\gamma(\mathbb{R}) \subset A$. Define

$$\mathrm{Per}(g, A) := \bigcup_{\gamma \in \mathcal{P}(g,A)} \gamma(\mathbb{R}),$$

$$\mathcal{H}(A) := \{\, g \in \mathcal{R}^\infty(M) \mid \forall \gamma \in \mathcal{P}(g, A),\ \gamma \text{ is hyperbolic}\,\},$$

$$\mathcal{F}^2(A) := \mathrm{int}_{C^2}\, \mathcal{H}(A).$$

Let $\mathcal{G}_1$ be as in Theorem 6.1,

Theorem E.

If $g \in \mathcal{G}_1 \cap \mathcal{F}^2(A)$, then $\Lambda := \overline{\mathrm{Per}(g, A)}$ is a hyperbolic set for ϕ^g.

Proof: Let ℓ be the injectivity radius of g. For each $\alpha \in \mathcal{A} := \mathcal{P}(g, A)$ let $T = T(\alpha)$ be the period of α and choose $0 = t_0 < t_1 < t_2 < \cdots < t_m = T(\alpha)$ such that $t_{i+1} - t_i \in [\frac{1}{4}\ell, \frac{1}{2}\ell]$. Then $\alpha|_{[t_i,t_{i+1}]}$ is injective. Let

$$\mathcal{N}(i, \alpha) := \{\, v \in T_{\alpha(i)}SM \mid \langle v, \dot\alpha(t_i)\rangle_g = 0\,\}. \tag{65}$$

Choose an orthonormal symplectic basis $\mathcal{B}(i, \alpha)$ for $\mathcal{N}(i, \alpha)$. Let $\xi^\alpha : \mathbb{Z} \to Sp(n)$ be the periodic sequence of period m such that ξ^α_i is the matrix of $d\phi^g_{t_{i+1}-t_i} : \mathcal{N}(i, \alpha) \to \mathcal{N}(i+1, \alpha)$ in the basis $\mathcal{B}(i, \alpha)$ and $\mathcal{B}(i + 1, \alpha)$. We use the following

9.1. **Lemma.** *The family $\xi = \{\, \xi^\alpha \,\}_{\alpha \in \mathcal{A}}$ is stably hyperbolic.*

Then, from theorem 8.1 we obtain a hyperbolic splitting on $\mathcal{P}(g, A)$. The hyperbolicity condition implies the continuity of the splitting in $\mathrm{Per}(g, A)$ (see [16, prop. 6.4.4] for diffeomorphisms). Then the splitting extends continuously to the closure $\Lambda = \overline{\mathrm{Per}(g, A)}$ and the extension is also hyperbolic.

□

Proof of Lemma 9.1: If ξ is not stably hyperbolic, then there is a periodically equivalent family η with $d(\eta,\xi)$ arbitrarily small which is not hyperbolic. Modifying η if necessary, we can assume that $\{\,\alpha\in\mathcal{A}\,|\,\eta^\alpha\ne\xi^\alpha\,\}=\{\,\alpha_0\,\}$ is a single sequence and η^{α_0} is not hyperbolic. Since $g\in\mathcal{G}_1$ and $d(\xi^{\alpha_0},\eta^{\alpha_0})$ is arbitrarily small, by theorem 7.1 there is a metric $g_1\in\mathcal{R}^\infty(M)$, which is C^∞, such that g_1 is C^2 arbitrarily near g (and hence $g_1\in\mathcal{H}(A)$), the same α_0 is a periodic orbit for g_1, $g_1=g$ on $\alpha(\mathbb{R})$ (hence the same subspaces $\mathcal{N}(i,a)$ satisfy (65) for g_1), and $\eta_i^{\alpha_0}=d\phi^{g_1}_{t_{i+1}-t_i}:\mathcal{N}(i,\alpha_0)\to\mathcal{N}(i+1,\alpha_0)$ for all $0\le i<m(\alpha_0)$. Since η^{α_0} is not hyperbolic and $\alpha_0(\mathbb{R})\subset A$, this contradicts the fact $g_1\in\mathcal{H}(A)$.

□

The linearized Poincaré map P_c of a prime closed geodesic c is a symplectic map. If c is not hyperbolic denote by $z_j=\pm\exp(2\pi\lambda_j)$, $\lambda_j\in[0,\frac12]$, $j=1,\dots,\ell\le n$ the eigenvalues of P_c with norm 1. The numbers $0\le\lambda_1<\cdots<\lambda_\ell\le\frac12$ are called *Poincaré exponents of* c. Following Rademacher [31], we say that a riemannian metric is *strongly bumpy* if all the eigenvalues of the linearized Poincaré map of every prime closed geodesic are simple and if any finite set of the disjoint union of the Poincaré exponents of the prime closed geodesics is algebraically independent.

For $2\le k\le\infty$, let $\mathcal{B}_k$ be the set of strongly bumpy metrics in $\mathcal{R}^k(M)$.

9.2. **Rademacher's theorem.** [31] *For any* $2\le k\le\infty$*:*

(i) $\mathcal{B}_k$ *is residual in* $\mathcal{R}^k(M)$.
(ii) *If* $g\in\mathcal{B}_k$ *then* g *has infinitely many geometrically distinct closed geodesics.*

Let $\mathcal{K}$ be the set of metrics g in $\mathcal{R}^2(M)$ such that

- The metric g is strongly bumpy: $g\in\mathcal{B}_2$.
- All heteroclinic points of hyperbolic closed geodesics of g are transversal.

By theorems 9.2 and 2.1, for any $2\le k\le\infty$, the set $\mathcal{K}\cap\mathcal{R}^k(M)$ is residual in $\mathcal{R}^k(M)$.

Given a continuous flow ϕ_t on a topological space X a point $x\in X$ is said *wandering* if there is an open neighbourhood U of x and $T>0$ such that $\phi_t(U)\cap U=\emptyset$ for all $t>T$. Denote by $\Omega(\phi_t|_X)$ the set of non-wandering points for (X,ϕ_t). Recall

9.3. **Smale's spectral decomposition theorem for flows.** [33, 16]

If Λ *is a locally maximal hyperbolic set for a flow* ϕ_t*, then there exists a finite collection of basic sets* $\Lambda_1,\dots\Lambda_N$ *such that the non-wandering set of the restriction* $\phi_t\big|_\Lambda$ *satisfies*

$$\Omega\big(\phi_t\big|_\Lambda\big)=\bigcup_{i=1}^N\Lambda_i.$$

Now let $\mathcal{D}:=\mathcal{K}\cap\mathcal{G}_1$, where $\mathcal{G}_1$ is from Theorem 6.1,

Theorem D.

If $g \in \mathcal{D} \cap \mathcal{F}^2(M)$, then $\Lambda = \overline{\mathrm{Per}(g)}$ contains a non-trivial hyperbolic basic set.

Proof: Since $\mathcal{D} \subset \mathcal{G}_1$, applying Theorem E to $A = M$ we get that Λ is a hyperbolic set. By proposition 6.4.6 in [16], there exists an open neighbourhood U of Λ such that the set

$$\Lambda_U := \bigcap\nolimits_{t\in\mathbb{R}} \phi_t^g(\overline{U})$$

is hyperbolic. Since $\Lambda = \overline{\mathrm{Per}(g)}$, its non-wandering set is $\Omega(\phi_t|_\Lambda) = \Lambda$. By the definition of Λ_U, $\Lambda \subseteq \Lambda_U$ and hence $\Lambda = \Omega(\phi_t|_\Lambda) \subseteq \Omega(\phi_t|_{\Lambda_U})$. By corollary 6.4.20 in [16], the periodic orbits are dense in the non-wandering set $\Omega(\phi_t|_{\Lambda_U})$ of the locally maximal hyperbolic set Λ_U. Thus $\Lambda \subseteq \Omega(\phi_t|_{\Lambda_U}) \subseteq \overline{\mathrm{Per}(g)} = \Lambda$. By theorem 9.3, the set $\Lambda = \Omega(\phi_t|_{\Lambda_U})$ decomposes into a finite collection of basic sets. Since the number of periodic orbits in Λ is infinite, at least one of the basic sets Λ_i is not a single periodic orbit, i.e. it is non-trivial.

□

Appendix A. Arc Spaces.

Let X be an algebraic variety on $\mathbb{R}^N$. Define the *path space* on X as

$$\mathcal{C}(X) := \{\, (a_n)_{n\in\mathbb{N}} \in \prod\nolimits_{n\in\mathbb{N}} \mathbb{R}^N \;\big|\; \exists \gamma \in C^\infty(\mathbb{R},\mathbb{R}^N),\ \gamma(\mathbb{R}) \subset X,\ \tfrac{1}{n!}\gamma^{(n)}(0) = a_n, \forall\, n \in \mathbb{N} \,\}$$

Let $F = (f_1, \dots, f_q)$ be generators of the ideal $I(X) = \{\, f \in \mathbb{R}[x_1, \dots, x_N] \mid f|_X \equiv 0 \,\}$. Recall that the *arc space* $\mathcal{L}(X)$ is

$$\mathcal{L}(X) := \{\, (a_n)_{n\in\mathbb{N}} \in \prod\nolimits_{n\in\mathbb{N}} \mathbb{R}^N \;\big|\; F(\textstyle\sum_{k=0}^n a_k t^k) \equiv 0 \,\},$$

where the equality $\equiv$ is as formal power series. The *jet space* $\mathcal{L}_n(X)$ is

$$\mathcal{L}_n(X) := \left\{\, (a_k)_{k\in\mathbb{N}} \in \prod\nolimits_{k=0}^n \mathbb{R}^N \;\middle|\; F(\textstyle\sum_{k=0}^n a_k t^k) = 0 \pmod{t^{n+1}} \,\right\}.$$

Then $\mathcal{L}_n(X)$ is an algebraic variety. Let $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$ be the projection $(a_k)_{k\in\mathbb{N}} \mapsto (a_k)_{k=0}^n$. Then $\pi_n(\mathcal{L}(X))$ is a constructible set in $\mathcal{L}_n(X)$ (see [8, p. 202]). Let $\overline{\pi_n(\mathcal{C}(X))}$ be the Zariski closure of $\pi_n(\mathcal{C}(X))$.

A.1. **Proposition.**

(i) *dim* $\overline{\pi_n(\mathcal{C}(X))} \le (n+1)$ *dim* X.
(ii) *The fibers of* $\pi_{n+1}(\mathcal{C}(X)) \to \pi_n(\mathcal{C}(X))$ *have dimension* $\le$ *dim* X.

Proof:

By Lemma A.2 it is enough to proof the proposition for an algebraic variety X in $\mathbb{C}^N$. Observe that item (ii) implies item (i). We prove item (ii).

Fix $\overline{a} = (a_0, \dots, a_n) \in \pi_n(\mathcal{C}(X))$. Define

$$Z_{n+1} := \left\{\, (t,x) \in \mathbb{C} \times \mathbb{C}^N \;\middle|\; F(a_0 + \dots + a_n t^n + t^{n+1} x) = 0 \,\right\}.$$

For $t \in \mathbb{C}$, let

$$Z_{n+1}(t) := \{ \, x \in \mathbb{C}^N \mid (t,x) \in Z_{n+1} \, \}.$$

Let $\mathbb{F}_{\overline{a}} := \theta_n^{-1}(\overline{a})$ be the fiber of $\theta_n : \pi_{n+1}(\mathcal{C}(X)) \to \pi_n(\mathcal{C}(X))$ over $\overline{a}$. The limit W_{n+1} at $t=0$ of the 1-parameter family of varieties $Z_{n+1}(t)$ exists (see [10, pp. 71–72]):

$$W_{n+1} := \lim_{t\to 0} Z_{n+1}(t),$$

i.e. if $Z^*_{n+1} := \{ \, (t,x) \in Z_{n+1} \mid t \neq 0 \, \}$, then $Z^*_{n+1} \cup W_{n+1}$ is the Zariski closure of Z^*_{n+1}.

Claim 1: $\quad \mathbb{F}_{\overline{a}} \subset W_{n+1}$.

Indeed, let $a_{n+1} \in \mathbb{F}_{\overline{a}}$. Since $(a_0, \dots, a_n, a_{n+1}) \in \pi_{n+1}(\mathcal{C}(X))$ there is $\gamma \in C^\infty(\mathbb{R}, \mathbb{R}^N)$ such that $F \circ \gamma \equiv 0$ and

$$\gamma(t) = a_0 + \cdots + a_n t^n + a_{n+1} t^{n+1} + \mathcal{O}(t^{n+2}), \quad t \in \mathbb{R}.$$

Let $x_t := \frac{1}{t^n}\left[\gamma(t) - \sum_{k=0}^n a_k t^k\right] = a_{n+1} + \mathcal{O}(t) \in Z_{n+1}(t)$. This implies that $a_{n+1} \in W_{n+1}$.

The following claim finishes the proof:

Claim 2: $\quad \dim W_{n+1} \leq \dim X$.

For $t \neq 0$, we have that the variety $Z_{n+1}(t)$ is isomorphic to X by the invertible change of variables $Z_{n+1}(t) \ni z \longleftrightarrow x \in X$: $x = a_0 + a_1 t + \cdots + a_n t^n + t^{n+1} z$. Therefore $\dim Z_{n+1}(t) = \dim X$, when $t \neq 0$.

Consider $\mathbb{C}^N = \mathbb{C}^N \times \{1\} \subset \mathbb{CP}^N = \mathbb{C}^N \cup \mathbb{CP}^{N-1}$ and the corresponding projective varieties $\mathbb{Z}_{n+1}(t)$, $\mathbb{Z}^*_{n+1} = \cup_{t\neq 0} \mathbb{Z}_{n+1}(t)$, $\mathbb{W} = \lim_{t\to 0} \mathbb{Z}_{n+1}(t)$. Then $W_{n+1} = \mathbb{W}_{n+1} \cap \mathbb{C}^N$.

Claim 2 follows form the fact that $\overline{\mathbb{Z}_{n+1}} = \mathbb{Z}^*_{n+1} \cup \mathbb{W}_{n+1}$ is a flat family (see [10, prop. II-29]) and the fact that the dimension of the fibers of a flat family is constant (e.g. [15, pp. 256 – 257]). Another proof is the following:

Since for a generic fiber $t \neq 0$, $\dim \mathbb{Z}_{n+1}(t) = \dim X$, we have that $\dim \mathbb{Z}^*_{n+1} = \dim X + 1$. If $\dim W_{n+1} > \dim X$, then $\dim \mathbb{W}_{n+1} \geq \dim W_{n+1} \geq \dim X + 1$. Therefore $\mathbb{W}_{n+1}$ contains an irreducible component of $\overline{\mathbb{Z}^*_{n+1}}$. This is incompatible with $\mathbb{W}_{n+1} = \lim_{t\to 0} \mathbb{Z}_{n+1}(t)$ (see [10, prop. II-2, p. 75 – 76] [6].

□

A.2. **Lemma.** *Let $X \subset \mathbb{R}^N$ be an algebraic variety an let $\mathbb{X} \subset \mathbb{C}^N$ be the algebraic variety defined by the same polynomials as X. Then $\dim_{\mathbb{R}}(X) \leq \dim_{\mathbb{C}}(X)$.*

Proof: Let $\mathbb{T}$ be a stratum of $\mathbb{X}$ and $T := \mathbb{T} \cap \mathbb{R}^d$. Then $\mathbb{T}$ is a complex submanifold of $\mathbb{C}^N$. In particular, its tangent spaces are closed under multiplication by $\sqrt{-1}$. Then the

[6] Observe that a priori $\mathbb{W}_{n+1}$ could have all its irreducible components of maximal dimension in the hyperplane at infinity $\mathbb{CP}^{N-1}$ and then $\dim W_{n+1} < \dim \mathbb{W}_{n+1}$. Since the function $f(t) := \dim \mathbb{Z}_{n+1}(t)$ is upper semi-continuous (see [14, p. 139]), $\dim \mathbb{W}_{n+1} \geq \limsup_{t\to 0} \dim \mathbb{Z}_{n+1}(t) = \dim X$. Then the argument above also shows that $\dim \mathbb{W}_{n+1} = \dim X$.

2-form $\Omega(\mathbf{u},\mathbf{v}) = \mathrm{Im}(\langle\!\langle \mathbf{u},\mathbf{v}\rangle\!\rangle)$ is non-degenerate on $\mathbb{T}$, because $\Omega(\mathbf{u},\mathbf{u}\sqrt{-1}) = -\sum |u_j|^2 \neq 0$ if $\mathbf{u} \neq 0$. Let $x \in T \subset \mathbb{T}$. Since the tangent space $T_xT \subset \mathbb{R}^d$, $\Omega|_{T_xT} \equiv 0$, i.e. T_xT is an isotropic subspace for Ω. Therefore $\dim_{\mathbb{R}} T_xT \leq \frac{1}{2}\dim_{\mathbb{R}} T_x\mathbb{T} = \dim_{\mathbb{C}} T_x\mathbb{T}$.

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