Ground States are generically a periodic orbit

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USP, Sao Paulo
December 11, 2013

Expanding map

X compact metric space.

$T : X \to X$ an expanding map i.e.

$T \in C^0, \quad \exists d \in \mathbb{Z}^+, \quad \exists 0 < \lambda < 1, \quad \exists e_0 > 0 \quad$ s.t.

$\forall x \in X$ the branches of T^{-1} are λ-Lipschitz, i.e.

$\forall x \in X \quad \exists S_i : B(x, e_0) \to X$, $i = 1, \ldots, \ell_x \leqslant d$,

$$d\big(S_i(y), S_i(z)\big) \leqslant \lambda\, d(y, z),$$

$$\begin{cases} T \circ S_i = I_{B(x,e_0)}, \\ S_i \circ T|_{B(S_i(x),\,\lambda e_0)} = I_{B(S_i(x),\,\lambda e_0)}. \end{cases}$$

Main Theorem

X compact metric space.
$T : X \to X$ expanding map, $F \in Lip(X, \mathbb{R})$.
A maximizing measure is a T-invariant Borel probability μ on X such that

$$\int F \, d\mu = \max \left\{ \int F \, d\nu \;\middle|\; \nu \text{ invariant Borel probability } \right\}.$$

Theorem

If X is a compact metric space and $T : X \to X$ is an expanding map then there is an open and dense set $\mathcal{O} \subset Lip(X, \mathbb{R})$ such that for all $F \in \mathcal{O}$ there is a single F-maximizing measure and it is supported on a periodic orbit.

- Bousch, Jenkinson: There is a residual set $\mathcal{U} \subset C^0(X, \mathbb{R})$ s.t. $F \in \mathcal{U} \implies \mathcal{M}(F)$ has a unique measure and it has full support.
- Yuan & Hunt: Periodic maximizing measures are stable. (i.e. same maximizing measures for perturbations of the potential in Hölder or Lipschitz topology.)
  Non-periodic maximizing measures are not stable in Hölder or Lipchitz.
- Contreras, Lopes, Thieullen:
  Generically in $C^\alpha(X, \mathbb{R})$ there is a unique maximizing measure.
  If $F \in C^\alpha(X, \mathbb{R})$, then F can be approximated in the C^β topology $\beta < \alpha$ by G with the maximizing measure supported on a periodic orbit.

- Bousch: Proves a similar result for Walters functions:
  $\forall \varepsilon > 0 \; \exists \delta > 0$

$$\forall n \in \mathbb{N}, \quad \forall x, y \in X, \quad d_n(x,y) < \delta \implies |S_n f(x) - S_n f(y)| < \varepsilon.$$

$$d_n(x,y) := \sup_{i=0,\dots,n} d\big(T^i(x), T^i(y)\big).$$

- Quas & Siefken: prove a similar result for super-continuous functions.
  (functions whose local Lipschitz constant converges to 0 at a given rate: here X is a Cantor set or a shift space).

Lax Operator

$$\mathcal{M}(T) := \{\ T\text{-invariant Borel probabilities}\ \}$$

$F \in Lip(X, \mathbb{R})$, $\qquad \mathcal{L}_F : Lip(X, \mathbb{R}) \to Lip(X, \mathbb{R})$:

$$\mathcal{L}_F(u)(x) := \max_{y \in T^{-1}(x)} \{\, \alpha + F(x) + u(x)\,\},$$

where $\quad \alpha := -\max_{\mu \in \mathcal{M}(T)} \int F\, d\mu.$

Set of maximizing measures

$$\mathcal{M}(F) := \Big\{ \mu \in \mathcal{M}(T) \;\Big|\; \int F\, d\mu = -\alpha(F) \Big\}.$$

Calibrated sub-action = Fixed point of Lax Operator.

$$\mathcal{L}_F(u) = u$$

write

$$\overline{F} := F + \alpha + u - u \circ T.$$

REMARKS:

1. $$-\alpha(\overline{F}) = \max_{\mu \in \mathcal{M}(T)} \int \overline{F}\, d\mu = 0.$$
2. $$\overline{F} \leqslant 0.$$
3. $$\mathcal{M}(\overline{F}) = \mathcal{M}(F) = \Big\{ \mu \in \mathcal{M}(T) \;\Big|\; \operatorname{supp}(\mu) \subset [\,\overline{F} = 0\,] \Big\}.$$

Proposition

If F is Lipschitz then
there exists a Lipschitz calibrated sub-action.

Proof.

1. Prove that $\mathrm{Lip}\,(\mathcal{L}_F(u)) \leqslant \lambda\,(\,\mathrm{Lip}(u) + \mathrm{Lip}(F))$.
2. Then $\mathcal{L}_F$ leaves invariant the space

$$\mathbb{E} := \left\{ u \in Lip(X, \mathbb{R}) \mid \mathrm{Lip}(u) \leqslant \frac{\lambda\,\mathrm{Lip}(F)}{1-\lambda} \right\}.$$

3. $\mathbb{E}/_{\{\text{constants}\}}$ is compact & convex.
   $\mathcal{L}_F$ is continuous on $\mathbb{E}$.
   Schauder Thm. $\Longrightarrow$ $\mathcal{L}_F$ has a fixed pt. on $\mathbb{E}$.

□

REMARKS

1. If u is a calibrated sub-action:
   Every point $z \in X$ has a calibrating pre-orbit $(z_k)_{k \leqslant 0}$ s.t.

$$\begin{cases} T^i(z_{-i}) = z_0 = z, & \forall i \geqslant 0; \\ u(z_{k+1}) = u(z_k) + \alpha + F(z_k), & \forall k \leqslant -1. \end{cases}$$

   Equivalently, since $T(z_k) = z_{k+1}$,

$$\overline{F}(z_k) = 0 \qquad \forall k \leqslant -1.$$

2. If ν maximizing measure $\implies$ $\text{supp}(\nu) \subset [\overline{F} = 0]$.
   If $z \in \text{supp}(\nu)$ $\implies$ $\exists$ pre-orbit of $z \subseteq \text{supp}(\nu)$.

3. We will obtain a periodic orbit $\mathcal{O}(y)$ s.t.
   every calibrating pre-orbit has α-limit in $\mathcal{O}(y)$.

   By item 2 this will imply that
   every maximizing measure has support in $\mathcal{O}(y)$.

Shadowing Lemma

Definition

1. $(x_n)_{n\in\mathbb{N}} \subset X$ is a δ-pseudo-orbit if $d\big(x_{n+1}, T(x_n)\big) \leqslant \delta, \quad \forall n \in \mathbb{N}$.
2. A point $y \in X$ ε-shadows a pseudo-orbit $(x_n)_{n\in\mathbb{N}}$ if $d\big(T^n(y), x_n\big) < \varepsilon, \quad \forall n \in \mathbb{N}$.

Proposition (Shadowing Lemma)

If $(x_k)_{k\in\mathbb{N}}$ *is a* δ*-pseudo-orbit*

$\Longrightarrow$ $\exists y \in X$ *whose orbit* ε*-shadows* (x_k) *with* $\varepsilon = \frac{\delta}{1-\lambda}$.

If (x_k) *is periodic*

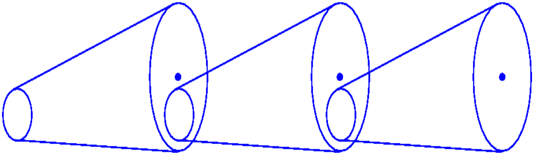
$\Longrightarrow$ y *is a periodic point with the same period.*

Proof.

$a = \frac{\lambda\delta}{1-\lambda}$.

$\{y\} = \bigcap_{k=0}^{\infty} S_0 \circ \cdots \circ S_k\big(B(x_{k+1}, a)\big)$.

where the inverse branch S_k is chosen such that $S_k(T(x_k)) = x_k$. □



The perturbation

- Original argument: Yuan & Hunt.
- Present argument: Quas & Siefken.
- Adapted to pseudo-orbits.

$$Per(T) := \bigcup_{p\ n\mathbb{N}^+} Fix(T^p) = \text{ periodic points.}$$

For $y \in Per(T)$:

$$\mathbb{P}_y := \{\, F \in Lip(X, \mathbb{R}) \mid \exists F - \text{maxim. meas. supported on } \mathcal{O}(y) \,\}$$

$$\overset{\circ}{\mathbb{P}}_y := \text{int}\, \mathbb{P}_y \qquad \text{on } Lip(X, \mathbb{R}).$$

Proposition

Let $F, u \in Lip(X, \mathbb{R})$ *with* $\mathcal{L}_F(u) = u$,
$\overline{F} := F + \alpha(F) + u - u \circ T$, *and* $M \in \mathbb{N}^+$.

Suppose that
$\forall \delta > 0 \quad \exists\, p(\delta)$*-periodic* δ*-pseudo-orbit* $(x_k)_{k=1}^{p}$
in $[\overline{F} = 0]$,
with at most M *jumps*,

such that for $\gamma_\delta := \min_{1 \leqslant i < j \leqslant p(\delta)} d(x_i, x_j)$,

$$\lim_{\delta \to 0} \frac{\gamma_\delta}{\delta} = +\infty.$$

Then

$$F \in closure\Big(\bigcup_{y \in Per(T)} \mathring{\mathbb{P}}_y \Big).$$

Proof:

1. Let $\mathcal{O}(y)$:= periodic orbit $\frac{\delta}{1-\lambda}$-shadowing $(x_k)_{k=1}^{p}$.

2. Let G be a small perturbation of $\overline{F} - \varepsilon d(x, \mathcal{O}(y))$

   [equivalently of $F - \varepsilon\, d(x, \mathcal{O}(y))$]

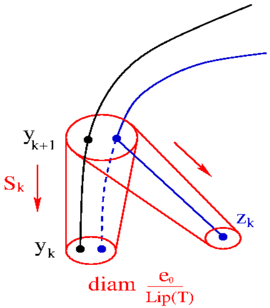
   It is enough to prove that $G \in \mathbb{P}_y$.

3. $\mathcal{O}(y)$ shadows the pseudo-orbit (x_k) on $[\overline{F} = 0]$. Then $\mathcal{O}(y)$ has $\overline{F} - \varepsilon\, d$ action nearby 0 (proportional to δ).
   Also $\mathcal{O}(y)$ has G-action neary 0.

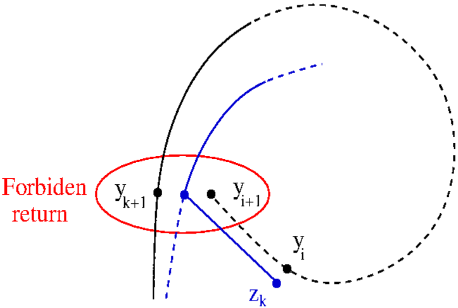
4. It is enough to prove that if $\mathcal{L}_G(v) = v$ any calibrating pre-orbit for v has α-limit = $\mathcal{O}(y)$.

5. Let $(z_k)_{k \leqslant 0}$ be a calibrating orbit for (v, G).
Since G is nearby $\overline{F} - \varepsilon\, d$, (z_k) eventually approaches $\mathcal{O}(y)$.
The time (z_k) shadows $\mathcal{O}(y)$ is OK.
It is enough to prove that (z_k) only spends finite time far from $\mathcal{O}(y)$.

6. Since (x_k) has no forbidden intermediate returns, then (y_k) doesn't have either.



y_{k+1}
S_k
y_k
z_k
diam $\frac{e_0}{Lip(T)}$



Forbiden return
y_{k+1}
y_{i+1}
y_i
z_k

7. The first time z_K that (z_k) does not continue shadowing $\mathcal{O}(y)$ it separates from $\mathcal{O}(y)$ because:
   - It is at least at distance $\frac{e_0}{Lip(T)}$ from the point y_K in $\mathcal{O}(y)$ that would continue the shadowing: because the inverse branch S_i is injective on $B(T(y_K), e_0)$.
   - It is far from other points in the orbit $\mathcal{O}(y)$ because otherwise its iterate $z_{K+1} = T(z_K)$ would be near two points y_i, y_j in $\mathcal{O}(y)$. Then $d(y_i, y_j)$ was small, i.e. a forbidden intermediate small return.
8. There is $a < 0$ such that for all such z_K, $\overline{G}(z_K) < a$. But $\overline{G} \leqslant 0$ and on a calibrating pre-orbit

$$\sum_{i=L}^{0} \overline{G}(z_i) = v(z_0) - v(z_L) \geqslant -2 \left\| \overline{G} \right\|_{sup}$$

is uniformly bounded.
Therefore the quantity of such z_K must be finite.

Zero entropy

Theorem (Morris)

Let X be a compact metric space and $T : X \to X$ an expanding map. There is a residual set $\mathcal{G} \subset Lip(X, \mathbb{R})$ such that if $F \in \mathcal{G}$ then there is a unique F-maximizing measure and it has zero metric entropy.

Proof.

1. Use estimates of Bressaud & Quas to obtain a close return in $\text{supp}(\mu)$ which is not too long in time.
   Construct a periodic orbit L_n with it. It has an action proportional to the distance of the return.
2. Use $f_n(x) := f(x) - \varepsilon\, d(x, L_n)$
   If a measure ν is nearby the closed orbit L_n, then it has small entropy.
   If it is far from L_n then it is not minimizing for the perturbed function f_n.
   Those f_n form a dense set.

□

Proof of the Main Theorem

We prove that $\mathcal{O} := \bigcup_{y \in Per(T)} \overset{\circ}{\mathbb{P}}_y$ is open and dense in $Lip(X, \mathbb{R})$. It is clearly open.

Suppose it is not dense. Then there is an open subset $\emptyset \neq \mathcal{U} \subset Lip(X, \mathbb{R})$ disjoint from $\mathcal{O}$.

By Morris Theorem we can choose $F \in \mathcal{U}$ such that there is a unique (ergodic) F-maximizing measure μ and

$$h_\mu(T) = 0.$$

μ maximizing $\Longrightarrow$ for any calibrating sub-action u, $\mathrm{supp}(u) \subset [\overline{F} = 0]$.

μ is ergodic $\Longrightarrow$ there is a generic point q for μ, i.e. for any continuous function $f : X \to \mathbb{R}$

$$\int f \, d\mu = \langle f \rangle(q) = \lim_N \frac{1}{N} \sum_{i=0}^{N-1} f(T^i(q)).$$

Many returns

By the perturbation proposition with $M = \#jumps = 2$, there is $Q > 0$ and $\delta_0 > 0$ such that if $(x_k)_{k\geqslant 0} \subset \mathcal{O}(q)$ is a p-periodic δ-pseudo-orbit with at most 2 jumps made with elements of the positive orbit of q (which is in $[\overline{F} = 0]$) and $0 < \delta < \delta_0$. Then

$$\gamma = \min_{1\leqslant i<j<p} d(x_i, x_j) < Q\,\delta.$$

i.e. every closed pseudo-orbit in $\mathcal{O}(q)$ with at most 2 jumps must have an intermediate return with proposition at most Q.

Main idea: This will contradict the zero entropy of μ.

Fix a point $w \in \text{supp}(\mu)$ for which Brin-Katok theorem holds:

$$h_\mu(T) = -\lim_{L \to +\infty} \frac{1}{L} \log \mu\big(V(w, L, \varepsilon)\big),$$

where $V(w, L, \varepsilon)$, $L \in \mathbb{N}$, $\varepsilon > 0$ is the dynamic ball

$$V(w, L, \varepsilon) := \{\, x \in X \mid d(T^k x, T^k w) < \varepsilon,\ \forall k = 0, \ldots, L \,\}.$$

Since T is an expanding map, for $\varepsilon < e_0$ small we have

$$V(w, L, \varepsilon) = S_1 \circ \cdots S_L\big(B(T^L w, \varepsilon)\big),$$

for an appropriate sequence of inverse branches S_i.
Thus

$$V(w, L, \varepsilon) \subseteq B(w, \lambda^L \varepsilon).$$

Main idea:

The measure of $V(w, L, \varepsilon)$ can be estimated by the proportion of the orbit of q which is spent on it.

If the measure of $V(w, L, \varepsilon)$ decreases exponentially with L it contradicts $h_\mu(T) = 0$.

We estimate the measure of the ball $B(w, \lambda^L \varepsilon) \supset V(w, L, \varepsilon)$.

By the perturbation proposition: Two consecutive visits of the orbit of q in the ball $B(w, \lambda^L \varepsilon)$ give rise to (exponentially) many intermediate returns (or approximations) which are outside the ball.
Thus the measure of the ball decreases exponentially with L.

Let N_0 be such that $2Q^{-N_0} < \delta_0$.
For $N > N_0$ let $0 \leqslant t_1^N < t_2^N < \cdots$ be all the Q^{-N} returns to w, i.e.

$$\{t_1^N, t_2^N, \dots\} = \{\, n \in \mathbb{N} \mid d(T^n q, w) \leqslant Q^{-N} \,\}.$$

Proposition

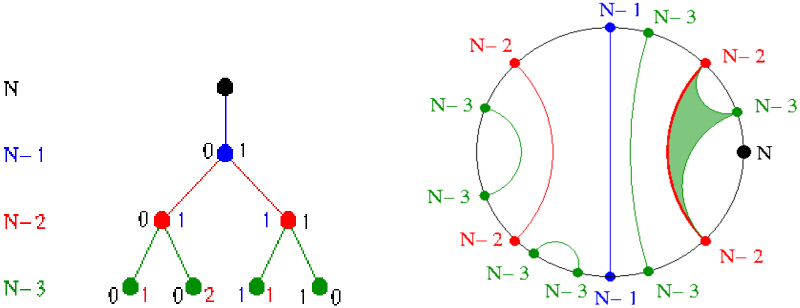
For any $\ell \geqslant 1$, $\quad t_{\ell+1}^N - t_\ell^N \geqslant \sqrt{2}^{N-N_0-1}$.

From this

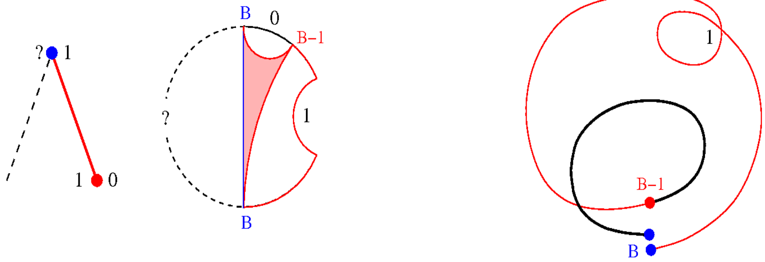
$$\mu\big(B(w, Q^{-N})\big) \leqslant \frac{1}{\sqrt{2}^{N-N_0-1}}.$$

And then $\mu\big(V(w,L,\varepsilon)\big) \leqslant \mu\big(B(w,\lambda^L\varepsilon)\big)$ decreases exponentially with L.

Proof of the proposition



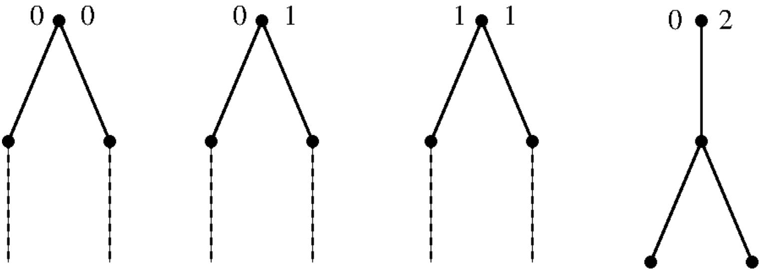
An example of a distribution of returns implied by the perturbation lemma and the tree representing it.
The shadow will be explained later.



If the return implied by the proposition contains one of the endpoints of the mother periodic pseudo-orbit we observe that it divides the mother pseudo-orbit in two child pseudo-orbits. We draw lines connecting the ends of these pseudo-orbits and shadow the internal part of the disk $\mathbb{D}$ which does not contain an interval in the circle . We treat the shadow as one node in the tree. Therefore

$$t^N_{\ell+1} - t^N_{\ell} \geqslant \#\{\text{ nodes in the tree }\}.$$

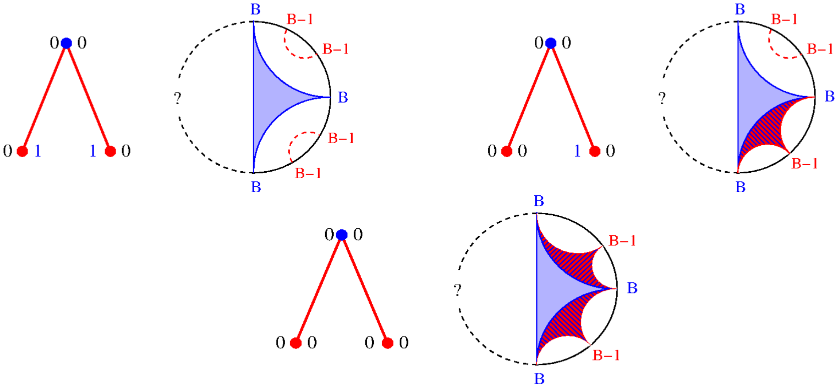
- Each pseudo-orbit with at most two jumps and $\delta = Q^{-B}$ gives rise to a new node in the tree and a new intermediate return. An also subsequent pseudo-orbits with $\delta = Q^{-B+1}$.
- A pseudo-orbit with 3 jumps is not continued in the tree. This only happens in a node in the tree with numbers $(0, 2)$. At the side with 0 there is a pseudo-orbit with one jump. So the tree may not brach but always continues.
- We enumerate all the possibilities and show that in two steps every node has at least two grandchildren nodes.



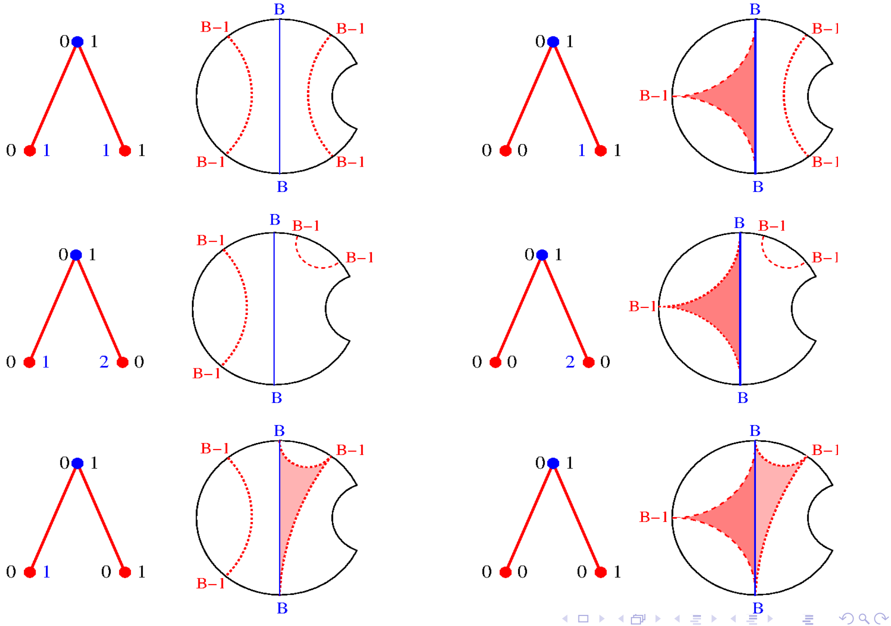
- The process continues as long as $Q^{-M} < \delta_0$, i.e. $N_0 < M < N$.
- The number of nodes duplicates every 2 steps in the tree.

$$\#\{ \text{ nodes } \} \geqslant 2^{\frac{N-N_0-1}{2}} = \sqrt{2}^{N-N_0-1}.$$

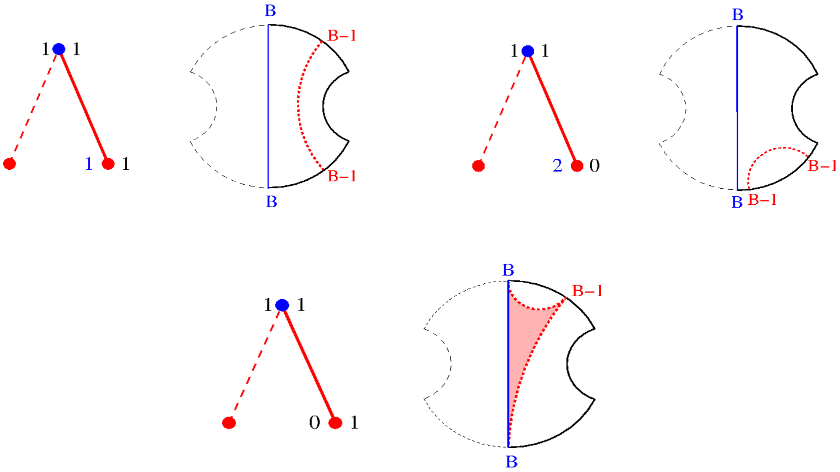
Nodes with numbers $(0, 0)$



Nodes with numbers (0,1)



Nodes with numbers (1,1)



Nodes with numbers (0,2)