Ergod. Th. & Dynam. Sys. (1999), **19**, 901–952
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Convex Hamiltonians without conjugate points

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(*Received* 24 *April* 1997 *and accepted in revised form* 10 *February* 1998)

Abstract. We construct the Green bundles for an energy level without conjugate points of a convex Hamiltonian. In this case we give a formula for the metric entropy of the Liouville measure and prove that the exponential map is a local diffeomorphism. We prove that the Hamiltonian flow is Anosov if and only if the Green bundles are transversal. Using the Clebsch transformation of the index form we prove that if the unique minimizing measure of a generic Lagrangian is supported on a periodic orbit, then it is a hyperbolic periodic orbit.

We also show some examples of differences with the behaviour of a geodesic flow without conjugate points, namely: (non-contact) flows and periodic orbits without invariant transversal bundles, segments without conjugate points but with crossing solutions and non-surjective exponential maps.

0. *Introduction*

Let M be a closed connected Riemannian manifold and T^*M its cotangent bundle. By a convex Hamiltonian on T^*M we shall mean a C^3 function $H : T^*M \to \mathbb{R}$ satisfying the following conditions.

(a) *Convexity:* For all $q \in M$, $p \in T_q^*M$, the Hessian matrix $(\partial^2 H/\partial p_i \partial p_j)(q, p)$ (calculated with respect to linear coordinates on T_q^*M) is positive definite.

(b) *Superlinearity:*
$$\lim_{|p|\to\infty} \frac{H(q, p)}{|p|} = +\infty.$$

The *Hamiltonian equation* for H is defined as

$$q' = H_p, \quad p' = -H_q, \tag{1}$$

where H_q and H_p are the partial derivatives with respect to q and p. Observe that the Hamiltonian function H is constant along the solutions of (1). Its level sets $\Sigma = H^{-1}\{e\}$ are called *energy levels* of H. Then the compactness of M and the superlinearity hypothesis imply that the energy levels are compact. Since the Hamiltonian vectorfield (1) is Lipschitz, the solutions of (1) are defined on all of $\mathbb{R}$. Denote by ψ_t the corresponding *Hamiltonian flow* on T^*M.

Our main interest in this paper are the *disconjugate* orbits of the Hamiltonian flow, i.e. orbits without conjugate points. Let $\pi : T^*M \to M$ be the canonical projection and define the *vertical subspace* on $\theta \in T^*M$ by $V(\theta) = \ker(d\pi)$. Two points $\theta_1, \theta_2 \in T^*M$ are said to be *conjugate* if $\theta_2 = \psi_\tau(\theta_1)$ for some $\tau \neq 0$ and $d\psi_\tau(V(\theta_1)) \cap V(\theta_2) \neq \{0\}$.

In this work we generalize some results of the theory of geodesics without conjugate points. A first step is to construct the *Green subspaces* for disconjugate orbits.

PROPOSITION A. *Suppose that the orbit of $\theta \in T^*M$ does not contain conjugate points and $H(\theta) = e$ is a regular value of H. Then there exist two φ-invariant Lagrangian subbundles $\mathbb{E}, \mathbb{F} \subset T(T^*M)$ along the orbit of θ given by*

$$\mathbb{E}(\theta) = \lim_{t\to+\infty} d\psi_{-t}(V(\psi_t(\theta))),$$
$$\mathbb{F}(\theta) = \lim_{t\to+\infty} d\psi_{t}(V(\psi_{-t}(\theta))).$$

Moreover, $\mathbb{E}(\theta) \cup \mathbb{F}(\theta) \subset T_\theta\Sigma$, $\mathbb{E}(\theta) \cap V(\theta) = \mathbb{F}(\theta) \cap V(\theta) = \{0\}$, $\langle X(\theta)\rangle \subseteq \mathbb{E}(\theta) \cap \mathbb{F}(\theta)$ *and* $\dim \mathbb{E}(\theta) = \dim \mathbb{F}(\theta) = \dim M$, *where* $X(\theta) = (H_p, -H_q)$ *is the Hamiltonian vectorfield and* $\Sigma = H^{-1}\{e\}$.

These bundles were constructed for disconjugate geodesics of Riemannian metrics by Green [**19**] and of Finsler metrics by Foulon [**18**]. In the Finsler case this is done by defining a new connection $\widetilde{\nabla}$ such that the solutions γ of the Euler–Lagrange equation satisfy $\widetilde{\nabla}_{\dot\gamma}(\dot\gamma) \equiv 0$. This approach leads to extensions of many important theorems in Riemannian geometry to Finsler geometry. See Chern [**6**] and the references therein for a survey and an introduction to this field.

If the energy level Σ does not have conjugate points, the Green bundles $\mathbb{E}$ and $\mathbb{F}$ can be defined on all $\theta \in \Sigma$. Nevertheless, in general they are neither continuous (cf. [**4**]) nor transversal in $T\Sigma$ (i.e. $\dim \mathbb{E} \cap \mathbb{F} = 1$).

An example of the relationship between the transversality of the Green subspaces and hyperbolicity appears in the following.

PROPOSITION B. *Let Γ be a periodic orbit of ψ_t without conjugate points. Then Γ is hyperbolic (on its energy level) if and only if $\mathbb{E}(\theta) \cap \mathbb{F}(\theta) = \langle X(\theta)\rangle$ for some $\theta \in \Gamma$, where $\langle X(\theta)\rangle$ is the one-dimensional subspace generated by the Hamiltonian vectorfield $X(\theta)$. In this case $\mathbb{E}$ and $\mathbb{F}$ are its stable and unstable subspaces.*

We can compare Proposition A with the following theorem by Paternain and Paternain [**32**].

THEOREM 0.1. (Paternain [**32**]) *Suppose that e is a regular value of H, and that $\psi_t|_\Sigma$ admits a continuous invariant Lagrangian subbundle E. Then:*

(a) $E(\theta) \cap V(\theta) = \{0\}\ \forall \theta \in \Sigma$;

(b) $\pi(\Sigma) = M$;

(c) Σ *contains no conjugate points.*

It is not difficult to see (e.g. [**24**]) that when the flow $\psi_t|_\Sigma$ is Anosov, the stable and unstable subbundles of $T\Sigma$ are Lagrangian. Paternain's theorem then implies that Anosov–Lagrangian flows do not have conjugate points. This result was proven for geodesic flows by Klingenberg [**23**]. An extension of the Klingenberg result appears in Mañé [**24**].

Define

$$B(\theta) = \left\{ \xi \in T_\theta\Sigma \;\middle|\; \sup_{t\in\mathbb{R}} |d\psi_t(\theta)\xi| < +\infty \right\}.$$

An important property (cf. Proposition 1.11) in the proof of Proposition B is that if the orbit of θ is disconjugate then $B(\theta) \subseteq \mathbb{E}(\theta) \cap \mathbb{F}(\theta)$. This in turn implies that $\langle X(\theta)\rangle \subseteq \mathbb{E}(\theta) \cap \mathbb{F}(\theta)$ and $\mathbb{E}(\theta) \cup \mathbb{F}(\theta) \subseteq T_\theta\Sigma$ (cf. Corollary 1.12). The global version of Proposition B is given by the following.

THEOREM C. *Let $\Sigma = H^{-1}\{e\}$ be a regular energy level without conjugate points. Then the following statements are equivalent:*

(a) *$\psi_t|_\Sigma$ is Anosov;*

(b) *for all $\theta \in \Sigma$, $\mathbb{E}(\theta)$ and $\mathbb{F}(\theta)$ are transversal in $T_\theta\Sigma$;*

(c) *for all $\theta \in \Sigma$, $\mathbb{E}(\theta) \cap \mathbb{F}(\theta) = \langle X(\theta)\rangle$;*

(d) *if $\theta \in \Sigma$, $v \in T_\theta\Sigma$, $v \notin \langle X(\theta)\rangle$ then $\sup_{t\in\mathbb{R}} |d\psi_t(\theta)\cdot v| = +\infty$.*

Where $\langle X(\theta)\rangle \subset T_\theta\Sigma$ is the one-dimensional subspace generated by the vectorfield $X(\theta)$ of ψ.

This theorem was proven by Eberlein (cf. [**13**]) for geodesic flows. For further applications of this theorem see [**9**] or [**11**]. An interesting ingredient in its proof is the characterization of recurrent quasi-hyperbolic $\mathbb{R}$-actions of vector bundles. Given a vector bundle $\pi : \boldsymbol{E} \to B$ consider a continuous $\mathbb{R}$-action $\Lambda : \mathbb{R} \to \mathrm{Isom}(\boldsymbol{E}, \boldsymbol{E})$, where $\Lambda_t : \boldsymbol{E} \to \boldsymbol{E}$ is a bundle map which is a linear isomorphism on each fiber and $\Lambda_{s+t} = \Lambda_s \circ \Lambda_t$. The action Λ induces a continuous flow $\phi_t : B \hookleftarrow$ such that $\phi_t \circ \pi = \pi \circ \Lambda_t$. We say that Λ is *quasi-hyperbolic* if

$$\sup_{t\in\mathbb{R}} |\Lambda_t(\xi)| = +\infty \quad \text{for all } \xi \in \boldsymbol{E},\ \xi \neq 0,$$

and we say that it is *hyperbolic* if there exists a continuous splitting $\boldsymbol{E} = E^s \oplus E^u$ and $C > 0$, $\lambda > 0$ such that:

(i) $|\Lambda_t(\xi)| \leq C\mathrm{e}^{-\lambda t}$ for all $t > 0$, $\xi \in E^s$;

(ii) $|\Lambda_{-t}(\xi)| \leq C\mathrm{e}^{-\lambda t}$ for all $t > 0$, $\xi \in E^u$.

It is easy to see that hyperbolic actions are quasi-hyperbolic. The converse is false by counterexamples found in Robinson [**34**] and Franks and Robinson [**14**]. But it is true with an additional recurrency hypothesis (that in our case is always true because the Lagrangian flow preserves the Liouville measure).

Define the *non-wandering* set $\Omega(\phi)$ of ϕ as the set of points $b \in B$ such that for every neighbourhood U of b there exists $T > 0$ such that $\phi_T(U) \cap U \neq 0$. The following theorem is a slight modification of a theorem in Freire [**15**], written there for quasi-Anosov flows (i.e when $\Lambda_t = d\phi_t$ and $\boldsymbol{E}$ is a continuous invariant subbundle of TB transversal to the vectorfield X, see Theorem 3.1).

THEOREM 0.2. *Let $\pi : \boldsymbol{E} \to B$ be a continuous vector bundle and $\Lambda : \mathbb{R} \to \mathrm{Isom}(\boldsymbol{E}, \boldsymbol{E})$ a continuous $\mathbb{R}$-action with induced flow ϕ_t: $\pi \circ \Lambda_t = \phi_t \circ \pi$. If Λ is quasi-hyperbolic and $\Omega(\phi|_B) = B$, then Λ is hyperbolic.*

For completeness of the exposition we present a proof of Theorem 0.2 in §3. The reason for using this version of Freire's result is that our Hamiltonian flows are not of contact type and *a priori* they may not preserve any continuous bundle transversal to the vectorfield $X(\theta)$. Indeed, in the Appendix we show an example of a convex Hamiltonian without conjugate points where this phenomenon occurs. The solution is to study the behaviour of an (orbitwise) reparametrization of the flow which preserves a continuous transverse bundle $\boldsymbol{E} \subset T(T^*M)$. This reparametrization cannot be made global but instead we use the quasi-hyperbolicity of the action $\Lambda_t(\theta) = P(\psi_t(\theta)) \circ d\psi_t(\theta) \circ P(\theta)$ where $P(\theta) : T_\theta T^*M \to \boldsymbol{E}(\theta)$ is the projection along the direction of the vectorfield $X(\theta)$.

Generic Lagrangians.
A C^4 function $L : TM \to \mathbb{R}$ is called a (convex autonomous) *Lagrangian* if it satisfies the following properties.

(a) *Convexity:* For all $x \in M$, $v \in T_xM$, the Hessian matrix $(\partial^2 L/\partial v_i \partial v_j)(x, v)$ (calculated with respect to linear coordinates on T_xM) is positive definite.

(b) *Superlinearity:*
$$\lim_{|v|\to\infty} \frac{L(x, v)}{|v|} = +\infty,$$
uniformly on $x \in \mathbb{R}$.

Observe that by the compactness of M, the *Euler–Lagrange* equation, which in local coordinates is
$$\frac{d}{dt}\frac{\partial L}{\partial v}(x, \dot{x}) = \frac{\partial L}{\partial x}(x, \dot{x}), \tag{2}$$
generates a complete flow $\varphi : TM \times \mathbb{R} \to TM$ defined by
$$\varphi_t(x_0, v_0) = (x(t), \dot{x}(t)),$$
where $x : \mathbb{R} \to M$ is the maximal solution of (2) with initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$.

A Lagrangian flow is conjugated to a Hamiltonian flow by the Legendre transform $\mathcal{F}_L : TM \to T^*M$ defined by
$$\mathcal{F}_L(x, v) = \left(x, \frac{\partial L}{\partial v}(x, v)\right).$$
The corresponding Hamiltonian is given by $H(q, p) = E(\mathcal{F}_L^{-1}(q, p))$, where $E : TM \to \mathbb{R}$ is the *energy* function, defined by
$$E(x, v) := \left.\frac{\partial L}{\partial v}\right|_{(x,v)} \cdot v - L(x, v). \tag{3}$$
Conversely, given a convex Hamiltonian H on T^*M, the corresponding convex Lagrangian is given by the Legendre transform of the Hamiltonian $\mathcal{F}_H : T^*M \to T^{**}M \approx TM$ which can be seen as
$$L(x, v) = \max_p\{p(v) - H(x, p)\} = (pH_p - H) \circ (\mathcal{F}_H)^{-1}.$$

Observe that item (b) in Theorem 0.1 implies that the energy e of an Anosov level for the Lagrangian flow satisfies

$$e > e_0 := -\max_{p \in M} L(p, 0),$$

because e_0 is the minimal energy level e for which $\pi(\Sigma) = M$. Define

$$c(L) := -\min \left\{ \int_{TM} L \, d\mu \;\middle|\; \mu \text{ is a } \varphi - \text{invariant probability} \right\}.$$

It is proven in [**28**] (see also [**8**]) that $c(L) \geq e_0$.

We say that a φ-invariant probability μ is *minimizing* if

$$c(L) = -\int L \, d\mu,$$

and we say that it is *uniquely ergodic* if it is the only invariant probability measure supported inside $\operatorname{supp}(\mu)$.

Action minimizing curves do not contain conjugate points. This fact is known since Morse's earlier works (cf. [**30**]). We present a simple proof of this fact in §4 and we use Proposition B to prove the following theorem, which was stated as Theorem III in Mañé's unfinished work [**28**].

THEOREM D. *There exists a generic set $\mathcal{O} \subseteq C^\infty(M, \mathbb{R})$ such that for all $\phi \in \mathcal{O}$ the Lagrangian $L + \phi$ has a unique minimizing measure and this measure is uniquely ergodic. When this measure is supported on a periodic orbit or a fixed point, this orbit (point) Γ is hyperbolic and its stable and unstable manifolds intersect transversally $W^s(\Gamma) \pitchfork W^u(\Gamma)$.*

In Mañé [**28**], it is conjectured that there exists a generic set $\mathcal{O}$ such that this unique minimizing measure is supported on a periodic orbit or an equilibrium point.

The first statement of Theorem D was proved by Mañé in [**26**]. Here we prove the generic hyperbolicity of the minimizing periodic orbit and the transversality property $W^s(\Gamma) \pitchfork W^u(\Gamma)$. The hyperbolicity of Γ was announced in Mañé [**28**] and the transversality $W^s(\Gamma) \pitchfork W^u(\Gamma)$ was conjectured in Mañé [**27**]. We use the Clebsch transformation [**7**] to derive formulas for the index form on an orbit without conjugate points. Then we use similar arguments to the Rauch comparison theorem to obtain the transversality of the Green bundles of the periodic orbit. Finally, Proposition B implies the hyperbolicity of the periodic orbit.

The metric entropy.

The machinery developed for the proof of Theorem C can be used to obtain formulas for the metric entropy of a Hamiltonian flow without conjugate points. Oseledec's theory [**31**] gives a splitting

$$T_\theta \Sigma = E^s(\theta) \oplus E^c(\theta) \oplus E^u(\theta)$$

for a set of points $\theta \in \Lambda \subseteq \Sigma$ of total measure, such that

$$\lim_{t\to\pm\infty} \frac{1}{t} \log |d\psi_t(\theta)\xi| < 0 \quad \text{for } \xi \in E^s(\theta),$$
$$\lim_{t\to\pm\infty} \frac{1}{t} \log |d\psi_t(\theta)\xi| = 0 \quad \text{for } \xi \in E^c(\theta),$$
$$\lim_{t\to\pm\infty} \frac{1}{t} \log |d\psi_t(\theta)\xi| > 0 \quad \text{for } \xi \in E^u(\theta).$$

Following Freire and Mañé [**16**], we see in Proposition 1.14 that

$$E^u(\theta) \subseteq \mathbb{F}(\theta) \subseteq E^u(\theta) \oplus E^c(\theta),$$
$$E^s(\theta) \subseteq \mathbb{E}(\theta) \subseteq E^s(\theta) \oplus E^c(\theta).$$

Given a Riemannian metric on M, define the *horizontal subspace* as the kernel of the connection map. Then there exist symmetric linear isomorphisms $\mathbb{U} : H(\theta) \to V(\theta)$ such that $\mathbb{F}(\theta) = \text{graph}(\mathbb{U}(\theta))$ for all $\theta \in \Sigma$. They satisfy the Ricatti equation (7)

$$\dot{\mathbb{U}} + \mathbb{U}H_{pp}\mathbb{U} + \mathbb{U}H_{pq} + H_{qp}\mathbb{U} + H_{qq} = 0,$$

where H_{pp}, H_{qp}, H_{pq} and H_{qq} are linear maps which coincide with the second derivatives of H in local coordinates, and $\dot{\mathbb{U}}$ is the covariant derivative of $\mathbb{U}$ defined as

$$\dot{\mathbb{U}}(\theta)v = \lim_{h\to 0} \frac{1}{h}[\tau_h \mathbb{U}(\psi_h(\theta))v - \mathbb{U}(\theta)v] \in V(\theta) \approx T^*_\theta M,$$

where $\tau_h : T^*_{\pi\psi_h(\theta)}M \to T^*_{\pi(\theta)}M$ is the parallel transport.

For $\theta \in \Lambda$ define the unstable Jacobian by

$$\chi(\theta) := \lim_{T\to+\infty} \frac{1}{T} \log |\det d\psi_T|_{E^u(\theta)}|.$$

We prove in Proposition 1.14 that

$$\chi(\theta) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T \text{tr}[H_{pq} + H_{pp}\mathbb{U}](\psi_t(\theta))\, dt,$$

where tr[ ] is the trace. From Ruelle's inequality [**35**], for any invariant measure μ for $\psi_t|_\Sigma$, we obtain that the metric entropy satisfies

$$h_\mu(\psi|_\Sigma) \le \int_\Sigma \chi\, d\mu \le \int_\Sigma \text{tr}[H_{pq} + H_{pp}\mathbb{U}]\, d\mu.$$

Pesin's formula states that for a smooth invariant measure μ of a $C^{1+\alpha}$ diffeomorphism $f : N \hookleftarrow$, we have that

$$h_\mu(f) = \int_N \chi\, d\mu.$$

In our case we obtain the following.

THEOREM E. *Let m be the Liouville measure on an energy level Σ without conjugate points. Then*

$$h_m(\psi|_\Sigma) = \int_\Sigma \text{tr}[H_{pq} + H_{pp}\mathbb{U}]\, dm.$$

This theorem was proved for geodesic flows by Freire and Mañé [**16**], Finsler metrics by Foulon [**18**] and mechanical Lagrangians by Innami [**22**].

The exponential map.

The quasi-hyperbolicity in the case $\mathbb{E}(\theta) \pitchfork \mathbb{F}(\theta)$ is obtained from Proposition 1.10 which states the horizontal growth of vertical vectors, i.e.

$$\lim_{t\to+\infty} \|d\pi \circ d\psi_t|_{V(\theta)}\| = +\infty.$$

The bounds obtained in Proposition 1.10 are not uniform, from an example in Ballmann *et al* [**4**]. In §6 we give a uniform lower bound for $\|d\pi \circ d\psi_t|_{V(\theta)}\|$, $\theta \in \Sigma$, $|t| \geq 1$. This bound was given by Freire and Mañé [**16**] in the case of geodesic flows.

Define the exponential map of an energy level Σ as $\exp_q : T_q^*M \to M$, $\exp_q(t\theta) = \pi(\psi_t(\theta))$, where $\theta \in \Sigma \cap T_q^*M$ and q is a regular value of $\pi : \Sigma \to M$. This map is not differentiable at $\mathbf{0} \in T_q^*M$, but its Lagrangian version is differentiable with $d\exp_q(\mathbf{0}) = I$. An important question is whether the lift of this map to the universal cover, $\exp_q : T_q^*M \to \tilde{M}$, is a diffeomorphism in an energy level without conjugate points and, in particular, if the universal cover $\tilde{M}$ is homeomorphic to $\mathbb{R}^n$. We show in Corollaries 1.18 and 1.13 that if the energy level Σ has no conjugate points then $\exp_q$ is a local diffeomorphism and $\pi(\Sigma) = M$.

Let $\Theta = p\,dq$ be the Liouville's 1-form (4) in T^*M. Given a regular energy level Σ and $q \in \pi(\Sigma)$, let $\Sigma_q := (\pi|_\Sigma)^{-1}\{q\}$. Since $\tilde{M}$ is simply connected, in order to obtain a global diffeomorphism $\exp_q : T_q^*M \to \tilde{M}$ it is sufficient to show that $\exp_q$ is a covering map. For this it is enough that $\Theta(X) > 0$. Indeed, in §6 we prove the following.

THEOREM F. *Let $H : T^*M \to \mathbb{R}$ be convex and $\Sigma = H^{-1}\{e\}$ a regular energy level. Suppose that $q \in \pi(\Sigma)$ satisfies the following.*

(a) *The positive orbit of all $\theta \in \Sigma_q$ has no conjugate points.*

(b) $\inf\{|\Theta(X(\vartheta))| \mid \vartheta \in \psi_{\mathbb{R}^+}(\Sigma_q)\} > 0$.

Then the (Lagrangian) exponential map $\exp_q : T_q^*\tilde{M} \to \tilde{M}$ *associated to (ψ_t, Σ) is a diffeomorphism.*

We give an example (cf. Example A.4) in the appendix in which $\exp_q$ is not surjective and where all the orbits in Σ of ψ_t starting at $\pi^{-1}\{q\}$ have no conjugate points. This shows that condition (a) is not sufficient in Theorem F. By Theorem X in Mañé [**28**] (see also [**8**]), this cannot happen for high energy levels. Example A.1 in the appendix also shows that the norm of the derivative $\|d_\theta(\exp_q)\|$ may not be bounded below on an energy level without conjugate points. We do not know if the exponential map is a diffeomorphism when the whole energy level has no conjugate points.

The condition $\Theta(X) = pH_p > 0$ is written $\Theta(X) = vL_v > 0$ in Lagrangian form. It is satisfied by Finsler Lagrangians, $L(x, v) = \frac{1}{2}\|v\|_x^2$, mechanical Lagrangians $L(x, v) = \frac{1}{2}\langle v, v\rangle_x - \phi(x)$ (with $\Theta(X) = 1$, $H(q, p) = \frac{1}{2}\langle p, p\rangle_q + \phi(q)$) and all convex Hamiltonians on sufficiently high energy levels. It fails for magnetic Lagrangians $L(x, v) = \frac{1}{2}\langle v, v\rangle_x - \phi(x) + \eta_x(v)$ (with $d\eta \neq 0$) on lower energy levels. If one adds a closed 1-form to a Lagrangian, the Euler–Lagrange equation, and hence the Lagrangian flow, do not change. This may help to make a given Lagrangian flow satisfy condition (b). See Remark 6.5 and [**9**] for a characterization of the energy levels for which this condition holds for a convex Lagrangian. See [**10**] for other proofs of Theorem F and its converse.

The obstruction to obtain a bound on $\|d_{\psi_t(\theta)}(\exp_q)\|$ is the following angle $\sphericalangle(X(\psi_t(\theta)), d\psi_t(V(\theta)\cap T_\theta\Sigma))$. If the orbit of θ is disconjugate this angle is always non-zero by Proposition 1.16. In Example A.3 of the Appendix, this angle reaches zero *before* the first conjugate point appears. Moreover, there are crossings of solutions on Σ starting at $\pi(\theta)$ before the first conjugate point.

In §1 we prove Proposition A, Theorem E and we show the proof of item (c) in Theorem 0.1. In §2 we study the transversal behaviour of the Hamiltonian flow and prove Proposition B. In §3 we show the proof of Theorem 0.2 and we prove Theorem C. In §4 we give a characterization of the index form and in §5 we prove Theorem D. In §6 we give a uniform lower bound for $d\pi\circ d\psi_t|_{V(\theta)}$ and prove Theorem F. Finally, we add some examples in the Appendix.

1. *The Green bundles*

For the proof of the basic results in this section see Abraham and Marsden [**1**]. There is a canonical symplectic structure on T^*M given by a closed non-degenerate 2-form $\omega = d\Theta$, where Θ is the Liouville's 1-form, defined by

$$\Theta_\theta(\xi) = \theta(d\pi(\theta)\xi), \tag{4}$$

where $\pi : T^*M \to M$ is the canonical projection. The Hamiltonian vectorfield X of a function $H : T^*M \to \mathbb{R}$ is defined by $\omega(X,\cdot) = -dH$. The Hamiltonian flow ψ_t preserves the function H and the 2-form ω. The level sets of H are called *energy levels*. Given a local chart $q : U \subseteq M \to \mathbb{R}^n$ it induces a *natural chart* $(q,p) : T^*U \to \mathbb{R}^n\times\mathbb{R}^n$ of T^*M. In natural charts the Liouville's 1-form is written as $\Theta = p\,dq$. Hence, in natural charts, $\omega = dp\wedge dq$ and the Hamiltonian equations are given by

$$\dot q = H_p, \quad \dot p = -H_q. \tag{5}$$

In general, a set of local coordinates (q,p) of T^*M is called *symplectic* if the canonical 2-form is written as $\omega = dp\wedge dq$. A subspace $E\subset T(T^*M)$ is said to be *Lagrangian* if $\dim E = n = \dim M$ and $\omega(x,y) = 0$ for all $x, y\in E$.

Fix a Riemannian metric on M and the corresponding induced metric on T^*M. Then $T_\theta T^*M$ splits as a direct sum of two Lagrangian subspaces: the vertical subspace $V(\theta) = \ker(d\pi(\theta))$ and the horizontal subspace $H(\theta)$ given by the kernel of the connection map. Using the isomorphism $K : T_\theta T^*M \to T_{\pi(\theta)}M\times T^*_{\pi(\theta)}M$, $\xi\mapsto (d\pi(\theta)\xi, \nabla_\theta(\pi\xi))$, we can identify $H(\theta)\approx T_{\pi(\theta)}M\times\{0\}$ and $V(\theta)\approx\{0\}\times T^*_{\pi(\theta)}M\approx T_{\pi(\theta)}M$. If we choose local coordinates along $t\mapsto\pi\psi_t(\theta)$ such that $t\mapsto(\partial/\partial q_i)(\pi\psi_t(\theta))$ are parallel vectorfields, then this identification becomes $\xi\leftrightarrow(dq(\xi), dp(\xi))$. Let $E\subset T_\theta T^*M$ be an n-dimensional subspace such that $E\cap V(\theta) = \{0\}$. Then E is a graph of some linear map $S : H(\theta)\to V(\theta)$. It can be checked that E is Lagrangian if and only if in symplectic coordinates S is symmetric.

Proof of Proposition A. Take $\theta\in T^*M$ and $\xi = (h,v)\in T_\theta T^*M = H(\theta)\oplus V(\theta)\approx T_{\pi(\theta)}M\oplus T_{\pi(\theta)}M$. Consider a variation

$$\alpha_s(t) = (q_s(t), p_s(t)),$$

such that for each $s \in]-\varepsilon,\varepsilon[$, α_s is a solution of the Hamiltonian H such that $\alpha_0(0)=\theta$ and $(d/ds)\alpha_s(0)|_{s=0}=\xi$. Writing $d\psi_t(\xi)=(h(t),v(t))$, we obtain the Hamiltonian Jacobi equations

$$\dot{h}=H_{pq}h+H_{pp}v, \quad \dot{v}=-H_{qq}h-H_{qp}v, \tag{6}$$

where the covariant derivatives are evaluated along $\pi(\alpha_o(t))$, and H_{qq}, H_{qp}, H_{pp} and H_{pq} are linear operators on $T_{\pi(\theta)}M$, which in local coordinates coincide with the matrices of partial derivatives $(\partial^2 H/\partial q_i\partial q_j)$, $(\partial^2 H/\partial q_i\partial p_j)$, $(\partial^2 H/\partial p_i\partial p_j)$ and $(\partial^2 H/\partial p_i\partial q_j)$. Moreover, since the Hamiltonian H is convex, then H_{pp} is positive definite.

We now derive the Ricatti equation. Let E be a Lagrangian subspace of $T_\theta T^*M$. Suppose that for t in some interval $]-\varepsilon,\varepsilon[$ we have that $d\psi_t(E)\cap V(\psi_t(\theta))=\{0\}$. Then we can write $d\psi_t(E)=\text{graph } S(t)$, where $S(t): H(\psi_t\theta)\to V(\psi_t\theta)$ is a symmetric map. That is, if $\xi\in E$ then

$$d\psi_t(\xi)=(h(t),S(t)h(t)).$$

Using equation (6) we have that

$$\dot{S}h+S(H_{pq}h+H_{pp}Sh)=-H_{qq}h-H_{qp}Sh.$$

Since this holds for all $h\in H(\psi_t(\theta))$ we obtain the Ricatti equation:

$$\dot{S}+SH_{pp}S+SH_{pq}+H_{qp}S+H_{qq}=0. \tag{7}$$

Take the symmetric map $S(t)$ for which $d\psi_t(V(\theta))=\text{graph}(S(t))$, $t>0$. Let $Y(t): T_{\pi(\theta)}M\approx V(\theta)\to H(\psi_t\theta)\approx T_{\pi\psi_t\theta}M$, $t\in\mathbb{R}$, be the family of isomorphisms $Y(t)w=d\pi\cdot d\psi_t(\theta)(0,w)$, where $(0,w)\in H(\theta)\oplus V(\theta)$. Then for $t>0$, $Y(t)$ satisfies

$$\dot{Y}=(H_{pq}+H_{pp}S)Y, \quad Y(0)=0,\ \dot{Y}(0)=H_{pp}, \tag{8}$$

where $\dot{Y}(0)$ is calculated from (8) using that

$$\lim_{t\to 0}S(t)Y(t)=I.$$

Given $u\in T_{\pi(\theta)}M$, define

$$h_1(t)=Y(t)u, \quad v_1(t)=S(t)Y(t)u,\ t>0.$$

Then (h_1,v_1) is a solution of equation (6) with $h_1(0)=0$, $v_1(0)=u$. Let $\xi\in T_\theta T^*M$ and let $(h(t),v(t))=d\psi_t(\xi)\in H\oplus V$. Since $d\psi_t$ preserves the symplectic form, we have that

$$\langle h(t),v_1(t)\rangle-\langle v(t),h_1(t)\rangle=\langle h(0),u\rangle.$$

Hence,

$$\langle Y^*(t)S(t)h(t),u\rangle-\langle Y^*(t)v(t),u\rangle=\langle h(0),u\rangle$$

for all $u\in T_{\pi(\theta)}M$. Therefore,

$$v(t)=S(t)h(t)-(Y^*(t))^{-1}h(0). \tag{9}$$

Since

$$\dot{h} = H_{pq}h + H_{pp}v$$

we get

$$\dot{h} = (H_{pq} + H_{pp}S)h - H_{pp}(Y^*)^{-1}h(0). \tag{10}$$

Differentiating $Y^{-1}h$ and using equation (8), we obtain

$$h(t) = Y(t)Y(c)^{-1}h(c) + Y(t)\int_t^c Y^{-1}(s)H_{pp}(Y^*(s))^{-1}h(0)\,ds, \quad \text{for } 0 < t < c. \tag{11}$$

Now consider the map $d\psi_t$ restricted to the subspace $d\psi_{-c}(V(\psi_c\theta))$, $c > 0$. Its projection $Z_c(t)$ to the horizontal subspace is given by the solutions of (6) with $h(c) = 0$. Then $Z_c(t)$ is the family of isomorphisms given by

$$Z_c(t) = Y(t)\int_t^c Y^{-1}(s)H_{pp}(Y^*(s))^{-1}\,ds, \quad 0 < t < c, \tag{12}$$

because $Z_c(t)$ satisfies (6) for $0 < t < c$, $Z_c(c) = 0$ and

$$Z_c(0) = \lim_{t\to 0^+} Z_c(t) = I.$$

Observe that from its definition, the linear map $Z_c(t)$ is defined for all $t \in \mathbb{R}$ and the no-conjugate points condition implies that $Z_c(t)$ is an isomorphism for all $t \neq c$. Moreover, since $Z_c(t)$ is a matrix solution of the Jacobi equation (6), then $\dot{Z}_c(0)$ exists, and taking the limit in (12), we have that the linear isomorphism

$$\dot{Z}_d(0) - \dot{Z}_c(0) = H_{pp}(\theta)\int_c^d Y(s)^{-1}H_{pp}(Y^*(s))^{-1}\,ds \tag{13}$$

is symmetric and the second factor on the right is positive definite for all $0 < c < d$. In particular

$$Z_d(t) - Z_c(t) = Y(t)H_{pp}(\theta)^{-1}[\dot{Z}_d(0) - \dot{Z}_c(0)], \quad 0 < c < d. \tag{14}$$

For $\varepsilon > 0$ define

$$Z_{-\varepsilon}(t) := Y(t)N_c + Z_c(t), \quad t \in \mathbb{R}, \tag{15}$$

where $N_c := -Y(-\varepsilon)^{-1}Z_c(-\varepsilon)$. We have that $Z_{-\varepsilon}(t)$ is the solution of the Jacobi equation (6) satisfying $Z_{-\varepsilon}(-\varepsilon) = 0$, $Z_{-\varepsilon}(0) = I$ and

$$\dot{Z}_{-\varepsilon}(0) - \dot{Z}_c(0) = H_{pp}(\theta)N_c. \tag{16}$$

CLAIM 1.1. *$Y^{-1}(t)Z_c(t)$ is symmetric for all $t \neq 0$. In particular N_c is symmetric for all $c > 0$.*

Proof. Using (10), we have that

$$\dot{Z}_c = AZ_c + H_{pp}(Y^*)^{-1},$$
$$\dot{Y} = AY + 0, \quad t \neq 0,$$

where $A = H_{pq} + H_{pp}S$. Then

$$\dot{Z}_{-\varepsilon} = AZ_{-\varepsilon} + H_{pp}(Y^*)^{-1}.$$

Using (9) we have that the functions

$$h_2(t) = Z_{-\varepsilon}(t)u, \quad v_2(t) = S(t)h_2(t) - (Y^*(t))^{-1}u, \tag{17}$$

are solutions of (6) with $h_2(-\varepsilon) = 0$, $h_2(0) = u$. Similarly, the functions

$$h_3(t) = Z_c(t)w, \quad v_3(t) = S(t)h_3(t) - (Y^*(t))^{-1}w, \tag{18}$$

are solutions of (6) with $h_3(0) = w$, $h_3(c) = 0$.

Since the flow preserves the symplectic form, we have that the following expressions do not depend on t for all $u, w \in T_{\pi(\theta)}M$:

$$\langle h_2(t), v_3(t)\rangle - \langle v_2(t), h_3(t)\rangle$$
$$\langle Z_{-\varepsilon}u, (SZ_c - (Y^*)^{-1})w\rangle - \langle (SZ_{-\varepsilon} - (Y^*)^{-1})u, Z_c w\rangle$$
$$Z_c^* S Z_{-\varepsilon} - Y^{-1}Z_{-\varepsilon} - Z_c^* S Z_{-\varepsilon} + Z_c^*(Y^*)^{-1} = Z_c^*(Y^*)^{-1} - Y^{-1}Z_{-\varepsilon}.$$

Using formula (15), this is equal to

$$(Y^{-1}Z_c)^* - Y^{-1}Z_c - N_c = -N_c^*, \tag{19}$$

where the right-hand side corresponds to its value when $t = -\varepsilon$. Since $Z_c(c) = 0$, evaluating the equation at $t = c$, we get that N_c is symmetric for all $c > 0$. Using (19) again, we obtain that $Y^{-1}Z_c$ is symmetric for all $t \neq 0$. □

CLAIM 1.2. *N_c is positive definite for all $c > 0$.*

Proof. It is enough to show it for N_c^{-1}. We have that

$$\frac{d}{dt} - Z_c(t)^{-1}Y(t)\Big|_{t=0} = Z_c^{-1}\dot{Z}_c Z_c^{-1} Y - Z_c^{-1}\dot{Y}|_{t=0} = -H_{pp}.$$

Therefore, $-Z_c(t)^{-1}Y(t)$ is positive definite for small $t < 0$. By Claim 1.1, this matrix is symmetric for all $t < 0$. Since for $t < 0$ its determinant never vanishes, it is positive definite for all $t < 0$, in particular for $t = -\varepsilon$, N_c^{-1} is positive definite. □

Define a partial order on the symmetric isomorphisms of $T_{\pi(\theta)}M$ by writing $A \rhd B$ if $H_{pp}(\theta)^{-1}(A - B)$ is positive definite. From (13) we have that the family $\mathcal{F} = \{\dot{Z}_c(0) - \dot{Z}_1(0) \mid c > 1\}$ is monotone increasing. Moreover, by (16) and Claim 1.2 we have that

$$\begin{aligned} \dot{Z}_c(0) - \dot{Z}_1(0) &\lhd H_{pp}N_c + (\dot{Z}_c(0) - \dot{Z}_1(0)) \\ &\lhd \dot{Z}_{-\varepsilon}(0) - \dot{Z}_1(0), \end{aligned}$$

so that the family $\mathcal{F}$ is bounded above. It can be seen that this implies the existence of a least upper bound which is the limit of the family. We obtain the first part of the following.

CLAIM 1.3.
(a) $\lim_{c\to+\infty} \dot Z_c(0) - \dot Z_1(0) = Q$ *exists and is symmetric.*
(b) $\lim_{c\to+\infty} Z_c(t) = D(t)$ *exists (uniformly for bounded t intervals).*
Moreover,

$$\mathbf{h}(t) = D(t),$$
$$\mathbf{v}(t) = S(t)D(t) + (Y^*(t))^{-1}, \quad t \neq 0$$

is a matrix solution of (6) *such that* $D(0) = I$, $\dot D(0) = Q + \dot Z_1(0)$ *and* $\det D(t) \neq 0$ *for all* $t \in \mathbb{R}$.

Part (b) is a consequence of (14) and the continuous dependence of the solutions of equation (6) on the initial data.

We have that the subspace

$$\mathbb{E}_\theta(0) := \{(u, \mathbf{v}(0)u) \mid u \in T_{\pi(\theta)}M\} \subset H(\theta) \oplus V(\theta)$$

is the limit of the subspaces

$$E_c(\theta) := d\psi_{-c}(V(\psi_c(\theta))) = \{(h_3(\theta), v_3(\theta)) \mid u \in T_{\pi(\theta)}M\},$$

where $h_3(t)$ and $v_3(t)$ are given by (18). Moreover,

$$\mathbb{E}_\theta(t) := d\psi_t(\mathbb{E}_\theta(0)) = \{(\mathbf{h}(t)u, \mathbf{v}(t)u) \mid u \in T_{\pi(\theta)}M\}$$

satisfies $\mathbb{E}_\theta(t) \cap V(\psi_t\theta) = \{0\}$ and

$$\mathbb{E}_\theta(t) = \lim_{c\to+\infty} d\psi_{t-c}(V(\psi_c(\theta))) = \lim_{d\to+\infty} d\psi_{-d}(V(\psi_d(\psi_t\theta))) = \mathbb{E}_{\psi_t\theta}(0).$$

Since the vertical subspace is Lagrangian, then $E_c(\theta)$ is Lagrangian. By the continuity of the symplectic form, the subspaces $\mathbb{E}_\theta(t)$ are Lagrangian.

To obtain the 'unstable' Green bundle $\mathbb{F}(\theta)$, observe that the flow of the Hamiltonian $\overline{H}(\theta) = -H(\theta)$ is the flow ψ_t of H with the time reversed. In this case $\overline{H}_{pp}$ is negative definite, but similar arguments apply to obtain the subbundle $\mathbb{F}$. Applying the lemma to $\overline{H}$, we obtain the subbundle $\mathbb{F}$. Moreover, in this case the family $\mathcal{G} = \{\dot Z_{-c}(0) - \dot Z_{-1}(0) \mid c > 1\}$ is monotone decreasing and bounded below by $\dot Z_\varepsilon(0) - \dot Z_{-1}(0)$. This finishes the proof of the first part of Proposition A, we defer the proof of the second statement to Corollary 1.12. □

Let

$$\begin{aligned}\mathbb{S}(t) :&= \boldsymbol{v}(t)\boldsymbol{h}(t)^{-1}\\ &= S(t) + Y^*(t)^{-1}D(t)^{-1}.\end{aligned} \tag{20}$$

Then $\mathbb{S}(t)$ is a symmetric solution of the Ricatti equation (7), which is defined on all $t \in \mathbb{R}$. The symmetry of $\mathbb{S}(t)$ can be checked either because $\mathbb{E}(t) = \operatorname{graph}(\mathbb{S}(t))$ is a Lagrangian subspace or because

$$Y(t)^{-1}D(t) = \lim_{c\to\infty} Y(t)^{-1}Z_c(t)$$

is symmetric and this implies that $(Y^*)^{-1}D^{-1}$ is symmetric.

Let $\mathbb{S}(\theta)$, $\mathbb{U}(\theta)$ be the symmetric solutions of the Ricatti equation (7) corresponding to the Green bundles $\mathbb{E}(\theta) = \text{graph}(\mathbb{S}(\theta))$ and $\mathbb{F}(\theta) = \text{graph}(\mathbb{U}(\theta))$. Let $K_c(\theta) : H(\theta) \to V(\theta)$ be the symmetric linear map such that $\text{graph}(K_c(\theta)) = d\psi_{-c}(V(\psi_c(\theta)))$. Define a partial order on the symmetric isomorphisms of $T_{\pi(\theta)}M$ by writing $A \succ B$ if $A - B$ is positive definite.

PROPOSITION 1.4. *For all* $\varepsilon > 0$*:*

(a) *if* $d > c > 0$ *then* $K_{-\varepsilon} \succ K_d \succ K_c$*;*

(b) *if* $d < c < 0$ *then* $K_\varepsilon \prec K_d \prec K_c$*;*

(c) $\lim_{d\to+\infty} K_d = \mathbb{S}$, $\lim_{d\to-\infty} K_d = \mathbb{U}$*;*

(d) $\mathbb{S} \preccurlyeq \mathbb{U}$.

Proof. Let $(Z_c(t), V_c(t))$ be the matrix solution of the Jacobi equation (6) with $Z_c(0) = I$, $Z_c(c) = 0$, $c > 0$. From (9) we have that

$$V_c(t) = S(t)Z_c(t) - (Y^*(t))^{-1}. \tag{21}$$

Hence, using (14) and (13), we have that

$$V_d(t) - V_c(t) = S(t)[Z_d(t) - Z_c(t)] \tag{22}$$

$$= S(t)Y(t) \int_c^d Y^{-1} H_{pp} (Y^*)^{-1}\, ds. \tag{23}$$

When $t \to 0$ we have that $S(t)Y(t) \to I$. Therefore,

$$V_d(0) - V_c(0) = \int_c^d (Y^{-1} H_{pp} (Y^*)^{-1})\, ds. \tag{24}$$

Let $K_c(t) = V_c(t)Z_c(t)^{-1}$ be the corresponding solution of the Ricatti equation (7). Since $Z_c(0) = I$, we have that $K_c(0) = V_c(0)$. By (24), $K_d(0) - K_c(0)$ is positive definite if $d > c > 0$, hence, the sequence $K_d(0)$, $d > 0$ is strictly increasing.

From (17) we have that equation (22) also holds for $c = -\varepsilon$. Using (15), we have that

$$\begin{aligned} V_d(t) - V_{-\varepsilon}(t) &= S(t)(Z_d(t) - Z_{-\varepsilon}(t)) \\ &= S(t)(-Y(t)N_d). \end{aligned}$$

Hence, $V_d(0) - V_{-\varepsilon}(0) = -N_d$ is negative definite. Since $Z_{-\varepsilon}(0) = I$, then $K_{-\varepsilon}(0) = V_{-\varepsilon}(0)$. Therefore, $K_{-\varepsilon} \succ K_d$ for all $d > 0$.

This completes the proof of item (a). Item (c) has already been proven on Claim 1.3 above. Item (b) is proven similarly using the concave Hamiltonian $\overline{H} = -H$ and item (d) is a corollary of items (a), (b) and (c). □

Remark 1.5. If a bounded open segment $\{\psi_t(\theta) \mid a^- < t < a^+\}$ has no conjugate points, the arguments above apply to obtain limit solutions $\lim_{c\to a^\pm} Z_c(t) = D^\pm(t)$, $D^\pm(0) = I$, $\det D(t) \neq 0$, Lagrangian subspaces $E^\pm(\theta) = \lim_{t\to a^\pm} d\psi_{-t}(V(\psi_t\theta)) = d\psi_{a^\pm}(V(\psi_{a^\pm}(\theta)))$ and the monotonicity properties of Proposition 1.4. Nevertheless, in general $E^\pm(\theta) \not\subset T_\theta\Sigma$ (see Remark 1.17).

Remark 1.6. The existence of the Green bundles, the solutions $\boldsymbol{h}$, $\boldsymbol{v}$, $\mathbb{U}$, $\mathbb{S}$ and the monotonicity properties in Proposition 1.4 do not need the uniform boundedness of the operators H_{pp}, H_{pq}, H_{qq}. But in the unbounded case we cannot guarantee either that $X(\theta) \in \mathbb{E}(\theta) \cap \mathbb{F}(\theta)$ or $\mathbb{E}(\theta) \cup \mathbb{F}(\theta) \subset T_\theta \Sigma$.

PROPOSITION 1.7.

(a) *Any symmetric solution $S_\theta(t) : H(\psi_t(\theta)) \to V(\psi_t(\theta))$ of the Ricatti equation* (7) *which is defined on $t > 0$ is uniformly bounded on $t \geq 1$, i.e. there exists $A > 0$ such that*

$$\|S_\theta(t)\| < A \quad \textit{for all } \theta \in \Sigma,\ t > 1.$$

(b) *Any measurable symmetric solution of the Ricatti equation defined on the energy level Σ is uniformly bounded.*

An immediate consequence is the following.

COROLLARY 1.8.

(a) *The solution $S_\theta(t)$ of equation* (8) *given by* $\mathrm{Image}(Y_\theta(t), S_\theta(t)Y_\theta(t)) = d\psi_t V(\theta)$ *is uniformly bounded for $|t| > 1$.*

(b) *The solutions $\mathbb{S}_\theta(t)$, $\mathbb{U}_\theta(t)$ corresponding to the Green bundles are uniformly bounded on Σ.*

We shall need the following lemma, whose proof is elementary.

LEMMA 1.9. *The function $w(r) = R\coth(Rt - d)$, $R > 0$, satisfies:*

(i) $\dot{w} + w^2 - R^2 = 0$;

(ii) $\dot{w} < 0$, $w(-t + d/R) = -w(t + d/R)$;

(iii) $\lim_{t\to+\infty} w(t) = R$, $\lim_{t\to-\infty} w(t) = -R$;

(iv) $\lim_{t\to(d/R)^+} w(t) = +\infty$, $\lim_{t\to(d/R)^-} w(t) = -\infty$.

Proof of Proposition 1.7. We first prove (a). By the spectral theorem, it is enough to prove that $x^* S_\theta(t) x$ is uniformly bounded for $\|x\| = 1$ and $t > 1$. Since $H_{pq}^* = H_{qp}$ and $S^* = S$, we have that

$$\dot{S} + S^* H_{pp} S + H_{pq}^* S + S^* H_{pq} + H_{qq} = 0.$$

Let $C := (H_{pp})^{-1} H_{pq}$, $D := H_{qq} - C^* H_{pp} C - \dot{C}$ and $V_\theta(t) := S_\theta(t) + C(\psi_t(\theta))$. Then

$$\dot{V} + V^* H_{pp} V + D = 0.$$

Since $C(\theta)$ and $D(\theta)$ are continuous, then they are uniformly bounded on the energy level. Hence, it is enough to prove that $x^* V_\theta(t) x$ is uniformly bounded on $t > 1$ and $\theta \in \Sigma$. Observe that $V_\theta(t)$ is not necessarily symmetric. Let $M > 0$ be such that

$$y^* H_{pp}(\theta) y \geq \frac{1}{M}\|y\|^2 \quad \text{for all } \theta \in \Sigma. \tag{25}$$

Let $R > 0$ be such that

$$|y^* D(\theta) y| < MR^2 \quad \text{for all } \theta \in \Sigma, \quad \|y\| = 1. \tag{26}$$

We claim that

$$|y^* V_\theta(t) y| \leq MR\coth(R) \quad \text{for all } \|y\| = 1,\ t > 1,\ \theta \in \Sigma.$$

We prove the claim. Let $\theta_0 \in \Sigma$, $\|x\| = 1$. Write $V(t) := V_\theta(t)$, $H_{pp}(t) := H_{pp}(\psi_t\theta)$. Suppose that there exists $t_0 \in \mathbb{R}$ such that

$$x^* V(t_0) x =: M\alpha > MR \coth(R).$$

There exists $d_0 \in \mathbb{R}$ such that $R\coth(Rt_0 - d_0) = \alpha$. Observe that $d_0 > 0$ and $t_0 > d_0/R > 0$. Write $w(t) := R\coth(Rt - d_0)$. Then $w(t)$ is a solution of $\dot w + w^2 - R^2 = 0$ for $t > d_0/R$. In particular,

$$M\dot w + Mw^2 - MR^2 = 0.$$

Let $f(t) := x^* V(t) x - Mw(t)$. Then $f(t_0) = 0$ and

$$f'(t) + (x^* V(t)^* H_{pp}(t) V(t) x - Mw(t)^2) + (x^* D(t) x + MR^2) = 0. \tag{27}$$

Using the Schwartz inequality and (25), we have that

$$\begin{aligned} Mw(t_0)^2 = M\alpha^2 &= \frac{1}{M}(M\alpha)^2 = \frac{1}{M}\langle x, V(t_0)x\rangle^2 \\ &\le \frac{1}{M}|V(t_0)x|^2 \le (V(t_0)x)^* H_{pp}(t_0)(V(t_0)x) \\ Mw(t_0)^2 &\le x^* V(t_0)^* H_{pp}(t_0) V(t_0) x. \end{aligned} \tag{28}$$

Then (28), (26) and (27) imply that $f'(t_0) < 0$. The same argument can be applied each time that $f(t) = 0$. Therefore, $x^* V(t) x \le Mw(t)$ for all $t > t_0$ and

$$x^* V(t) x \ge Mw(t) \quad \text{for all } d_0/R < t < t_0.$$

Then

$$\lim_{t\to(d_0/R)^+} x^* V(t) x \ge \lim_{t\to(d_0/R)^+} Mw(t) = +\infty.$$

Since $d_0/R > 0$, this contradicts the existence of $V(t)$ for $t > 0$. Hence, such t_0 does not exist and

$$x^* V(t) x \le MR\coth(R) \quad \text{for all } t > 0.$$

Now suppose that there exists $t_1 \in \mathbb{R}$, $\|z\| = 1$ such that

$$x^* V(t_1) x < -MR.$$

We compare $v(t) = x^* V(t) x$ with $Mw_1(t)$, where $w_1(t) = R\coth(Rt - c_0)$ is such that $Mw_1(t_1) = v(t_1)$. Observe that $w_1(t)$ is defined for $t < c_0/R$ and in this case $c_0 > 0$ and $c_0/R > t_1 > 0$. The same argument as above shows that $v(t) \le Mw_1(t)$ for $t_1 \le t < c_0/R$. Since $\lim_{t\to(c_0/R)^-} w_1(t) = -\infty$, this contradicts the fact that $V(t)$ is defined for all $t > 0$.

The proof of (b) is similar to that of (a). In this case we can change the $MR\coth(R)$ bound by MR. We allow $t_0 \in \mathbb{R}$ and $d_0 \in \mathbb{R}$ to be non-positive if necessary. The second inequality has also the same proof, but here we allow $t_1 \in \mathbb{R}$ and $c_0 \in \mathbb{R}$. □

The following proposition states that $\lim_{t\to\pm\infty} \|d\pi \circ d\psi_t(\theta)|_{V(\theta)}\| = +\infty$ for all $\theta \in \Sigma$. This limit is not uniform in $\theta \in \Sigma$. A uniform lower bound for this norm is given in §6.

PROPOSITION 1.10. *For all $R > 0$ there exists $T > T(R, \theta) > 0$ such that $|Y_\theta(t)v| > R|v|$ for all $|t| > T$ and all $v \in T_\theta M \setminus \{0\}$.*

Proof. Let

$$M(t) := \int_t^\infty Y^{-1}(s) H_{pp}(s) (Y^*(s))^{-1}\, ds.$$

From (12), using that $D(t) = \lim_{c\to\infty} Z_c(t)$, we have that

$$D(s) = Y(s) M(s) \quad \text{for } s > 0. \tag{29}$$

Consider the solutions $S(t)$, $\mathbb{S}(t)$ of the Ricatti equation given on (8) and (20). From (20) and (29) we have that

$$\mathbb{S}(t) - S(t) = Y^*(t)^{-1} M(t)^{-1} Y(t)^{-1} \quad \text{for } t > 0.$$

Since the solutions $S(t)$, $\mathbb{S}(t)$ are defined on all $t > 0$, by Proposition 1.7, there exist $t_0 > 0$ and $k > 0$ such that $\|S(t)\| < k$ and $\|\mathbb{S}(t)\| < k$ for all $t > t_0$. Then for $|x| = 1$ and $t > t_0$,

$$|\langle M(t)^{-1} Y(t)^{-1} x, Y(t)^{-1} x\rangle| = |\langle S(t)x, x\rangle| + |\langle \mathbb{S}(t)x, x\rangle| \le 2k.$$

Let $\lambda(t)$ be the largest eigenvalue of $M(t)$, $t > 0$. Since $M(t)$ is positive definite then $\lambda(t) > 0$ and $\|M(t)\| = \lambda(t)$. Moreover,

$$2k \ge |\langle M(t)^{-1} Y(t)^{-1} x, Y(t)^{-1} x\rangle| \ge \frac{1}{\lambda(t)} |Y(t)^{-1} x|^2,$$

$$|Y(t)^{-1} x| \le \sqrt{2k}\, \|M(t)\|^{1/2} \quad \text{for all } |x| = 1,\ t > t_0.$$

Then if $|v| = 1$, we have that

$$|Y(t)v| \ge \frac{1}{\|Y(t)^{-1}\|} \ge \frac{1}{\sqrt{2k}\|M(t)\|^{1/2}} \quad \text{for } t > t_0.$$

Since $M(t) \to 0$ when $t \to +\infty$, given $R > 0$ there exists $T > t_0$ such that $(2k\|M(t)\|)^{1/2} < 1/R$ for all $t > T$ and hence, $|Y(t)v| > R$ for all $t > T$ and $|v| = 1$.

Using the Hamiltonian $\overline{H} := -H$, we get the result for $t < -T$. □

For $\theta \in TM$, define

$$B(\theta) := \left\{ \xi \in T_\theta T^* M \;\middle|\; \sup_{t\in\mathbb{R}} |d\psi_t(\theta) \cdot \xi| < +\infty \right\}.$$

PROPOSITION 1.11. *If the ψ-orbit of $\theta \in TM$ does not contain conjugate points, then $B(\theta) \subseteq \mathbb{E}(\theta) \cap \mathbb{F}(\theta)$.*

An example showing that in general $\mathbb{E}(\theta) \cap \mathbb{F}(\theta) \not\subset B(\theta)$ is given in Ballmann *et al* [**4**].

Proof. Let $\xi \in B(\theta)$ and let $\zeta_\Gamma \in E_T(\theta) := d\psi_{-T}(V(\psi_T(\theta)))$ be such that $d\pi(\theta) \cdot \zeta_T = d\pi(\theta) \cdot \xi$, in particular $\xi - \zeta_T \in V(\theta)$. Since $\lim_{T\to+\infty} E_T(\theta) = \mathbb{E}(\theta)$ and $\mathbb{E}(\theta) \pitchfork V(\theta)$ then there exists $\zeta := \lim_{T\to+\infty} \zeta_T \in \mathbb{E}(\theta)$. Moreover, $d\pi \circ d\psi_T(\xi - \zeta_T) = d\pi \circ d\psi_T(\xi)$. Since $d\pi \circ d\psi_T(\xi)$ is bounded on $T > 0$, by Proposition 1.10, $\lim_{T\to+\infty}(\xi - \zeta_T) = 0$. Since $\zeta_T \to \zeta \in \mathbb{E}(\theta)$, then $\xi \in \mathbb{E}(\theta)$.

Similarly, if $\eta_T \in F_T(\theta) := d\psi_T(V(\psi_{-T}(\theta)))$ is such that $d\pi(\theta) \cdot \eta_T = d\pi(\theta) \cdot \xi$, then $\lim_{T\to+\infty} \eta_T \in \mathbb{F}(\theta)$ and $\lim_{T\to+\infty}(\xi - \eta_T) = 0$. □

COROLLARY 1.12. *If the orbit of* θ *has no conjugate points then:*
(a) $\langle X(\theta)\rangle \subseteq \mathbb{E}(\theta)\cap\mathbb{F}(\theta)$;
(b) $\mathbb{E}(\theta)\cup\mathbb{F}(\theta)\subseteq T_\theta\Sigma$.

Proof. The property $\langle X(\theta)\rangle \subseteq \mathbb{E}(\theta)\cap\mathbb{F}(\theta)$ is a direct consequence of Proposition 1.11. It remains to show that $\mathbb{E}(\theta)\cup\mathbb{F}(\theta)\subset T_\theta\Sigma$. Observe that since $-dH=\omega(X,\cdot)$, then

$$T_\theta\Sigma=\{\xi\in T_\theta T^*M \mid \omega(X(\theta),\xi)=0\}.$$

Since $\mathbb{F}(\theta)$ is Lagrangian and $X(\theta)\in\mathbb{F}(\theta)$, then $\omega(X(\theta),\xi)=0$ for all $\xi\in\mathbb{F}(\theta)$. Hence, $\mathbb{F}(\theta)\subseteq T_\theta\Sigma$. Similarly, $\mathbb{E}(\theta)\subseteq T_\theta\Sigma$. □

COROLLARY 1.13. *If* M *is connected and* Σ *is a regular energy level without conjugate points, then* $\pi(\Sigma)=M$.

Proof. For $\theta\in\Sigma$, $d\pi(\theta):\mathbb{F}(\theta)\to T_{\pi(\theta)}M$ is an isomorphism. By Corollary 1.12, $\mathbb{F}(\theta)\subset T_\theta\Sigma$. Hence, $\pi:\Sigma\to M$ is an open map. Since M is connected then $\pi(\Sigma)=M$. □

The metric entropy.
Let Λ be the set of total measure having the Oseledec's splitting $T_\theta T^*M=E^s(\theta)\oplus E^c(\theta)\oplus E^u(\theta)$.

PROPOSITION 1.14. *For all* $\theta\in\Lambda$:
(a) $E^s(\theta)\subseteq\mathbb{E}(\theta)\subseteq E^s(\theta)\oplus E^c(\theta)$;
(b) $E^u(\theta)\subseteq\mathbb{F}(\theta)\subseteq E^u(\theta)\oplus E^c(\theta)$;
(c) $\chi(\theta)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T \operatorname{tr}[H_{pq}+H_{pp}\mathbb{U}](\psi_t(\theta))\,dt$.

Proof. We only prove

$$E^u(\theta)\subseteq\mathbb{F}(\theta)\subseteq E^u(\theta)\oplus E^c(\theta), \tag{30}$$

which shall be used for the proof of Theorem E; the other inclusions are similar.

We first prove the second inclusion. Observe that for $\overline{v}=(0,v)\in V(\theta)$ we have that $d\psi_t(\theta)\overline{v}=(Y_\theta(t)v,*)$. Then, by Proposition 1.10, we have that $V(\theta)\cap E^s(\theta)=\{0\}$. For every $\theta\in\Lambda$ there exists a subspace $R(\theta)\subset E^u(\theta)\oplus E^c(\theta)$ and a linear map $L(\theta):R(\theta)\to E^s(\theta)$ such that $V(\theta)=\operatorname{graph} L(\theta)$.

Let μ be an ergodic invariant measure for ψ_t. Given $\varepsilon>0$ let $K\subset\Lambda$ be a compact subset such that $\mu(K)>1-\varepsilon$ and also:
(a) $\sup\{\|L(\theta)\| \mid \theta\in K\}<+\infty$;
(b) there exists $T>0$ and $0<\eta<1<\lambda<\eta^{-1}$ such that for $t>T$ and $\theta\in K$,

$$\|d\psi_t|_{E^s(\theta)}\|<\eta^t \quad\text{and}\quad \|d\psi_t^{-1}|_{E^u(\theta)\oplus E^c(\theta)}\|<\lambda^t.$$

Let $V_{-s}(\theta):=d\psi_s(V(\psi_{-s}(\theta)))$. Then

$$\begin{aligned} V_{-s}(\theta)&=d\psi_s(\operatorname{graph} L(\psi_{-s}(\theta)))\\ &=\operatorname{graph}[d\psi_s\circ L(\psi_{-s}(\theta))\circ(d\psi_s|_{R(\psi_{-s}(\theta))})^{-1}].\end{aligned}$$

For $\theta \in K$ take a sequence $s_n \to +\infty$ such that $\psi_{-s_n}(\theta) \in K$. Since $d\psi_{s_n}(R(\psi_{-s_n}(\theta))) \subseteq E^u(\theta) \oplus E^c(\theta)$, by properties (a) and (b), we have that

$$\|d\psi_{s_n} \circ L(\psi_{-s_n}(\theta)) \circ (d\psi_{-s_n}|_{d\psi_{s_n} R(\psi_{-s_n}(\theta))})\| \le \eta^{s_n}\lambda^{s_n} \sup_{\theta \in K} \|L(\theta)\|.$$

Since $\eta^{s_n}\lambda^{s_n} \to 0$ when $n \to +\infty$, we have that

$$\mathbb{F}(\theta) = \lim_{n\to+\infty} V_{-s_n}(\theta) = \lim_{n\to+\infty} d\psi_{s_n}(R(\psi_{-s_n}(\theta))) \subseteq E^u(\theta) \oplus E^c(\theta).$$

Therefore, the second inclusion in (30) is satisfied for $\theta \in K$. Since $\mu(K) \ge 1 - \varepsilon$ and ε and μ are arbitrary, this inclusion holds for a set of total measure in Λ.

We now prove the first inclusion. There exists (cf. [**1**, Theorem 3.1.19]) a continuous Riemannian metric $\langle\, ,\rangle$ on T^*M and a continuous family of linear isomorphisms $J(\theta) : T_\theta T^*M \hookleftarrow$ such that $J(\theta)^2 = -I$ and the symplectic form is written as $\omega(x, y) = \langle x, J(\theta)y\rangle$ for all $x, y \in T_\theta T^*M$ and $\theta \in T^*M$. From now on we use this metric. Let $\{e_1(\theta), \dots, e_n(\theta)\}$ be an orthonormal basis of $\mathbb{F}(\theta)$ and define $e_{n+i}(\theta) = Je_i(\theta)$ for $i = 1, \dots, n$. Since the subspace $\mathbb{F}(\theta)$ is Lagrangian, the subspace $J\mathbb{F}(\theta) = \operatorname{span}\{e_{n+1}(\theta), \dots, e_{2n}(\theta)\}$ is the orthogonal complement of $\mathbb{F}(\theta)$ with respect to $\langle\ ,\ \rangle$. The matrices of ω and $d\psi_t(\theta)$ with respect to the family of basis $\{e_1(\theta), \dots, e_{2n}(\theta)\}$ are

$$J = \begin{bmatrix} & I \\ -I & \end{bmatrix} \quad \text{and} \quad d\psi_t(\theta) = \begin{bmatrix} A_t(\theta) & C_t(\theta) \\ & B_t(\theta) \end{bmatrix}.$$

Since $d\psi_t$ preserves the symplectic form we have that $(d\psi_t)^* J (d\psi_t) = J$. Hence, $B_t^*(\theta) = A_t(\theta)^{-1}$ and $B_t^*(\theta)C_t(\theta)$ is symmetric.

Since $\mathbb{F}(\theta) \subseteq E^u(\theta) \oplus E^c(\theta)$, then

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|B_t^{*-1}(\theta)w\| = \lim_{t\to\pm\infty} \frac{1}{t} \log \|A_t(\theta)w\| \ge 0$$

for all $w \in \mathbb{R}^n$, $\theta \in \Lambda$. This implies that

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|B_t^{-1}(\theta)w\| \ge 0. \tag{31}$$

Suppose that $E^u(\theta) \not\subset \mathbb{F}(\theta)$. Take $v \in E^u(\theta) \setminus \mathbb{F}(\theta)$. Then $v = w + z$ with $z \in \mathbb{F}(\theta)$ and $0 \ne w \in J\mathbb{F}(\theta)$. We have that $d\psi_{-t}v = u_{-t} + (*, B_t^{-1}(\theta)w)$ with $u_{-t} \in \mathbb{F}(\psi_{-t}(\theta)) \subseteq E^u(\psi_{-t}(\theta)) \oplus E^c(\psi_{-t}(\theta))$. Since $B_t^{-1}(\theta)w \in J\mathbb{F}(\psi_{-t}(\theta))$ and $\mathbb{F}$ and $J\mathbb{F}$ are orthogonal, we have that $\|d\psi_{-t}(v)\| \ge \|B_t^{-1}(\theta)w\|$. Hence, from (31) we obtain

$$\lim_{t\to+\infty} \frac{1}{t} \log \|d\psi_{-t}(\theta)v\| \ge 0.$$

This contradicts the choice $v \in E^u(\theta)$.

Finally, we prove (c). Let $\pi_\theta : \mathbb{F}(\theta) \to H(\theta)$ be the restriction of $d\pi$. Then $\pi_\theta^{-1}(v) = (v, \mathbb{U}(\theta)v)$ and by Corollary 1.8

$$\sup_{\theta\in\Sigma} \|\pi_\theta^{-1}\| < +\infty. \tag{32}$$

Fix $\theta \in \Lambda$, let $Z_\theta(t) : V(\theta) \to H(\psi_t(\theta))$ be defined by

$$Z_\theta(t)v = \pi_{\psi_t(\theta)}\, d\psi_t(\theta)\, (v, \mathbb{U}(\theta)v)\,.$$

Then

$$d\psi_t(\theta)|_{\mathbb{F}(\theta)} = \pi^{-1}_{\psi_t(\theta)} \circ Z_\theta(t) \circ \pi_\theta.$$

By (30) and (32), we have that

$$\begin{aligned}\chi(\theta) &= \lim_{T\to+\infty} \frac{1}{T} \log|\det d\psi_t(\theta)|_{\mathbb{F}(\theta)}| \\ &= \lim_{T\to+\infty} \frac{1}{T} \log|\det Z_\theta(t)|.\end{aligned}$$

Since the isomorphisms $h(t) = Z_\theta(t)$, $v(t) = \mathbb{U}(\psi_t(\theta))Z_\theta(t)$ satisfy the Jacobi equation (6), we have that

$$\dot{Z}_\theta = (H_{pq} + H_{pp}\mathbb{U})Z_\theta.$$

Since $\det Z_\theta(t) \neq 0$ for all $t \in \mathbb{R}$, then $\det Z_\theta(t)$ has constant sign and

$$\frac{d}{dt}|\det Z_\theta(t)| = \frac{d}{dt}\det Z_\theta(t) = \operatorname{tr}(H_{pq} + H_{pp}\mathbb{U})|\det Z_\theta(t)|,$$

$$\frac{d}{dt}\log|\det Z_\theta(t)| = \operatorname{tr}[H_{pq} + H_{pp}\mathbb{U}],$$

$$\lim_{T\to+\infty} \frac{1}{T} \log|\det d\psi_t(\theta)|_{\mathbb{F}(\theta)}| = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \operatorname{tr}[H_{pq} + H_{pp}\mathbb{U}]\,dt. \qquad \square$$

The exponential map.

The following proposition proves item (c) in Theorem 0.1. We present a modern proof that shall be needed in the sequel.

PROPOSITION 1.15. (Hartman [**20**]) *Let $H : T^*M \to \mathbb{R}$ be a convex Hamiltonian. Then:*

(a) *a half-open segment $\{\psi_t(\theta) \mid t \in [0, a[\}$, $a \in]0, +\infty[\cup \{+\infty\}$ has no conjugate points if and only if there exists a Lagrangian subspace $E \subset T_\theta T^*M$ such that $d\psi_t(E) \cap V(\psi_t(\theta)) = \{0\}$ for all $t \in]0, a[$;*

(b) *a closed segment $\{\psi_t(\theta) \mid t \in [0, a]\}$ has no conjugate points if and only if there exists a Lagrangian subspace $E \subset T_\theta T^*M$ such that $d\psi_t(E) \cap V(\psi_t(\theta)) = \{0\}$ for all $t \in [0, a]$;*

(c) *if a closed segment $\{\psi_t(\theta) \mid t \in [0, a]\}$ has no conjugate points, then there exists $\delta > 0$ such that the segment $\{\psi_t(\theta) \mid t \in [-\delta, a + \delta]\}$ has no conjugate points.*

Proof. (a) If the half-open segment is disconjugate, then the Lagrangian subspace $E = V(\theta)$ satisfies $d\psi_t(V(\theta)) \cap V(\psi_t(\theta)) = \{0\}$.

Conversely, suppose that such Lagrangian subspace E is given. Let $E(t) := d\psi_t(E)$. Then the projection $d\pi : E(t) \to H(\psi_t(\theta))$ is an isomorphism. Let $S(t) : H(\psi_t(\theta)) \to V(\psi_t(\theta))$ be the linear isomorphism such that $E(t) = \operatorname{graph}(S(t))$ for $t \in]0, a[$. Since $E(t)$ is Lagrangian, then $S(t)$ is symmetric and it satisfies the Ricatti equation (7). Let $Z(t) : E(0) \to H(\psi_t(\theta))$ be given by $Z(t)w = d\pi\, d\psi_t(\theta)w$. Then

$$\dot{Z} = (H_{pq} + H_{pp}S)Z, \qquad Z(0) = d\pi|_E. \tag{33}$$

Let $w \in E$ and $(h_1(t), v_1(t)) = d\psi_t(w) \in E(t)$, then $h_1(t) = Z(t)w$, $v_1(t) = S(t)Z(t)w$. Let $\xi \in T_\theta T^*M$ and $(h(t), v(t)) = d\psi_t(\xi)$. Suppose that $h(c) = 0$, $c \in]0, a[$. Using $h_1(t)$, $v_1(t)$, the same arguments as in (9), (10), (11) give that

$$h(t) = Z(t)Z(c)^{-1}h(c) + Z(t)\int_c^t Z^{-1}(s)H_{pp}(Z^*(s))^{-1}(Z(c)^*v(c))\,ds. \tag{34}$$

Since $h(c) = 0$, then

$$\langle Z(t)^{-1}h(t), Z(c)^*v(c)\rangle = \int_c^t \langle H_{pp}(Z(s)^*)^{-1}Z(c)^*v(c), (Z(s)^*)^{-1}Z(c)^*v(c)\rangle\,ds. \tag{35}$$

In particular $h(t) \neq 0$ for all $t \in]0, a[\setminus\{c\}$. Hence, the segment $]0, a[$ does not have conjugate points.

It remains to prove the case $c = 0$. Let $0 < b < a$. We prove that θ is not conjugate to $\psi_b(\theta)$. We already know that $\psi_b(\theta)$ has no conjugate points in $\{\psi_t(\theta) \mid 0 < t < a\}$. For $0 < d < b$ let $E_d = d\psi_{b-d}(V(\psi_d(\theta))) = \mathrm{graph}(K_d)$, where $K_d : H(\psi_b(\theta)) \to V(\psi_b(\theta))$ is the corresponding solution of the Ricatti equation (7). Let $\varepsilon > 0$ be such that $b < b + \varepsilon < a$ and let $E_{b+\varepsilon} = \psi_{-\varepsilon}(V(\psi_{b+\varepsilon}(\theta))) = \mathrm{graph}(K_{b+\varepsilon})$. By Proposition 1.4(b) and Remark 1.5, the family K_d is monotone decreasing on d and has the (lower) bound $K_{b+\varepsilon}$. Hence, there exists $\mathbb{E} := \lim_{d\to 0^+} E_d = d\psi_b(V(\theta))$ and $\mathbb{E} \cap V(\psi_b(\theta)) = \{0\}$.

(c) If the closed segment $[0, a]$ has no conjugate points, then $d\psi_t(V(\theta)) \cap V(\psi_t(\theta)) = \{0\}$ for $0 < t \leq a$. Then $d\psi_t(V(\theta)) \cap V(\psi_t(\theta)) = \{0\}$ for $0 < t \leq a + \delta$ for some $\delta > 0$. By item (a), using $E(t) = d\psi_t(V(\theta))$, the segment $[0, a+\delta]$ has no conjugate points. The same argument using $F(t) = d\psi_{a+\delta-t}(V(\psi_{a+\delta}(\theta)))$ shows that there exists $\delta_1 > 0$ such that the segment $[-\delta_1, a + \delta]$ has no conjugate points.

(b) Suppose that the segment $[0, a]$ is disconjugate. By item (c) there exists $\delta > 0$ such that the segment $[-\delta, a + \delta]$ is disconjugate. Now apply item (a) to the segment $[-\delta/2, a + \delta[$. Conversely, if there is a Lagrangian subspace $E \subset T_\theta T^*M$ with $d\psi_t(E) \cap V(\psi_t(\theta)) = \{0\}$ for all $0 \leq t \leq a$, then the argument of equation (35) applies for all $c \in [0, a]$. □

It may be impossible to find such a Lagrangian subspace E of Proposition 1.15(a),(b) satisfying $E \subset T_\theta\Sigma$ as explained in Remark 1.17. Compare this with Theorem 0.1. *If* there exists a *continuous* invariant Lagrangian bundle $E(\theta)$, one can always suppose that $E(\theta) \subset T_\theta\Sigma$ by taking $(E(\theta) \cap T_\theta\Sigma) \oplus \langle X(\theta)\rangle$. *Then* it must satisfy the transversality condition 0.1(a).

PROPOSITION 1.16. *For $\theta \in \Sigma$ define $W(\theta) := (V(\theta) \cap T_\theta\Sigma) \oplus \langle X(\theta)\rangle$. Suppose that the orbit of θ has no conjugate points. Then for all $t \neq 0$, $d\psi_t(W(\theta)) \cap V(\psi_t(\theta)) = \{0\}$.*

Proof. Suppose that there exists $b > 0$ such that $(0, w_b) \in d\psi_b(W(\theta)) \cap V(\psi_b(\theta)) \neq \{0\}$. Let $(k_0, w_0) := d\psi_{-b}(0, w_b) \in W(\theta)$. Since the segment $\{\psi(\theta) \mid t \in [0, +\infty[\}$ has no conjugate points, then $k_0 \neq 0$. Write $H_p(t) := H_p(\psi_t(\theta))$, $H_q(t) := H_q(\psi_t(\theta))$. Since $W(\theta) = (V(\theta) \cap T_\theta\Sigma) \oplus \langle X(\theta)\rangle$ and $X(\theta) = (H_p(0), -H_q(0))$, then $k_0 = \alpha H_p(0)$ for some $\alpha \neq 0$. Multiplying w_b by $1/\alpha$, we can assume that $\alpha = 1$. Let

$$(h(t), v(t)) := d\psi_t(X(\theta)) - (k_0, w_0) \in d\psi_t(W(\theta)) \subseteq T_{\psi_t(\theta)}\Sigma.$$

Then

$$h(0) = H_p(0) - k_0 = 0, \quad v(0) \neq 0;$$
$$h(b) = d\pi[X(\psi_t(\theta)) - (0, w_b)] = H_p(b). \tag{36}$$

Since $(h(0), v(0)) \in T_\theta\Sigma$, then

$$0 = dH(h(0), v(0)) = H_q h(0) + H_p v(0) = H_p(0) \cdot v(0). \tag{37}$$

Let $\mathbb{E}(t) = \mathbb{E}(\psi_t(\theta))$ be the stable Green bundle of $\psi_{\mathbb{R}}(\theta)$, let $\mathbb{S}(t)$ be from (20) and let $Z(t) := \boldsymbol{h}(t) : H(\psi_t(\theta)) \to V(\psi_t(\theta))$ be from Claim 1.3. So that $Z(t)w = d\pi\, d\psi_t(w, \mathbb{S}(0)w)$ and $Z(t)$ satisfies (33). Using formula (34) for $c = 0$, (with $Z(t) = \boldsymbol{h}(t)$, $Z(0) = I$ and $h(0) = 0$) we obtain that

$$h(t) = Z(t) \int_0^t Z^{-1}(s) H_{pp}(s) (Z^*(s))^{-1} v(0)\, ds.$$

Hence, since $v(0) \neq 0$, we have that

$$\langle Z(b)^{-1} h(b), v(0)\rangle = \int_0^b \langle H_{pp}(Z^*(s)^{-1} v(0)), (Z^*(s)^{-1} v(0))\rangle\, ds > 0. \tag{38}$$

Since $X(\theta) \in \mathbb{E}(\theta)$, then

$$\begin{aligned}(H_p(b), -H_q(b)) &= X(\psi_b(\theta)) = d\psi_b(X(\theta)) = d\psi_b(H_p(0), \mathbb{S}(0) H_p(0)) \\ &= (Z(b)H_p(0), \mathbb{S}(b) \cdot Z(b) H_p(0)).\end{aligned}$$

From the first component we get that $Z(b)^{-1} H_p(b) = H_p(0)$. Using (36) we have that $Z(b)^{-1} h(b) = H_p(0)$. Replacing this equation on (38), we obtain that

$$\langle H_p(0), v(0)\rangle > 0.$$

This contradicts equation (37). □

Remark 1.17. The same proof applies to the following statement: if $H : T^*M \to \mathbb{R}$ is convex and there is a Lagrangian subspace $E \subset T_\theta\Sigma$ such that $d\psi_t(E) \cap V(\psi_t(\theta)) = \{0\}$ for $t \in [0, a]$, then $d\psi_t(W(\theta)) \cap V(\psi_t(\theta)) = \{0\}$ for all $t \in]0, a]$.

In particular this hypothesis holds if $\{\psi_t(\theta) \mid t \geq 0\}$ or $\{\psi_t(\theta) \mid t \leq 0\}$ have no conjugate points. Just take the corresponding Green bundles.

Example A.2, in the appendix, does not satisfy the conclusion of Proposition 1.16 on $t \in [0, \pi]$, but nevertheless the segment $\{\psi_t(\theta) \mid t \in [0, \pi]\}$ has no conjugate points. This implies that the Lagrangian subspace given by Proposition 1.15(b) may not satisfy $E \subset T_\theta\Sigma$.

COROLLARY 1.18. *Let $H : T^*M \to \mathbb{R}$ be convex, $\Sigma = H^{-1}\{e\}$, $q \in \pi(\Sigma)$ and suppose that the orbits $\{\psi_t(\theta) \mid t \geq 0\}$ have no conjugate points for all $\theta \in \pi^{-1}\{q\} \cap \Sigma$. Then the exponential map $\exp_q : T_q^*M \to M$ is an immersion.*

Proof. We have that $\exp_q(t\theta) = \pi \circ \psi_t(\theta)$ for $\theta \in \Sigma_q = \pi^{-1}\{q\} \cap \Sigma$. Consider the splitting

$$T_{t\theta}(T_q^*M) \approx T_q^*M = T_\theta(\Sigma_q) \oplus \langle\theta\rangle = (V(\theta) \cap T_\theta\Sigma) \oplus \langle\theta\rangle.$$

The derivative $d\exp_q(t\theta) : T_q^*M \to T_{\pi(\psi_t\theta)}M$ is given by

$$d\exp_q(t\theta)|_{V(\theta)\cap T_\theta\Sigma} = d\pi \circ d\psi_t(\theta)|_{V(\theta)\cap T_\theta\Sigma}, \tag{39}$$

$$d\exp_q(t\theta)(\theta) = d\pi(X(\psi_t(\theta))) = d\pi \circ d\psi_t(X(\theta)). \tag{40}$$

Therefore

$$\begin{aligned} d\exp_q(t\theta)(T_q^*M) &= d\pi \circ d\psi_t((V(\theta)\cap T_\theta\Sigma) \oplus \langle X(\theta)\rangle) \\ &= d\pi \circ d\psi_t(W(\theta)). \end{aligned} \tag{41}$$

Since $\dim W(\theta) = n$, then $\dim d\psi_t(W(\theta)) = n$. By Proposition 1.16 $d\psi_t(W(\theta)) \cap V(\psi_t(\theta)) = \{0\}$. Hence, $d\pi : d\psi_t(W(\theta)) \to T_{\pi(\psi_t\theta)}M$ is a linear isomorphism. From (41), $d\exp_q(t\theta)$ is a linear isomorphism for all $t > 0$, $\theta \in \Sigma_q$. □

2. *The transversal behaviour*

Given $\theta \in \Sigma$, consider $H_p(\theta) = d\pi X(\theta) \neq 0$. Let $\boldsymbol{N}(\theta) \subseteq T_\theta\Sigma$ be defined by

$$\boldsymbol{N}(\theta) := \{\xi \in T_\theta\Sigma \mid \langle d\pi\,\xi, d\pi X(\theta)\rangle_{\pi(\theta)} = 0\}. \tag{42}$$

Then $\boldsymbol{N}(\theta) \oplus \langle X(\theta)\rangle = T_\theta\Sigma$. Fix $\theta_0 \in \Sigma$ and a smooth coordinate system $(q_1, \dots, q_n, t)$ of $M \times \mathbb{R}$ along the projection $(\pi(\psi_t(\theta_0)), t)$ of the orbit of θ_0 such that:

(2a) $\partial/\partial q_1|_{(\pi(\psi_t\theta_0),t)} = d\pi X(\psi_t(\theta_0))$;

(2b) $\partial/\partial q_2, \dots, \partial/\partial q_n$ is an orthonormal basis for $d\pi \boldsymbol{N}(\psi_t(\theta_0)) = \langle \partial/\partial q_1\rangle^\perp$ along $\pi(\psi_t(\theta_0), t)$.

Write $p_i = dq_i|_{T_qM}$, $i = 1, 2, \dots, n$. From (2a) we have that

$$1 = \frac{d}{dt}q_1\bigg|_{(\pi(\psi_t\theta_0),t)} = H_{p_1}(\psi_t(\theta_0)) \neq 0.$$

Then the equation $H(q_1, q_2, \dots, q_n; p_1, p_2, \dots, p_n) = h$ can be solved locally for p_1:

$$p_1 = -K(Q, P, T; h),$$

where $P = (p_2, \dots, p_n)$, $Q = (q_2, \dots, q_n)$, $T = q_1$. This solution can be extended to a simply connected neighbourhood W of the orbit $\{(\psi_t(\theta_0), t) \mid t \in \mathbb{R}\} \subseteq T^*M \times \mathbb{R}$.

From now on we omit the $\mathbb{R}$-coordinate in $W \subseteq T^*M \times \mathbb{R}$. Let ϕ_T be the reparametrization of the flow ψ_t on W such that it preserves the foliation $q_1 = T =$ constant. In particular $d\phi_T(\theta_0)$ preserves the transversal bundle $\boldsymbol{N}(\phi_T(\theta_0))$ along the orbit of θ_0. Define $\tau(T, \theta)$ by $\phi_T(\theta) = \psi_{\tau(T,\theta)}(\theta)$. Then $\tau(T, \theta_0) = T$ for all $T \in \mathbb{R}$. The following reduction appears in Arnold [**2**].

LEMMA 2.1. *The orbits $\phi_T(\theta) = (T = q_1, Q(T), p_1(T), P(T); \tau(T, \theta)) \in W$ satisfy the equations*

$$\frac{dQ}{dT} = \frac{\partial K}{\partial P}, \qquad \frac{dP}{dT} = -\frac{\partial K}{\partial Q},$$

where $Q = (q_2, \dots, q_n)$, $P = (p_2, \dots, p_n)$, $T = q_1$ and $K(Q, P; T)$ is defined by $H(T, q_2, \dots, q_n; -K, p_2, \dots, p_n) = h$.

Proof. Consider the canonical 1-form $\Theta = p\,dq$ on T^*M and the symplectic form $\omega = d\Theta = dp \wedge dq$. On $T\Sigma$ we have that $\omega(X(\theta), \cdot) = -dH = 0$. Let $(d/dT)\phi_T(\theta) = (Y(\phi_T(\theta)), (d/dT)\tau(T,\theta))$ be the vectorfield of ϕ_T. Then $Y(\phi_T(\theta))$ is a multiple of the vectorfield $X(\phi_T(\theta))$ of ψ_t, and hence, $\omega(Y(\phi_T(\theta)), \cdot) = 0$ on $T\Sigma$. We have

$$\Theta = p\,dq = P\,dQ - K\,dT,$$

$$\omega|_{T\Sigma} = d\Theta|_{T\Sigma} = \sum_{i=2}^{n} [dP_i \wedge dQ_i - K_{Q_i}\,dQ_i \wedge dT - K_{P_i}\,dP_i \wedge dT].$$

The matrix of $\omega|_{T\Sigma}$ is given by

$$\begin{array}{cl} \begin{bmatrix} 0 & -I & -K_Q^* \\ I & 0 & -K_P^* \\ K_Q & K_P & 0 \end{bmatrix} & \begin{array}{l} \}\ Q \\ \}\ P \\ \}\ T = q_1. \end{array} \end{array}$$

Then the vector $(K_P, -K_Q, 1 = dq_1/dT) =: Z(\phi_T(\theta))$ satisfies $\omega(Z(\phi_T(\theta)), \cdot) = 0$ on $\Sigma \approx \{(T, Q, p_1, P) \mid p_1 = -K(Q, P, T)\}$. Since the form ω has maximal rank on Σ, we have that $Z(\phi_T(\theta)) = Y(\phi_T(\theta))$ for $\theta \in W$. This proves the lemma. $\square$

Since for $\vartheta \in W$, $\phi_T(\vartheta) = \psi_{\tau(T,\vartheta)}(\vartheta)$, then

$$\begin{aligned} d\phi_T(\vartheta) \cdot \xi &= d\psi_{\tau(T,\vartheta)} \cdot \xi + \left(\left.\frac{\partial \tau}{\partial \vartheta}\right|_{(T,\vartheta)} \cdot \xi \right) \left.\frac{d}{dt}\psi_t(\vartheta)\right|_{\tau(T,\vartheta)} \\ &= d\psi_{\tau(T,\vartheta)} \cdot \xi + \alpha(T,\xi) X(\phi_T(\vartheta)), \end{aligned} \tag{43}$$

where $\alpha(T,\xi) = \partial\tau/\partial\vartheta|_{(T,\vartheta)} \cdot \xi$.

Let $\Lambda(\psi_t(\theta_0)) : T_{\psi_t(\theta_0)}\Sigma \to \boldsymbol{N}(\psi_t(\theta_0))$ be the projection along the direction of the vectorfield, i.e.

$$\Lambda\xi = \xi + \beta(\xi) X(\psi_t(\theta_0)) \text{ with } \beta(\xi) \in \mathbb{R} \text{ such that } \Lambda\xi \in \boldsymbol{N}(\psi_t(\theta_0)).$$

Through the proof of Lemma 2.2, all the quantities will be understood in local coordinates $T = q_1$, $Q = (q_2, \dots, q_n)$, $P = (p_2, \dots, p_n)$ of $W \cap \Sigma$. In particular, $T_{\phi_T(\theta_0)}\Sigma = \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ for all $T \in \mathbb{R}$, with coordinates $(\partial/\partial T, \partial/\partial Q, \partial/\partial P)$. Since along the orbit of θ_0 we have that

$$\dot{q} = (\dot{T}, \dot{Q}) = (1, \mathbf{0}_{n-1}) = (H_{p_1}, H_P) \quad \text{along} \quad \phi_T(\theta_0) = (T, \mathbf{0}_{n-1}, \mathbf{0}_{n-1}),$$

then the subspace $\boldsymbol{N}(\phi_T(\theta_0))$ in (42) is written as $\mathcal{N} := \{0\} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ in coordinates. Define $\mathbb{P} : T_\vartheta\Sigma = \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \mathcal{N} = \{0\} \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ as the projection

$$\mathbb{P}(h_1, H, V) := (0, H, V).$$

Using $\beta = -h_1$ on (43) and $H_p = (1, \mathbf{0})$, the projection Λ is written as

$$\Lambda(h_1, H, V) = (0, H, V + h_1 \cdot H_Q),$$

where $H_Q := (H_{q_2}, \dots, H_{q_n})$. Define the matrices

$$\mathbb{H}(T) = \begin{bmatrix} H_{pq} & H_{pP} \\ -H_{Qq} & -H_{QP} \end{bmatrix}_{(2n-1)\times(2n-1)}$$

$$\mathbb{K}(T) = \begin{bmatrix} \mathbf{0}_{1\times n} & \mathbf{0}_{1\times(n-1)} \\ K_{Pq} & K_{PP} \\ -K_{Qq} & -K_{QP} \end{bmatrix}_{(2n-1)\times(2n-1)}$$

along the orbit of θ_0.

LEMMA 2.2. *Along the orbit $\theta_T := \phi_T(\theta_0)$ of θ_0, we have that:*
(a) *the matrix $K_{PP}(\theta_T)$ is positive definite;*
(b) *for $\xi \in \boldsymbol{N}(\theta_T)$; $\mathbb{K}(\theta_T)\xi = \Lambda(\theta_T)\mathbb{H}(\theta_T)\xi$;*
(c) *the operator $\mathbb{K}(\theta_T)|_{N(\theta_T)}$ is uniformly bounded on $T \in \mathbb{R}$.*

Proof. (b) Write $\theta_T = \phi_T(\theta_0) = \psi_T(\theta_0) = (T, \mathbf{0}, \mathbf{0})$. From Lemma 2.1, the Jacobi equation for ϕ_T is written as

$$\frac{d}{dT}[\mathbb{P}\, d\phi_T\, \xi] = \mathbb{K}[\mathbb{P}\, d\phi_T\, \xi],$$

where $d\phi_T\xi = (h_1(T), H(T), V(T)) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$, for any $\xi \in T_{\theta_0}\Sigma = \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$. Also, from (6),

$$\frac{d}{dt}[d\psi_t\xi] = \mathbb{H}[d\psi_t\xi].$$

Since $d\phi_T(\theta_0)$ preserves the foliation $q_1 = \text{constant}$, then it preserves the subspace $\mathcal{N}$. Therefore

$$\mathbb{P}\, d\phi_T\, \xi = d\phi_T\xi = \Lambda\, d\phi_T\xi \quad \text{for } \xi \in \mathcal{N}.$$

In particular, the first coordinate of $d\phi_T\xi$ is $h_1(T) \equiv 0$ and also $(d/dT)h_1 \equiv 0$. Hence, if $\xi \in \mathcal{N}$, then

$$\begin{aligned} \frac{d}{dT}[\mathbb{P}\, d\phi_T\, \xi] &= \frac{d}{dT}[d\phi_T\xi] \\ &= \frac{d}{dT}[d\psi_T\xi + \alpha(T)\, X(\theta_T)] \\ &= \mathbb{H}(d\psi_T\xi) + \dot\alpha(T)\, X(\theta_T) + \alpha(T)\, \frac{d}{dT}X(\theta_T) \\ &= \mathbb{H}[d\phi_T\xi - \alpha(T)X(\theta_T)] + \dot\alpha(T)X(\theta_T) + \alpha(T)\, \frac{d}{dT}X(\theta_T), \end{aligned}$$

where $\alpha(T) = \alpha(T, \xi)$ is from (43).

Since $(d/dT)X(\theta_T) = \mathbb{H}\, X(\theta_T)$, then

$$\frac{d}{dT}[\mathbb{P}\, d\phi_T\, \xi] = \mathbb{H}[d\phi_T\xi] + \dot\alpha(T)\, X(\theta_T).$$

Hence,

$$\mathbb{K}\, d\phi_T\xi = \mathbb{H}[d\phi_T\xi] + \dot\alpha(T)X(\theta_T) \quad \text{for } \xi \in \mathcal{N}. \tag{44}$$

Observe that this equation must hold in all the $2n-1$ coordinates of $T_\vartheta\Sigma$. Since Image($\mathbb{K}$) $\subseteq \mathcal{N}$, then $\Lambda\mathbb{K} = \mathbb{K}$. So

$$\begin{aligned} \mathbb{K}(d\phi_T\zeta) &= \Lambda\, \mathbb{K}(d\phi_T\zeta) = \Lambda[\mathbb{H}\, d\phi_T\zeta + \dot\alpha(T,\zeta)X(\theta_T)] \\ &= \Lambda\, \mathbb{H}(d\phi_T\zeta) \quad \text{for } \zeta \in \mathcal{N}. \end{aligned}$$

Since $d\phi_T : \boldsymbol{N}(\theta_0) = \mathcal{N} \to \boldsymbol{N}(\theta_T) = \mathcal{N}$ is surjective, this proves item (b).

(a) Let $V \in \mathbb{R}^{n-1}$, then

$$V^* \mathbb{K}_{PP} V = [(0, V^*), \mathbf{0}_{n-1}]\, \mathbb{K} \begin{bmatrix} \mathbf{0}_n \\ V \end{bmatrix}.$$

Since $\xi = (0_n, V) \in \mathcal{N}$, we can use equation (44) to obtain

$$\begin{aligned} V^* K_{PP} V &= [(0, V^*), \mathbf{0}_{n-1}] \left[\mathbb{H} \begin{bmatrix} \mathbf{0}_n \\ V \end{bmatrix} + \dot{\alpha}(T, d\phi_{-T}\xi) \begin{bmatrix} H_p^* \\ -H_Q^* \end{bmatrix} \right] \\ &= 0 + V^* H_{PP} V + \dot{\alpha}((0, V^*) \cdot H_p^*) \\ &= V^* H_{PP} V + \dot{\alpha} \langle (1, \mathbf{0}_{n-1}), (0, V^*) \rangle \\ &= V^* H_{pp} V > 0. \end{aligned}$$

We now prove (c). Since the operator $\mathbb{H}(\vartheta)$ is uniformly bounded on $\vartheta \in \Sigma$, from (b) it is enough to prove that the projection Λ is bounded on the orbit of θ_0. For it is enough to see that the angle $\sphericalangle(X(\theta_T), \boldsymbol{N}(\theta_T))$ is uniformly bounded away from 0. Given a vector

$$\xi = (h, v) = \sum_{i=1}^{n} \left(h_i \frac{\partial}{\partial q_i} + v_i \frac{\partial}{\partial p_i} \right) \in \boldsymbol{N}(\theta_T),$$

written in our coordinates (2a), (2b), define the norm $|\xi| := \left(\sum_{i=1}^n h_i^2 + v_i^2 \right)^{1/2}$. Suppose that $|\xi| = 1$. Since $\xi \in \boldsymbol{N}(\theta_T)$, then $H_p \cdot h = 0$. Moreover, $H_p = \partial/\partial q_1$ and hence,

$$\begin{aligned} \frac{|X(\theta_T) \cdot \xi|}{|X(\theta_T)|} &= \frac{|H_p \cdot h - H_q \cdot v|}{(|H_p|^2 + |H_q|^2)^{1/2}} = \frac{|H_q||v|}{\sqrt{1 + |H_q|^2}} \\ &\le 1 \cdot \max_{0 \le x \le A} \frac{x}{\sqrt{1 + x^2}} = \frac{A}{\sqrt{1 + A^2}} < 1, \end{aligned}$$

where $A := \sup_{T \in \mathbb{R}} |H_q(\theta_T)| < +\infty$ with the norm in our coordinates $\partial/\partial p_i$. □

COROLLARY 2.3. *Along the orbit $\theta_T = \phi_T(\theta_0)$, we have the following.*

(a) *The orbit of θ_0 under ϕ_T has no conjugate points.*

(b) *Existence of the Green bundles for $d\phi_s|_{\boldsymbol{N}(\theta_T)}$:*

$$\begin{aligned} \mathbb{E}^\top(\theta_T) &= \lim_{s \to +\infty} d\phi_{-s}(V(\theta_{T+s}) \cap \boldsymbol{N}(\theta_{T+s})) \\ \mathbb{F}^\top(\theta_T) &= \lim_{s \to -\infty} d\phi_{-s}(V(\theta_{T+s}) \cap \boldsymbol{N}(\theta_{T+s})). \end{aligned}$$

Moreover, $\mathbb{E}^\top(\theta) = \mathbb{E}(\theta) \cap \boldsymbol{N}(\theta)$ and $\mathbb{F}^\top(\theta) = \mathbb{F}(\theta) \cap \boldsymbol{N}(\theta)$ for all $\theta \in \Sigma$.

(c) *Horizontal growth of iterates of vertical vectors: for all $R > 0$ there exists $S = S(R, \theta_T) > 0$ such that for all $|s| > S(R, \theta_T)$ and all $\xi \in \boldsymbol{N}(\theta_T) \cap V(\theta_T)$ we have that $|d\pi(d\phi_s(\theta_T) \cdot \xi)| > R\,|\xi|$.*

(d) *Define*

$$B^\top(\theta_T) = \left\{ \xi \in \boldsymbol{N}(\theta_T) \;\middle|\; \sup_{s \in \mathbb{R}} |d\pi \cdot d\psi_s \cdot \xi| < +\infty \right\}.$$

Then $B^\top(\theta_T) \subseteq \mathbb{E}^\top(\theta_T) \cap \mathbb{F}^\top(\theta_T)$.

(e) *Let* $\Psi_s := d\phi_s|_{\boldsymbol{N}} = \Lambda \circ d\psi_s|_{\boldsymbol{N}}$, *then*

$$d\phi_s|_{\boldsymbol{N}(\theta_T)} = \Lambda(\theta_{T+s}) \circ d\Psi_s \circ \Lambda(\theta_T) = \Psi_s,$$
$$\Psi_{s+t} = \Psi_s \circ \Psi_t.$$

Proof. Let $U(\theta_T) := V(\theta_T) \cap \boldsymbol{N}(\theta_T) = V(\theta_T) \cap T_{\theta_T}\Sigma$. From (43) we have that

$$d\phi_T(\theta_0) \cdot \xi = d\psi_T(\theta_0) \cdot [\xi + \alpha(T, \xi) X(\theta_0)].$$

Hence, $d\phi_T(U(\theta_0)) \subseteq d\psi_T(W(\theta_0))$, where $W(\theta) := (V(\theta) \cap T_\theta\Sigma) \oplus \langle X(\theta)\rangle$. From Proposition 1.16,

$$U(\theta_T) \cap d\phi_T(U(\theta_0)) \subseteq V(\theta_T) \cap d\psi_T(W(\theta_0)) = \{0\} \quad \text{for all } T \neq 0.$$

This implies item (a).

From Lemma 2.2, the first part of item (b) and items (c) and (d) have the same proofs as Propositions A, 1.10 and Proposition 1.11, respectively.

Since $d\phi_s$ and $d\psi_s$ satisfy (43) and $d\phi_s$ preserves the bundle $\boldsymbol{N}(\theta_T)$, we have that

$$d\phi_s|_{\boldsymbol{N}(\theta_T)} = \Lambda(\theta_{T+s}) \circ d\psi_s|_{\boldsymbol{N}(\theta_T)} = \Lambda(\theta_{T+s}) \circ d\psi_s \circ \Lambda(\theta_T) = \Psi_s.$$

Moreover,

$$\Psi_{s+t} = d\phi_{s+t}|_{\boldsymbol{N}(\theta_T)} = d\phi_s|_{\boldsymbol{N}(\theta_{T+t})} \circ d\phi_t|_{\boldsymbol{N}(\theta_T)} = \Psi_s \circ \Psi_t.$$

This proves item (e).

Finally, we have that

$$\begin{aligned}\mathbb{E}^\top(\theta_0) &= \lim_{T\to+\infty} [d\phi_{-T}(V(\theta_T) \cap \boldsymbol{N}(\theta_T))] \\ &= \Lambda(\theta_0) \lim_{T\to+\infty} [d\psi_{-T}(V(\theta_T) \cap T_{\theta_T}\Sigma)] \\ &\subseteq \Lambda(\theta_0) \cdot \mathbb{E}(\theta_0) \subseteq \mathbb{E}(\theta_0),\end{aligned}$$

because $X(\theta_0) \in \mathbb{E}(\theta_0)$. Thus, $\mathbb{E}^\top(\theta_0) \subseteq \mathbb{E}(\theta_0) \cap \boldsymbol{N}(\theta_0)$. Since $\dim \mathbb{E}^\top(\theta_0) = \dim U = n - 1 = \dim(\mathbb{E}(\theta_0) \cap \boldsymbol{N}(\theta_0))$, we have that $\mathbb{E}^\top(\theta_0) = \mathbb{E}(\theta_0) \cap \boldsymbol{N}(\theta_0)$. The equation $\mathbb{F}^\top(\theta_0) = \mathbb{F}(\theta_0) \cap \boldsymbol{N}(\theta_0)$ is proven similarly. This completes the proof of item (b). □

Proof of Proposition B. Let $\theta_0 \in \Gamma$ and let $\tau > 0$ be the period of θ_0. Suppose that Γ is hyperbolic, then by Proposition 1.14, $\mathbb{E}(\theta_0) \subseteq E^s(\theta_0) \oplus \langle X(\theta_0)\rangle$ and $\mathbb{F}(\theta_0) \subseteq E^u(\theta_0) \oplus \langle X(\theta_0)\rangle$. Hence, $\mathbb{E}(\theta_0) \cap \mathbb{F}(\theta_0) = \langle X(\theta_0)\rangle$.

Now observe that if we construct the coordinates $(q_1, \ldots, q_n)$ in (2a), (2b) so that they are periodic with period τ, then $d\phi_\tau(\theta_0) : \boldsymbol{N}(\theta_0) \hookleftarrow$ is the derivative of the Poincaré map of ψ_t defined on the cross-section $\{\vartheta \mid q_1(\vartheta) = 0\}$.

Suppose that $\mathbb{E}(\theta_0) \cap \mathbb{F}(\theta_0) = \langle X(\theta_0)\rangle$. By Corollary 2.3(a) and (b) we have that $B^\top(\theta_0) \subseteq \mathbb{E}^\top(\theta_0) \cap \mathbb{F}^\top(\theta_0) = \langle X(\theta_0)\rangle \cap \boldsymbol{N}(\theta_0) = \{0\}$. This implies that Γ is hyperbolic. □

3. *Quasi-hyperbolic actions*

Let B be a compact metric space and $\pi : \boldsymbol{E} \to B$ a vector bundle provided with a continuous norm $|\cdot|_p$ on each fibre $\pi^{-1}\{p\}$. Let Ψ be an $\mathbb{R}$-action $\Psi : \mathbb{R} \to \mathrm{Isom}(\boldsymbol{E})$, i.e.

(i) there exists a continuous flow ψ_t on B such that $\pi \circ \Psi_t = \psi_t \circ \pi$;
(ii) $\Psi_t : \boldsymbol{E}(p) \to \boldsymbol{E}(\psi_t(p))$ is a linear isomorphism, where $\boldsymbol{E}(p) := (\pi|_{\boldsymbol{E}})^{-1}\{p\}$, $p \in B$;
(iii) $\Psi_{s+t} = \Psi_s \circ \Psi_t$.

We say that the action Ψ_t is *quasi-hyperbolic* if

$$\sup_{t\in\mathbb{R}} |\Psi_t(\xi)| = +\infty \quad \text{for all } \xi \in \boldsymbol{E}, \xi \neq 0.$$

We say that Ψ is *hyperbolic* if there exists an invariant continuous splitting $\boldsymbol{E}(p) = E^u(p) \oplus E^s(p)$, $p \in B$ and $C > 0$, $\lambda > 0$ such that:
(3a) $\Psi_t(E^u(p)) = E^u(\psi_t(p))$, $\Psi_t(E^s(p)) = E^s(\psi_t(p))$, for all $x \in B$, $t \in \mathbb{R}$;
(3b) $|\Psi_t(\xi)| \leq C\mathrm{e}^{-\lambda t}|\xi|$ for all $t > 0$, $\xi \in E^u(p)$, $p \in B$;
(3c) $|\Psi_{-t}(\xi)| \leq C\mathrm{e}^{-\lambda t}|\xi|$ for all $t > 0$, $\xi \in E^u(p)$, $p \in B$.

The aim of this section is to prove the following.

THEOREM 0.2. *If the action Ψ is quasi-hyperbolic and the non-wandering set $\Omega(\psi|_B) = B$, $\psi \circ \pi = \pi \circ \Psi$, then $\Psi|_{\boldsymbol{E}}$ is hyperbolic.*

Applying this theorem to the case in which $B = M$ is a compact manifold, ψ_t is a differentiable flow on M, $\Psi_t = d\psi_t$ and $\boldsymbol{E}$ is a $d\psi_t$-invariant continuous bundle such that $\boldsymbol{E} \oplus \langle (d/dt)\psi_t \rangle = TM$, we get the following.

THEOREM 3.1. (Freire [**15**]) *If M is a closed manifold and ψ is a quasi-Anosov flow on M such that $\Omega(\psi) = M$, then ψ is Anosov.*

The proof of Theorem 0.2 is similar to that of Theorem 3.1. We include it here for completeness.

Suppose that the action of Ψ is quasi-hyperbolic. For $p \in B$ define

$$E^s(p) := \left\{ v \in \boldsymbol{E}(p) \;\middle|\; \sup_{t>0} |\Psi_t(p)(v)| < +\infty \right\},$$
$$E^u(p) := \left\{ v \in \boldsymbol{E}(p) \;\middle|\; \sup_{t<0} |\Psi_t(p)(v)| < +\infty \right\}.$$

Observe that by the quasi-hyperbolicity we have that

$$E^s(p) \cap E^u(p) = \{0\} \quad \text{for all } p \in B.$$

LEMMA 3.2. *There exists $\tau > 0$ such that for all $p \in B$,*

$$\|\Psi_\tau|_{E^s(p)}\| < \tfrac{1}{2}, \quad \|\Psi_{-\tau}|_{E^u(p)}\| < \tfrac{1}{2}$$

for all $p \in M$.

Proof. Suppose that the lemma is false for E^s. The proof for E^u is similar. Then there exist sequences $x_n \in B$, $v_n \in E^s(x_n)$, $|v_n| = 1$ such that $|\Psi_n(v_n)| \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. We claim that there exists C such that

$$\sup_{t\geq 0} \|\Psi_t(p)|_{E^s(p)}\| < C < +\infty.$$

for all $p \in B$.

Suppose the claim is true. Let $w_n := \Psi_n(v_n)$ and $y_n := \psi_n(x_n)$. We have that $\frac{1}{2} \le |w_n| \le C$ for all n and

$$|\Psi_t(w_n)| = |\Psi_{t+n}(v_n)| \le C \quad \text{for all } t \ge -n.$$

Since B is compact there exists a convergent subsequence of (y_n, w_n). If $y_n \xrightarrow[n]{} y$ and $w_n \xrightarrow[n]{} w$, we have that $y \in B$, $w \in \boldsymbol{E}(y)$, $|w| \ge \frac{1}{2}$ and

$$|\Psi_t(w)| \le C \quad \text{for all } t \in \mathbb{R}.$$

This is a contradiction.

Now we prove the claim. Suppose it is false. Then there exists $x_n \in B$, $t_n \ge 0$, $v_n \in E^s(x_n)$, $|v_n| = 1$, such that

$$\sup_n |\Psi_{t_n}(v_n)| = +\infty. \tag{45}$$

Let $s_n > 0$ be such that

$$|\Psi_{s_n}(v_n)| > \tfrac{1}{2} \sup_{s \ge 0} |\Psi_s(v_n)| \ge \tfrac{1}{2} |\Psi_{t_n}(v_n)|.$$

By (45) we have that $s_n \xrightarrow[n]{} +\infty$. Let $y_n := \psi_{s_n}(x_n)$ and

$$w_n := \frac{\Psi_{s_n}(v_n)}{|\Psi_{s_n}(v_n)|}.$$

Then $|w_n| = 1$ and if $t > -s_n$ we have that

$$|\Psi_t(w_n)| = \frac{|\Psi_{t+s_n}(v_n)|}{|\Psi_{s_n}(v_n)|} \le \frac{2|\Psi_{t+s_n}(v_n)|}{\sup_{s \ge 0} |\Psi_s(v_n)|} \le 2.$$

Since $|w_n| = 1$ and $y_n \in \boldsymbol{M}$, there exists a convergent subsequence $(y_n, w_n) \to (y, w)$. We would have that $y \in B$, $w \in \boldsymbol{E}(y)$, $|w| = 1$ and

$$|\Psi_t(w)| \le 2 \quad \text{for all } t \in \mathbb{R}.$$

This is a contradiction. □

LEMMA 3.3. *There exists $K > 0$ such that for all $x \in B$ and $v \in \boldsymbol{E}(x)$,*

$$|\Psi_t(v)| \le K(|v| + |\Psi_s(v)|) \quad \textit{for all } 0 \le t \le s.$$

Proof. Suppose it is false. Then there exist $x_n \in B$, $0 \ne v_n \in \boldsymbol{E}(x_n)$ and $0 \le t_n \le s_n$, such that

$$|\Psi_{t_n}(v_n)| \ge n(|v_n| + |\Psi_{s_n}(v_n)|).$$

Then $t_n \to +\infty$ and $s_n - t_n \to +\infty$ when $n \to +\infty$. We can assume that $|\Psi_{t_n}(v_n)| = \sup_{0 \le t \le s_n} |\Psi_t(v_n)|$. Let

$$w_n := \frac{\Psi_{t_n}(v_n)}{|\Psi_{t_n}(v_n)|}.$$

For $-t_n < t < s_n - t_n$, we have that

$$|\Psi_t(w_n)| = \frac{|\Psi_{t+t_n}(v_n)|}{|\Psi_{t_n}(v_n)|} \le \frac{1}{|\Psi_{t_n}(v_n)|} \sup_{0 \le t \le s_n} |\Psi_t(v_n)| = 1.$$

Taking a subsequence, if necessary, we can assume that $x_n \to x$ and $w_n \to w \in \boldsymbol{E}(x)$. Then $|\Psi_t(w)| \le 1$ for all $t \in \mathbb{R}$, with $|w| = 1$. This is a contradiction. □

Proof of Theorem 0.2. Standard methods (cf. Hirsch–Pugh–Shub [**21**]) show that the continuity of the strong stable and unstable bundles is redundant in the definition of a hyperbolic action. So it remains to prove that for all $x \in B$, we have that $E^s(x) \oplus E^u(x) = \boldsymbol{E}(x)$. Given $x \in B$, since $x \in \Omega(\psi|_B)$, there exist $x_n \to x$, $s_n \to +\infty$ such that $\psi_{s_n} x_n \to x$. Let E_n be a subspace of $\boldsymbol{E}(x_n)$ such that

$$\dim E_n = \dim \boldsymbol{E}(x) - \dim E^s(x), \quad E^s(x) \oplus \lim_n E_n = \boldsymbol{E}(x).$$

We claim that there exists $C > 0$ such that

$$\|\Psi_{-t}|\Psi_{s_n}(E_n)\| \le C \quad \text{for all } 0 \le t \le s_n.$$

Suppose that the claim is true. Taking a subsequence if necessary, we can assume that the limit $\lim_n \Psi_{s_n}(E_n)$ exists. Then

$$\|\Psi_{-t}|\lim_n \Psi_{s_n}(E_n)\| \le C \quad \text{for all } t > 0.$$

Then $\lim_n \Psi_{s_n}(E_n) \subset E^u(x)$ and $\dim E^u(x) + \dim E^s(x) \ge \dim E_n + \dim E^s(x) = \dim \boldsymbol{E}(x)$. Since $E^s(x) \cap E^u(x) = \{0\}$, this completes the proof.

Now we prove the claim. Suppose it is false, then there exist $v_n \in \Psi_{s_n}(E_n)$, $|v_n| = 1$ and $0 < t_n < s_n$ such that $|\Psi_{-t_n}(v_n)| \ge n$. By Lemma 3.3, we have that

$$n \le |\Psi_{-t_n}(v_n)| \le K(|\Psi_{-s_n}(v_n)| + |v_n|).$$

Hence, $|\Psi_{-s_n}(v_n)| \ge (n - K)/K$. From Lemma 3.3 we also have that

$$\frac{|\Psi_t(\Psi_{-s_n}(v_n))|}{|\Psi_{-s_n}(v_n)|} \le K + \frac{K}{|\Psi_{-s_n}(v_n)|} \quad \text{for } 0 < t < s_n.$$

Let $w_n := \Psi_{-s_n}(v_n)/|\Psi_{-s_n}(v_n)|$. The estimates above give that

$$|\Psi_t(w_n)| \le K + \frac{K^2}{n - K} \quad \text{for } 0 < t < s_n.$$

If $w_n \to w$, then $|w| = 1$ and $w \in \lim E_n$, thus, $w \notin E^s(x)$. But $|\Psi_t(w)| \le K$ for all $t \ge 0$. This is a contradiction. □

Proof of Theorem C. For $\theta \in \Sigma$, let

$$\boldsymbol{N}(\theta) := \{\xi \in T_\theta\Sigma \mid \langle d\pi \cdot \xi, d\pi\, X(\theta)\rangle_{\pi(\theta)} = 0\},$$

and let $\Lambda(\theta) : T_\theta\Sigma \to \boldsymbol{N}(\theta)$ be the projection along the direction of $X(\theta)$:

$$\Lambda(\theta) \cdot \xi = \xi + \beta(\xi) X(\theta) \quad \text{such that } \Lambda(\theta) \cdot \xi \in \boldsymbol{N}(\theta).$$

Let $\Psi_s(\theta) = \Lambda(\psi_s(\theta)) \circ d\psi_s|_{\boldsymbol{N}(\theta)}$. By Corollary 2.3(e), Ψ_s defines an $\mathbb{R}$-action on the vector bundle $\pi : \boldsymbol{N} \to \Sigma$.

Suppose that the Green bundles satisfy $\mathbb{E} \cap \mathbb{F} = \langle X\rangle$ on Σ. Then by Corollary 2.3(b) and 2.3(d),

$$B^\top(\theta) \subseteq \mathbb{E}^\top(\theta) \cap \mathbb{F}^\top(\theta) = \mathbb{E}(\theta) \cap \mathbb{F}(\theta) \cap \boldsymbol{N}(\theta) = \{0\}.$$

Therefore, the action Ψ_s is quasi-hyperbolic. Since ψ_s preserves the Liouville measure, which is positive on open sets, then $\Omega(\psi|_\Sigma) = \Sigma$. Hence, Ψ_s is a hyperbolic action.

This implies that ϕ_t is Anosov. By the methods in Hirsch–Pugh–Shub [**21**], we get that ψ_t is Anosov. We outline the proof below.

For $\theta \in \Sigma$ let $\mathcal{E}^u(\theta) \subseteq \boldsymbol{N}(\theta)$ be the subspace given by item (3b) in the definition of hyperbolic action. Then $\mathcal{E}^u$ is a continuous subbundle of $T\Sigma$. Define

$$\mathcal{F} := \{L : \mathcal{E}^u \to \mathbb{R} \text{ continuous } \mid L(\theta) : \mathcal{E}^u(\theta) \to \mathbb{R} \text{ is linear } \forall \theta \in \Sigma\},$$

with $\|L\| = \sup_{\xi \in \mathcal{E}^u \setminus \{0\}} |L(\xi)|/|\xi|$.

To each functional in $\mathcal{F}$ associate a subbundle W_L of $T\Sigma$ by

$$W_L(\theta) := \text{graph}(L(\theta)) = \{\xi + (L(\theta) \cdot \xi) X(\theta) \mid \xi \in \mathcal{E}^u(\theta)\}.$$

Let $\tau > 0$ be such that $C e^{-\lambda\tau} < e^{-\lambda} < 1$, where C and λ are from (3b). Consider the following 'graph transformation' $T : \mathcal{F} \to \mathcal{F}$, corresponding to $W_{TL} = d\psi_\tau(W_L)$ and defined by

$$d\psi_\tau(\xi + L(\xi)X(\theta)) = \Psi_\tau(\xi) + [T(L)(\Psi_\tau(\xi))]X(\psi_\tau(\theta)), \quad \theta \in \Sigma,\ \xi \in \mathcal{E}^u(\theta).$$

We claim that T is a contraction. Indeed

$$\begin{aligned} \left| \frac{T(L_1 - L_2)(\Psi_\tau \xi)}{\|\Psi_\tau \xi\|} \right| \|X(\psi_\tau \theta)\| &\leq \frac{\|d\psi_\tau[(L_1(\xi) - L_2(\xi))X(\theta)]\|}{e^\lambda \|\xi\|} \\ &\leq \frac{|(L_1 - L_2)(\xi)|}{e^\lambda \|\xi\|} \|X(\psi_\tau \theta)\|. \end{aligned}$$

Hence, $\|T(L_1) - T(L_2)\| \leq e^{-\lambda} \|L_1 - L_2\|$.

The fixed point L^* of T gives the (continuous) strong unstable bundle of ψ_t. Indeed, if $E^u = W_{L^*}$, then E^u is clearly $d\psi_t$-invariant. Moreover, if $\zeta \in E^u(\theta)$, then $\zeta = \xi + (L^*\xi)X(\theta)$ with $\xi = \Lambda\zeta \in \mathcal{E}^u(\theta)$. Since L^* is continuous then there exists $Q_1 > 0$ such that $|\zeta| \leq Q_1|\xi| = Q_1|\Lambda\zeta|$ for all $\zeta \in \mathcal{E}^u$. Since $\boldsymbol{N}(\theta)$ is transversal to $\langle X(\theta)\rangle$ and both are continuous subbundles of $T\Sigma$, (Σ compact), then the angle $\sphericalangle(X(\theta), \boldsymbol{N}(\theta))$ is bounded below and hence, there exists $Q_2 > 0$ such that $(1/Q_2)|\Lambda\zeta| = (1/Q_2)|\xi| < |\zeta|$. Then

$$\begin{aligned} |d\psi_t(\zeta)| &= |\Psi_t(\xi) + (T(L^*) \cdot \xi) X(\psi_t \theta)| \\ &\geq \frac{1}{Q_2} |\Psi_t(\xi)| \geq \frac{C^{-1} e^{\lambda t}}{Q_2} |\xi| \geq \frac{e^{\lambda t}}{C Q_1 Q_2} |\zeta|. \end{aligned}$$

Finally,

$$\dim E^u = \dim \text{graph}(L^*) = \dim \mathcal{E}^u = n - 1.$$

The existence of a continuous strong stable subbundle of dimension $n - 1$ is proven similarly. □

4. *The index form*

Let $L = pH_p - H$ be the Lagrangian associated to H. Consider $\partial L/\partial v(x, v) : T_{(x,v)}(T_x M) \approx T_x M \to \mathbb{R}$ as an element $L_v \in T^*M$. The Legendre transform $\mathcal{F}_L(x, v) = (x, L_v)$ identifies $q = x$, $p = L_v$. Similarly, using $\mathcal{F}_L^{-1} = \mathcal{F}_H$, we get that $v = H_p$ under the same identification. Also

$$H = vL_v - L = pH_p - L(q, H_p).$$

Hence, $H_q = -L_q = -L_x$. With these identifications the Hamiltonian equations (1) become

$$\begin{aligned} \frac{d}{dt}x = v \quad &\longleftrightarrow \quad \frac{d}{dt}q = H_p, \\ \frac{d}{dt}L_v = L_x \quad &\longleftrightarrow \quad \frac{d}{dt}p = -H_q. \end{aligned} \tag{46}$$

Hence, identifying $L_x = -H_q \in T_{(p,q)}(T^*M)$, the Euler–Lagrange equation (46) is understood as a first-order differential equation on T^*M, where $(d/dt)L_v$ is the tangent vector to the path $t \mapsto L_v(\gamma(t), \dot{\gamma}(t))$ in T^*M.

We derive the Jacobi equation in this Lagrangian setting. Let $\gamma(t)$ be a solution of the Euler–Lagrange equation (2). Considering a variation $f(s,t)$ of $\gamma(t) = f(0,t)$ made of solutions $t \mapsto f(s,t)$ of the Euler–Lagrange equation (2) and taking the covariant derivative D/ds, we obtain the *Jacobi equation*

$$\frac{D}{dt}(L_{vx}k + L_{vv}\dot{k}) = L_{xx}k + L_{xv}\dot{k}, \tag{47}$$

where $k = (\partial f/\partial s)(0,t)$, $\dot{k} = (D/dt)(\partial f/\partial s)$ and the derivatives of L are evaluated on $\gamma(t) = f(0,t)$. Here we have used that $(D/ds)(\partial F/\partial t) = (D/dt)(\partial F/\partial s)$ for the variation map $F(s,t) = L_v(f(s,t), (\partial f/\partial t)(s,t)) \in T^*M$, where D/ds and D/dt are the covariant derivatives on the Riemannian manifold T^*M. The linear operators L_{xx}, L_{xv}, L_{vv} coincide with the corresponding matrices of partial derivatives in local coordinates. The solutions of (47) satisfy

$$D\varphi_t(k(0), \dot{k}(0)) = (k(t), \dot{k}(t)) \in T_{\gamma(t)}(TM),$$

where φ_t is the Lagrangian flow on TM. A solution of (47) is called a *Jacobi field*.

Let Ω_T be the set of continuous piecewise C^2 vectorfields ξ along $\gamma_{[0,T]}$. Define the *index form* on Ω_T by

$$I(\xi,\eta) = \int_0^T (\dot{\xi}L_{vv}\dot{\eta} + \dot{\xi}L_{vx}\eta + \xi L_{xv}\dot{\eta} + \xi L_{xx}\eta)\,dt, \tag{48}$$

which is the second variation of the action functional for variations $f(s,t)$ with $\partial f/\partial s \in \Omega_T$. For general results on this form see Duistermaat [**12**].

The following transformation of the index form is taken from Hartman [**20**] and was originally due to Clebsch [**7**]. Let $\theta \in T^*M$ and suppose that the orbit of θ, $\psi_t(\theta)$, $0 \le t \le T$ does not have conjugate points. Let $E \subset T_\theta T^*M$ be a Lagrangian subspace such that $d\psi_t(E) \cap V(\psi_t(\theta)) = \{0\}$ for all $0 \le t \le T$. Such a subspace E always exists by Proposition 1.15(b). Let $E(t) := d\psi_t(E)$ and let $H(t), V(t)$ be a matrix solution of the Hamiltonian Jacobi equation (6) such that $\det H(t) \neq 0$ and $E(t) = \mathrm{Image}(H(t), V(t)) \subset T_{\psi_t(\theta)}(T^*M)$ is a Lagrangian subspace. In particular $H(t)$ satisfies the Lagrangian Jacobi equation (47). We have that

$$\begin{bmatrix} H(t) \\ V(t) \end{bmatrix} = D\mathcal{F}_L(\varphi_t(\theta)) \cdot \begin{bmatrix} H(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ L_{vx} & L_{vv} \end{bmatrix} \begin{bmatrix} H \\ H' \end{bmatrix}. \tag{49}$$

From (49) and (47), we have that

$$V = L_{vv}H' + L_{vx}H, \quad V' = L_{xv}H' + L_{xx}H. \tag{50}$$

Moreover, since $E(t)$ is a Lagrangian subspace, then the Hamiltonian solution of the Ricatti equation (7), $S(t) = V(t)H(t)^{-1}$, is symmetric, i.e.

$$H(t)^*V(t) = V(t)^*H(t). \tag{51}$$

Let $\eta \in \Omega_T$ and define $\zeta \in \Omega_T$ by $\eta(t) = H(t)\zeta(t)$. Then the integrand of $I(\eta, \eta)$ in (48) is

$$(L_{vv}H\zeta' + V\zeta) \cdot (H'\zeta + H\zeta') + (V'\zeta + L_{xv}H\zeta') \cdot H\zeta.$$

Since $(L_{vv}H\zeta') \cdot (H'\zeta) = (H\zeta') \cdot (L_{vv}H'\zeta)$ and $(L_{xv}H\zeta') \cdot (H\zeta) = (H\zeta') \cdot (L_{vx}H\zeta)$, the integrand can be written as

$$L_{vv}H\zeta' \cdot H\zeta' + H\zeta' \cdot V\zeta + V\zeta \cdot H'\zeta + V\zeta \cdot H\zeta' + V'\zeta \cdot H\zeta.$$

Using (51), we have that $V^*H\zeta' \cdot \zeta = H^*V\zeta' \cdot \zeta = V\zeta' \cdot H\zeta$. Hence, the integrand in (48) is $L_{vv}H\zeta' \cdot H\zeta' + (V\zeta \cdot H\zeta)'$. Since η and ζ are continuous, we have that

$$I(\eta, \eta) = \int_0^T (L_{vv}H\zeta' \cdot H\zeta')\, dt + H\zeta \cdot V\zeta\big|_0^T \quad \text{if } \eta = H\zeta \in \Omega_T. \tag{52}$$

Let $\mathcal{F}_H(q, p) = (q, H_p)$ be the Legendre transform of the Hamiltonian. Then $\mathcal{F}_L$ and $\mathcal{F}_H$ are inverse maps, in particular

$$(D\mathcal{F}_H)^{-1} = \begin{bmatrix} I & 0 \\ H_{pq} & H_{pp} \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ L_{vx} & L_{vv} \end{bmatrix} = D\mathcal{F}_L,$$

and hence, $L_{vv} = (H_{pp})^{-1}$. We obtain that

$$I(\xi, \eta) = \int_0^T (H\zeta')^*(H_{pp})^{-1}(H\rho')\, dt + (H\zeta)^*(V\rho)\big|_0^T \tag{53}$$

for $\xi = H\zeta \in \Omega_T$, $\eta = H\rho \in \Omega_T$. This formula can also be written as

$$I(\xi, \eta) = \int_0^T (H\zeta')^*(H_{pp})^{-1}(H\rho')\, dt + \xi^* S\eta\big|_0^T, \tag{54}$$

where $S(t) = V(t)H(t)^{-1}$ is the corresponding solution to the Ricatti equation (7).

COROLLARY 4.1. *If $\theta \in T^*M$ and the segment $\{\psi_t(\theta) \mid t \in [0, T]\}$ has no conjugate points then the index form is positive definite on*

$$\Gamma_T = \{\xi : [0, T] \to TM \mid \xi(t) \in T_{\pi\psi_t\theta}M, \xi \textit{ is piecewise } C^2, \xi(0) = 0, \xi(T) = 0\}.$$

Moreover, $I_T(\eta, \eta) = 0$ if and only if η is a Jacobi field on Γ_T.

Proof. Let $\xi \in \Gamma_T$, $\xi \neq 0$. Write $\xi(t) = H(t)\zeta(t)$. Since $\det H(t) \neq 0$, $\zeta(0) = 0$, $\zeta(T) = 0$ and $\zeta(t) \not\equiv 0$, then $\zeta' \not\equiv 0$. Now use formula (53). □

We now extend formula (53) to the case in which $H(t)$ may be singular (or $\zeta'(t)$, $\eta'(t)$ may be discontinuous) at a finite set of points. Let $\{t_1, \ldots, t_N\}$ be the points in $[0, T]$ such that $\det H(t_i) = 0$. Then

$$I(\xi,\eta) = \int_0^T (H\zeta')^*(H_{pp})^{-1}(H\rho')\,dt + (H\zeta)^*(V\rho)|_0^T - \sum_{i=1}^N (H\zeta)^*(V\rho)|_{t_i^-}^{t_i^+}, \qquad (55)$$

where $\xi = H\zeta$, $\eta = H\rho$. But now ζ, ρ are piecewise C^2 but may be discontinuous at $t_1, \ldots, t_N$. Alternatively, we can use a sum of formulas (53) or (55) using different Lagrangian subspaces $d\psi_t(E_i)$ and corresponding solutions (H_i, V_i) on disjoint subintervals $]t_i, t_{i+1}[\subset [0, T]$.

Observe that for any convex Hamiltonian on T^*M and any $\theta \in M$ there exists $\varepsilon > 0$ such that the segment $\{\psi_t(\theta) \mid |t| < \varepsilon\}$ has no conjugate points. Indeed, let $Y_\theta(t)v = d\pi\, d\psi_t(\theta)(0, v)$, where $(0, v) \in V(\theta)$. Then from the Jacobi equation (6) we have that

$$\frac{D}{dt} Y_\theta(t)\bigg|_{t=0} = H_{pp}(\theta).$$

Since $Y_\theta(0) = 0$ and $H_{pp}(\theta)$ is non-singular, then $Y_\theta(t)$ is an isomorphism for $|t| < \varepsilon$, some $\varepsilon > 0$. In particular $d\psi_t(\theta)V(\theta) \cap V(\theta) = \{0\}$ for $|t| < \varepsilon$.

We say that a curve $\gamma(t) \in M$, $t \in [0, T]$ is *minimizing* if it minimizes the action functional

$$\int_0^T L(\delta(t), \delta'(t))\,dt$$

over all absolutely continuous curves $\delta(t) \in M$, $0 \le t \le T$, such that $\delta(0) = \gamma(0)$ and $\delta(T) = \gamma(T)$.

COROLLARY 4.2. *If $\theta \in T^*M$ and the segment $\{\pi\psi_t(\theta) \mid t \in [0, S[\}$ is minimizing, then it has no conjugate points.*

Proof. Suppose it is false. Let $\psi_T(\theta)$, $T < S$ be the first conjugate point in $\{\pi\psi_t(\theta) \mid t \in [0, S[\}$. Then there exists $\xi = (0, v) \in V(\theta)$ such that $d\psi_T(\theta)\xi \in V(\psi_T(\theta))$. Let $\eta(t) = d\pi\, d\psi_t(\theta)\xi$. Using the limit of (53) on the interval $[0, T[$ we have that $I(\eta, \eta)|_0^T = 0$.

Let $\varepsilon > 0$ be such that (cf. Proposition 1.15) the segment $\{\psi_t(\theta) \mid T - \varepsilon \le t \le T + \varepsilon\}$ has no conjugate points. Let $\zeta(t) = Y_{\psi_{T+\varepsilon}(\theta)}(t - T - \varepsilon)w = d\pi\, d\psi_{t-T-\varepsilon}(\psi_{T+\varepsilon}(\theta))(0, w)$, $T - \varepsilon \le t \le T + \varepsilon$, where w is such that $\zeta(T - \varepsilon) = \eta(T - \varepsilon)$. Let $\widehat{\eta} = \eta(t)$ if $T - \varepsilon \le t \le T$ and $\widehat{\eta}(t) = 0$ if $T \le t \le T + \varepsilon$. Then using $H(t) = Y_{\psi_{T+\varepsilon}(\theta)}(t - T - \varepsilon)$ on (53), we have that

$$I(\widehat{\eta}, \widehat{\eta})|_{T-\varepsilon}^{T+\varepsilon} > I(\zeta, \zeta)|_{T-\varepsilon}^{T+\varepsilon}.$$

Extend ζ and $\widehat{\eta}$ by $\zeta(t) = \widehat{\eta}(t) = \eta(t)$ for $0 \le t \le T - \varepsilon$. Then

$$I(\zeta, \zeta)|_0^{T+\varepsilon} < I(\widehat{\eta}, \widehat{\eta})|_0^{T+\varepsilon} = 0.$$

Now consider the variation $f(s, t) = \exp_{\pi\psi_t(\theta)} s\zeta(t)$. We have that $f(s, 0) = \pi(\theta)$, $f(s, T + \varepsilon) = \pi(\psi_{T+\varepsilon}(\theta))$, $\partial f/\partial s(0, t) = \zeta(t)$ and if

$$A(s) = \int_0^{T+\varepsilon} L\left(f(s,t), \frac{\partial f}{\partial t}(s,t)\right) dt$$

is the action of $f(s,t)$, then $A'(0) = 0$ because $f(0,t) = \pi\psi_t(\theta)$ satisfies the Euler–Lagrange equation and

$$A''(0) = I(\zeta,\zeta) < 0.$$

Therefore, the segment $\{\pi\psi(\theta) \mid 0 \le t \le T + \varepsilon\}$ is not minimizing. $\square$

COROLLARY 4.3. *If Γ_T is as in Corollary 4.1, then*

$$\text{index } I_T|_{\Gamma_T} = \sum_{0<t\le T} \dim[d\psi_t(V(\theta)) \cap V(\psi_t(\theta))],$$

where the sum is over the discrete set of points conjugate to θ.

A proof of this corollary can be obtained from the following remarks.

1. By Corollary 4.1, if $0 < t \le T$ then $I_t(\eta,\eta) = 0$ only if η is a Jacobi field on Γ_t. Hence, the index of I_t can only change with t when $\psi_t(\theta)$ is conjugate to θ.
2. If $I_t(\zeta,\zeta) < 0$ and $0 < t \le T$, then the extension $\widehat{\zeta}(s) = \zeta(s)$ on $0 \le s \le t$ and $\widehat{\zeta}(s) = 0$ on $0 \le s \le T$ satisfy $I_T(\widehat{\zeta},\widehat{\zeta}) = I_t(\zeta,\zeta) < 0$. Hence, the function $t \mapsto \text{index}(I_t|_{\Gamma_t})$ is non-decreasing.
3. The proof of Corollary 4.2 constructs a linearly independent vectorfield $\zeta \in \Gamma_T$ with $I_T(\zeta,\zeta) < 0$ for each linearly independent vector $\xi \in V(\theta) \cap d\psi_{-t}(V(\psi_t(\theta)))$, $0 < t \le T$.

5. *Proof of Theorem D*

We begin by quoting the following theorem by Mañé [**26**]. Given a (periodic) Lagrangian $L : TM \times S^1 \to \mathbb{R}$ and $\omega \in H^1(M,\mathbb{R})$, let $\mathcal{M}^\omega(L)$ be the set of minimizing measures of $L + \overline{\omega}$, where $\overline{\omega}$ is any 1-form in the class ω.

THEOREM 5.1. (Mañé [**26**])

(a) *For every $\omega \in H^1(M,\mathbb{R})$ there exists a residual subset $\mathcal{O}(\omega) \subset C^\infty(M \times S^1)$ such that $\psi \in \mathcal{O}(\omega)$ implies $\#\mathcal{M}^\omega(L+\psi) = 1$.*

(b) *There exist residual subsets $\mathcal{O} \subset C^\infty(M \times S^1)$ and $\mathcal{H} \subset H^1(M,\mathbb{R})$ such that $\psi \in \mathcal{O}$ and $\omega \in \mathcal{H}$ imply $\#\mathcal{M}^\omega(L+\psi) = 1$.*

Now take an autonomous Lagrangian L and $\omega = 0 \in H^1(M,\mathbb{R})$. Then the item (a) implies the first part of Theorem D. Let $\mathcal{O}$ be the residual subset given by this theorem. Let $\mathcal{A}$ be the subset of $\mathcal{O}$ of potentials ψ for which the measure on $\mathcal{M}(L+\psi)$ is supported on a periodic orbit. Let $\mathcal{B} := \mathcal{O} \setminus \mathcal{A}$ and let $\mathcal{A}_1$ be the subset of $\mathcal{A}$ on which the minimizing periodic orbit is hyperbolic. We prove that $\mathcal{A}_1$ is relatively open on $\mathcal{A}$. For, let $\psi \in \mathcal{A}_1$ and

$$\mathcal{M}(L+\psi) = \{\mu_\gamma\},$$

where μ_γ is the invariant probability measure supported on the hyperbolic periodic orbit γ for the flow of $L+\psi$. We claim that if $\phi_k \in \mathcal{A}$, $\phi_k \to \psi$ and $\mathcal{M}(L+\phi_k) = \{\mu_{\eta_k}\}$, then $\eta_k \to \gamma$. Indeed, since L is superlinear, the velocities in the support of the minimizing measures $\mu_k := \mu_{\eta_k}$ are bounded (cf. [**25, 29**]), and hence, there exists a subsequence $\mu_k \to \nu$ converging weakly* to a some invariant measure ν for $L+\psi$. Then if $\nu \ne \mu_\gamma$,

$$\lim_k S_{L+\phi_k}(\mu_k) = S_{L+\psi}(\nu) > S_{L+\psi}(\mu_\gamma). \tag{56}$$

Thus, if δ_k is the analytic continuation of the hyperbolic periodic orbit γ to the flow of $L + \phi_k$ in the original energy level $c(L + \psi)$, since $\lim_k S_{L+\psi_k}(\mu_{\delta_k}) = S_{L+\psi}(\mu_\gamma)$, for k large we have that,

$$S_{L+\phi_k}(\mu_{\delta_k}) < S_{L+\phi_k}(\mu_{\eta_k});$$

which contradicts the choice of η_k. Therefore, $\nu = \mu_\gamma$. For energy levels h near to $c(L + \psi)$ and potentials ϕ near to ψ, there exist hyperbolic periodic orbits $\gamma_{\phi,h}$ which are the continuation of γ. Now, on a small neighbourhood of a hyperbolic orbit there exists a unique invariant measure supported on it, and it is in fact supported in the periodic orbit. Thus, since $\eta_k \to \gamma$, then η_k is hyperbolic. Hence, $\phi_k \in \mathcal{A}_1$ and $\mathcal{A}_1$ contains a neighbourhood of ϕ in $\mathcal{A}$.

Let $\mathcal{U}$ be an open subset of $C^\infty(M, \mathbb{R})$ such that $\mathcal{A}_1 = \mathcal{U} \cap \mathcal{A}$. We shall prove below that $\mathcal{A}_1$ is dense in $\mathcal{A}$. This implies that $\mathcal{A}_1 \cup \mathcal{B}$ is generic. Let $\mathcal{V} := \mathrm{int}(C^\infty(M, \mathbb{R}) \setminus \mathcal{U})$, then $\mathcal{U} \cup \mathcal{V}$ is open and dense in $C^\infty(M, \mathbb{R})$. Moreover, $\mathcal{V} \cap \mathcal{A} = \emptyset$ because $\mathcal{A} \subseteq \overline{\mathcal{A}_1} \subseteq \overline{\mathcal{U}}$ and $\mathcal{V} \cap \mathcal{A} \subseteq \mathcal{A} \setminus \overline{\mathcal{U}} = \emptyset$. Since $\mathcal{O} = \mathcal{A} \cup \mathcal{B}$ is generic and

$$\begin{aligned}(\mathcal{U} \cup \mathcal{V}) \cap (\mathcal{A} \cup \mathcal{B}) &= (\mathcal{U} \cap \mathcal{A}) \cup ((\mathcal{U} \cup \mathcal{V}) \cap \mathcal{B}) \\ &\subseteq \mathcal{A}_1 \cup \mathcal{B},\end{aligned}$$

then $\mathcal{A}_1 \cup \mathcal{B}$ is generic.

Note. The perturbation needed to achieve hyperbolicity in the case of a periodic orbit and a singularity follow the same spirit. However, to prove it in the case of a singular point is much easier because the Jacobi equation (the linear part of the flow) is autonomous. We suggest it is read first.

We have to prove that $\mathcal{A}_1$ is dense in $\mathcal{A}$, i.e. given that a Lagrangian L has a unique minimizing measure supported on a periodic orbit γ, then there exists a perturbation by an arbitrarily small C^∞-potential ϕ, such that the new Lagrangian $L + \phi$ has a unique minimizing measure supported on a hyperbolic periodic orbit.

Fix $\rho \in \mathcal{A}$ and let Γ be the periodic orbit in $\mathcal{M}(L + \rho)$, we can assume that $\rho = 0$. By the graph property (cf. [**25, 29**]), the projection $\pi|_\Gamma : \Gamma \to M$, $\pi(x, v) = x$ is injective. In particular $\pi(\Gamma) \subseteq M$ is a simple closed curve. Choose coordinates $\overline{x} = (x_1, \dots, x_n) : U \to S^1 \times \mathbb{R}^{n-1}$ on a tubular neighbourhood of $\pi(\Gamma)$ such that $\overline{x}(\Gamma) = S^1 \times \{\mathbf{0}\}$ and $\{\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_n\}$ is an orthonormal frame over the points of $\pi(\Gamma)$. In particular $\partial/\partial x_1$ is parallel to $\pi(\Gamma)$. Define $\widetilde{L} = L + \phi$ with $\phi(x_1, \dots, x_n) = \frac{1}{2} f(\overline{x})\varepsilon(x_2^2 + x_3^2 + \dots + x_n^2)$, where $f(\overline{x})$ is a C^∞ non-negative bump function with support in U which is one on a small neighbourhood of $\pi(\Gamma)$. Clearly, ϕ can be made C^∞ arbitrarily small. On $\pi(\Gamma)$ we have that $\widetilde{L}_{vv} = L_{vv}$, $\widetilde{L}_{vx} = L_{vx}$ and $\widetilde{L}_{xx} = L_{xx} + \varepsilon \left[\begin{smallmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I \end{smallmatrix}\right]$. Observe that since $\widetilde{L} \geq L$ then $\mathcal{M}(L + \phi) = \{\mu_\Gamma\}$.

Let H and $\widetilde{H} = H - \phi$ be the Hamiltonians associated to L and $\widetilde{L}$ respectively. Let Γ be the corresponding periodic orbit for H and $\widetilde{H}$ on T^*M. We have to prove that the periodic orbit Γ is hyperbolic for the flow of $\widetilde{H}$. Since it is minimizing, by Corollary 4.2 the orbit Γ has no conjugate points. By Proposition B it is enough to prove that the Green bundles $\widetilde{\mathbb{E}}$, $\widetilde{\mathbb{F}}$ satisfy $\widetilde{\mathbb{E}}(\theta) \cap \widetilde{\mathbb{F}}(\theta) = \langle \widetilde{X}(\theta) \rangle$ on a point $\theta \in \Gamma$, where $\widetilde{X}$ is the Hamiltonian vectorfield of $\widetilde{H}$.

Fix $\theta \in \Gamma$ and let $\widetilde{Z}_T(t) = d\pi \circ d\widetilde{\psi}_t \circ [d\pi|_{d\widetilde{\psi}_{-T}(V(\widetilde{\psi}_T(\theta)))}]^{-1}$. Let $(\widetilde{Z}_T, \widetilde{V}_T)$ and $\widetilde{K}_T(t) = \widetilde{V}_T(t)(\widetilde{Z}_T(t))^{-1}$, $0 \le t < T$, be the corresponding solutions of the Jacobi equation (6) and the Ricatti equation (7) respectively: $\mathrm{graph}(\widetilde{K}_T(t)) = d\widetilde{\psi}_t(d\widetilde{\psi}_{-T}(V(\widetilde{\psi}_T\theta)))$, $\widetilde{Z}_T(T) = 0$, $\widetilde{Z}_T(0) = I$.

Define $\boldsymbol{N}(\theta) := \{w \in T_{\pi(\theta)}M \mid \langle w, \dot{\gamma}\rangle = 0\}$. Then $\boldsymbol{N}(\theta)$ is the subspace of $T_{\pi(\theta)}M$ generated by the vectors $\partial/\partial x_2, \dots, \partial/\partial x_n$. Let $v_0 \in \boldsymbol{N}(\theta)$, $|v_0| = 1$ and let $\xi^T(t) := \widetilde{Z}_T(t)v_0$. Denote by $\widetilde{I}_T$ and I_T the index forms on $[0, T]$ for $\widetilde{L}$ and L respectively. Using the solution $(\widetilde{Z}_T, \widetilde{V}_T)$ on formula (53), we obtain that

$$\widetilde{I}_T(\xi^T, \xi^T) = -(\widetilde{Z}_T(0)v_0)^*(\widetilde{V}_T(0)v_0) = -v_0{}^*\widetilde{K}_T(0)v_0. \tag{57}$$

Moreover, in the coordinates $(x_1, \dots, x_n; \partial/\partial x_1, \dots, \partial/\partial x_n)$ on TU we have that

$$\begin{aligned}\widetilde{I}_T(\xi^T, \xi^T) &= \int_0^T (\dot{\xi}^T\widetilde{L}_{vv}\dot{\xi}^T + 2\dot{\xi}^T\widetilde{L}_{xv}\xi^T + \xi^T\widetilde{L}_{xx}\xi^T)\,dt \\ &= \int_0^T (\dot{\xi}^T L_{vv}\dot{\xi}^T + 2\dot{\xi}^T L_{xv}\xi^T + \xi^T L_{xx}\xi^T)\,dt + \int_0^T \varepsilon\sum_{i=2}^n |\xi_i^T|^2\,dt. \end{aligned}\tag{58}$$

We have that $\widetilde{Z}_T(0) = I$ and for all $t > 0$, $\lim_{t\to\infty}\widetilde{Z}_T(t) = \widetilde{\boldsymbol{h}}(t)$ with $\widetilde{\boldsymbol{h}}(t)$ given on Claim 1.3 for $\widetilde{H}$. Writing $\pi_N(\xi) = (\xi_2, \xi_3, \dots \xi_n)$ then $|\pi_N\widetilde{\boldsymbol{h}}(0)v_0| = |v_0| = 1$ because $v_0 \in \mathbf{N}(\theta)$. Hence, there exists $\lambda > 0$ and $T_0 > 0$ such that $|\pi_N\xi^T(t)| = |\pi_N\widetilde{Z}_T(t)v_0| > \frac{1}{2}$ for all $0 \le t \le \lambda$ and $T > T_0$. Therefore,

$$\widetilde{I}_T(\xi^T, \xi^T) \ge I_T(\xi^T, \xi^T) + \frac{\varepsilon\lambda}{4}. \tag{59}$$

Let $(\boldsymbol{h}(t), \boldsymbol{v}(t)) = d\psi_t \circ (d\pi|_{\mathbb{E}(\theta)})^{-1}$ be the solution of the Jacobi equation for H given by Claim 1.3 and let $\mathbb{S}(\psi_t(\theta)) = \boldsymbol{v}(t)\boldsymbol{h}(t)^{-1}$ be the corresponding solution of the Ricatti equation, with $\mathrm{graph}[\mathbb{S}(\psi_t(\theta))] = \mathbb{E}(\psi_t(\theta))$. Using formula (53), and writing $\xi^T(t) = \boldsymbol{h}(t)\zeta(t)$, we have that

$$\begin{aligned}I_T(\xi^T, \xi^T) &= \int_0^T (\boldsymbol{h}\dot{\zeta})^* H_{pp}^{-1}(\boldsymbol{h}\dot{\zeta})\,dt + 0 - (\boldsymbol{h}(0)\zeta(0))^*(\boldsymbol{v}(0)\zeta(0)), \\ I_T(\xi^T, \xi^T) &\ge -v_0^*\mathbb{S}(\theta)v_0. \end{aligned}\tag{60}$$

From (57), (59) and (60), we get that

$$v_0^*\mathbb{S}(\theta)v_0 \ge v_0^*\widetilde{K}_T v_0 + \frac{\varepsilon\lambda}{4}.$$

From Proposition 1.4, we have that $\lim_{T\to+\infty}\widetilde{K}_T(0) = \widetilde{\mathbb{S}}(\theta)$, where $\mathrm{graph}(\widetilde{\mathbb{S}}(\theta)) = \widetilde{\mathbb{E}}(\theta)$, the stable Green bundle for $\widetilde{H}$. Therefore,

$$v_0^*\mathbb{S}(\theta)v_0 \ge v_0^*\widetilde{\mathbb{S}}(\theta)v_0 + \frac{\varepsilon\lambda}{4}. \tag{61}$$

Similarly, for the unstable Green bundles we obtain that

$$v_0^*\mathbb{U}(\theta)v_0 + \lambda_2 \le v_0^*\widetilde{\mathbb{U}}(\theta)v_0 \quad \text{for } v_0 \in \boldsymbol{N}(\theta), |v_0| = 1, \tag{62}$$

for some $\lambda_2 > 0$ independent of v_0.

From Proposition 1.4 we have that $\mathbb{U}(\theta) \succcurlyeq \mathbb{S}(\theta)$. From (61) and (62) we get that $\widetilde{\mathbb{U}}|_N \succ \mathbb{U}|_N \succcurlyeq \mathbb{S}|_N \succ \widetilde{\mathbb{S}}|_N$. Since $\widetilde{\mathbb{E}}(\theta) = \text{graph}(\widetilde{\mathbb{S}}(\theta))$ and $\widetilde{\mathbb{F}}(\theta) = \text{graph}(\widetilde{\mathbb{U}}(\theta))$, we get that $\widetilde{\mathbb{E}}(\theta) \cap \widetilde{\mathbb{F}}(\theta) \subseteq \langle \widetilde{X}(\theta) \rangle$. Then Proposition B shows that Γ is a hyperbolic periodic orbit for $L + \phi$.

This proves that $\mathcal{A}_1$ is dense in $\mathcal{A}$. Let $\mathcal{A}_2$ be the subset of $\mathcal{A}_1$ of potentials ψ for which the minimizing hyperbolic periodic orbit Γ has transversal intersections $W^s(\Gamma) \pitchfork W^u(\Gamma)$. Then $\mathcal{A}_2$ is relatively open on $\mathcal{A}_1$. Indeed, if $\phi_n \in \mathcal{A}_1$ and $\phi_n \to \psi \in \mathcal{A}_2$ and $\mathcal{M}(L+\phi_n) = \{\mu_{\gamma_n}\}$, $\mathcal{M}(L+\psi) = \{\mu_\Gamma\}$, we have seen that the γ_ns are continuations of the hyperbolic orbit Γ for nearby flows (on nearby energy levels): $\gamma_n = \Gamma_{\phi_n, h_n}$. Since compact subsets (fundamental domains) of the invariant manifolds $W^s(\Gamma_{\phi,h})$ depend continuously in the C^1 topology on (ϕ, h), then the transversality property $W^s(\Gamma_{\phi,h}) \pitchfork W^u(\Gamma_{\phi,h})$ persists on a neighbourhood of (ψ, h_0). In particular, it holds for $\gamma_n = \Gamma_{\phi_n, h_n}$ for (ϕ_n, h_n) sufficiently close to (ψ, h_0).

By the same arguments as above, it is enough to prove that $\mathcal{A}_2$ is dense in $\mathcal{A}$. For it we need the following lemma.

LEMMA 5.2. *Let Γ be a hyperbolic periodic orbit without conjugate points of a convex Hamiltonian. Then for all $\theta \in W^s(\Gamma)$ there exists $S = S(\theta) > 0$ such that the segment $\{\psi_t(\theta) \mid t \geq S\}$ has no conjugate points.*

Proof. Let $\mathbb{E}$ be the stable Green subspace of Γ. Then $\mathbb{E}(\vartheta) \cap V(\vartheta) = \{0\}$ for all $\vartheta \in \Gamma$. Since the weak stable manifold $W^s(\Gamma)$ is tangent to $\mathbb{E}$; then for all $\theta \in W^s(\Gamma)$ there exists $S = S(\theta) > 0$ such that $E(t) := T_{\psi_t(\theta)} W^s(\Gamma)$ satisfies $E(t) \cap V(\psi_t(\theta)) = \{0\}$. Moreover, $\dim E(t) = n$ and for all $u \in E(t)$, $\lim_{s \to +\infty} d\psi_s(u) \in \langle X(\theta) \rangle$. Since ψ_t preserves the symplectic form ω, we have that $\omega(u, v) = \lim_{s \to +\infty} \omega(d\psi_s(u), d\psi_s(v)) = 0$ for all $u, v \in E(t)$. Hence, $E(t)$ is Lagrangian and $E(t) \cap V(\psi_t(\theta)) = \{0\}$ for all $t > S$. Then the Lemma follows from Proposition 1.15. $\square$

Now we prove that $\mathcal{A}_2$ is dense in $\mathcal{A}$. Let $\rho \in \mathcal{A} \setminus \mathcal{A}_2$ and make the perturbation $\widetilde{L} = (L + \rho) + \phi$ explained above, so that $\rho + \phi \in \mathcal{A}_1$. We can assume that $\rho + \phi = 0$ and also that $\rho + \phi \notin \mathcal{A}_2$, otherwise there is nothing to prove. Write $\mathcal{M}(L) = \{\mu_\Gamma\}$ and let $\theta \in W^s(\Gamma) \cap W^u(\Gamma)$ be such that $\dim(T_\theta W^s(\Gamma) \cap T_\theta W^u(\Gamma)) > 1$. Observe that the α and ω limit sets of θ are Γ. In particular, the orbit of θ has no autoaccumulation points. Let $S = S(\theta) > 0$ be from Lemma 5.2 and $\epsilon > 0$ small. Then there exists $\tau > S(\theta)$ and a neighbourhood $W \subset M$ of $\pi(\psi_t(\theta))$ such that $\{t \in \mathbb{R} \mid \pi\psi_t(\theta) \in W\} =]\tau, \tau + 3\epsilon[$ and $W \cap \pi(\Gamma) = \varnothing$. Let $U \subset M \times \mathbb{R}$ be a tubular neighbourhood of $\{(\pi\psi_t(\theta); t) \mid t > S(\theta)\}$. Choose coordinates $\overline{y} = (y_1, \dots, y_n; t) : U \to \mathbb{R}^n \times \mathbb{R}$ such that $\overline{y}(\pi\psi_t(\theta), t) = (\alpha(t), \mathbf{0}; t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$, and $\{\partial/\partial y_1, \partial/\partial y_2, \dots, \partial/\partial y_n\}$ is an orthonormal frame over the points $\pi\psi_t(\theta)$, $t > S(\theta)$. Let $\varphi(y_1, \dots, y_n) = \frac{1}{2} f(\overline{y})\delta(y_2^2 + y_3^2 + \cdots + y_n^2)$, where $f(\overline{y})$ is a C^∞ bump function with support in W which is one on a small neighbourhood of $\{\pi\psi_t(\theta) \mid t \in [\tau + \epsilon, \tau + 2\epsilon]\}$. By choosing a small $\delta > 0$ the function φ can be made C^∞ arbitrarily small. Choose δ small enough such that the orbit Γ remains hyperbolic. Then we still have that $\mathcal{M}(L + \phi) = \{\mu_\Gamma\}$ and the orbit of θ is the same for both flows. Write $\widetilde{L} = L + \rho + \phi$, $\widehat{L} = \widetilde{L} + \psi$ and $\widetilde{I}_T$, $\widehat{I}_T$ the corresponding index forms on $\{\psi_t(\theta) \mid t \in [\tau, T]\}$. Since $\{\psi_t(\theta) \mid t > S(\theta)\}$ has no conjugate points, the stable Green subspace $\widehat{\mathbb{E}}(\psi_t(\theta))$, $t > S(\theta)$ and the stable solution of the Ricatti equation

$\widehat{\mathbb{S}}(\psi_t(\theta))$ exist. The same arguments as in equations (58)–(61) give for $\xi^T = \widehat{Z}_T(t)v_0$, $v_0 \in \boldsymbol{N}(\psi_\tau(\theta))$, $|v_0| = 1$, $\widehat{Z}_T(t) = d\pi \circ d\widehat{\psi}_{t-T} \circ [d\pi|_{d\psi_{-T}(V(\widehat{\psi}_{T+\tau}(\theta)))}]^{-1}$, that

$$\widehat{I}_T(\xi^T, \xi^T) = \widetilde{I}_T(\xi^T, \xi^T) + \int_{\tau+\epsilon}^{\tau+2\epsilon} \delta \sum_{i=2}^{n} |\xi_i^T|^2 \, dt,$$

$$v_0^* \mathbb{S}(\psi_\tau(\theta)) v_0 \geq v_0 \widehat{\mathbb{S}}(\psi_\tau(\theta)) v_0 + \frac{\delta \widehat{\lambda}}{4},$$

for some $\widehat{\lambda} > 0$. In particular

$$\widehat{\mathbb{E}}(\psi_\tau(\theta)) \cap \widetilde{\mathbb{E}}(\psi_\tau(\theta)) = \langle X(\psi_\tau \theta) \rangle. \tag{63}$$

By the same arguments as for Proposition 1.14, we have that $\widehat{\mathbb{E}}(\psi_\tau(\theta)) = T_{\psi_\tau(\theta)}(W^u_{\widetilde{L}}(\psi_\tau\theta))$. Observe that $T_{\psi_\tau(\theta)}(W^u_{\widetilde{L}}(\psi_\tau\theta))$ did not change. Write $T_{\psi_\tau(\theta)}(W^u_{\widetilde{L}}(\psi_\tau\theta)) = \mathbb{V} \oplus \mathbb{W}$, with $\mathbb{V} \subseteq \widetilde{\mathbb{E}}(\psi_\tau\theta)$ and $\mathbb{W} \cap \widetilde{\mathbb{E}}(\psi_\tau\theta) = \{0\}$. By diminishing δ if necessary we can get (63) and $\widehat{\mathbb{E}}(\psi_\tau\theta) \cap \mathbb{W} = \{0\}$. Thus,

$$T_{\widehat{\psi}_\tau(\theta)} W^s_{\widetilde{L}}(\Gamma) \cap T_{\widehat{\psi}_\tau(\theta)} W^u_{\widetilde{L}}(\Gamma) = \widehat{\mathbb{E}}(\psi_\tau\theta) \cap T_{\psi_\tau(\theta)} W^u(\Gamma) = \langle \widehat{X}(\psi_\tau\theta) \rangle.$$

Finally, by a finite number of these perturbations on a fundamental domain of $W^s(\Gamma)$ one can remove all tangencies of $W^s(\Gamma)$ and $W^u(\Gamma)$ and obtain a potential in $\mathcal{A}_2$ arbitrarily C^∞-near $\rho \in \mathcal{A}$. □

The case of a singularity.
Suppose that the minimizing measure μ is supported on a singularity $(x_0, 0)$ of the Lagrangian flow. From the Euler–Lagrange equation (2) we get that $L_x(x_0, 0) = 0$. Differentiating the energy function (3) we see that $(x_0, 0)$ is a singularity of the energy level $c(L)$. Moreover, the minimizing property of μ implies that x_0 is a minimum of the function $x \mapsto L_{xx}(x, 0)$. In particular, $L_{xx}(x_0, 0)$ is positive semidefinite in linear coordinates in $T_{x_0}M$.

Choose coordinates $\overline{y} = \{y_1, \ldots, y_n\}$ on a neighborhood of x_0 such that $\overline{y}(x_0) = \mathbf{0}$ and $L_{vv}(x_0, 0)$ is the identity on the basis $\{\partial/\partial y_1|_{x_0}, \ldots, \partial/\partial y_n|_{x_0}\}$. Use the coordinates $(\{y_1, \ldots, y_n\}, \{\partial/\partial y_1, \ldots, \partial/\partial y_n\})$ on a neighbourhood of x_0. On the orbit $\varphi_t(x_0, 0) \equiv (x_0, 0)$ the matrices L_{xx}, L_{xv}, L_{vv} are constant with respect to the time t. The Jacobi equation (47) at $(x_0, 0)$ becomes

$$\ddot{k} = L_{xx}(x_0, 0)\, k,$$

where $L_{vv}(x_0, 0) = I$. Let f be a non-negative C^∞ function on $\mathbb{R}^n$ with support on $|y| < \varepsilon < 1$ and $f \equiv 1$ on a neighbourhood of $\mathbf{0}$. Let $\phi(y) = \frac{1}{2}\delta f(y)(y_1^2 + \cdots + y_n^2)$. Adjusting $\delta > 0$ we can make ϕ arbitrarily C^∞-small. Let $\widetilde{L} := L + \phi$. Then the atomic measure μ supported on $(x_0, 0)$ is still the unique minimizing measure for $\widetilde{L}$ and the Jacobi equation for the orbit $\widetilde{\varphi}_t(x_0, 0) \equiv (x_0, 0)$ is

$$\ddot{k} = (L_{xx}(x_0, 0) + \delta I)\dot{k}. \tag{64}$$

Since $L_{xx}(x_0, 0)$ is positive semidefinite, then $A := (L_{xx}(x_0, 0) + \delta I)$ is positive definite. Equation (64) is linear

$$\begin{bmatrix} k' \\ k'' \end{bmatrix} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} \begin{bmatrix} k \\ k' \end{bmatrix} \tag{65}$$

with constant coefficients, where $k \equiv 0$ is a hyperbolic singularity.

To obtain the transversality property $W^s \pitchfork W^u$ at homoclinic orbits we can use the same arguments as above if we have a replacement for Lemma 5.2. The arguments of Lemma 5.2 apply if we show that the stable subspace of $(x_0, 0)$ is transversal to the vertical subspace. It is enough to prove that the eigenvectors (x, y) of the (non-singular) matrix in (65) have coordinate $x \neq 0$. These eigenvectors satisfy $y = \lambda x$, $Ax = \lambda^2 x$, $(\lambda \in \mathbb{R})$. Hence, if $x = 0$, then $(x, y) = (0, 0)$ is not an eigenvector. □

6. *The exponential map*

The following lemma was proven for geodesic flows by Freire and Mañé [**16**] and has intrinsic interest (compare with Proposition 1.10).

LEMMA 6.1. *Let H be a convex Hamiltonian. Choose a Riemannian metric $\|\cdot\|$ on M. If ψ_t has no conjugate points on $\Sigma = H^{-1}\{e\}$, then there exists $B > 0$ such that*

$$\|Y_\theta(t) \cdot v\| > B\|v\|$$

for all $\theta \in \Sigma$, $|t| > 1$ and $v \in T_\theta M$, where $Y_\theta(t) \cdot v = d\pi \circ d\psi_t(\theta) \cdot (0, v)$, $(0, v) \in H(\theta) \oplus V(\theta)$.

We outline the proof of Theorem F. In the case in which $\Theta(X) > 0$ on Σ one can define the vectorfield $Y = 1/|\Theta(X)|X$. This vectorfield preserves the 1-form Θ on $T\Sigma$. Since Σ has no conjugate points, then $\pi(\Sigma) = M$, (cf. Corollary 1.13) and we can define a Hamiltonian on T^*M by $\mathbb{H}(t\theta) = t^2$, where $\theta \in \Sigma$. This new Hamiltonian $\mathbb{H}$ is convex and has no conjugate points in Σ. The Hamiltonian vectorfield $\mathbb{Y}$ of $\mathbb{H}$ is called the (convex) *symplectification* of the contact vectorfield Y. The transversal bundle $\boldsymbol{N}(\theta) = \ker \Theta|_{T_\theta\Sigma}$ is invariant under the flow of $\mathbb{Y}$. Let $E_q : T_q\tilde{M} \to \tilde{M}$ be the exponential map for $(\mathbb{Y}, \Sigma)$. The derivative $dE_q(t\theta)$ is given in formulas (39) and (40). The norm of the directional derivative in formula (40) is clearly bounded below. By Lemma 6.1, the norm of the derivative $dE_q|_{V(\theta)\cap T_\theta\Sigma}$ in formula (39) is uniformly bounded away from zero. Since $V(\theta) \cap T_\theta\Sigma \subset \boldsymbol{N}(\theta)$, and the angle $\sphericalangle(\boldsymbol{N}(\theta), X(\theta))$ is bounded below, we obtain that $\|dE_q\|$ is bounded away from zero. This implies that $E_q : T_q\tilde{M} \to \tilde{M}$ is a covering map and hence, a diffeomorphism. In particular, the universal cover $\tilde{M}$ of M is homeomorphic to $\mathbb{R}^n$. By Corollary 1.18, the exponential map $\exp_q$ for X is a local diffeomorphism. Once we know that E_q is a diffeomorphism, it is easy to show that $\exp_q$ is bijective and hence, a diffeomorphism.

Proof of Lemma 6.1. Since $Y_\theta(t)$ satisfies (8) (with H_{qp} and H_{pp} uniformly bounded on Σ) and by Corollary 1.8, $S_\theta(t)$ is uniformly bounded for $|t| > 1$, then there exists $D_1 > 1$ such that

$$\|\dot{Y}_\theta(t) \cdot w\| \leq D_1\|Y_\theta(t) \cdot w\| \quad \text{for all } |t| > 1,\ w \in V(\theta),\ \theta \in \Sigma.$$

Let

$$A := (1 + D_1)^2 \sup_{\theta\in\Sigma} \|H_{pp}(\theta)^{-1}\|.$$

Let $H_\theta(t) := Y_{\psi_{-2}(\theta)}(t+2) \circ (Y_{\psi_{-2}(\theta)}(2))^{-1}$, then $\det H_\theta(t) \neq 0$ for $t > -2$, $H_\theta(0) = I$ and there exists a linear homomorphism $V_\theta(t) : H(\theta) \to V(\psi_t(\theta))$ such that (H_θ, V_θ) is a matrix solution of the Jacobi equation (6) and Image$(H_\theta(t), V_\theta(t)) = d\psi_{t+2}(V(\psi_{-2}(\theta)))$ is a Lagrangian subspace. Let

$$D_2 := \max_{\substack{-2\le t\le 2\\ \theta\in\Sigma}} \|H_\theta(t)^*(H_{pp})^{-1}H_\theta(t)\| < +\infty.$$

Let $\rho : \mathbb{R} \to [0,1]$ be a C^∞-function such that $\rho(t) = 0$ for $|t| \ge 1$ and $\rho(0) = 1$. Let

$$C := D_2 \int_{-1}^{1} |\rho'(t)|^2\,dt.$$

Let

$$0 < B < (AC)^{1/2}. \tag{66}$$

Suppose that $\|Y_\theta(T)v_0\| < B\|v_0\|$ for some $\theta \in \Sigma$, $v_0 \in T_{\pi(\theta)}M$, $\|v_0\| = 1$ and $|T| > 1$. Assume that $T > 1$, the case $T < -1$ is proven similarly. Let $J(t) \in H(\psi_t(\theta)) = T_{\pi\psi_t(\theta)}M$ be defined by $J(t) = 0$ for $-1 \le t \le 0$, $J(t) = Y_\theta(t)v_0$ for $0 \le t \le T$ and $J(t) = (T+1-t)\tau_t J(T)$ for $T \le t \le T+1$, where $\tau_t : T_{\pi\psi_T(\theta)}M \to T_{\pi\psi_t(\theta)}M$ is the parallel transport along the path $t \mapsto \pi\psi_t(\theta)$. Then $J(t)$ is continuous and piecewise C^2.

Let $v(t) \in V(\theta) \approx T_{\pi(\theta)}M$ be defined by $J(t) = Y_\theta(t)v(t)$. Then $v(t) = 0$ for $-1 \le t \le 0$ and $v(t) = v_0$ for $0 \le t \le T$. Let $\{e_1, \dots, e_n\}$ be a basis of $T_{\pi(\theta)}M$ and write $v(t) = \sum_{i=1}^n v_i(t)e_i$, $Y_i(t) = Y(t)e_i$. Then the covariant derivative along $\pi\psi_t(\theta)$ of $J(t) = Y(t)v(t)$ is

$$\frac{D}{dt}(Yv) = \frac{D}{dt}\left(\sum_{i=1}^n v_i(t)Y_i(t)\right) = \sum_i \left(\frac{DY_i}{dt}\right) v_i + \sum_i \left(\frac{dv_i}{dt}\right) Y_i,$$

$$Yv' = \frac{D}{dt}(Yv) - \frac{DY}{dt}v.$$

For $T \le t \le T+1$, since $J(t) = Y_\theta(t)v(t) = (T+1-t)\tau_t J(T)$, we have that

$$\begin{aligned}\|Yv'\| &\le \left\|\frac{D}{dt}(Yv)\right\| + \left\|\frac{DY}{dt}v\right\|\\ &\le \left\|\frac{D}{dt}J(t)\right\| + D_1\|J(t)\|\\ &\le \|J(T)\| + D_1\|J(T)\|\\ &\le (1+D_1)B.\end{aligned}$$

Using $Y_\theta(t)$ on formula (55) we have that

$$\begin{aligned}I(J,J) &= \int_T^{T+1} (Yv')^*(H_{pp})^{-1}(Yv')\,dt\\ &\le (1+D_1)^2B^2\|H_{pp}^{-1}\| = AB^2.\end{aligned}$$

Let $Z(t) \in H(\psi_t(\theta)) = T_{\pi\psi_t(\theta)}M$ be defined by $Z(t) = \rho(t)H_\theta(t)v_0$. Then $Z(t) = Y_\theta(t)\xi(t)$ for $t \ne 0$, with $\xi(t) = 0$ for $|t| > 1$. Since $d\psi_t(\theta)\cdot V(\theta) =$ Image$(Y_\theta(t), S_\theta(t)Y_\theta(t))$, we have that $S_\theta(t)Y_\theta(t) \to I$ when $t \to 0$. Using $Y_\theta(t)$ on formula (55), we have that the only non-zero term is

$$I(Z,J) = -Z(0^+)\cdot v_0 + Z(0^-)\cdot 0 = -\|v_0\|^2 = -1.$$

Write $Z(t) = H(t)\zeta(t)$, $\zeta(t) = \rho(t)v_0$. Using $H(t)$ and $V(t)$ on formula (53), we have that

$$\begin{aligned} I(Z, Z) &= \int_{-1}^{1} (H\zeta')^*(H_{pp})^{-1}(H\zeta')\, dt \\ &= \int_{-1}^{1} (\rho'(t))^2 (H(t)v_0)^*(H_{pp}^{-1})(H(t)v_0)\, dt \\ &\leq D_2 \int_{-1}^{1} |\rho'(t)|^2\, dt = C. \end{aligned}$$

Since by (48) the index form is symmetric, for $\lambda \in \mathbb{R}$ we have that

$$I(Z - \lambda J, Z - \lambda J) \leq C + 2\lambda + \lambda^2 AB^2.$$

Since by (66) $4 - 4AB^2C > 0$, this polynomial in λ has two real roots. Therefore, $I(Z - \lambda J, Z - \lambda J) < 0$ for some value of λ. Hence, there must be conjugate points in the orbit segment $\{\psi_t(\theta) \mid t \in [-1, T + 1]\}$. □

PROPOSITION 6.2. *Let e be a regular value of H. If $\Theta(X) > 0$ on the energy level $\Sigma = H^{-1}\{e\}$ and the flow of X has no conjugate points, then there exists a Hamiltonian $\mathbb{H}$ on T^*M with vectorfield Z such that:*

(i) *$\mathbb{H}$ is convex and $\mathbb{H}^{-1}\{e\} = \Sigma$;*
(ii) *$Z = 1/\theta(X) \cdot X$ on Σ;*
(iii) *the flow ζ_t of Z has no conjugate points on Σ;*
(iv) *ζ_t preserves the 1-form Θ on $T\Sigma$;*
(v) *if $N(\theta) := (d/ds)s\theta|_{s=1}$, then for $\theta \in \Sigma$ we have*

$$d\zeta_t(\theta) \cdot N(\theta) = N(\zeta_t(\theta)) + tZ(\zeta_t(\theta)).$$

Remark 6.3. This proposition holds if $\Theta(X) \neq 0$ on a connected compact forward invariant set $K \subset \Sigma$. In this case we obtain an open connected neighbourhood U of K such that $\inf_{\vartheta \in U} |\Theta(X(\vartheta))| > 0$ and a convex Hamiltonian $\mathbb{H}$ without conjugate points, defined on a neighbourhood of U in T^*M. To apply this on Theorem F, we can take $K =$ closure of $\psi_{\mathbb{R}^+}((\pi|_\Sigma)^{-1}\{q\}) \subseteq \Sigma$.

Remark 6.4. Given $q \in \pi\Sigma$, let $S(q) := T_q^*M \cap \Sigma$ be the energy sphere at q. Then it is impossible to have $\Theta(X(p)) < 0$ on all $p \in S(q)$ and the condition $\Theta(X(q, p)) > 0$ is equivalent to $H(q, 0) < e$, i.e. the zero section $M \times 0$ lies inside the (vertical convex hull of the) energy level σ. Indeed, $S(q)$ is convex and if $H(q, 0) < e$ then, since for $p \in S(q)$ the vector $H_p(q, p)$ is outwards normal to $S(q)$, then $\Theta(X(q, p)) = p \cdot H_p(q, p) > 0$. Otherwise, if $H(q, 0) \geq e$, then there exists a supporting hyperplane E for $S(q)$ in T_q^*M containing $(q, 0)$. Let $p \in E \cap S(q)$. Then $\Theta(X(q, p)) = p \cdot H_p(q, p) = 0$ because the vector p is tangent to $S(q)$ at (q, p).

Remark 6.5. By Remark 6.4, the condition $\Theta(X) \neq 0$ is equivalent to $H(q, 0) < e$, $\forall q \in M$. We can extend the result of Theorem F to the case $H(q, d_q f) < e$, $\forall q \in M$ for some differentiable function $f : M \to \mathbb{R}$ because the flow of the new Hamiltonian

$H_{df}(q,p) := H(q, p + d_q f)$ is conjugate to the flow of H on $H = e$ and satisfies $H_{df}(q,0) < e$, $\forall q \in M$. See Theorem A in [**9**] for a characterization of the energy levels where the last condition holds. See [**10**] for other proofs of Theorem F and its converse.

Proof of Proposition 6.2. Recall that $\omega(X,\cdot) = -dH$. Let $\Theta = p\,dq$ be the canonical 1-form on T^*M. Suppose that $\Theta(X) > 0$ on Σ. Let Y be the vectorfield on Σ given by $Y = 1/\Theta(X)\cdot X$. Then Y preserves the 1-form Θ on $T\Sigma$ because the Lie derivative

$$
\begin{aligned}
L_Y\Theta &= di_Y\Theta + i_Y\,d\Theta = d[\Theta(Y)] + \omega(Y,\cdot)\\
&= d(1) - \frac{1}{\Theta(X)}dH = 0 \text{ on } T\Sigma.
\end{aligned}
$$

Let $\phi_t(q,p)$ be the flow of Y on Σ. Define the symplectification $\widetilde{\phi}_t$ of ϕ_t (cf. Arnold [**2, 3**]) on the cotangent bundle without its zero section $T^*M\setminus M$ by $\widetilde{\phi}_t(q,\lambda p) = \lambda\phi_t(q,p)$, $(q,p)\in\Sigma$. Observe that $\widetilde{\phi}_t$ preserves the canonical 1-form Θ. Indeed, we have that $\pi\circ\phi_t = \pi\circ\widetilde{\phi}_t$ and if $\phi_t(q,p) = (q_t,p_t)$,

$$
\begin{aligned}
\Theta(q_t,\lambda p_t)\cdot(d\widetilde{\phi}_t w) &= (\lambda p_t)\cdot d\pi(d\widetilde{\phi}_t w)\\
&= \lambda\,[p_t\cdot(d\pi\, d\phi_t w)]\\
&= \lambda\,[p\cdot(d\pi\, w)] = \Theta(q,\lambda p)\cdot w.
\end{aligned}
$$

Let $\widetilde{Y}$ be the vectorfield of $\widetilde{\phi}_t$. We have that

$$
L_{\widetilde{Y}}\Theta = d[\Theta(\widetilde{Y})] + \omega(\widetilde{Y},\cdot) = 0.
$$

Therefore, the function $\widetilde{H} = \Theta(\widetilde{Y})$ is a Hamiltonian for $\widetilde{Y}$. But $\widetilde{Y}$ has conjugate points on Σ and $\widetilde{H}$ is not convex. In fact, $\widetilde{H}$ is homogeneous of degree one on the fibres of T^*M.

Let $\mathbb{H} = \frac{1}{2}\widetilde{H}^2$ and let Z be the Hamiltonian vectorfield of $\mathbb{H}$. We have that $Z = \widetilde{H}\cdot\widetilde{Y}$. Since $\widetilde{H}|_\Sigma \equiv 1$, then Z is also an extension of Y. Let ζ_t be the flow of Z. Observe that

$$
\zeta_t(q,sp) = s\widetilde{\phi}_{t\widetilde{H}(q,sp)}(q,p) = s\widetilde{\phi}_{ts}(q,p), \quad \text{for } (q,p)\in\Sigma.
$$

Taking the derivative $d/ds|_{s=1}$, we obtain

$$
d\zeta_t(\theta)\cdot N(\theta) = N(\zeta_t(\theta)) + t\,Y(\zeta_t(\theta)) \quad \text{for } \theta\in\Sigma.
$$

Write $\zeta(t,\theta) = \psi(s(t,\theta),\theta)$ with $s(t,\theta) > 0$, $\theta\in\Sigma$. Then

$$
D\zeta_t = D\psi_s + X(\psi_s(\theta))\frac{\partial s}{\partial\theta} \quad \text{on } T\Sigma. \tag{67}
$$

Let $\mathbb{E}\subset T\Sigma$ be the stable Green bundle for X on Σ. Since $X(\theta)\in\mathbb{E}(\theta)$ and $\mathbb{E}$ is ψ_t-invariant, then equation (67) shows that the flow ζ_t of Z preserves the Lagrangian bundle $\mathbb{E}\subset T\Sigma$, which satisfies $\mathbb{E}(\theta)\cap V(\theta) = \{0\}$ for all $\theta\in\Sigma$. By Proposition 1.15, ζ_t has no conjugate points on Σ.

The following lemma completes the proof of Proposition 6.2.

LEMMA 6.6. *The Hamiltonian $\mathbb{H}$ is convex, i.e. $\mathbb{H}_{pp}$ is positive definite.*

Proof. Let $q \in \pi(\Sigma)$ and $S(q) := T_q^*M \cap \Sigma$. Since $\mathbb{H}$ is homogeneous of degree two, then $\mathbb{H}_{pp}$ is homogeneous of degree zero and it is enough to prove that $\mathbb{H}_{pp}(q, p)$ is positive definite for $p \in S(q)$. Let $p_0 \in S(q)$. Let $h : T_{p_0}S(q) \times \mathbb{R} \to T_q^*M$ be $h(x, \lambda) := x + \lambda p_0$. Let $G = \mathbb{H} \circ h$ and $K = \widetilde{H} \circ h$. Then $G = \frac{1}{2}K^2$. Since $D^2 h = 0$, then $D^2\mathbb{H}(q_0)$ is positive definite if $D^2G(\mathbf{0}, 1)$ is positive definite. Since $K(\mathbf{0}, \lambda) = \lambda$, we have that

$$G_{\lambda\lambda}(\mathbf{0}, 1) = [K_\lambda^2 + K \cdot K_{\lambda\lambda}]|_{(0,1)} = 1.$$

Since in the coordinates (x, λ) the vectors $\partial/\partial x_i|_{(\mathbf{0},1)}$, $i = 1, \dots, n-1$ are tangent at p_0 to $S(q)$ and $\widetilde{H} \equiv 1$ on $S(q)$, then $K_x(\mathbf{0}, 1) = \mathbf{0}$. Hence,

$$\begin{aligned} G_{xx}(\mathbf{0}, 1)(v, v) &= |K_x \cdot v|^2 + K K_{xx}(v, v) \\ &= K_{xx}(v, v). \end{aligned}$$

Observe that K is homogeneous of degree one. Therefore, $K_x(\mathbf{0}, \lambda) = K_x(\mathbf{0}, 1)$ for all $\lambda > 0$. Hence, $K_{x\lambda}(\mathbf{0}, 1) = 0$, so that

$$G_{x\lambda}(\mathbf{0}, 1) \cdot (v, \mu) = \mu\,[K_\lambda(K_x \cdot v) + K(K_{x\lambda} \cdot v)] = 0.$$

Then

$$\begin{aligned} D^2G(\mathbf{0}, 1)(v, \mu)^{(2)} &= \mu^2 G_{\lambda\lambda} + 2\mu(G_{\lambda x} \cdot v) + G_{xx}(v, v) \\ &= \mu^2 + K_{xx}(\mathbf{0}, 1) \cdot (v, v). \end{aligned}$$

So it is enough to see that $K_{xx}(\mathbf{0}, 1)$ is positive definite.

Let $v \in T_{p_0}S(q)$ be a unit vector and define $f : \mathbb{R} \to \mathbb{R}$ by $f(s, t) = H(t(p_0 + sv))$. Observe that $H_p(p_0) \cdot v = 0$ and $\partial f/\partial t|_{(0,1)} = p_0 \cdot H_p(p_0) = \Theta(X) > 0$. Let $t :]-\varepsilon, \varepsilon[\to \mathbb{R}$ be a local C^2-function such that $f(s, t(s)) = e$ and $t(0) = 1$. Differentiating this equation with respect to s we obtain

$$H_p[t'(s)(p_0 + sv) + t(s)v] = 0, \tag{68}$$

$$H_{pp}[t'(s)(p_0 + sv) + t(s)v]^{(2)} + H_p[t''(s)(p_0 + sv) + 2t'(s)v] = 0. \tag{69}$$

Evaluating (68) at $s = 0$, $t(0) = 1$, we have that $t'(0)(H_p \cdot p_0) = 0$. Since $H_p \cdot p_0 = \Theta(X) > 0$, then $t'(0) = 0$. Evaluating (69), we get that

$$t''(0) = \frac{-v^* H_{pp} v}{\Theta(X(p_0))} < -\delta$$

for some $\delta > 0$ uniform for all $(q, p_0) \in \Sigma$, $v \in T_{p_0}S(q)$, $|v| = 1$.

Since $K(sv, 1) = \widetilde{H}(p_0 + sv) = 1/t(s)$ and $t'(0) = 0$, $t(0) = 1$, then

$$v^* K_{xx} v = -t''(0) > \delta$$

for all $v \in T\Sigma$, $|v| = 1$. □

Proof of Theorem F. Through the proof of Theorem F we work with the lifted Hamiltonian to the cotangent bundle of the universal cover $\tilde{M}$ of M. Let ζ_t be the flow of the Hamiltonian $\mathbb{H}$ of Proposition 6.2. Let $q \in \tilde{M}$ and let $E_q : T_q^*M \to \tilde{M}$ be the exponential map associated to the flow ζ_t and the energy level Σ. We first prove that

E_q is a diffeomorphism. The derivative $dE_q(t\theta)$ is non-singular by Corollary 1.18 and is given by formulas (39) and (40). We claim that the norm $\|dE_q(t\theta)\|$ is uniformly bounded below. By Lemma 6.1, the norm $\|dE_q(t\theta)|_{V(\theta)\cap T_q\Sigma}\|$ is uniformly bounded below for $t > \epsilon$. From equation (40), $\|dE_q(t\theta)|_{\langle\theta\rangle}\|$ is also uniformly bounded below. It remains to see that the angle $\sphericalangle[dE_q(t\theta)(V(\theta)\cap T_q\Sigma), d\pi(X(\zeta_t\theta))]$ is uniformly bounded below.

Since ζ_t preserves the 1-form Θ, the bundle $\mathbf{K}(\theta) = \ker\Theta|_{T_\theta\Sigma}$ is invariant under $d\zeta_t$. Since $V(\theta)\cap T_q\Sigma \subset \mathbf{K}(\theta)$, then

$$dE_q(t\theta)(V(\theta)\cap T_q\Sigma) \subseteq d\pi\, d\zeta_t(\theta)(\mathbf{K}(\theta)) \subseteq d\pi(\mathbf{K}(\zeta_t(\theta))). \tag{70}$$

For $\theta = (q, p) \in \Sigma$ write $H_p(q, p) = \alpha p + k$, with $\langle k, p\rangle = 0$. Since $\Theta(X) = p\cdot H_p \neq 0$ on Σ there exists $0 \le \lambda < 1$ such that $|k| \le \lambda|H_p|$ on Σ. Let $h \in d\pi(\mathbf{K}(\zeta_t\theta))$ with $|h| = 1$, i.e. $\langle p, h\rangle = 0$, $|h| = 1$. Then

$$\begin{aligned}\langle h, H_p\rangle &= \alpha\langle p, h\rangle + \langle k, h\rangle = \langle k, h\rangle,\\ |\langle h, H_p\rangle| &\le 1\cdot\lambda|H_p|.\end{aligned}$$

Hence, $\cos\sphericalangle(h, H_p) \le \lambda < 1$. Therefore, the angle $\sphericalangle[d\pi(\mathbf{K}(\theta)), d\pi(X(\theta))]$ is uniformly bounded below on $\theta\in\Sigma$. By (70), $\sphericalangle[dE_q(t\theta)(V(\theta)\cap T_\theta\Sigma), d\pi(X(\zeta_t\theta))] > \sphericalangle[d\pi(\mathbf{K}(\zeta_t\theta)), d\pi(X(\zeta_t\theta))]$ is uniformly bounded below. This proves that $\|dE_q(t\theta)\|$ is uniformly bounded below on $t\in\mathbb{R}^+$, $\theta\in\Sigma$. Hence, the map $E_q : T_q\tilde{M} \to \tilde{M}$ is a differentiable covering map. Since $\tilde{M}$ is simply connected, it must be a diffeomorphism.

Now let $\exp_q : T_q\tilde{M} \to \tilde{M}$ be the exponential map of (ψ_t, Σ). Since ζ_t is a reparametrization of ψ_t for $t > 0$, $\theta\in\Sigma$, there exists $\tau(t,\theta) > 0$ such that $\psi_t(\theta) = \zeta_{\tau(t,\theta)}(\theta)$. By Corollary 1.18, $\exp_q$ is a local diffeomorphism. It remains to prove that it is bijective. Since $|\Theta(X)| = |X|/|Y|$ is bounded then $t\mapsto\tau(t,\theta)$ is a (bijective) homeomorphism of $\mathbb{R}^+$. This implies that the map $t\theta\mapsto\tau(t,\theta)\theta$, with $\theta\in T_q^*M\cap\Sigma$, is a homeomorphism of T_q^*M. Since E_q is bijective and $\exp_q(t\theta) = E_q(\tau(t,\theta)\theta)$, then $\exp_q$ is bijective. □

Appendix: examples

We learned the following formalism for twisted Hamiltonians on surfaces from G. Paternain and M. Paternain. Let M be a closed orientable Riemannian surface with Riemannian metric $\langle\,,\rangle_x$. Let $K : TTM \to TM$ be the connection map $K\xi = \nabla_{\dot{x}}v$, where $\xi = d/dt(x(t), v(t))$. Let $\pi : TM \to M$ be the canonical projection. Let ω_0 be the symplectic form in TM obtained by pulling back the canonical symplectic form via the Legendre transform associated to the Riemannian metric, i.e.

$$\omega_0(\xi,\zeta) = \langle d\pi\xi, K\zeta\rangle - \langle d\pi\zeta, K\xi\rangle.$$

Denote by Ω the area 2-form on M. Given a smooth function $F : M \to \mathbb{R}$, define a new symplectic form ω_F on TM by

$$\omega_F = \omega_0 + (F\circ\pi)(\pi^*\Omega).$$

This is called a *twisted symplectic structure* on TM. Let $H : TM \to \mathbb{R}$ be the Hamiltonian

$$H(x, v) = \tfrac{1}{2}\langle v, v\rangle_x.$$

Consider the Hamiltonian vectorfield X_F corresponding to (H, ω_F), i.e.

$$\omega_F(X_F(\theta), \cdot) = dH. \tag{71}$$

Define $Y : TM \to TM$ as the bundle map such that

$$\Omega_x(u, v) = \langle Y(u), v\rangle_x. \tag{72}$$

Then $Y(u) = iu$ is the angle of rotation $+\pi/2$. From (71) and (72) we have that

$$dH_\theta(\xi) = \omega_0(X_F(\theta), \xi) + F(\pi(\theta))\langle Y(d\pi X_F(\theta)), d\pi\xi\rangle_{\pi(\theta)}. \tag{73}$$

Hence, if $\xi = (\xi_1, \xi_2) = (d\pi\xi, K\xi) \in H(\theta) \oplus V(\theta)$ and $X_F = (X_1, X_2) \in H(\theta) \oplus V(\theta)$, equation (73) becomes

$$\begin{aligned} \langle \theta, \xi_2\rangle &= \langle X_1, \xi_2\rangle - \langle X_2, \xi_1\rangle + F\langle Y(X_1), \xi_1\rangle, \\ X_F(\theta) &= (\theta, FY(\theta)). \end{aligned} \tag{74}$$

In particular, the orbits of the Hamiltonian flow are of the form $(\gamma(t), \dot\gamma(t))$ for $\gamma : \mathbb{R} \to M$, with

$$\frac{D}{dt}\dot\gamma = F(\gamma)Y(\dot\gamma). \tag{75}$$

If $\int_M F\Omega = 0$ this flow can be seen as a Lagrangian flow as follows. The condition $\int_M F\Omega = 0$ implies that the cohomology class of $F\Omega$ is zero. Hence, there exists a 1-form η on M such that $d\eta = F\Omega$. Consider the Lagrangian

$$L(x, v) = \tfrac{1}{2}\langle v, v\rangle_x + \eta_x(v).$$

The corresponding Euler–Lagrange equation is

$$\frac{D}{dt}\langle\dot\gamma, \cdot\rangle = d\eta_x(\dot\gamma) = \langle FY(\dot\gamma), \cdot\rangle,$$

which gives the same differential equation (75). This flow corresponds to a magnetic field with Lorentz force $F(x)Y(v)$.

Observe that the theory developed in the previous sections also applies to the case in which $\int_M F\Omega \neq 0$. Indeed, let $q : U \subseteq M \to \mathbb{R}^2$ be a local chart defined on a simply connected domain $U \subset M$. Then the 2-form $F\Omega$ is exact on U and there exists a 1-form $\eta|_U$ such that $d\eta = F\Omega$ on U. The Hamiltonian flow on $TU \subset TM$ is the Lagrangian flow of $L(x, v) = \frac{1}{2}\langle v, v\rangle_x + \eta_x(v)$, $x \in U$. Let (q, p) be the natural chart on $T^*U \subseteq T^*M$ corresponding to the chart q and let $\mathcal{F}^{-1}$ be the inverse of the Legendre transform corresponding to L. Let $(q, w) = \mathcal{F}^{-1} \circ (q, p)$. Since $\mathcal{F}^*(d\Theta) = \omega_F$, then (q, w) is a symplectic chart for ω_F and sends the vertical bundle of $T_{q(U)}\mathbb{R}^n$ to the vertical bundle of $T_U M$.

We derive the Jacobi equation. Let $(c(t), \dot c(t))$ be an orbit of the flow. Consider a variation $\gamma_s(t) = f(s, t)$, where $f(0, t) = c(t)$ and the paths $t \mapsto f(s, t)$ are solutions of (75):

$$\frac{D}{dt}\dot\gamma_s = FY(\dot\gamma_s). \tag{76}$$

Denote the variational field of f by $J(t) = \partial f/\partial s(0, t)$. Using the formula

$$\frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} = \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}$$

and taking the covariant derivative of equation (76) in the direction $J(t)$, we obtain

$$\ddot{J} + R(\dot{c}, J)\dot{c} = \frac{D}{ds}[FY(\dot{\gamma}_s)].$$

Since $Y(v)$ is linear on the fibers of TM and $\dot{J} = (D/ds)(\partial f/\partial t) = (D/dt)(\partial f/\partial s)$, we have that

$$\begin{aligned}\frac{D}{ds}FY(\dot{\gamma}_s) &= [\nabla_J(FY)](\dot{\gamma}_s) + FY(\dot{J}),\\ &= (\nabla_J F)Y(\dot{c}) + FY(\dot{J}).\end{aligned}$$

Hence, we obtain the Jacobi equation

$$\ddot{J} + R(\dot{c}, J)\dot{c} - (\nabla_J F)Y(\dot{c}) - FY(\dot{J}) = 0. \tag{77}$$

Consider the orthonormal basis $\{\dot{c}, Y(\dot{c}) = i\dot{c}\}$ of $T_{c(t)}M$. We have that

$$\begin{aligned}\frac{D}{dt}\dot{c} &= FY(\dot{c}),\\ \frac{D}{dt}Y(\dot{c}) &= (\nabla_{\dot{c}}Y)(\dot{c}) + Y\left(\frac{D}{dt}\dot{c}\right)\\ &= -F\dot{c}.\end{aligned}$$

Write

$$J = x\dot{c} + yY(\dot{c}).$$

Then

$$\begin{aligned}\dot{J} &= (\dot{x} - Fy)\dot{c} + (\dot{y} + Fx)Y(\dot{c}),\\ \ddot{J} &= (\ddot{x} - 2F\dot{y} - \dot{F}y - F^2x)\dot{c} + (\ddot{y} + 2F\dot{x} + \dot{F}x - F^2y)Y(\dot{c}),\\ FY(\dot{J}) &= -F(\dot{y} + Fx)\dot{c} + F(\dot{x} - Fy)Y(\dot{c}).\end{aligned}$$

Replacing these formulas in equation (77) we obtain

$$\ddot{x} - (Fy)' = 0, \tag{78}$$

$$\ddot{y} + Ky + F\dot{x} = y(\nabla_{Y(\dot{c})}F), \tag{79}$$

where $K = \langle R(\dot{c}, Y(\dot{c}))\dot{c}, Y(\dot{c})\rangle$ is the sectional curvature along $c(t)$. Fix the energy level $\Sigma = H^{-1}\{\frac{1}{2}\} = SM$. If $(J, \dot{J}) \in T_{\dot{c}(t)}\Sigma \subset H(\theta) \oplus V(\theta)$, we have that $dH(J, \dot{J}) = \langle \dot{c}, \dot{J}\rangle = 0$. Hence,

$$\dot{x} - Fy = 0. \tag{80}$$

This implies equation (78), and from (79) and (80), we get

$$\ddot{y} + (K + F^2 - \nabla_{Y(\dot{c})}F)y = 0. \tag{81}$$

Hence, the Jacobi equations on $T_{\dot{c}(t)}\Sigma$ are given by (80) and (81).

Example A.1. A convex Hamiltonian without conjugate points and no continuous invariant transversal bundle.

Take $K \equiv -1$, $F \equiv 1$. Then equations (80) and (81) become

$$\begin{aligned} \dot{x} &= y, \\ \ddot{y} &= 0. \end{aligned} \tag{82}$$

The solutions have the form

$$\begin{aligned} x(t) &= \tfrac{1}{2}at^2 + bt + c, \\ y(t) &= at + b, \\ \dot{y}(t) &= a. \end{aligned} \tag{83}$$

The derivative of the Hamiltonian flow ψ_t on $(c(t), \dot{c}(t))$ is given by

$$\begin{aligned} d\psi_t(J(0), \dot{J}(0)) &= (J(t), \dot{J}(t)) \in H(\psi_t\theta) \oplus V(\psi_t\theta) \\ &= (x\dot{c} + yY(\dot{c}); 0\dot{c} + (\dot{y} + x)Y(\dot{c})) \\ &= (x(t)\dot{c} + y(t)Y(\dot{c}); z(t)Y(\dot{c})), \end{aligned} \tag{84}$$

where $z(t) = \dot{y}(t) + x(t)$ and $\theta = (c(0), \dot{c}(0))$.

Every orbit $(c, \dot{c})$ has no conjugate points because the only solution of (83) with $x(0) = y(0) = 0$ and $y(T) = 0 = x(T)$, $T \neq 0$ is $x(t) = y(t) \equiv 0$. The Green bundles coincide and are generated by $X(\theta) = (\dot{c}; -Y(\dot{c}))$ and $(Y(\dot{c}); \mathbf{0})$. The last vector can be found by solving $x(0) = 0$, $y(0) = 1$, $x(T) = y(T) = 0$ and letting $T \to \pm\infty$.

Now suppose that there exists an invariant continuous subbundle $\boldsymbol{N}(\theta) \subseteq T_\theta\Sigma$ which is transversal to the vectorfield $X(\theta)$. Let $\theta \in \Sigma$ be a recurrent point and let $t_n \underset{n}{\to} +\infty$ be such that $\psi_{t_n}(\theta) \underset{n}{\to} \theta$. Let $\xi \in \boldsymbol{N}(\theta) \setminus \{0\}$. If $d\psi_t(\xi) = (x(t), y(t); z(t)) = (J(t); \dot{J}(t))$, then

$$\begin{aligned} \lim_{n\to\pm\infty} \frac{1}{|x(t_n)|} d\psi_{t_n}(\xi) &= \lim_n \frac{1}{x(t_n)} (x(t_n), y(t_n); z(t_n)) \\ &= (1, 0; 1) = X(\theta). \end{aligned} \tag{85}$$

By the continuity of $\boldsymbol{N}$ we have that

$$X(\theta) = \lim_n \frac{1}{x(t_n)} d\psi_t(\xi) \in \boldsymbol{N}(\theta).$$

This contradicts the transversality hypothesis.

The same argument as in equation (85) shows that for every $\theta \in \Sigma$, the angle $\sphericalangle(d\psi_t(V(\theta) \cap T_\theta\Sigma), X(\psi_t\theta)) \xrightarrow[t\to+\infty]{} 0$.

In this case the trajectories of the Hamiltonian flow are the unit tangent vectors of the horospheres of the Riemannian metric, i.e.

$$\psi_t(\theta) = -Y(h_t(Y(\theta))), \tag{86}$$

where h_t is the horocycle flow. In particular, the exponential map $\exp_q : T_q\tilde{M} \to \tilde{M}$ is a diffeomorphism for all $q \in M$. This flow has no periodic orbits because the horocycles $W^{uu} \subset \Sigma = T^1\tilde{M}|_{\pi_1(M)}$ are homeomorphic to $\mathbb{R}$.

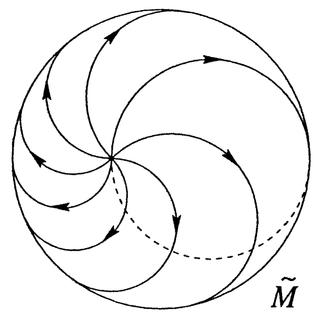

FIGURE A.1.

We prove (86). Let $\theta \in T_q\tilde{M}$, $\|\theta\| = 1$. Let $\phi_t : T\tilde{M} \hookleftarrow$ be the geodesic flow of $\tilde{M}$. Consider a sphere

$$S_T(\theta) = \{q \in \tilde{M} \mid d_{\tilde{M}}(q, \pi\phi_T(\theta)) = T\}$$

and a parametrization by arc length $\gamma_T(s)$ of $S_T(\theta)$ with $\gamma_T(0) = \pi(\theta)$. Then $(D/ds)\dot{\gamma}_T(s)$ is orthogonal to $\dot{\gamma}_T(s)$. Moreover, it has constant norm, because it is the image of $(D/ds)\dot{\gamma}_T(0)$ by the isometry given by a rotation about the center $\pi\phi_T(\theta)$ of $S_T(\theta)$. Therefore, γ_T satisfies

$$\frac{D}{ds}\dot{\gamma}_T = \lambda_T Y(\dot{\gamma}_T)$$

for some constant $\lambda_T > 0$. Passing to the limit when $T \to +\infty$ we obtain that the reparametrizations by arc length γ of the horospheres satisfy

$$\frac{D}{ds}\dot{\gamma} = \mu Y(\dot{\gamma}), \tag{87}$$

with $\mu = \lim_{T\to+\infty} \lambda_T$, (in fact $T \to \lambda_T$ is strictly decreasing). We claim that $\mu = 1$. Indeed, the solutions of (87) of are closed circles if μ is big enough. The infimum μ_0 of such μs corresponds to the horospheres. For $\mu > \mu_0$ the flow has conjugate points corresponding to the period (length) of these closed circles (see Example A.3). From the Jacobi equation (81) with $F = \mu$, we see that $\Sigma = H^{-1}\{\frac{1}{2}\}$ has conjugate points for $\mu > 1$ and the flow is Anosov for $\mu < 1$. Hence, $\mu_0 = 1$.

Indeed, for $\mu < 1$ the solution of (80) and (81) are $y = A\mathrm{e}^{\lambda t} + B\mathrm{e}^{-\lambda t}$, $x = [\mu(A/\lambda)\mathrm{e}^{\lambda t} - (B/\lambda)\mathrm{e}^{-\lambda t} + C]$, with $\lambda = \sqrt{1-\mu^2}$. Writing $\xi = (x(0), y(0); \dot{x}(0), \dot{y}(0))$ we have that $d\psi_t(\xi) = (x(t), y(t); \dot{x}(t), \dot{y}(t))$. For $A \neq 0$ the function $t \mapsto \|d\psi_t(\xi)\|$ has exponential growth and for $B \neq 0$, $t \mapsto \|d\psi_{-t}(\xi)\|$ has exponential growth. As in §3, this implies that the flow is Anosov for $\mu < 1$. For $\mu > 1$, the Jacobi equations (80), (81) exhibit conjugate points, by the same arguments as for equations (89) and (90).

Example A.2. A periodic orbit without conjugate points and no transversal invariant subspace.

We construct a convex Lagrangian which has the Jacobi equations (82) on a periodic orbit. Let $\theta \in \Sigma$ be such a periodic point with period $\tau > 0$. Changing the coordinates $(x, y; z)$ on $T_\theta\Sigma$ of equation (84) to $(x, y; \dot{y}) = (x, y; z - x)$, the matrix of the derivative

$d\psi_\tau(\theta)$ is

$$d\psi_\tau(\theta) = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ & 1 & t \\ & & 1 \end{bmatrix} \begin{matrix} \}x \\ \}y \\ \}\dot{y} \end{matrix} \approx \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix} \quad \text{in Jordan form.}$$

Let $\{e_1, e_2, e_3\}$ be the basis of $T_\theta\Sigma$ in which $d\psi_\tau(\theta)$ is written in canonical form. The transformations $d\psi_\tau$ and

$$d\psi_\tau - I = \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

have the same invariant subspaces. The vectorfield corresponds to the unique eigenspace $\langle e_1\rangle$ and the only invariant subspaces are $\langle e_1\rangle = \langle X(\theta)\rangle$, $\langle e_1, e_2\rangle = \mathbb{E}(\theta) = \mathbb{F}(\theta)$ and $\langle e_1, e_2, e_3\rangle = T_\theta\Sigma$. None of them are transversal to $X(\theta) = e_1$.

Now we make the construction. Consider the torus $\mathbb{T}^2 = \mathbb{R}^2/_{5\mathbb{Z}^2}$ with the flat metric and polar coordinates (r, θ) centered at $(0, 0) \in \mathbb{R}^2/_{5\mathbb{Z}^2}$ defined on $r \le 2$. Let $F : \mathbb{T}^2 \to \mathbb{R}$ be a C^∞-function such that $F(r, \theta) = F(r)$, $F(r = 1) = 1$, $(\partial F/\partial r)|_{r=1} = -1$, and $\int_{\mathbb{T}^2} F\Omega = 0$. Let η be a 1-form in $\mathbb{T}^2$ such that $d\eta = F\Omega$. Define $L(x, v) = \frac{1}{2}\langle v, v\rangle_x + \eta_x(v)$. At $r = 1$, the Euler–Lagrange equation (75) is

$$\frac{D}{dt}\dot{\gamma} = F(\gamma)Y(\dot{\gamma}) = Y(\dot{\gamma}) \quad \text{at } r = 1.$$

Hence, $r = 1$, $\dot{\theta} = -1$ is a periodic solution. Moreover, at $r = 1$ we have that $F = 1$, $\nabla_{Y(\dot{c})}F = -\partial F/\partial r = +1$. Hence, the Jacobi equations (80), (81) are the same as (82).

Example A.3. A segment without conjugate points with crossing solutions.

Let $W = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 4\}$ and consider the Lagrangian

$$L(x, v) = \tfrac{1}{2}\langle v, v\rangle + \eta_x(v),$$

where $\langle\, ,\rangle$ is the Euclidean metric and $d\eta = -\Omega$. This Lagrangian corresponds to $F|_W = -1$ in the formalism above and can be embedded into a convex Lagrangian on a closed surface. The Euler–Lagrange equation (75) is

$$\dot{v} = -iv. \tag{88}$$

The solutions of (88) on the energy level $|v| = 1$, are all the circles contained in W with constant angular velocity $\dot{\theta} = -1$. Fix $\mathcal{C}$: $R(t) = 1$, $\theta(t) = \pi - t$ in polar coordinates in W with center $(0, 0)$. Let $\vartheta = (0, 1)_{(-1,0)}$ be the initial vector of this solution. Then the circles which are rotations of $\mathcal{C}$ fixing the initial point $p = (-1, 0)$ are also solutions of (88) (see Figure A.3) with the same energy $E = \frac{1}{2}|v| = \frac{1}{2}$. The solution $\mathcal{C}$ has conjugate points, but the conjugate points appear at $t = 2\pi$. In particular the segment $\{\psi_t(\vartheta) \mid 0 \le t \le 3\pi/2\}$ has no conjugate points. But the intersections with other solutions starting at $p = (-1, 0)$ appear at $t < 3\pi/2$ and at $t = \pi$ there is a vector $\xi \in V(\vartheta) \cap T_\vartheta\Sigma$ such that $d\pi(d\psi_t\xi) = d\pi X(\psi_t\vartheta) = (0, -1)$. Proposition 1.16 says that these phenomena cannot occur if all the positive (negative) orbits of ϑ have no conjugate points. The claims stated above can be proved using the Jacobi equations (80) and (81):

$$\dot{x} + y = 0, \quad \ddot{y} + y = 0, \tag{89}$$

whose solutions are

$$
\begin{aligned}
x(t) &= a\cos t + b\sin t + c, \\
y(t) &= a\sin t - b\cos t.
\end{aligned} \tag{90}
$$

Thus, if $x(0) = y(0) = 0$ and $\dot{y}(0) \neq 0$ then $(x(t), y(t)) \neq 0$ for all $0 < t < 2\pi$. Moreover, if $a = -1$, $b = 0$, $c = +1$, then $x(0) = y(0) = 0$, $y(\pi) = 0$, $x(\pi) = +2$.

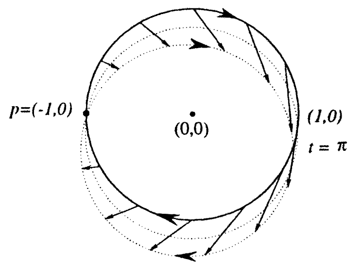

FIGURE A.3.

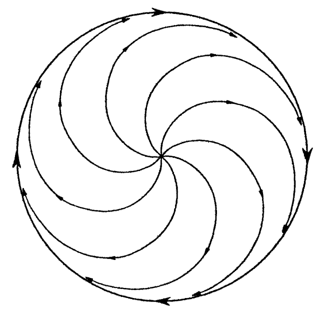

FIGURE A.4.

Example A.4. A non-surjective exponential map without conjugate points.

Consider the flat torus $\mathbb{T}^2 = \mathbb{R}^2/_{6\mathbb{Z}} = [-3,3]^2/_{6\mathbb{Z}}$ and polar coordinates (r,θ) on $\{(x,y) \in \mathbb{T}^2 | x^2 + y^2 < 3\}$ centered at $\mathbf{0} = (0,0)$. Define the Lagrangian

$$L(x,v) = \tfrac{1}{4}\langle v, v\rangle_x + \eta_x(v) + \phi(x),$$

where ϕ is a C^∞-function on $\mathbb{T}^2$ such that $\phi(r,\theta) = \frac{1}{4}r^2 + \cos\pi r$ on $r \leq 2$ and $\phi(r,\theta) \geq 1$ outside $r \leq 2$; and $\eta(r,\theta) = -\frac{1}{2}r^2 f(r)\, d\theta$ with $f(r)$ a C^∞-function with support on $r \leq \frac{5}{2}$ such that $f(r) = 1$ on $r \leq 2$ and $f(r) \leq 1$.

Observe that the path $\mathcal{C} : r(t) \equiv 1$, $\theta(t) = t$ is *static* (cf. Mañé [**28**]). Indeed, the Lagrangian

$$L(r,\theta;\dot r,\dot\theta) = \tfrac{1}{4}(\dot r^2 + r^2\dot\theta^2) - \tfrac{1}{2}r^2\dot\theta + \tfrac{1}{4}r^2 + \cos\pi r \quad \text{on } r \leq 2,$$

is minimized on $\dot r = 0$, $\dot\theta = 1$, $r = 1$, with $L = -1$; and on $r \geq 2$, we have that $\phi(r) \geq 1$ and $f \leq 1$, so that

$$\tfrac{1}{4}r^2\dot\theta^2 + \eta(v) \geq r^2(\tfrac{1}{4}\dot\theta^2 - \tfrac{1}{2}\dot\theta) \geq -\tfrac{1}{4}r^2 \geq -1.$$

Hence, $L \geq \phi(r) \geq 1$ on $r \geq 2$.

By Theorem VII in Mañé [**28**] (see also [**8**]), there exists a semistatic (hence, forward minimizing) solution $\gamma(t)$ such that $\gamma(0) = (0,0)$ and the ω-limit of $(\gamma,\dot\gamma)$ is the (hyperbolic) closed orbit $\mathcal{C}$. By the rotational symmetry of the Lagrangian, for every one-dimensional subspace $\mathcal{L} \subset T_{(0,0)}\mathbb{T}^2$ we obtain a semistatic curve γ, which is tangent to $\mathcal{L}$, with $\gamma_{\mathcal{L}}(0) = (0,0)$ and ω-limit$(\gamma_{\mathcal{L}},\dot\gamma_{\mathcal{L}}) = \mathcal{C}$. Since the semistatic solutions are minimizing, none of them $\gamma_{\mathcal{L}}|_{[0,+\infty[}$ have conjugate points. Moreover, by Theorem X in Mañé [**28**] (also [**8**]), $\mathcal{C}$ and all the $(\gamma_{\mathcal{L}},\dot\gamma_{\mathcal{L}})$'s are in the same energy level $E = c(L)$. By the graph property (Theorem VII, Mañé [**28**] and also [**8**]), the $\gamma_{\mathcal{L}}|_{]0,+\infty[}$s do not intersect $\pi(\mathcal{C})$ nor do they intersect each other.

Acknowledgements. G. Contreras was partially supported by CNPq-Brazil and PUC-Rio. R. Iturriaga was partially supported by Conacyt-Mexico, grant 0324P-E.

We want to thank G. Paternain, M. Paternain and R. O. Ruggiero for useful conversations and J. Sotomayor for directing us to reference [**20**]. We also want to thank CMAT, Uruguay, IMPA, Brazil, CIMAT, Mexico and PUC-Rio, Brazil, for their hospitality.

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