THE HAUSDORFF DIMENSION OF THE HARMONIC CLASS ON NEGATIVELY CURVED SURFACES

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ABSTRACT. We study the regularity of the Hausdorff dimension of the harmonic class of a surface M of negative curvature as a function of the riemannian metric. We prove that it is a C^{r-3} function of the metric in the Banach manifold of C^r riemannian metrics on M. We also prove regularity results for some asymptotic quantities associated to the Brownian motion on $\tilde{M}$.

Introduction

In the last years there has been increasing interest in potential theory on simply connected manifolds $\tilde{M}$ of bounded negative curvature. Anderson, Schoen and Sullivan [**An**], [**AS**], [**S**] have proven that the Dirichlet problem on $\tilde{M}$ can be solved for continuous data on the sphere at infinity $S(\infty)$ of $\tilde{M}$. In [**Ki**], Kifer gives a probabilistic proof of this result, relating it to the Brownian motion on $\tilde{M}$. When $\tilde{M}$ is the universal cover of a closed manifold of negative curvature M, Ledrappier [**Le**] related some asymptotic quantities associated to the Brownian motion on $\tilde{M}$ with ergodic quantities associated to the geodesic flow of M and obtained rigidity results for the metric on M (see theorem 1.1). For example, if $(\rho,\theta)\in\mathbb{R}^+\times\{\,v\in T_x\tilde{M}\mid |v|=1\,\}$ are the geodesic polar coordinates about x of a point $z=\exp_x\rho\,\theta\in\tilde{M}$, and $A(x,z)$ is the function defined by $dV(z)=A(x,z)\,d\rho\,d\theta$, where dV is the volume element of $\tilde{M}$, then for almost every Brownian path $\tilde{\omega}(t)$ on $\tilde{M}$ we have the same limit

$$\lambda=\lim_{t\to\infty}\frac{\log\,A\big(x,\tilde{\omega}(t)\big)}{d\big(x,\tilde{\omega}(t)\big)}\,,$$

where $d(x,y)$ is the distance function on $\tilde{M}$. We restrict ourselves to the case of the universal cover of a closed surface and consider λ as a function of the riemannian metric g. We prove that the map $g\mapsto\lambda(g)$ is C^{r-3} when g varies in the C^r topology.

Solving the Dirichlet problem on $\tilde{M}$ for boundary data on $S(\infty)$ gives rise to harmonic measures ω_x associated to each point x of $\tilde{M}$. All these measures are

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absolutely continuous with respect to each other and define a measure class on $S(\infty)$. Since in the case of surfaces the sphere at infinity has a natural C^1 structure, the Hausdorff dimension of the harmonic class $HD(\omega_g)$ is well defined. It gives a measure of the deviation of g from a metric of constant curvature (cf. Katok [**Ka**]). We prove that the map $g \mapsto HD(\omega_g)$ is C^{r-3} varying g in the C^r topology.

The actual condition that we need on the riemannian metric g on M is that the geodesic flow of g is Anosov. This allows some sets of positive curvature but not conjugate points (cf. Klingenberg [**Kli**] or Mañé [**Ma**]). We state the theorems in this setting.

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1. Notations and statements of results.

Let (M, g) be a closed surface of genus $g \geq 2$ endowed with a riemannian metric whose geodesic flow is Anosov, for example, a metric with variable negative curvature $-b^2 \leq K \leq -a^2$. Let $\pi : \tilde{M} \to M$ be its universal cover with the metric induced by π and (M, g). Let $S_g M$ (resp. $S_g \tilde{M}$) be the unit tangent bundle of (M, g) (resp. $(\tilde{M}, \widetilde{g})$ the lift of g) with the natural projection $p : S_g M \to M$ (resp. $\tilde{p} : S_g \tilde{M} \to \tilde{M}$). Let $\mathbf{\Gamma} = \pi_1(M)$ be the group of deck transformations of $\tilde{M}$.

a. The harmonic class.

Two geodesics γ and η in $\tilde{M}$ are said to be equivalent if $\sup_{t \geq 0} d\big(\gamma(t), \eta(t)\big) < +\infty$. The space of equivalence classes is called the *sphere at infinity* and is denoted by $S(\infty)$ (see e.g. [**BGS**]). For $\tilde{X}$ in $S\tilde{M}$ let $\gamma_{\tilde{X}}$ be the geodesic in $\tilde{M}$ defined by $\big(\gamma_{\tilde{X}}(0), \gamma'_{\tilde{X}}(0)\big) = \tilde{X}$. Denote by $\tau : S\tilde{M} \to S(\infty)$ the map which associates to each $\tilde{X}$, the class of $\gamma_{\tilde{X}}$. For x in $\tilde{M}$, the restriction τ_x of τ to $S_x \tilde{M} = \tilde{p}^{-1}\{x\}$ is a homeomorphism between $S_x \tilde{M}$ and $S(\infty)$. The *cone topology* on $\tilde{M} \cup S(\infty)$ is obtained by adding to the topology of $\tilde{M}$ and $S(\infty)$ the open sets $C(A, R) := \tau(A) \cap \cap_{t > R} \exp_{\tilde{x}}(tA)$, where A is an open subset of $S_{\tilde{x}} \tilde{M}$.

Let $\tilde{\phi} : S\tilde{M} \times \mathbb{R} \to S\tilde{M}$ be the geodesic flow of $(\tilde{M}, \widetilde{g})$, $\tilde{\phi}_t(\tilde{X}) = \big(\gamma_{\tilde{X}}(t), \gamma'_{\tilde{X}}(t)\big)$ and $\phi : SM \times \mathbb{R} \to SM$ be the geodesic flow of (M, g). Given $\tilde{X} \in S\tilde{M}$, the *weak stable manifold* of $\tilde{X}$ is defined by

$$\widetilde{W}^s(\tilde{X}) := \big\{ \tilde{Y} \in S\tilde{M} \,\big|\, \sup_{t \geq 0} d\big(\tilde{\phi}_t(\tilde{X}), \tilde{\phi}_t(\tilde{X})\big) < +\infty \big\} .$$

$\widetilde{W}^s(\tilde{X})$ is a C^1 submanifold of $S\tilde{M}$ homeomorphic to $\mathbb{R}^2$. The *stable foliation* $\widetilde{\mathcal{F}}^s = \big\{ \widetilde{W}^s(\tilde{X}) \,|\, \tilde{X} \in S\tilde{M} \big\}$ is $\mathbf{\Gamma}$-invariant and projects onto the stable foliation $\mathcal{F}^s = \big\{ W^s(X) \,|\, X \in SM \big\}$, $W^s(\pi X) := \pi\big(\widetilde{W}^s(\tilde{X})\big)$, for the Anosov flow ϕ_t on

SM. Since dim $M = 2$, the foliations $\widetilde{\mathcal{F}}^s$, $\mathcal{F}^s$ are C^1 (see [**HPS**] or [**HK**]). The *strong unstable manifold*

$$\widetilde{W}^{uu}(\tilde{X}) = \big\{ \tilde{Y} \in S\tilde{M} \;\big|\; \lim_{t \to +\infty} d\big(\tilde{\phi}_{-t}(\tilde{X}), \tilde{\phi}_{-t}(\tilde{Y})\big) = 0 \,\big\}$$

is the negative horosphere passing through $\tilde{X}$, which is a $\mathbf{\Gamma}$-invariant embedded submanifold of $S\tilde{M}$ homeomorphic to $\mathbb{R}$ and projects onto the strong unstable manifold $W^{uu}(\pi\tilde{X}) = \pi\big(\widetilde{W}^{uu}(\tilde{X})\big)$ for $\pi(\tilde{X})$. They form a foliation $\widetilde{\mathcal{F}}^{uu} = \big\{ \widetilde{W}^{uu}(\tilde{X}) \,|\, \tilde{X} \in S\tilde{M} \,\big\}$, called the strong unstable foliation or the horospheric foliation which is transversal to the stable foliation. The spheric foliation $\mathcal{S} = \{ S_x\tilde{M} \,|\, x \in \tilde{M} \}$ is also transversal to $\widetilde{\mathcal{F}}^s$. The restrictions $\tau : \widetilde{W}^{uu}(\tilde{X}) \to S(\infty) - \{\tau(-\tilde{X})\}$ and $\tau_x : S_x\tilde{M} \to S(\infty)$ are homeomorphisms whose transition maps $\tau|_{\widetilde{W}^{uu}(\tilde{X})} \circ \big(\tau|_{\widetilde{W}^{uu}(\tilde{Y})}\big)^{-1}$, $\tau|_{\widetilde{W}^{uu}(\tilde{X})} \circ \tau_y^{-1}$, $\tau_x \circ \tau_y^{-1}$ are the holonomy maps of the stable foliation, i.e. e.g. the diagram

$$\begin{tikzcd} \widetilde{W}^{uu}(\tilde{X}) \arrow[rr, "\text{holonomy}"] \arrow[dr, "\tau"'] & & \widetilde{W}^{uu}(\tilde{Y}) \arrow[dl, "\tau"] \\ & S(\infty) & \end{tikzcd}$$

commutes. Since the holonomy maps of $\mathcal{F}^s$ are C^1, this gives a natural C^1 structure to $S(\infty)$.

The Laplacian operator on $\tilde{M}$ is the operator $\Delta\varphi = \operatorname{div}\big(\operatorname{grad}(\varphi)\big)$ on $C^2(\tilde{M}, \mathbb{R})$, where $\langle \operatorname{grad}(\varphi), \tilde{X} \rangle = \tilde{X}(\varphi)$, $\forall \tilde{X} \in T\tilde{M}$ and $\operatorname{div}(F)$ is the trace of $Y \mapsto \nabla_Y F$, the riemannian connection on a vectorfield $F : \tilde{M} \to T\tilde{M}$. The Dirichlet problem $\Delta\varphi = 0$, $\varphi|_{S(\infty)} = f$ can be solved for any $f : S(\infty) \to \mathbb{R}$ continuous (see [**An**], [**S**], [**AS**] or [**Ki**]). Let $Hf = \varphi$ be the solution to the problem. For $x \in \tilde{M}$, the *harmonic measure at* x is the unique Borel measure ω_x on $S(\infty)$ such that

$$(Hf)(x) = \int_{S(\infty)} f \, d\omega_x$$

for any $f \in C^0\big(S(\infty), \mathbb{R}\big)$. All these measures are absolutely continuous with respect to each other. Their equivalence class is called the *harmonic class* of $\tilde{M}$.

Given a subset K of a separable metric space (Ω, d), the *Hausdorff dimension* of K is defined to be

$$\begin{aligned} HD(K) &:= \inf\{\, \delta > 0 \,|\, m_\delta(K) = 0 \,\}\,, \\ m_\delta(K) &:= \liminf_{\delta \to 0} \big\{ \textstyle\sum_{V \in \mathcal{O}} (\operatorname{diam}\, V)^\delta \big\}\,, \end{aligned}$$

where the infimum on $m_\delta(K)$ is taken over all open covers $\mathcal{O}$ of K with diam $\mathcal{O} < \epsilon$. Given a Borel probability measure μ on (Ω, d), the Hausdorff dimension of μ is defined to be

$$HD(\mu) := \inf\big\{ HD(\Lambda) \,|\, \mu(\Lambda) = 1 \,\big\}\,.$$

This number is constant in an equivalence class of (absolutely continuous) probabilities.

Since C^1 maps preserve Hausdorff dimension and $HD(\cup_{n=1}^{\infty} K_n) = \sup_{n\in\mathbb{N}} HD(K_n)$, we can define the *Hausdorff dimension of the harmonic class* to be $HD(\omega) := HD(\omega_x \circ \tau_y^{-1}) = HD(\omega_x \circ (\tau|_{\widetilde{W}^{uu}(\tilde{Z})})^{-1})$ for any $x,\ y \in \tilde{M},\ \tilde{Z} \in S\tilde{M}$. We write $HD(\omega_g)$ when we want to make explicit the dependence of $HD(\omega)$ on the riemannian metric g of M.

Kifer and Ledrappier [**KL**] proved that for a simply connected complete riemannian manifold $\tilde{M}$ of bounded negative sectional curvatures $-b^2 \le K \le -a^2$, the Hausdorff dimensions $HD(\omega_x \circ \tau_x^{-1})$ (which a priori depend on $x \in \tilde{M}$ because the maps $\tau_x \circ \tau_y^{-1}$ are only Hölder continuous) are all positive. Actually, they are all equal by remark 2.2.

Let $R^r(M)$ be the Banach manifold of C^r riemannian metrics on M with the C^r topology and let $A^r(M)$ be the open subset of C^r metrics whose geodesic flow is Anosov, in particular, metrics with negative curvature. Here we prove

Theorem A.
The map $A^r(M) \ni g \mapsto HD(\omega_g) \in \mathbb{R}$ is C^{r-3}, $r \ge 3$.

b. The Brownian motion.
Let (M, g) be as above. Denote by $\widetilde{\Omega} = C^0([0, +\infty[, \tilde{M})$ the space of continuous paths on $\tilde{M}$ with the topology given by uniform convergence on compact subsets. For $x \in \tilde{M}$ let $\boldsymbol{P}_x$ be the Borel probability on $\widetilde{\Omega}_x := [\,\tilde{\omega} \in \widetilde{\Omega} \,|\, \tilde{\omega}(0) = x\,]$ defined by

$$\boldsymbol{P}_x\big[\tilde{\omega} \in \widetilde{\Omega} \,|\, \tilde{\omega}(0) = x,\ \tilde{\omega}(t) \in A\,\big] = \int_A p(t,x,y)\, dm_g(y)$$

for any $t > 0$ and any Borel subset $A \subset \tilde{M}$, where m_g is the volume element of $\tilde{M}$ and $P : \mathbb{R} \times \tilde{M} \times \tilde{M} \to \mathbb{R}$ is the fundamental solution of the heat equation on $\tilde{M}$:

$$\frac{\partial p}{\partial t} + \nabla p(t,\cdot,y) = 0\,,$$
$$\lim_{t \downarrow 0} \int_{\tilde{M}} p(t,x,y)\, f(y)\, dm_g(y) = f(x)\,,$$

for any continuous function $f : \tilde{M} \to \mathbb{R}$. Since the *heat kernel* satisfies (see [**Ch**], th. VIII.4, VIII.5):

(i) $p(t,y,x) = p(t,x,y) \ge 0, \qquad \forall t \ge 0, \quad \forall x, y \in \tilde{M}.$
(ii) $\int_{\tilde{M}} p(t,x,y)\, dm_g(y) = 1, \qquad \forall g \ge 0, \quad \forall x,\ y \in \tilde{M}.$
(iii) $\int_{\tilde{M}} p(s,x,y)\, p(t,y,z)\, dm_g(y) = p(s+t,x,z), \quad \forall s,t \ge 0, \quad \forall x,\ y,\ z \in \tilde{M}$;

we have that the family $\mathcal{P} = \{ \boldsymbol{P}_x \mid x \in \tilde{M} \}$ of probability measures defines a continuous Markov process on $\tilde{M}$ called the *Brownian motion* on $\tilde{M}$. The induced probabilities $P_{\pi(\tilde{X})} = \boldsymbol{P}_{\tilde{x}} \circ \pi$ on $\Omega = C^0\big([0,+\infty[, M\big)$ define the Brownian motion on M.

Since the geodesic flow ϕ_t is Anosov, $\tilde{M}$ can not have conjugate points and $\exp_x T_x\tilde{M} \to \tilde{M}$ is a diffeomorphism for every $x \in M$ (see [**Kli**] or [**Ma**]). For $x \in \tilde{M}$, we consider geodesic polar coordinates about x, i.e. we identify $T_x\tilde{M}$ with $]0,+\infty[\times S_x\tilde{M} \cup \{0\}$ and a point $z \in \tilde{M}$ is described by the polar coordinates of $\exp_x^{-1}(z)$. For $\tilde{\omega} \in \widetilde{\Omega}$, denote by $\big(r(\tilde{\omega},t), \theta(\tilde{\omega},t)\big)$ the geodesic polar coordinate about x of the point $\tilde{\omega}(t)$. For $x \in \tilde{M}$, let λ_x be the Lebesgue measure on $S_x\tilde{M}$ and denote by $A_g(x,z)$ the function on $\tilde{M} \times \tilde{M}$ such that

$$dm_g(\exp_z t\xi) = A\big(x,(t,\xi)\big)\, dt\, d\lambda_x(\xi)$$

for $\xi \in S_x\tilde{M}$. Let $V_g(x,t)$ be the volume of the ball of radius t about x:

$$V_g(x,t) = \int_0^t \left(\int_{S_x\tilde{M}} A_g\big(x,(s,\xi)\big)\, d\lambda_x(\xi) \right) ds\,.$$

The following theorem has been proved by several people:

1.1. Theorem.

[**Pr**] *For all $x \in \tilde{M}$, $\boldsymbol{P}_x$-a.e. $\tilde{\omega} \in \widetilde{\Omega}$, $\theta(\tilde{\omega},t)$ converges as t goes to infinity towards some limit $\theta(\tilde{\omega},+\infty) \in S_x\tilde{M}$.*
The induced measure on $S(\infty) \approx S_x\tilde{M}$ by $\theta(\cdot,+\infty) : (\Omega, \boldsymbol{P}_x) \to S_x\tilde{M}$ is the harmonic measure ω_x.

[**Va**] *There exists a number $\alpha > 0$ such that for all $x \in \tilde{M}$, $\boldsymbol{P}_x$-a.e. $\tilde{\omega}$, $\lim\limits_{t\to+\infty} \frac{1}{t}\, r(\tilde{\omega},t) = \alpha(g)$.*

[**Kai**] *There exists a number $\beta > 0$ such that for all $x \in \tilde{M}$, $\boldsymbol{P}_x$-a.e. $\tilde{\omega}$, $\lim\limits_{t\to+\infty} -\frac{1}{t}\log\, p(t,x,\tilde{\omega}(t)) = \beta(g)$.*

[**Le**] *There exists a number $\gamma > 0$ such that for all $x \in \tilde{M}$, $\boldsymbol{P}_x$-a.e. $\tilde{\omega}$, $\lim\limits_{t\to+\infty} \frac{1}{t}\log A\big(x,\tilde{\omega}(t)\big) = \gamma(g)$.*

[**Le**] *In general $\beta \le \gamma$ and $\beta \le \alpha\, h$, where h is the topological entropy of the geodesic flow on SM.*

[**Le**] *Each of the equalities $\beta = \gamma$ or $\beta = \alpha h$ hold if and only if the surface M has constant curvature.*

We prove the following slightly more general result than theorem A:

Theorem B.

(i) *The map $A^r(M) \ni g \mapsto \frac{\beta(g)}{\alpha(g)} \in \mathbb{R}$ is C^{r-2}.*

(ii) *The map $A^r(M) \ni g \mapsto \frac{\gamma(g)}{\alpha(g)} \in \mathbb{R}$ is C^{r-3}.*

(iii) *The Hausdorff dimension of the harmonic measure is $HD(\omega_g) = \frac{\beta(g)}{\gamma(g)}$.*

Since for surfaces the harmonic measure on $S_x\tilde{M} \approx S(\infty)$ is absolutely continuous with respect to the Lebesgue measure only in the case of constant (negative) curvature and in this case $HD(\omega) = 1$, then $HD(\omega_g)$ can be seen as a measure of the deviation of g from a metric of constant curvature (cf. [**Ka**]).

2. Equilibrium states.

Given a Hölder continuous function $F : SM \to \mathbb{R}$, there exists a unique ϕ-invariant probability measure μ_F on SM, called the *equilibrium state* of (ϕ, F) such that it maximizes the functional

$$\nu \mapsto h_\nu(\phi_1) + \int F\, d\mu$$

over all the ϕ-invariant Borel probability measures on SM, where $h_\nu(\phi_1)$ is the entropy of ϕ_1 with respect to ν (see [**BR**]).

For $X \in SM$ define the local stable and strong unstable manifolds of X by

$$W^s_\epsilon(X) = \big\{\, Y \in SM \mid d\big(\phi_t(X), \phi_t(Y)\big) \le \epsilon, \quad \forall t \ge 0 \,\big\}$$
$$W^{uu}_\epsilon(X) = \big\{\, Y \in SM \mid d(X, Y) \le \epsilon \text{ and } \lim_{t\to+\infty} d\big(\phi_{-t}(X), \phi_{-t}(Y)\big) = 0 \,\big\}.$$

If $\epsilon > 0$ is sufficiently small then they are transversal embedded discs in SM with $\dim W^s_\epsilon(X) = 2$, $\dim W^{uu}_\epsilon(X) = 1$. For $\epsilon > 0$ small there exists a partition ξ of SM with $\operatorname{diam} \xi < \epsilon$ such that it is *subordinate* to $\mathcal{F}^{uu}$, i.e. $\xi(X) \subset W^{uu}_\epsilon(X)$ for all $X \in SM$ (see [**LY**]) and such that it is a *measurable partition*, i.e. the quotient space SM/ξ is separated by a countable number of measurable sets (see [**Ro**]). Then (cf. [**Ro**]) there exists a system of *conditional measures* associated to it, i.e. for μ-a.e. $X \in SM$ there exists a probability measure $\mu_X = \mu_{\xi(X)}$ on $\xi(X)$ such that for any Borel set A on SM, the function $X \mapsto \mu_{\xi(X)}\big(A \cap \xi(X)\big)$ is measurable and $\mu(A) = \int_{SM} \mu_{\xi(X)}\big(A \cap \xi(X)\big)\, d\mu(X)$.

If μ^F is an equilibrium state and $\mathcal{L}$ is the holonomy map of the stable foliation $\mathcal{F}^s$ from (a subset of) $\xi(X)$ to $\xi\big(\phi_t(X)\big)$: $\mathcal{L}(Y) = W^s(Y) \cap \xi\big(\phi_t(X)\big)$, then the measures $\mu^F_{\xi(X)}$ and $\mu^F_{\xi(\phi_t(X))} \circ \mathcal{L}^{-1}$ are equivalent on $\xi(X) \cap \mathcal{L}^{-1}\big(\xi(\phi_t(X))\big)$. It follows that the measure ν on $\xi(X)$ defined by $\nu(A) = \mu^F\big(\cup_{Y\in A} W^s_\epsilon(Y)\big)$ is equivalent to $\mu^F_{\xi(X)}$.

Observe that if $Hf = \varphi$ is the solution of $\Delta\varphi = 0$, $\varphi|_{S(\infty)} = f$ on $\tilde{M}$ and $\Gamma \in \mathbf{\Gamma}$, then $H(f \circ \Gamma) = (Ff) \circ \Gamma$ so that the harmonic measures satisfy $\omega_{\Gamma(x)} = \omega_x \circ \tilde{\Gamma}^{-1}$, where $\tilde{\Gamma}$ is the extension of Γ to $S(\infty)$. Since for $\tilde{X} \in S\tilde{M}$, $\tau|_{W^{uu}(D\Gamma\cdot\tilde{X})} \circ D\Gamma = \tilde{\Gamma} \circ \tau|_{W^{uu}(\tilde{X})}$, we have that the measures $\nu_{\tilde{X}} := \omega_{p(\tilde{X})} \circ \tau_{W^{uu}(\tilde{X})}$ satisfy $\tilde{\nu}_{D\Gamma.\tilde{X}} = \tilde{\nu}_{\tilde{X}} \circ D\Gamma$ and hence the system $\{\, \tilde{\nu}_{\tilde{X}} \mid \tilde{X} \in S\tilde{M} \,\}$ projects to a family of measures $\{\nu_X \mid X \in SM, \}$, $\nu_{\pi\tilde{X}} \circ D\pi = \tilde{\nu}_{\tilde{X}}$, that we call the *horospheric harmonic measure* on SM.

2.1. Theorem. [**Le**]

[1] *The horospheric harmonic measures are equivalent to the conditional measures on local strong unstable manifolds of the equilibrium state* μ^F *of the function*

$$F(\pi\tilde{X}) = \log K\big(\gamma_{\tilde{X}}(0), \gamma_{\tilde{X}}(1), \tau(\tilde{X})\big), \tag{1}$$

where $K : \tilde{M} \times \tilde{M} \times S(\infty) \to \mathbb{R}$ *is the Poisson kernel of* $\tilde{M}$ *(see section §4):* $\mu^F_{\xi(X)} \approx \nu_X$ *for all* $X \in SM$.

[2] *We have, for the Brownian motion in* $\tilde{M}$, *that* $\gamma = \alpha \int J^u \, d\mu^F$, *where*

$$J^u(X) = \tfrac{d}{dt}\big[\log\big|\det D\phi_t|_{T_X W^{uu}(X)}\big|\big]_{t=0}.$$

In particular $\frac{\gamma}{\alpha}$ *is the positive Lyapunov exponent of* $(SM, \{\phi_t, t \in \mathbb{R}\}, \mu^F)$.

[3] *We have* $\beta = \alpha\, h_\mu(\phi)$.

Since $D\pi$ is a C^1 map and the Hausdorff dimension $HD\big(\mu^F_{\xi(X)}\big)$ is constant for μ-a.e. $X \in SM$ (cf. [**M**]), then we have that $HD(\omega_g) = HD(\nu_X) = HD\big(\mu^F_{\xi(X)}\big)$, μ-a.e. $X \in SM$. Ledrappier, Manning and Young (cf. [**LY**], [**M**], [**Y**]) proved that $HD\big(\mu^F_{\xi(X)}\big) = h_\mu(\phi)/\lambda(\mu)$, where $\lambda(\mu)$ is the positive Lyapunov exponent of (ϕ, μ). In particular we have that $HD(\omega_g) = \beta(g)/\gamma(g)$.

2.2. Remark. F. Ledrappier and L-S. Young [**LY**] proved that in higher dimensions, $\dim SM \geq 4$, the Hausdorff dimension of conditional measures on W^{uu}, $HD\big(\mu_{\xi(X)}\big)$, of invariant probabilities μ, are the same μ-a.e. $X \in SM$. This implies that in $\dim M > 2$, even when the holonomy maps of the stable foliation $\mathcal{F}^s$ are only Hölder continuous and hence the sphere at infinity has only a Hölder structure, the Hausdorff dimension of the harmonic class is well defined (and positive).

We are going to use the following

2.3. Theorem. [**C**]

Let X *be a* C^r *vectorfield on a compact manifold* N *whose flow is Anosov. Let* $\mathfrak{X}^r(N)$ *be the Banach space of* C^r *vectorfields on* N *and* $C^\alpha(N,\mathbb{R})$ *be the Banach space of* α*-Hölder continuous functions on* N. *Let* $\psi : \mathcal{V} \subset \mathfrak{X}^r(N) \to C^0(N,\mathbb{R})$ *be a continuous map from a neighbourhood* $\mathcal{V}$ *of* X *of vectorfields whose flows are Anosov. For* $Y \in \mathcal{V}$ *let* u_Y *be the topological equivalence of proposition 5.2, and suppose that the map* $F(Y) := \psi(Y) \circ u_Y$ *is such that* $F : \mathcal{V} \subset \mathfrak{X}^r(N) \to C^\alpha(N,\mathbb{R})$ *is* C^{s-1}, $s \leq r$. *For* $Y \in \mathcal{V}$, *let* μ_Y *be the equilibrium state for* $\big(Y, \psi(Y)\big)$ *and* $h(\mu_Y)$ *the metric entropy of* Y *with respect to* μ_Y. *Then there exists a neighbourhood* $\mathcal{U} \subset \mathcal{V}$ *of* X *in* $\mathfrak{X}^r(N)$ *such that the maps*

(i) $\mathcal{U} \ni Y \mapsto h(\mu_Y) \in \mathbb{R}$ *is* C^{s-1}.

(ii) $\mathcal{U} \ni Y \mapsto \mu_Y \in \big(C^\alpha(N,\mathbb{R})\big)^*$ *is* C^{s-1}.

(iii) $\mathcal{U} \ni Y \mapsto \lambda(\mu_Y) := \int \frac{d}{dt}\big[\log\big|\det D\big(\phi_t(Y)\big)|_{E^{uu}_Y(p)}\big|\big]_{t=0}\, d\mu_Y(p) \in \mathbb{R}$ *is* C^t *with* $t := \min\{s-1, r-2\}$.

where $(p,t) \mapsto \big(\phi_t(Y)\big)(p)$ *is the flow of* $Y \in \mathcal{U}$ *and* $E^{uu}_Y(p) = T_p W^{uu}_Y(p) \subset T_p N$ *is the unstable subspace for* Y *at* p.

Moreover, if $F : \mathcal{V} \subset \mathfrak{X}^r(N) \to C^\alpha(N,\mathbb{R})$ *is* C^{s-1} *and* $F : \mathcal{V} \subset \mathfrak{X}^r(N) \to C^\alpha(N,\mathbb{R})$ *is* C^s*, then the map* $\mathcal{U} \ni Y \mapsto P\big(F(Y)\big) \in \mathbb{R}$ *is* C^s*, where* $P\big(F(Y)\big)$ *is the pressure function* $F(Y)$ *for the flow of* Y*.*

Sketch of the proof of theorems A and B.

We will apply theorem 2.3 to our case: let $R^r(M)$ be the Banach manifold of C^r riemannian metrics on M. Given $g \in R^r(M)$, the geodesic flow of (M,g) is generated by a C^{r-1} vectorfield $X(g)$. Fix a riemannian metric $g_o \in \mathcal{A}^r(M)$ and a small neighbourhood $g_o \in \mathcal{V} \subset \mathcal{A}^r(M)$. Let $\Sigma M = S_{g_o}M$ be the g_o-unit tangent bundle. For $g \in \mathcal{V}$, using the orthogonal projection $S_gM \to \Sigma M$, conjugate the geodesic flow for g to a flow on ΣM with vectorfield $Y(g)$. Since this projection is differentiable, entropies and Lyapunov exponents for $Y(g)$ are the same as the corresponding ones for $X(g)$. We will prove (cf. 5.1) that the map $\mathcal{R}^r(M) \ni g \mapsto Y(g) \in \mathfrak{X}^{r-1}(\Sigma M)$ is C^∞. Let F_g be the function defined in theorem 2.1. On section §6 we will prove that the map $\mathcal{R}^r(M) \ni g \mapsto F_g \circ u_g \in C^\alpha(\Sigma M,\mathbb{R})$ is C^{r-2} for some $0<\alpha<1$ and the map $\mathcal{R}^r(M) \ni g \mapsto F_g \circ u_g \in C^0(\Sigma M,\mathbb{R})$ is C^{r-1}. Then using theorem 2.1 and theorem 2.3, we obtain theorems A and B.

□

3. Conformal equivalence.

Given an initial riemannian metric g_o on M, the existence of isothermal coordinates (see below) implies that we can find an oriented atlas on M in which locally we can write $g_o = f(x,y)\,(dx\otimes dx + dy\otimes dy)$, where f is a smooth scalar function. Writing $z = x+iy$ we obtain an analytic atlas. Indeed, for other isothermal chart (u,v), writing $w = u+iv$ and $g_o = h(u,v)\,(du\otimes du + dv\otimes dv)$ we have that the derivatives of the transition maps $w\circ z^{-1}$ must satisfy $\left[\frac{\partial(u,v)}{\partial(x,y)}\right]\left[\frac{\partial(u,v)}{\partial(x,y)}\right]^T = \frac{f(x,y)}{h(u,v)}\, Id$, which gives the Cauchy-Riemann equations for $\frac{dw}{dx}$.

This gives to M and $\tilde M$ the structure of riemann surfaces. The uniformization theorem [**FK**], implies that $\tilde M$ is conformally equivalent to $\mathbb{D} = \{\, z \in \mathbb{C} \mid |z| < 1 \,\}$ with the euclidean metric. We identify $\mathbb{D} \approx \tilde M$ so that the covering map $\pi : \mathbb{D} \to (M, g_o)$ is conformal, the deck transformations are holomorphic and the lifted metric $\widetilde{g_o}$ can be written as $\widetilde{g_o} = \rho_o\,(dx\otimes dx + dy \otimes dy)$, where $\rho_o : \mathbb{D} \to \mathbb{R}$ is a positive smooth function. Denote by $z = x+iy : \tilde M \to \mathbb{D}$ this coordinate system.

Consider the lift $\widetilde{g}$ to $\tilde M$ of another riemannian metric g on M. We look for coordinates $w : \tilde M \to \mathbb{D}$ such that $\widetilde{g} = \rho\,|dz + \mu\, d\overline{z}|^2$ where $\rho : \tilde M \to \mathbb{R}^+$ and

$\mu : \tilde{M} \to \mathbb{C}$ is a smooth function such that $|\mu(p)| < 1$ for all $p \in \tilde{M}$. Writing $\widetilde{g}$ in the coordinates $z = x + iy$ as

$$\widetilde{g} = \mathrm{A}\, dx \otimes dx + 2\,\mathrm{B}\, dx \otimes dy + \mathrm{D}\, dy \otimes dy\,,$$

we have $$|dz + \mu\, d\overline{z}|^2 = (1 + |\mu|^2)\ |dz|^2 + 2\,\Re(\overline{\mu}\, dz^2) = \lambda\, \widetilde{g}$$

with $\rho = \frac{1}{\lambda}$. Form this we get that

$$\alpha = \frac{\mathrm{A} - \mathrm{D}}{4}\,\lambda \quad , \quad \beta = \frac{\mathrm{B}}{2}\,\lambda \quad , \quad 1 + |\mu|^2 = \frac{\mathrm{A} + \mathrm{D}}{\mathrm{B}}\,\beta\,.$$

We choose the solution

$$\beta = 2\,\mathrm{B}\, \frac{(\mathrm{A} + \mathrm{D}) - 2\sqrt{\mathrm{AD} - \mathrm{B}^2}}{(\mathrm{A} + \mathrm{D})^2 - 4(\mathrm{AD} - \mathrm{B}^2)} = 2\,\mathrm{B}\, \frac{p - q}{p^2 - q^2}$$

$$\beta = \frac{2\,\mathrm{B}}{p + q} \qquad , \qquad \alpha = \frac{\mathrm{A} - \mathrm{D}}{p + q} \tag{2}$$

where $p := \mathrm{A}+\mathrm{D} > 0$ and $q := 2\sqrt{\mathrm{AD} - \mathrm{B}^2} > 0$ because the matrix $\boldsymbol{A} = \begin{bmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{B} & \mathrm{D} \end{bmatrix}$ is positive definite. Observe that we get that

$$1 + |\mu|^2 = \frac{2\,p}{p + q} = \frac{2}{1 + \frac{q}{p}} < 2$$

because $\frac{q}{p} > 0$, and then $|\mu| < 1$.

Let $C^k(r)$ be the Banach space of C^k functions $f : \{\, z \in \mathbb{C} |\ |z| < r \,\} \to \mathbb{D}$ with the C^k norm and let $C^0(\mathbb{D}, \mathbb{D})$ be the space of continuous functions of the open disc to itself with the C^0 norm.

3.1. Lemma. *For all $0 < r < 1$ and all $k \geq 0$, the map $\mu : \mathcal{R}^k(M) \to C^k(r) \cap C^0(\mathbb{D}, \mathbb{D})$ given by $g \mapsto \mu(g) \circ z^{-1}$ is C^∞.*

Proof:

From formula (2) it is clear that the map $\mu : \mathcal{R}^k(M) \to C^k(r)$ is C^∞. Observe that the equation $\widetilde{g} = \rho\ |dz + \mu\, d\overline{z}|^2$ has exactly two solutions for μ at each point, one with $|\mu| > 1$ and one with $|\mu| < 1$. We choose the one with $|\mu| < 1$. Let h be a deck transformation and write $w = h(z)$. The map h is holomorphic, so that $h_{\overline{z}} = 0$. Since h is a g-isometry, we have that

$$\begin{aligned} \rho\ |dz + \mu\, d\overline{z}| &= \rho(w)\ |dw + \mu(w)\, d\overline{w}| \\ &= (\rho \circ h)\ |h_z|\ \left| dz + (\mu \circ h)\, \frac{\overline{h_{\overline{z}}}}{h_z}\, d\overline{z} \right| . \end{aligned}$$

Therefore $$\mu \circ h = \mu\, \frac{h_z}{\overline{h_{\overline{z}}}}\ \cdot$$

Let $\mathcal{D}$ be a fundamental domain and choose $r_0 > 0$ such that $\mathcal{D} \subset [\,|z| < r_0\,]$. Choose any point $\tilde{p} \in \mathbb{D} = \tilde{M}$, then

$$|\mu(g_1)(\tilde{p}) - \mu(g_2)(\tilde{p})| = |\mu(g_1)(q) - \mu(g_2)(q)| \left| \frac{h_z(q)}{\overline{h}_{\overline{z}}(q)} \right|$$
$$= |\mu(g_1)(q) - \mu(g_2)(q)|\,,$$

where h is a deck transformation such that $\tilde{p} \in h(\mathcal{D})$ and $q \in \mathcal{D}$ is such that $h(q) = \tilde{p}$. Therefore

$$\|\mu(g_1) - \mu(g_2)\|_{C^0(\mathbb{D},\mathbb{D})} \leq \|\mu(g_1) - \mu(g_2)\|_{C^0(r)}\,.$$

This implies that $\mathcal{R}^k(M) \to C^0(\mathbb{D},\mathbb{D})$ is C^∞.

□

4. The Poisson kernel.

Given a riemannian metric g on M, the Laplace-Beltrami operator can be written in local coordinates as

$$\Delta = \sum_{i,j} g^{ij} \left(\frac{\partial^2}{\partial x^i\,\partial x^j} - \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right)$$

where

$$\Gamma^k_{ij} = \frac{1}{2} \sum_m \left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right) g^{mk}$$

are the Christoffel's symbols of g and $[g^{ij}] = [g_{ij}]^{-1}$ is the inverse matrix of the local representation of $g = \sum g_{ij}(dx^i \otimes dx^j)$. If we multiply a metric g on $\tilde{M}$ by a smooth function $\lambda : \tilde{M} \to \mathbb{R}^+$, the Laplace-Beltrami operator for λg takes the form:

$$\Delta_{\lambda g} = \sum_{i,j} (\lambda g)^{ij} \left(\frac{\partial^2}{\partial x^i\,\partial x^j} - \sum_k \Gamma^k_{ij}(\lambda g) \frac{\partial}{\partial x^k} \right)$$
$$= \frac{1}{\lambda} \Delta_g - \left(\frac{2 - \dim M}{2} \right) \frac{1}{\lambda^2} \sum_{m,k} \frac{\partial \lambda}{\partial x^m} g^{mk} \frac{\partial}{\partial x^k}$$
$$= \frac{1}{\lambda} \Delta_g\,.$$

so that the set of harmonic functions for g coincides with the set of harmonic functions for λg.

The Poisson kernel on $\tilde{M}$, $K : \tilde{M} \times \tilde{M} \times S(\infty) \to \mathbb{R}$ is defined as the Radon-Nikodym derivative of the harmonic measures:

$$K(x, y, \theta) := \frac{d\omega_y}{d\omega_x}(\theta).$$

Fix a riemannian metric g_o on M and its lift $\widetilde{g_o}$ to $\tilde{M}$. Suppose that its geodesic flow is Anosov. Fix an isothermal chart $z : (\tilde{M}, \widetilde{g_o}) \to (\mathbb{D}, e)$, where e is the euclidean metric (actually its conformal type) on $\mathbb{D}$.

4.1. Lemma. *There exists a neighbourhood $\mathcal{U}$ of g_o in the C^3-topology such that for all $g \in \mathcal{U}$ the chart z induces a homeomorphism $z : S_g(\infty) \to S^1 = \partial\mathbb{D} \subset \mathbb{C}$ of the sphere at infinity of g and S^1 by $z[\gamma_g] := \lim_{t\to+\infty} z \circ \gamma_g(t)$.*

Moreover,

(i) *The extension $z : \tilde{M} \cup S_g(\infty) \to \overline{\mathbb{D}}$ is a homeomorphism.*
(ii) *The map $z : S_{g_o}(\infty) \to S^1$ is Hölder continuous.*

Proof:
(i) In [**G**] it is proved that for any metric on M whose flow is Anosov, the map $z : M \cup S_g(\infty) \to \overline{\mathbb{D}}$ is a homeomorphism. It is also proved that any two Anosov geodesic flows for M are topologically equivalent.
(ii) Let $\phi : S_g\tilde{M} \times \mathbb{R} \to S_g\tilde{M}$ be the lift of the geodesic flow for g and let ρ be the g-distance on $S_g\tilde{M}$. Let $\psi : \Sigma\tilde{M} \times \mathbb{R} \to \Sigma\tilde{M}$ be the lift of the geodesic flow for the metric g_1 with constant curvature $K \equiv -1$ and let d be the hyperbolic distance on $\Sigma\tilde{M} = S_{g_1}\tilde{M}$. Let $h : S_gM \to \Sigma M$ be a topological equivalence of the geodesic flows for g and g_1, and let $\tilde{h} : S_g\tilde{M} \to \Sigma\tilde{M}$ be its lift. Since $S_g(\infty)$ is compact, it is enough to prove that for any $w \in S_g\tilde{M}$, the map $H : \widetilde{W}^{uu}(w, \phi) \approx S_g(\infty) \xrightarrow{z} S^1 \approx \widetilde{W}^{uu}\big(\tilde{h}(w), \psi\big)$ is Hölder continuous on a neighbourhood of w. We use local strong unstable manifolds:

$$\widetilde{W}^{uu}_\beta(p, \psi) := \big\{\, q \in \Sigma\tilde{M} \,\big|\, d(p, q) < \beta \text{ and } \lim_{t\to-\infty} d\big(\psi_t(p), \psi_t(q)\big) = 0 \,\big\}.$$

We have that $H = P \circ \tilde{h}$ where $P : D_\theta \subseteq \Sigma\tilde{M} \to \widetilde{W}^{uu}_\beta\big(\tilde{h}(w), \psi\big)$ is the projection along the flow lines of ψ, $D_\theta = \big\{\, z \in \widetilde{W}^{u}\big(\tilde{h}(w), \psi\big) \,\big|\, d\big(z, \tilde{h}(w)\big) < \theta \,\big\}$, and $\widetilde{W}^{u}\big(\tilde{h}(w), \psi\big) = \cup_{t\in\mathbb{R}}\, \psi_t\big(\widetilde{W}^{uu}(\tilde{h}(w), \psi)\big)$ is the weak unstable manifold of $\tilde{h}(w)$. Fix $\epsilon > 0$ small and such that if $p, q \in \widetilde{W}^{u}\big(\tilde{h}(w), \psi\big)$, $d(p, \tilde{h}(w)) < \epsilon$ and $d(q, \tilde{h}(w)) < \epsilon$, then there exists exactly one point in the intersection

$$\{w\} = \widetilde{W}^{uu}_{4\epsilon}(q, \psi) \cap \{\, \psi_t(p) \mid -4\epsilon \le t \le 4\epsilon \,\} \neq \emptyset.$$

Let $\theta := 12\,\epsilon$. For the hyperbolic metric g_1 we know that P is C^1. Let $B > 0$ be such that $d(P(x), P(y)) < B\,\rho(x, y)$ for all $x, y \in D_\theta$. We need the following:

Claim:

(a) *There exists $0 < a < A$ such that if $x \in S_g = \tilde{M}$ and $s(x) > 0$ is such that $\tilde{h}(\phi(x, 1)) = \psi(\tilde{h}(x), s(x))$, then $2a < s(x) < \frac{A}{2}$.*
(b) *There exists $0 < a < A$ such that if $x \in S_g\tilde{M}$, $T > 2$ and $s(x, T) > 0$ is such that $\tilde{h}(\phi(x, T)) = \psi(\tilde{h}(x), s(x, T))$, then $a\,T < s(x, T) < A\,T$.*

Proof: For (a) use the continuity of h and the compactness of S_gM. For (b), suppose that $T = n + \delta$, with $n \in \mathbb{Z}^+$ and $\delta \in [0,1[$. From item (a) we get that $2a\,\frac{t}{2} \le 2a\,n \le s(x,T) \le \frac{A}{2}\,n + \frac{A}{2} \le \frac{A}{2}\,2T$. $\square$

Let $\lambda > 0$ be such that $\rho\big(\phi_t(x),\phi_t(y)\big) \le e^{\lambda t}\,\rho(x,y)$ for all $t \ge 0$ and all $x, y \in S_g\tilde{M}$. Let $\eta > 0$ be such that $\rho(x,y) < 2\eta$ implies $d\big(\tilde{h}(x),\tilde{h}(y)\big) < \epsilon$. If $x, y \in \widetilde{W}^{uu}_{\delta}(w,\phi)$ with $\delta < e^{-3\lambda}\eta$, let $T := \min\{\, s > 0 \mid \rho\big(\phi_s(x),\phi_s(y)\big) = \eta \,\}$. This number exists by the expansivity of ϕ if η is small enough (in fact for any η using the conjugacy $\tilde{h}$). We have that $T > 3$ and $\rho(x,y) \ge \eta\,e^{-\lambda T}$.

There exist continuous functions $\sigma, \tau : \mathbb{R} \to \mathbb{R}$ such that $\sigma(0) = 0 = \tau(0)$ and $\tilde{h}(\phi(x,s)) = \psi(\tilde{h}(x),\sigma(x))$, $\tilde{h}(\phi(y),t) = \psi(\tilde{h}(y),\tau(t))$. By the claim, $aT \le \tau(T) \le AT$ and $aT \le \sigma(T) \le AT$. Since $x, y \in W^{uu}_{\delta}(w,\phi)$, then $\tilde{h}(x)$ and $\tilde{h}(y)$ are in the weak unstable manifold $W^u\big(\tilde{h}(w),\psi\big)$ of $\tilde{h}(w)$. Write $p := \tilde{h}(x)$, $q := \tilde{h}(y)$ and let

$$m := W^{uu}_{4\epsilon}\big(\tilde{h}(y)\big) \cap \{\, \psi_t\tilde{h}(x) \mid -4\epsilon \le t \le 4\epsilon \,\}.$$

Then

$$\begin{aligned}
d\big(\psi_{\tau(T)}(q),\psi_{\tau(T)}(m)\big) &\ge e^{\tau(T)}\,d(q,m) \ge e^{aT}\,d(q,m)\\
d\big(\psi_{s(T)}(p),\psi_{\tau(T)}(q)\big) &= d\big(\tilde{h}\phi_T(x),\tilde{h}\phi_T(y)\big) \le \epsilon\\
e^{aT}\,d(q,m) &\le d\big(\psi_{\sigma(T)}(p),\psi_{\tau(T)}(q)\big) + d\big(\psi_{\sigma(T)}(p),\psi_{\tau(T)}(m)\big)\\
&\le \epsilon + |\sigma(T)-\tau(T)| + d(p,m)\\
&\le \epsilon + 2A\,T + 4\epsilon\\
d(q,m) &\le (5\epsilon + 2AT)\,e^{-aT} \le e^{-\frac{a}{2}T}
\end{aligned}$$

if $T > T_0 := T_0(\epsilon,A,a) > 0$. If we choose $0 < \delta < e^{-\lambda T_0 \eta}$, then $x, y \in \widetilde{W}^{uu}_{\delta}(w,\phi)$ implies that $T > T_0$. In particular

$$d(q,m) \le e^{-\frac{a}{2}T} \le \big(\eta\,e^{-\lambda T}\big)^{\frac{a}{2\lambda}}\,\eta^{-\frac{a}{2\lambda}} \le \eta^{-\alpha}\,\rho(x,y)^{\alpha}$$

for $\alpha = \frac{a}{2\lambda}$. We have that $d(\tilde{h}(w),m) \le d(\tilde{h}(w),p) + d(p,m) \le \epsilon + 4\epsilon < 12\,\epsilon = \theta$. Then

$$\begin{aligned}
d\big(H(x),H(y)\big) &= d\big(P(p),P(q)\big) = d\big(P(m),P(q)\big)\\
&\le B\,d(q,m) \le B\,\eta^{-\alpha}\,\rho(x,y)^{\alpha}.
\end{aligned}$$

This proves lemma 4.1.

$\square$

Let $g \in \mathcal{U}$ be another metric on M and $\widetilde{g}$ its lift to $\tilde{M}$. Suppose that $f : (\overline{\mathbb{D}}, S^1) \to (\overline{\mathbb{D}}, S^1)$ is a homeomorphism which is differentiable on $\mathbb{D}$ and satisfies

$$f_{\overline{z}} = \mu(g)\,f_z\,,$$

where $\mu(g)$ is from section §3. Then for $w = f \circ z$, the metric $\widetilde{g}$ is written as $\widetilde{g} = \lambda\,|dw|^2$. By the remark above, the $\widetilde{g}$-harmonic functions on $\mathbb{D}$ in the coordinates w are the harmonic functions for the euclidean Laplacian on $\mathbb{D}$.

From now on we identify $\tilde{M} \approx \mathbb{D}$ and $S_g(\infty) \approx S^1$ for any metric g on $\mathcal{U}$, using z.

4.2. Lemma. *The Poisson kernel for $\widetilde{g}$ on $\mathbb{D} \cup S^1 \approx \tilde{M} \cup S_g(\infty)$ is given by*

$$k(x,y,\theta) = \mathbb{P}\big(f(x), f(y), f(\theta)\big)$$

for any $x, y \in \mathbb{D}$, $\theta \in S^1$, where $\mathbb{P}$ is the Poisson kernel for the euclidean Laplacian

$$\mathbb{P}(z,w,\theta) = \Re\left(\frac{e^{i\theta}+w}{e^{i\theta}-w}\cdot\frac{e^{i\theta}-z}{e^{i\theta}+z}\right).$$

Proof: For $z \in \mathbb{D}$, let ω_z be the $\widetilde{g}$-harmonic measure at z and λ_z be the euclidean harmonic measure at z. Let $\varphi : S^1 \to \mathbb{R}$ be a continuous function. By lemma 4.1, it corresponds to a continuous function $S_g(\infty) \to \mathbb{R}$. Let $\varphi(z)$ be its $\widetilde{g}$-harmonic extension to $\overline{\mathbb{D}}$, $\Delta_{\widetilde{g}}(\varphi) = 0$. Let $\phi(w) = \varphi(f^{-1}(w))$ be the function φ, written in the coordinates $w = f(z)$. Let Δ be the euclidean Laplacian on $\mathbb{D}$. Then $\Delta\phi = 0$ and hence

$$\begin{aligned}
\int \varphi\, d\omega_y = \varphi(y) = \phi(f(y)) &= \int_{S^1} \phi(\theta)\, d\lambda_{f(y)}(\theta) \\
&= \int_{S^1} \varphi\circ f^{-1}\, d\lambda_{f(y)} \\
&= \int_{S^1} \varphi\circ f^{-1}(\theta)\ \ \mathbb{P}\big(f(x), f(y), \theta\big)\ \ d\lambda_{f(x)}(\theta) \\
&= \int_{S^1} (\varphi\circ f^{-1})(k\circ f^{-1})\ \ d\lambda_{f(x)} = \int_{S^1} \varphi\cdot k\ \ d\omega_x
\end{aligned}$$

where $k(\theta) := \mathbb{P}\big(f(x), f(y), f(\theta)\big)$. Therefore

$$k(x,y,\theta) := \frac{d\omega_y}{d\omega_x}(\theta) = \mathbb{P}\big(f(x), f(y), f(\theta)\big).$$

□

5. Stability of the geodesic flow.

Fix a C^r riemannian metric $g_o \in \mathcal{A}^r(M) \subset \mathcal{R}^r(M)$ such that the geodesic flow of g_o is Anosov. Let ΣM be the g_o-unit tangent bundle $\Sigma M := \{\, v \in TM \mid g_o(v,v) = 1 \,\}$. Given another riemannian metric $g \in \mathcal{R}^r(M)$ and its unit tangent bundle $S_gM = \{\, v \in TM \mid g(v,v) = 1 \,\}$, define the map $F : S_gM \to \Sigma M$ by $F(v) = v\,\big(g_o(v,v)\big)^{-\frac{1}{2}}$. Let ψ_t be the geodesic flow for g and define $\varphi_t := F\circ\psi_t\circ F^{-1}$. Then F is a C^r conjugacy between ψ_t and ϕ_t.

Given a chart $x : U \subseteq M \to \mathbb{R}^2$, consider the chart $(\overline{x}, \overline{y}) = (x, dx) : TU \to \mathbb{R}^2\times\mathbb{R}^2$, with $\overline{y}(v) = (y_1, y_2)$ if $v = \sum y_i \frac{\partial}{\partial x^i}$. In this chart, the geodesic flow for g satisfies

$$\frac{dx^k}{dt} = y_k \quad , \quad \frac{dy_k}{dt} = -\sum_{ij} \Gamma^k_{ij}\, y_i\, y_j \quad , \quad k = 1, 2\,;$$

where
$$\Gamma^k_{ij} = \frac{1}{2}\sum_{\ell}\left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial_{\ell i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell}\right) g^{\ell k}\,,$$
are the Christoffel symbols for $g = \sum g_{ij}(dx^i \otimes dx^j)$ and $[g^{k\ell}] = [g_{ij}]^{-1}$.

Let $\lambda(s) = \big(p(s), \vec{v}(s)\big) \in T_{p(s)}M$ be an orbit of ψ_s. Then $F(\lambda(s)) = \big(p(s), \vec{v}(s)(g_o)^{-\frac{1}{2}}\big)$, $g_o := g_o\big(\vec{v}(s), \vec{v}(s)\big)$, and
$$\begin{aligned}\frac{d(F\circ\lambda)}{ds} &= \left(\frac{dp}{ds}\,,\ \frac{1}{\sqrt{g_o}}\frac{d\vec{v}}{ds} - \frac{\vec{v}}{2\,(g_o)^{\frac{3}{2}}}\,\frac{d}{ds}g_o\right)\\ &= \left(\vec{v}\,,\ -\frac{1}{\sqrt{g_o}}\sum_{ij}\Gamma^k_{ij}v^iv^j\frac{\partial}{\partial x^k} - \frac{\vec{v}}{2\,(g_o)^{\frac{3}{2}}}\,\frac{d}{ds}g_o\right)\end{aligned}$$
$$\begin{aligned}\frac{d}{ds}g_o\big(\vec{v}(s),\vec{v}(s)\big) &= \frac{d}{ds}\sum_{ij} g^o_{ij}(p(s))\ v^i(s)\ v^j(s)\\ &= \sum_{ijk}\frac{\partial g^o_{ij}}{\partial x^k}v^kv^iv^j - 2\sum_{ijk\ell} g^o_{ij}\Gamma^i_{k\ell}v^kv^\ell v^j\,.\end{aligned}$$
If $\vec{w}(s) = F\big(\vec{v}(s)\big)$, then $\vec{v}(s) = \frac{\vec{w}}{\sqrt{g(w,w)}}$ and $\sqrt{g_o(\vec{v},\vec{v})} = \frac{\sqrt{g_o(w,w)}}{\sqrt{g(w,w)}}$. We have
$$\frac{dw}{ds} = \left(\frac{w}{\sqrt{g}}\,,\ -\frac{1}{\sqrt{g}\sqrt{g_o}}\sum_{ij}\Gamma^k_{ij}w^iw^j\frac{\partial}{\partial x^k} - \frac{\vec{w}}{2\sqrt{g}\,(g_o)^{\frac{3}{2}}}\sum_{ijk}\left(\frac{\partial g^o_{ij}}{\partial x_k} - 2\sum_{\ell} g^o_{\ell k}\Gamma^k_{ij}\right)w^iw^jw^k\right)$$
This is the vectorfield of ψ_s, denote it by $X(g)$. Let $\mathfrak{X}^{r-1}(\Sigma M)$ be the Banach space of C^{r-1} vectorfields on ΣM with the C^r norm. The formula above proves the following lemma.

5.1. Lemma. *The map $\mathcal{R}^r(M) \to \mathfrak{X}^{r-1}(\Sigma M)$ is C^∞.*

We will need the following version of the structural stability theorem:

5.2. Proposition. [C]
Let $X \in \mathfrak{X}^{r-1}(\Sigma M)$ be an Anosov flow, then there exists a neighbourhood $\mathcal{V} \subset \mathfrak{X}^{r-1}(\Sigma M)$, $0 < \beta < 1$ and C^{r-2} maps $\mathcal{V} \to C^\beta_\phi(\Sigma M, \Sigma M) : Y \mapsto u_Y$ and $\mathcal{V} \to C^\beta_\phi(\Sigma M, [\frac{1}{2}, +\infty[) : Y \mapsto \gamma_Y$ such that $Y \circ u_Y = \gamma_Y\, D_\phi u_Y$.

Moreover, the corresponding maps $Y \mapsto u_Y$ and $Y \mapsto \gamma_Y$ for $\beta = 0$ are C^{r-1}.

Where $C^\beta_\phi(\Sigma M, \Sigma M)$ is the space of β-Hölder continuous functions $u : \Sigma M \to \Sigma M$ such that $\frac{d}{dt}u(\phi_t(p))|_{t=0}$ exists and it is β-Hölder continuous endowed with the norm $[\![u]\!]_\beta = \|u\|_\beta + \big\|\frac{d}{dt}(u\circ\phi_t)\big\|_\beta$ where $\|\ \|_\beta$ is the β-Hölder norm for a fixed C^r riemannian metric and $C^0_\phi(\Sigma M, \Sigma M)$ is the space of continuous functions $u : \Sigma M \to \Sigma M$ such that $\frac{d}{dt}(u \circ \phi_t)$ exists, with the norm $[\![u]\!]_0 = \|u\|_{\sup} + \left\|\frac{d}{dt}(u \circ \phi_t)\right\|_{\sup}$.

5.3. Corollary. *For $Y \in \mathcal{V}$ consider the map $\sigma_Y : M \to \mathbb{R}^+$ defined by $\psi_Y\big(u_Y(p), 1\big) = u_Y \circ \phi\big(p, \sigma_Y(p)\big)$, where ψ_Y is the flow of Y. Then*

(i) *The map $\mathcal{U} \to C^\beta(\Sigma M, \mathbb{R}^+) : Y \mapsto \sigma_Y$ is C^{r-2}.*
(ii) *In particular the maps $\mathcal{U} \to C^\beta(\Sigma M, \Sigma M) : Y \mapsto \psi_Y\big(u_Y(p), 1\big)$ is C^{r-2}.*
(iii) *The corresponding maps for $\beta = 0$ are C^{r-1}.*

Proof: From the equation $\psi\big(u(p), t\big) = u\big(\phi(p, s(t))\big)$we get that $\frac{ds}{dt} = \gamma\big(\phi_s(p)\big)$. Consider the map $F : \mathcal{U} \times C^\beta(\Sigma M, \mathbb{R}^+) \to C^\beta(\Sigma M, \mathbb{R}^+)$ given by $F(Y, \sigma)(p) = \int_0^\sigma \frac{1}{\gamma_Y(\phi_s(p))}\, ds$. Then the function σ_Y is characterized by $F(Y, \sigma_Y) \equiv 1$. Observe that $\left(\frac{\partial F}{\partial \sigma} \cdot \tau\right)(p) = \big(\gamma_Y(\phi_s(p))\big)^{-1} \tau(p)$ is invertible because $\gamma_Y\big(\phi_s(p)\big) > 0$. Since F is C^r, the implicit function theorem implies that $Y \mapsto \sigma_Y \in C^\beta(\Sigma M, \mathbb{R}^+)$ is C^{r-2}. The case $\beta = 0$ is similar.

For (ii) use the fact that $\mathfrak{X}^r(\Sigma M) \to C^\gamma(\Sigma M \times \mathbb{R}, \Sigma M) : Y \mapsto \psi_Y$ is C^{r-2} for $0 < \gamma < 1$ and $\mathfrak{X}^r(\Sigma M) \to C^0(\Sigma M \times \mathbb{R}, \Sigma M) : Y \mapsto \psi_Y$ is C^{r-1}.

□

6. Proof of Theorems A and B.

We will need the following generalization of a theorem by Ahlfors and Bers which will be proved on part II.

II.3.9. Theorem. *Given $\mu : \mathbb{D} \to \mathbb{D}$ measurable with $\|\mu\|_\infty < k < 1$ there exists a unique homeomorphism of the closed disk $f^\mu = f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ satisfying $f_{\overline{z}} = \mu\, f_z$, with generalized derivatives f_z, $f_{\overline{z}}$, such that $f(0) = 0$, $f(1) = 1$, $f(S^1) = S^1$. Moreover*

(i) *The map f is Hölder continuous on $\overline{\mathbb{D}}$ and if $0 < r < 1$ and $\mu \in C^n(|z| < r, \mathbb{D})$ then $f \in C^{n+\alpha}(|z| < r, \mathbb{D})$ for some $0 < \alpha = \alpha(k) < 1$.*
(ii) *For any $n \geq 1$ and any $0 < r < R < 1$, the map $\mathcal{L}_\infty(\mathbb{D}) \cap C^n(|z| < R, \mathbb{C}) \cap \big\{\, \|\mu\| < k \,\big\} \to C^{n+\alpha}(r) \cap C^\alpha(\mathbb{D}, \mathbb{D})$ given by $\mu \mapsto f^\mu$ is C^∞.*

We now prove theorems A and B. Let g be a riemannian metric in a small C^r neighbourhood of g_o. Then the map $F : S_g M \to \Sigma M$ of section §5 is a C^r conjugacy between the geodesic flow of g and the flow of $X(g)$. In particular F maps strong stable and strong unstable manifolds of the geodesic flows to strong stable and strong unstable manifolds of $X(g)$.

Let $\psi(g)$ be the geodesic flow of g and $\varphi(g) := F \circ \psi(g) \circ F^{-1}$. Let $\pi : TM \to M$ be the projection. Let $P_g : S_g M \to \mathbb{R}$ be $P_g(X) = \log K_g\big(\pi \tilde{X}, \pi\psi(g)(\tilde{X}, 1), \tau \tilde{M}\big)$ where $\tilde{X}$ is a lift of X under $p : T\tilde{M} \to TM$. Let μ_g be the equilibrium state of P_g for $\psi(g)$. Consider the measure $\nu_g := F^*(\mu_g)$, $\nu_g(A) = \mu_g(F^{-1}(A))$. We have for the metric entropies that $h_{\nu_g}(\varphi(g)) = h_{\mu_g}(\psi(g))$. Since the conjugacy

F is differentiable we have that the Lyapunov exponents of ν_g and μ_g coincide $\lambda^+(\nu_g) = \lambda^+(\mu_g)$.

In particular, the Hausdorff dimension of the conditional measures on local strong manifolds are equal:

$$HD^u(\nu_g) = HD^u(\mu_g) = \frac{h_{\mu_g}(\psi_g)}{\lambda^+(\mu_g)} = \frac{h_{\nu_g}(\varphi_g)}{\lambda^+(\nu_g)}.$$

For any $\varphi(g)$-invariant measure ν, we have that

$$h_\nu(\varphi_g) + \int_{\Sigma M} p_g \circ F^{-1}\, d\nu = h_\mu(\psi_g) + \int_{S_g M} P_g\, d\mu\,,$$

where $\nu = F^*(\mu)$, i.e. $\mu\big(F^{-1}(A)\big) := \nu(A)$. Therefore the maximum of these numbers is attained at $\nu = \nu_g = F^*(\mu_g)$. Hence ν_g is the equilibrium state of $G_g = P_g \circ F^{-1}$ for φ_g. We have that

$$\begin{aligned} G_g(X) &= P_g \circ F^{-1}(X) = P_g\left(\tfrac{X}{\|X\|_g}\right) \\ &= \log K_g\left(\pi\left(\tfrac{\tilde X}{\|X\|_g}\right), \pi\, \tilde\psi_h\left(\tfrac{\tilde X}{\|X\|_g}, 1\right)\,,\, \tau_g \tilde X\right) \\ &= \log K_g\left(\pi(\tilde X), \pi\tilde\varphi_g(\tilde X, 1), \tau_g \tilde X\right) \\ &= \log \mathbb{P}\left(f_g(\pi\tilde X), f_g\big(\pi\tilde\varphi_g(\tilde X,1)\big), f_g\tau_g\tilde X\right)\,, \end{aligned}$$

where $\mathbb{P}$ is the euclidean Poisson kernel on $\mathbb{D}$ and we consider $\pi : T\tilde M \to \tilde M \approx \mathbb{D}$. In order to apply proposition 5.2 we need to see that $g \mapsto G_g \circ u_{X(g)} \in C^\beta(\Sigma M, \mathbb{R})$ is C^{r-2} for some $\beta > 0$, where $u_{X(g)}$ is the topological equivalence of proposition 5.2.

Fix a fundamental domain of $p : \mathbb{D} \approx \tilde M \to M$ and its corresponding lift $q : M \to \mathbb{D}$. Since the C^r or C^α, $0 < \alpha < 1$ norms of maps are equivalent to sums of C^r or C^α norms of local restrictions of the maps, we don't bother with the discontinuities of this lift q. We have that

$$(3) \quad G_g \circ u_{X(g)}(V) = \log \mathbb{P}\left(f_g\, \pi\, q\, u_g(V),\ f_g\, \pi\, \tilde\varphi_g\big(q\, u_g(V), 1\big),\ f_g\, \tau_g\, q\, u_g(V)\,\right).$$

By the structural stability theorem, the $\tilde g_o$-geodesic of $q(V)$ and the $\tilde g$-geodesic of $q\big(u_g(V)\big)$ remain at bounded distance of each other. In particular, their limit on $\overline{\mathbb{D}}$ as $t \to +\infty$ is the same:

$$\Theta(V) := \tau_g\, q\, u_g(V) = \tau_{g_o}\, q(V) \qquad \text{for all } g \text{ near } g_o.$$

By lemma 4.1.(ii), the map $\Theta : \Sigma M \to S^1$ is Hölder continuous. By theorem II.3.9, for some $0 < \alpha < 1$, the map $g \mapsto f_g \in C^\alpha(S^1, S^1)$ is C^∞. Therefore for some $0 < \beta < 1$ the map $g \mapsto f_g \circ \Theta \in C^\beta(S^1, S^1)$ is C^∞.

By proposition 5.2 and lemma 5.1, for some $0 < \gamma < 1$, the map $g \mapsto u_g \in C^\gamma(\Sigma M, \Sigma M)$ is C^{r-2}. The maps $\pi : T\tilde M \to T\tilde M$ and $q : M \to \mathbb{D}$ are C^r and by theorem II.3.9 and lemma 3.1, the map $g \mapsto f_g \in C^{r-1+\alpha}(|z| < R, \mathbb{D})$ is C^∞ for some $0 < R < 1$ such that

$$\left\{\, w \mid \tilde d_g\big(w, q(M)\big) \le 4,\ \text{for some } g \in \mathcal{V}\,\right\} \subseteq \big[|z| < R\big]\,,$$

where $\mathcal{V}$ is a neighbourhood of g_o and $\tilde{d}_g$ is the $\tilde{g}$-distance in $\tilde{M} \approx \mathbb{D}$. Therefore the map $g \mapsto f_g \circ \pi \circ q \circ u_g \in C^\delta(\Sigma M, \mathbb{D})$ is C^{r-2} for some $0 < \delta < 1$. Observe that we used here the derivatives of f_g. For $\delta = 0$, this map is C^{r-1}.

By corollary 5.3, the map $g \mapsto \tilde{\psi}_g\big(q\, u_g(\cdot), 1\big) = q \circ \varphi_g\big(u_g(\cdot), 1\big) \in C^\beta(\Sigma M, \mathbb{D})$ is C^{r-2} for some $0 < \beta < 1$ and it is C^{r-1} for $\beta = 0$. Since $g \mapsto f_g \in C^{r-1+\alpha}\big(|z| < R, \mathbb{D}\big)$ is C^∞, we have that the second component of (3): $g \mapsto f_g \circ \pi \circ q \circ \varphi_g\big(u_g(\cdot), 1\big) \in C^\delta(\Sigma M, \mathbb{D})$ is C^{r-2} for some $0 < \delta < 1$ and it is C^{r-1} for $\delta = 0$.

Since $\mathbb{P}$ is C^∞, from equation (3) we get that the map $\mathcal{A}^r(M) \supseteq \mathcal{U} \ni g \mapsto G_g \circ u_{X(g)} \in C^\alpha(\Sigma M, \mathbb{R})$ is C^{r-2} for some $0 < \alpha < 1$ and it is C^{r-1} for $\alpha = 0$. Applying theorem 2.3, we have that $g \mapsto h(\nu_g) = h(\mu_g)$ is C^{r-2}, $g \mapsto \lambda^+(\nu_g) = \lambda^+(\mu_g)$ is C^{r-3} and also that $g \mapsto P(\varphi_g)$ is C^{r-1}, where $P(\varphi_g)$ is the pressure of G_g for ψ_g.

$\square$

II. REGULARITY OF QUASICONFORMAL MAPPINGS.

Our aim here is to prove (cf. Theorem 3.9) that if $f^\mu : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a μ-quasiconformal map normalized by $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$, then the map $\mu \mapsto f^\mu$ is C^∞; where μ varies in the space of C^k maps and f^μ in the space of $C^{k+\alpha}$ maps. We obtain similar results for solutions of non-homogeneous Beltrami equations (cf. corollary 3.8).

Bers [**B**] proved that f^μ is $C^{k+1+\alpha}$ if μ is $C^{k+\alpha}$. Ahlfors and Bers [**AB**] proved that the map $\mu \mapsto f^\mu$ is C^1 when μ is in $\mathcal{L}_\infty$ and f^μ is Hölder continuous. In order to get the second derivative, we are forced to deal with derivatives of non-homogeneous Beltrami equations.

The proof that the map $\mu \mapsto f^\mu$ is C^∞ relies in the fact that its derivative satisfies a non-homogeneous Beltrami equation and that the derivatives of such equations can be expressed again in terms of non-homogeneous Beltrami equations. In fact, if $f^\mu_{\overline{z}} = \mu\, f^\mu_z$, the derivative $\frac{d}{d\mu} f^\mu \cdot h = \omega$ satisfies [**AB**] $\omega_{\overline{z}} = \mu\, \omega_z + h\, f_z$. Consider the map $F(\mu, \sigma) = \omega^{\mu,\sigma}$, where $\omega = \omega^{\mu,\sigma}$ satisfies $\omega_{\overline{z}} = \mu\, \omega_z + \sigma$. Since F is linear on σ we will have that

$$\frac{\partial F}{\partial \sigma} \cdot h = F(\mu, h).$$

A formal computation shows that the derivative $\lambda = \frac{\partial F}{\partial \mu} \cdot h$ should satisfy $\lambda_{\overline{z}} = \mu\, \lambda_z + h\, \omega_z$. So that

$$\frac{\partial F}{\partial \mu} \cdot h = F(\mu, h \cdot F(\mu, \sigma)).$$

We will prove that F is C^1. Then a recursive argument will give that F is C^∞ and then $\mu \mapsto f^\mu$ is C^∞.

1. Preliminaries.

Given a C^1 function $f(x, y)$ defined on a region $\Omega \subseteq \mathbb{R}^2$ with values on $\mathbb{C}$, define the derivatives

$$f_z := \tfrac{1}{2}(f_x - i\, f_y) \qquad , \qquad f_{\overline{z}} := \tfrac{1}{2}(f_x + i\, f_y)\,. \tag{4}$$

If $f : \Omega \to \mathbb{C}$ is locally integrable then we say that f_z and $f_{\overline{z}}$ are the generalized derivatives of f if they are locally integrable and satisfy

$$\begin{aligned} \iint_\Omega f_z\, \varphi\, dx\, dy &= -\iint_\Omega f\, \varphi_z\, dx\, dy \\ \iint_\Omega f_{\overline{z}}\, \varphi\, dx\, dy &= -\iint_\Omega f\, \varphi_{\overline{z}}\, dx\, dy \end{aligned} \tag{5}$$

for all $\varphi \in C^1$ with compact support in Ω. The following lemma is well known:

1.1. Lemma. *If $f_{\overline{z}} \equiv 0$, then f is holomorphic.*

More precisely, there exists a holomorphic function which is almost everywhere equal to f. Define the following operators

$$
\begin{aligned}
(Ph)(w) &= -\frac{1}{\pi}\iint_{\mathbb{C}} h(z)\left[\frac{1}{z-w}-\frac{1}{z}\right] dx\,dy \quad , \quad z = x+iy\,,\\
(Hh)(w) &= -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{h(z)-h(w)}{(z-w)^2}\,dx\,dy \quad , \quad z = x+iy\,,\\
&= -\frac{1}{\pi}\lim_{\epsilon\to 0}\iint_{|z|>\epsilon} \frac{h(z)-h(w)}{(z-w)^2}\,dx\,dy\,.
\end{aligned}
$$

1.2. Lemma. *Suppose that $g \in \mathcal{L}_p(\mathbb{C})$, $p > 2$. Then Pg exists everywhere as an absolutely convergent integral and Hg exists almost everywhere as a Cauchy principal limit. The following relations hold:*

(6) $(Pg)_{\overline{z}} = g \quad , \quad (Pg)_z = Hg$.
(7) $|Pg(z_1) - Pg(z_2)| \leq K_p \, \|g\|_p \, |z_1 - z_2|^{1-\frac{2}{p}}$.
(8) $\|Hg\|_g \leq C_p \, \|g\|_p$.

Actually, (8) *holds for $p > 1$ and for $p = 2$ it can be replaced by*

(9) $\|Hg\|_p = \|g\|_2$.
(10) $\lim_{p\to 2} C_p = 1$.
(11) *Write $\partial g = g_z$, $\overline{\partial} g = g_{\overline{z}}$; then the operators ∂, $\overline{\partial}$ and H commute.*

The relation (8) is a deep result called Calderon-Zygmund's inequality. The proof of this lemma can be found in [**Ah**].

We now see the behaviour of the operator H on small discs. For $0 < R < 1$, define the operator

$$
\begin{aligned}
H_R h(w) &= -\frac{1}{\pi}\iint_{|z|<R} \frac{h(z)-h(w)}{(z-w)^2}\,dx\,dy \quad , \quad z = x+iy\\
&= -\frac{1}{\pi}\lim_{\epsilon\to 0}\iint_{\epsilon<|x|<R} \frac{h(z)-h(w)}{(z-w)^2}\,dx\,dy\,.\\
P_R h(w) &= -\frac{1}{\pi}\iint_{|z|<R} h(z)\,\left[\frac{1}{z-w}-\frac{1}{z}\right]\,dx\,dy\,.
\end{aligned}
$$

Define the norm

$$
\|h\|_{R,p} = \left(\iint_{|z|<R} |h(z)|^p \, dx\,dy\right)^{\frac{1}{p}}.
$$

Let $C^\alpha(D_R,\mathbb{C}) = C^\alpha(R)$ be a Banach space of α-Hölder continuous functions on the disc $D_R := \{\, z\in\mathbb{C}\,|\; |z|<R\,\}$, provided with the norm

$$\begin{aligned}
[[h]]_{R,p} &:= \|h\|_{R,\infty} + [h]_{R,\alpha}\ ,\\
\|h\|_{R,\infty} &:= \sup_{|z|<R} |h(z)|\ ,\\
[h]_{R,\alpha} &:= \sup_{|z-w|<1} \frac{|h(z)-h(w)|}{|z-w|^\alpha}\,.
\end{aligned}$$

Observe that

$$[[g\cdot h]]_{R,\alpha} \le 2\,[[g]]_{R,\alpha}\,[[h]]_{R,\alpha}\,.$$

For $R>0$, $p>2$, $n\ge 0$ define

$$W^{n,p}(R,0) := \big\{ h:\mathbb{C}\to\mathbb{C}\,|\, h\in C^{n-1}(\mathbb{C},\mathbb{C}),\ \|D^nh\|\in\mathcal{L}_p(D_R) \text{ and } h(z)=0 \text{ for } |z|>R\,\big\}\,,$$

$$W^{n,p}(R) := \big\{\, h:D_R\to\mathbb{C}\,|\, h\in C^{n-1}(D_R,\mathbb{C}), \|D^nh\|\in\mathcal{L}_p(D_R)\,\big\}\,.$$

$$\begin{aligned}
\|h\|_{W^{n,p}(R)} &:= \|h\|_{C^{n-1}(R)} + \|D^nh\|_{R,p}\ ,\\
\|h\|_{C^{n-1}(R)} &:= \textstyle\sum_{k=0}^{n-1} \left\|D^kh\right\|_{R,\infty}\ ,\\
\left\|D^kh\right\|_{R,\infty} &:= \sum_{i+j=k} \left\|\partial^i\overline{\partial}^j h\right\|_{R,\infty}\,.
\end{aligned}$$

On both $W^{n,p}(R,0)$ and $W^{n,p}(R)$ consider the norm $\|\ \|_{W^{n,p}(R)}$. Observe that for $0<R<1$, we have

$$[[h]]_{R,\alpha} \le \|h\|_{R,\infty} + \|Dh\|_{R,\infty} = \|h\|_{R,\infty} + \|\partial h\|_{R,\infty} + \left\|\overline{\partial}h\right\|_{R,\infty}\,.$$

1.3. Lemma.

(a) *For all $0<\alpha<1$ there exists $C(\alpha)>0$ such that*

$$[[H_Rh]]_{R,\alpha} \le C(\alpha)\,[[h]]_{R,\alpha}\,R^\alpha \text{ for all } 0<R<1.$$

Moreover, if $h\in C^\alpha(D_R,\mathbb{C})$, then P_Rh is $C^{1+\alpha}$ and

$$(P_R\,h)_{\overline{z}} = h \quad , \quad (P_R\,h)_z = Hh \text{ on } 0<R<1.$$

(b) *For all $0<\alpha<1$ there exists $D(\alpha)>0$ such that*

$$[[P_R\,h]]_{R,\alpha} \le D(\alpha)\,[[h]]_{R,\alpha}\,R^{1-\alpha}.$$

(c) *For all $p>2$ there exists $A(p)>1$ such that for all $0<R<1$ and $h\in W^{n,p}(R,0)$,*

$$\begin{aligned}
\|H_R\,h\|_{C^{n-1}(R)} &\le A(p)\,R^{1-\frac{2}{p}}\,\|h\|_{W^{n,p}(R)}\,,\\
\|D^nH_Rh\|_{R,p} &\le C_p\,\|D^nh\|_{R,p}\,.
\end{aligned}$$

In particular, the operator $H_R : W^{n,p}(R,0)\to W^{n,p}(R)$ is continuous and has norm $\|H_R\|_{W^{n,p}(R)} \le A(p)$.

(d) *For all $p > 2$ there exists $B(p,R) > 0$ such that for $h \in W^{n,p}(R,0)$*

$$\|P_R\, h\|_{W^{n+1,p}(R)} \leq B(p,R)\ \|h\|_{W^{n,p}(R)}\ .$$

in particular, the operator $P_R : W^{n,p}(R,0) \to W^{n+1,p}(R)$ is continuous.

The proof of part (a) of this lemma can be found in [**B**].

Proof:

(b) Let $p > 2$ be such that $\alpha = 1 - \frac{2}{p}$. By lemma 1.2 we have that

$$\begin{aligned}
[P_R\, h]_{R,\alpha} &\leq K_p\ \|h\|_{R,p} \leq K_p\ \|h\|_{R,\infty}\ \|1\|_{R,p} \\
&\leq K_p\, [[h]]_{R,\alpha}\, \pi^{\frac{1}{p}}\, R^{\frac{2}{p}} \\
\|P_R\, h\|_{R,\infty} &\leq |P_R\, h(0)\| + [P_R\, h]_{R,\alpha}\, R^{\alpha} \\
&\leq K_p\, [[h]]_{R,\alpha}\, \pi^{\frac{1}{p}}\, R^{\frac{2}{p}}\, R^{\alpha}\,,
\end{aligned}$$

because $P_R h(0) = 0$. Now observe that $\frac{2}{p} = 1 - \alpha$ and $R^{\alpha} < 1$ to get (b).

(c) Given $0 \leq k < n$, let $\delta^k h$ be a k-th partial derivative of h, $\delta^k h = \partial^i \overline{\partial}^j h$, $i + j = k$. Then

$$\delta^k h = P(\delta^k h_{\overline{z}}) + F\ , \tag{12}$$

where $F : \mathbb{C} \to \mathbb{C}$ is holomorphic. Since $\delta^k h \in \mathcal{L}_p(\mathbb{C})$ and by 1.2.(7), $P(\delta^k h_{\overline{z}})$ is $\mathcal{O}(|z|^2)$ when $|z| \to \infty$, then F is constant. In particular, for $\alpha = 1 - \frac{2}{p}$,

$$\begin{aligned}
[\delta^k h]_{R,\alpha} &= [P(\delta^k h_{\overline{z}})]_{R,\alpha} \leq K_p\ \|\delta^k h_{\overline{z}}\|_p = K_p\ \|\delta^k h_{\overline{z}}\|_{R,p} \\
[[\delta^k h]]_{R,\alpha} &\leq \|\delta^k h\|_{R,\infty} + K_p\ \|\delta^k h_{\overline{z}}\|_{R,p}\ .
\end{aligned}$$

By 1.2.(11), $\delta^k H h = H \delta^k h$ and

$$\begin{aligned}
\left\|\delta^k H h\right\|_{R,\infty} = \left\|H_R\, \delta^k h\right\|_{R,\infty} &\leq \left[\left[H_R\, \delta^k h\right]\right]_{R,\alpha} \\
&\leq C(\alpha)\, R^{\alpha}\ [[\delta^k h]]_{R,\alpha} \\
&\leq C(\alpha)\, R^{\alpha}\ \left(\left\|\delta^k h\right\|_{R,\infty} + \left\|\partial \delta^k h\right\|_{R,\infty} + \left\|\overline{\partial} \delta^k h\right\|_{R,\infty}\right) \quad \text{if } k \leq n-2\,, \\
&\leq C(\alpha)\, R^{\alpha}\ \left(\left\|\delta^{n-1} h\right\|_{R,\infty} + K_p\ \left\|\delta^{n-1} h_{\overline{z}}\right\|_{R,p}\right) \quad \text{if } k = n-1.
\end{aligned}$$

Adding over all k-th partial derivatives, we get

$$\|Hh\|_{C^{n-1}(R)} \leq C(\alpha)\ (3 + K_p)\ R^{\alpha}\ \|h\|_{W^{n,p}(R)}\ .$$

For $k = n$, we have that

$$\begin{aligned}
\|\delta^n H_R h\|_{R,p} &= \|H_R\, \delta^n h\|_{R,p} \leq C_p\ \|\delta^n h\|_{R,p} \\
\|D^n H_R\, h\|_{R,p} &\leq C_p\ \|D^n h\|_{R,p} \\
\|H_R h\|_{W^{n,p}(R)} &\leq (C_p + C(\alpha)\ (K_p + 3)\ R^{\alpha})\ \|h\|_{W^{n,p}(R)}\ .
\end{aligned}$$

(d) Let $\delta = \partial^i \overline{\partial}^j$, then

$$\begin{aligned} \partial^i \overline{\partial}^j \, P_R h &= \partial^i \overline{\partial}^{j-1} h && \text{if } j \geq 1, \\ &= \partial^{i-1} \overline{\partial}^j \, H h && \text{if } i \geq 1. \end{aligned}$$

Therefore

$$\begin{aligned} \|P_R h\|_{W^{n+1,p}(R)} &\leq \max\{\, \|h\|_{W^{n,p}(R)} \,, \|Hh\|_{W^{n,p}(R)} \,, \|P_R h\|_{R,p} \,, \|P_R h\|_{R,\infty} \,\} \\ &\leq B(p,R) \; \|h\|_{W^{n,p}(R)} \end{aligned}$$

for $B(p,R) := \sum\{\, 1 \,,\, A(p) \,,\, K_p \, \pi^{\frac{1}{p}} \, R \,,\, K_p \, R^{1-\frac{2}{p}} \,,\, \pi^{\frac{1}{p}} \, R^{\frac{2}{p}} \,\}$.

$\square$

For $\omega : \mathbb{C} \to \mathbb{C}$ and $p > 2$, let $\alpha = 1 - \frac{2}{p}$ and

$$\|\omega\|_{B_p} = \sup_{z_1 \neq z_2} \frac{|\omega(z_1) - \omega(z_2)|}{|z_1 - z_2|^{\alpha}} + \left(\iint_{\mathbb{C}} |\omega_z|^p \right)^{\frac{1}{p}} + \left(\iint_{\mathbb{C}} |\omega_{\overline{z}}|^p \right)^{\frac{1}{p}}$$

and define B_p as the space of maps $\omega : \mathbb{C} \to \mathbb{C}$ with $\omega(0) = 0$ and $\|\omega\|_{B_p} < \infty$, endowed with the norm $\| \;\; \|_{B_p}$.

1.4. Lemma. *Given $\mu \in \mathcal{L}_\infty(\mathbb{C})$, $\|\mu\|_\infty < k < 1$, $\sigma \in \mathcal{L}_p(\mathbb{C})$ with $k\, C_p < 1$. Then there exists a unique solution $\omega^{\mu,\sigma}$ of $\omega_{\overline{z}} = \mu\, \omega_z + \sigma$ with $\omega(0) = 0$ and $\omega_z \in \mathcal{L}_p(\mathbb{C})$. Moreover*

(i) *There exists $K = K(k,p)$ such that $\|\omega\|_{B_p} \leq K(k,p)\; \|\sigma\|_{\mathcal{L}_p}$.*
(ii) *If $\mu_n \to \mu$ almost everywhere, $\|\mu_n\|_\infty < k$ and $\sigma_n \to \sigma$ in $\mathcal{L}_p$, then $\omega^{\mu_n,\sigma_n} \to \omega^{\mu,\sigma}$ in B_p.*
(iii) *The unique solution of $\omega_{\overline{z}} = \mu\, \omega_z + \sigma$ such that $\omega(0) = a \in \mathbb{C}$ and $\omega_z \in \mathcal{L}_p$ is $\omega(z) = a + \omega^{\mu,\sigma}(z)$.*

The proof of all this lemma but item (iii) can be found in [**AB**]. Uniqueness in item (iii) is proved by substracting two such solutions and obtaining a solution of the homogeneous problem which is zero by (i).

1.5. Theorem. (Ahlfors-Bers) [AB]

Given $\mu : \mathbb{C} \to \mathbb{C}$ measurable with $\|\mu\|_\infty < k < 1$ and $p > 2$ with $k\, C_p < 1$. Then there exists a unique homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that $f_{\overline{z}} = \mu\, f_z$, $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$. Moreover,

(i) *f is $\alpha = 1 - \frac{2}{p}$ Hölder continuous on $S^2 = \mathbb{C} \cup \{\infty\}$.*
(ii) *f_z is locally of class $\mathcal{L}_p$.*
(iii) *$f_z \neq 0$ almost everywhere.*
(iv) *f^{-1} is $\alpha = 1 - \frac{2}{p}$ Hölder continuous and has generalized derivatives which are locally of class $\mathcal{L}_p$.*
(v) *$(f^{-1})_z$ and $(f^{-1})_{\overline{z}}$ are determined by the classical formulas.*
(vi) *f and f^{-1} transform measurable sets into measurable sets.*
(vii) *Integrals are transformed according to the classical rule.*

(viii) *If $\varphi_{\overline{z}} = \mu\,\varphi_z$ on a region $\Omega \subseteq \mathbb{C}$, then $\varphi \circ f^{-1}$ is holomorphic on $f(\Omega)$.*

The solution f of theorem 1.5 will be denoted by f^μ through the rest of the paper.

1.6. Lemma. [**AB**]
Let $f = f^\mu$, $\Omega \subseteq \mathbb{C}$ bounded and suppose that h_z, $h_{\overline{z}} \in \mathcal{L}_q(f(\Omega))$, $q > 2$. Then $h \circ f$ has generalized derivatives given by

$$(h \circ f)_z = (h_z \circ f)\, f_z + (h_{\overline{z}} \circ f)\, \overline{f}_z$$
$$(h \circ f)_{\overline{z}} = (h_z \circ f)\, f_{\overline{z}} + (h_{\overline{z}} \circ f)\, \overline{f}_{\overline{z}}$$

and

$$\|(h \circ f)_z\|_r \leq M \left(\|h_z\|_q + \|h_{\overline{z}}\|_q \right), \quad r = \frac{p\,q}{p+q-2},$$

where the norms are over the corresponding bounded regions Ω, $f(\Omega)$ and M is independent of h.

1.7. Corollary. *Let $f = f^\mu$ and suppose that $h_{\overline{z}} = \nu\, h_z$, then*

(i) *$(f^{-1})_{\overline{z}} = \lambda\,(f^{-1})_z$ with $\lambda = -\left(\frac{f_z}{\overline{f_{\overline{z}}}}\,\mu\right) \circ f^{-1}$, $\overline{f}_{\overline{z}} = \overline{(f_z)}$.*

(ii) *If $(h \circ f)_{\overline{z}} = \eta\,(h \circ f)_z$, then $\nu = \left(\frac{\eta - \nu}{1 - \eta\,\overline{\mu}}\,\frac{f_z}{\overline{f_{\overline{z}}}}\right) \circ f^{-1}$.*

(iii) *If $\overline{g(z)} = \frac{1}{f(1/\overline{z})}$, then $g_{\overline{z}} = \lambda\, g_z$, with $\overline{\lambda(z)} = \mu\,\left(\frac{1}{\overline{z}}\right)\,\frac{\overline{z}^2}{z^2}$.*

Write

$$\mathbb{D} := \left\{ z \in \mathbb{C} \,\middle|\, |z| < 1 \right\} \qquad , \qquad S^1 := \left\{ x \in \mathbb{C} \,\middle|\, |z| = 1 \right\}.$$

1.8. Corollary. *If $\overline{\mu(z)} = \mu\left(\frac{1}{\overline{z}}\right)\frac{\overline{z}^2}{z^2}$, then $F := f^\mu$ restricted to $\mathbb{D}$ is the unique solution of $F_{\overline{z}} = \mu F_z$ on $\mathbb{D}$ such that $F(0) = 0$, $F(1) = 1$ and $F(\mathbb{D}) = \mathbb{D}$. We have that $f = f^\mu$ satisfies $\overline{f(z)} = \frac{1}{f(1/\overline{z})}$. In particular, F is an $\alpha = 1 - \frac{2}{p}$ Hölder continuous homeomorphism of $\mathbb{D}$.*

Proof: By 1.7.(iii) and the uniqueness of the solution in theorem 1.5, we have that $\overline{f\left(\frac{1}{z}\right)} = \frac{1}{f(\overline{z})}$ and $f(0) = 0$, therefore $f(\mathbb{D}) \subset \mathbb{D}$ and it is a solution for F. If there exists another solution G on $\mathbb{D}$, then $H = G \circ F^{-1}$ is analytic on $\mathbb{D}$ and $H(0) = 0$, $H(1) = 1$. By Schwartz's lemma, $H(z) = \mathrm{e}^{i\theta} z$ for some $\theta \in [0, 2\pi[$. Since $H(1) = 1$, then $\theta = 0$, $H(z) = z$ and hence $G = F$.

□

Given $0 < R < \infty$, let $B_{R,p}$ be the Banach space of functions $\omega : \mathbb{C} \to \mathbb{C}$ with $\omega(0) = 0$ and finite norm $\|\ \|_{B_{R,p}}$:

$$\|\omega\|_{B_{R,p}} = \sup_{|z_1|,|z_2| \leq R} \frac{|\omega(z_1) - \omega(z_2)|}{|z_1 - z_2|^{1-\frac{2}{p}}} + \left(\iint_{|z| \leq R} |\omega_z|^p \; dx\, dy \right)^{\frac{1}{p}}.$$

The following theorems are due to Ahlfors and Bers:

1.9. Theorem. [**AB**]. *Suppose that* $\|\mu\|_\infty \le k$, $\|\mu\|_\infty \le k$, $\|\nu\|_\infty \le k$, $k\,C_p < 1$, $p > 2$. *Then for all* $R > 0$,

(a) $\|f^\mu - f^\nu\|_{B_{R,p}} \le c(R)\ \|\mu - \nu\|_\infty$, *with* $c(R)$ *depending only on* R, k, p.

(b) *If* $\mu_n \to \mu$ *almost everywhere, then* $\|f^{\mu_n} - f^\mu\|_{B_{R,p}} \to 0$.

1.10. Theorem. [**AB**]. *Let* $t = (t_1, \ldots, t_n)$ *and* $s = (s_1, \ldots, s_n)$ *be real vectors in* $\mathbb{R}^n$. *Suppose that for all* t *in some open set* Δ *we have*

$$\mu(s+t) = \mu(t) + \sum_{i=1}^{n} a_i(t)\, s_i + |s|\ \alpha(t,s)$$

with $\|\mu(t)\|_\infty \le k < 1$, $\|\alpha(t,s)\|_\infty \le c$ *and* $\alpha(t,s) \to 0$ *almost everywhere as* $s \to 0$. *Suppose further that the norms* $\|a_i(t+s)\|_\infty$ *are bounded and that* $a_i(t+s) \to a_i(t)$ *almost everywhere for* $s \to 0$. *Then* $\omega^{\mu(t)}$ *has a development*

$$f^{\mu(s+t)} = f^{\mu(t)} + \sum_{i=1}^{n} \omega_i(t)\, s_i + |s|\ \gamma(t,s)$$

with $\|\gamma(t,s)\|_{B_{R,p}} \to 0$ *for* $s \to 0$. *Where* $\omega_i(t)$ *is the solution of*

$$W_{\overline{z}} = \mu(t)\, W_z + a_i(t)\, f_z^{\mu(t)}$$

such that $W(0) = 0$, $W(1) = 0$ *and* $|W(z)| = \mathcal{O}(\left|f^{\mu(t)}\right|^2)$ *as* $z \to \infty$.

2. The local non homogeneous Beltrami equation.

From now on the functions μ are assumed to be measurable and with $\|\mu\|_\infty \leq k < 1$ for some fixed k and p is assumed to be $p > 2$ and such that $k\,C_p < 1$ unless otherwise stated.

2.1. Lemma. *Let $0 < R < 1$ and $p > 2$. Let $\mu,\ \sigma \in W^{n,p}(R)$ be such that $\mu(z) = \sigma(z) = 0$ for all $|z| \geq R$. Let ω be the solution of $\omega_{\overline{z}} = \mu\,\omega_z + \sigma$ such that $\omega(0) = 0$ and $\omega_z \in \mathcal{L}_p(\mathbb{C})$. Suppose that $k := \|\mu\|_\infty < 1$ and*

$$\Theta := \Theta(R, n, p, \|\mu\|_{W^{n,p}(R)}, k) = k\,C_p + 2^n\ \|\mu\|_{W^{n,p}(R)}\ A(p)\,R^{1-\frac{2}{p}} < 1$$

with $A(p)$ from lemma 1.3. Then $\omega \in W^{n+1,p}(R)$ and there exists $D(R,n) = D(R, n, p, \|\mu\|_{W^{n,p}(R)}, k) > 0$ such that

$$\|\omega\|_{W^{n+1,p}(R)} \leq D(R,n)\ \|\sigma\|_{W^{n,p}(R)}$$

Proof: Let q be a solution of

$$q = \mu\,H q + \sigma \tag{13}$$

in $\mathcal{L}_p(\mathbb{C})$. This is possible because the norm of the operator μH in $\mathcal{L}_p(\mathbb{C})$ is $\leq k\,C_p < 1$ and hence $(I - \mu H)$ is invertible in $\mathcal{L}_p(\mathbb{C})$. Let

$$\omega = Pq = P(I - \mu H)^{-1}\sigma\,. \tag{14}$$

Then we have that $\omega_z = Hq$, $\omega_{\overline{z}} = q = \mu\,Hq + \sigma$. Therefore ω is the unique solution of $\omega_{\overline{z}} = \mu\,\omega_z + \sigma$ with $\omega(0) = 0$, $\omega_z \in \mathcal{L}_p(\mathbb{C})$ of lemma 1.4.

Observe that we only need to use P_R and H_R in (14) because $q(z) \equiv 0$ on $|z| \geq R$ by (13) and μH sends $W^{n,p}(R,0)$ into itself.

Now we estimate the norm of the operator $(I - \mu H)^{-1}$ on $W^{n,p}(R,0)$:

$$\begin{aligned}
\|\mu\,H(\sigma)\|_{W^{n,p}(R)} &\leq \|\mu\|_{R,\infty}\ \|D^n H(\sigma)\|_{R,p} + 2^n\ \|\mu\|_{W^{n,p}(R)}\ \|H\sigma\|_{C^{n-1}(R)} \\
&\leq \|\mu\|_{R,\infty}\ C_p\ \|D^n\sigma\|_{R,p} + 2^n\ \|\mu\|_{W^{n,p}(R)}\ A(p)\,R^{1-\frac{2}{p}}\ \|\sigma\|_{W^{n,p}(R)} \\
&\leq \Theta\ \|\sigma\|_{W^{n,p}(R)}\ ,
\end{aligned}$$

where $\Theta := k\,C_p + 2^n\ \|\mu\|_{W^{n,p}(R)}\ A(p)\,R^{1-\frac{2}{p}} < 1$.

$$\begin{aligned}
\|\omega\|_{W^{n+1,p}(R)} &= \left\|P_R(I-\mu H)^{-1}\sigma\right\|_{W^{n+1,p}(R)} \\
&= \left\|P_R\big(\textstyle\sum_{k=0}^{\infty}(\mu H)^k\big)\,\sigma\right\|_{W^{n+1,p}(R)} \\
&\leq B(p,R)\,\big(\textstyle\sum_{k=0}^{\infty}\Theta^k\big)\ \|\sigma\|_{W^{n,p}(R)} \\
&\leq D(R,p)\ \|\sigma\|_{W^{n,p}(R)}
\end{aligned}$$

where $D(R, n, p, \|\mu\|_{W^{n,p}(R)}) = \frac{B(p,R)}{1-\Theta}$.

$\square$

2.2. Lemma. *Let* $\mathcal{W} := W^{n,p}(R,0) \cap \Big[\, \|\mu\|_{W^{n,p}(R)} < a\,,\ \|\mu\|_\infty < k < 1 \Big]$ *with* R *small enough such that* $\Theta(R,n,p,a,k) < 1$*, where* Θ *is from lemma 2.1. Then the map* $\mathcal{W} \times W^{n,p}(R,0) \to W^{n+1,p}(R)$*, given by* $(\mu,\sigma) \mapsto \omega^{\mu,\sigma}$*, is continuous.*

Proof: Let (μ,σ), $(\mu_o,\sigma_o) \in \mathcal{W} \times W^{n,p}(R,0)$. Let $\omega^o = \omega^{\mu_o,\sigma_o}$ and $\omega = \omega^{\mu,\sigma}$, i.e. $\omega^o_{\overline{z}} = \mu\,\omega^o_z + \sigma_o$ and $\omega_{\overline{z}} = \mu\,\omega_z + \sigma$ in $\mathcal{L}_p(\mathbb{C})$. We have that

$$(\omega-\omega^o)_{\overline{z}} = \mu\,(\omega-\omega^o)_z + (\mu-\mu_o)\,\omega^o_z + (\sigma-\sigma_o)$$

with

$$\|(\mu-\mu_o)\,\omega^o_z + (\sigma-\sigma_o)\|_{W^{n,p}(R)} \le 2^n\ \|\mu-\mu_o\|_{W^{n,p}(R)}\ \|\omega^o_z\|_{W^{n,p}(R)} + \|\sigma-\sigma_o\|_{W^{n,p}(R)}\,.$$

From lemma 2.1, we get that

$$\|\omega-\omega^o\|_{W^{n+1,p}(R)} \le D(\Theta,k)\ \Big(2^n\ \|\mu-\mu_o\|_{W^{n,p}(R)}\ \|\omega^o_z\|_{W^{n,p}(R)} + \|\sigma-\sigma_o\|_{W^{n,p}(R)}\Big)\,.$$

□

On $\mathcal{L}_\infty(\mathbb{C}) \cap W^{1,p}(R)$ consider the topology given by $\langle\mu_n\rangle \to \mu$ if $\mu_n \to \mu$ almost everywhere in $\mathbb{C}$ and $\|\mu_n-\mu\|_{W^{1,p}(R)} \to 0$.

2.3. Corollary. *Given* $0<R<1$, $0<k<1$, $L>0$ *with* $k\,C_p<1$*, there exists* $0<r<r(k,L)<R$ *such that the map* $\mu \mapsto \omega^\mu$ *:*

$$\mathcal{L}_\infty(\mathbb{C}) \cap W^{1,p}(R) \cap \big\{\,\mu \;\big|\; \|\mu\|_\infty < k,\ \|\mu\|_{W^{1,p}(R)} < L\,\big\} \to C^1\big(|z|<r,\mathbb{C}\big)$$

is continuous.

Proof: Let $\langle\mu_n\rangle$ be a sequence converging to μ_o in $\mathcal{L}_\infty(\mathbb{C}) \cap W^{1,p}(R)$ with $\|\mu_n\|_\infty < k$, $\|\mu_n\|_{W^{1,p}(R)} < L$. Let $\omega^n := f^{\mu_n} - f^{\mu_o}$, $f^o := f^{\mu_o}$. Since $\mu_n \to \mu_o$ a.e, then by theorem 1.5, $\|\omega^n\|_{B_{R,p}} \to 0$. In particular $\|\omega^n\|_{W^{1,p}(R)} \to 0$. We have that

$$\omega^n_{\overline{z}} = \mu_n\,\omega^n_z + (\mu_n-\mu_o)\,f^o_z.$$

Let $\lambda : \mathbb{C} \to [0,1]$ be a C^∞ function such that $\lambda(z) \equiv 1$ for $|z| \le r$ and $f(z) \equiv 0$ for $|z| \le 2r$. Choose r such that $\Theta(2r, n=1, p, L, k) < 1$ where Θ is from lemma 2.2 and $0 < 2r < R$. We have

$$(\lambda\,\omega^n)_{\overline{z}} = \widehat{\mu}_n\,(\lambda\,\omega^n)_z + (\lambda_{\overline{z}} - \mu_n\,\lambda_z)\,\omega^n + \lambda\,(\mu_n-\mu_o)\,f^o_z\,,$$

where $\widehat{\mu}_n(z) = \mu_n(z)$ for $|z|<2r$ and $\widehat{\mu}_n(z)=0$ for $\|z\| \ge 2r$. Since the sequence $\|\mu_n\|_{W^{1,p}(2r)}$ is bounded and $\|\omega^n\|_{W^{1,p}(R)} \to 0$, then $\|(\lambda_{\overline{z}} - \mu_n\,\lambda_z)\,\omega^n\|_{W^{1,p}(R)} \to 0$. Also $\|\lambda\,(\mu_n-\mu_o)\,f^o_z\|_{W^{1,p}(2r)} \to 0$. In particular

$$\|\omega^n\|_{C^1(D_r,\mathbb{C})} \le \|\lambda\,\omega^n\|_{C^1(D_{2r},\mathbb{C})} \le \|\lambda\,\omega^n\|_{W^{2,p}(2r)} \to 0\,.$$

□

3. Global non-homogeneous Beltrami equations.

3.1. Lemma. *If $0 < R < +\infty$ and $h \in \mathcal{L}_q(|z| < R)$ for some $q > 2$, then $h \in \mathcal{L}_p(|z| < R)$ for all $2 < p < q$, and*

$$\|h\|_{R,p} \le A \, \|h\|_{R,q} \, ,$$

where $A = A(R,q) = \max\left\{1, \sqrt{\pi}^{1-\frac{2}{q}} R^{1-\frac{2}{q}}\right\}$.

Proof: Let $\alpha = \frac{q}{p} > 1$ and $\frac{1}{\beta} + \frac{1}{\alpha} = 1$, then

$$\int_{|z|<R} |h|^p = \int_{|z|<R} |h|^{\frac{q}{\alpha}} \cdot 1 \le \left[\int_{|z|<R} |h|^q\right]^{\frac{1}{\alpha}} \left[\int_{|z|<R} 1\right]^{\frac{1}{\beta}} ,$$

$$\left(\|h\|_{R,p}\right)^p \le \left(\|h\|_{R,q}\right)^{\frac{q}{\alpha}} \left(\pi R^2\right)^{\frac{1}{\beta}}$$

$$\|h\|_{R,p} \le \|h\|_{R,q} \left(\pi R^2\right)^{\frac{1}{\beta p}} .$$

From this we get the lemma because $\frac{1}{\beta p} = \frac{1}{p} - \frac{1}{q} \in \left]0, \frac{1}{2}\left(1 - \frac{2}{q}\right)\right[$.

□

Given $\epsilon > 0$ and $q > 2$, define

$$\mathcal{D}_{\epsilon,q} := \left\{ \sigma : \mathbb{C} \to \mathbb{C} \,\middle|\, \sigma(z) \in \mathcal{L}_q(|z| < \tfrac{1}{\epsilon}) \text{ and } \sigma\left(\tfrac{1}{z}\right) \in \mathcal{L}_q(|z| < \epsilon) \right\},$$

with the norm

$$\|\sigma\|_{\mathcal{D}_{\epsilon,q}} := \left(\iint_{|z|<\frac{1}{\epsilon}} |\sigma(z)|^q \; dx\, dy\right)^{\frac{1}{q}} + \left(\iint_{|z|<\epsilon} \left|\sigma\left(\tfrac{1}{z}\right)\right|^q \; dx\, dy\right)^{\frac{1}{q}} .$$

3.2. Lemma. *For all $0 < \epsilon < +\infty$ and $q > 2$, we have that*

$$\mathcal{D}_{\epsilon,q} = \bigcap_{0<r<\infty} \bigcap_{2<p\le q} \mathcal{D}_{r,p} .$$

Moreover, given $0 < r < \infty$ and $2 < p \le q$, there exists $A = A(p,q,r,\epsilon) > 0$ such that

$$\|h\|_{\mathcal{D}_{r,p}} \le A \, \|h\|_{\mathcal{D}_{\epsilon,q}}$$

for all $h \in \mathcal{D}_{\epsilon,q}$.

Proof: One inclusion is trivial. For the otherone, let $r > \epsilon$, then $\left[|z| < \frac{1}{\epsilon}\right] \subset \left[|z| < \frac{1}{r}\right]$ and hence $\sigma \in \mathcal{L}_q(|z| < \frac{1}{\epsilon}) \subseteq \mathcal{L}_q(|z| < \frac{1}{r})$. We have that

$$\int\int_{\epsilon<|z|<r} \left|\sigma\left(\tfrac{1}{z}\right)\right|^q = \int\int_{\frac{1}{r}<|w|<\frac{1}{\epsilon}} \frac{1}{|w|^4} \, |\sigma(w)|^q \le \frac{1}{\epsilon^4} \int\int_{\frac{1}{r}<|w|<\frac{1}{\epsilon}} |\sigma(w)|^q$$

$$\le \frac{1}{\epsilon^4} \, \|\sigma\|_{\mathcal{D}_{\epsilon,q}} < +\infty ,$$

$$\int_{|z|<r} \left|\sigma\left(\tfrac{1}{z}\right)\right|^q = \int_{|z|<\epsilon} \left|\sigma\left(\tfrac{1}{z}\right)\right|^q + \int_{\epsilon<|z|<r} \left|\sigma\left(\tfrac{1}{z}\right)\right|^q < \left(1 + \frac{1}{\epsilon^4}\right) \, \|\sigma\|_{\mathcal{D}_{\epsilon,q}} .$$

Therefore $\sigma \in \mathcal{D}_{r,q}$ and $\|\sigma\|_{\mathcal{D}_{r,q}} \leq A_1 \|\sigma\|_{\mathcal{D}_{\epsilon,q}}$. By lemma 3.1, we have that $\sigma \in \mathcal{D}_{r,p}$ and $\|\sigma\|_{\mathcal{D}_{r,q}} \leq A_2(p,q,r,\epsilon)\, \|\sigma\|_{\mathcal{D}_{\epsilon,q}}$ for all $r > \epsilon$ and all $2 < p \leq q$. The case $0 < r < \epsilon$ is similar to this case.

□

3.3. Lemma. *Let $\epsilon > 0$, $q > 2$, $\sigma \in \mathcal{D}_{\epsilon,q}$. Then there exists a unique solution of $\Theta_{\overline{z}} = \sigma$, such that Θ is continuous, $\Theta(0) = 0$, $\Theta(1) = 0$ and*

$$\lim_{|z|\to\infty} \frac{\Theta(z)}{|z|^2} = 0\,.$$

The solution satisfies

(i) $|\Theta(z)| \leq 2\, K_q \, \|\sigma\|_{\mathcal{D}_{\epsilon,q}} \max\left\{ |z|^{1-\frac{2}{q}}, |z|^{1+\frac{2}{q}} \right\}$, *for all $z \in \mathbb{C}$, where K_q is from lemma 1.2.(7).*
(ii) $\Theta \in B_{R,p}$ *for all $R > 0$ and all $2 < p \leq q$.*
(iii) $\Theta_z \in \mathcal{D}_{r,p}$ *for all $2 < p < q$ and all $r > 0$.*
(iv) *For all $r > 0$ and $2 < p < q$ there exists $B = B(r,\epsilon,p,q) > 0$, such that $\|\Theta_z\|_{\mathcal{D}_{r,p}} < B\, \|\sigma\|_{\mathcal{D}_{\epsilon,q}}$.*

Proof: Let $a(z) = \sigma(z)$ for $|z| < \frac{1}{\epsilon}$ and $a(z) = 0$ for $|z| > \frac{1}{\epsilon}$, and let $b(z) = 0$ for $|z| > \epsilon$ and $b(z) = \sigma\left(\frac{1}{z}\right) \frac{z^2}{\overline{z}^2}$ for $|z| < \epsilon$. Since $\sigma \in \mathcal{D}_{\epsilon,p}$ we have that $a \in \mathcal{L}_q(\mathbb{C})$ and $b \in \mathcal{L}_q(\mathbb{C})$. Define

$$\begin{aligned} \Theta^a(z) &:= Pa(z) - z\, Pa(1) \\ \Theta^b(z) &:= -z^2\, Pb\left(\tfrac{1}{z}\right) + z\, Pb(1) \\ \Theta(z) &:= \Theta^a(z) + \Theta^b(z). \end{aligned}$$

We have that $\Theta^a_{\overline{z}} = a$, $\Theta^b_{\overline{z}} = \sigma$ on $|z| > \frac{1}{\epsilon}$ and $\Theta^b_{\overline{z}} = 0$ on $|z| < \frac{1}{\epsilon}$. Therefore $\Theta_{\overline{z}} = \sigma$. Also $\Theta(0) = \Theta(1) = 0$. Moreover,

$$\begin{aligned} |\Theta(z)| &\leq K_q \,\|a\|_q \left(|z|^{1-\frac{2}{q}} + |z|\right) + K_q\, \|b\|_q\, |z|^2 \left|\tfrac{1}{z}\right|^{1-\frac{2}{q}} + K_q\, \|b\|_q\, |z| \\ &\leq 2\, K_q\, \|\sigma\|_{\mathcal{D}_{\epsilon,q}} \max\left\{\, |z|^{1-\frac{2}{q}},\, |z|^{1+\frac{2}{q}} \,\right\}. \end{aligned}$$

Suppose that φ is another solution. Let $h = \varphi - \Theta$. Then h is analytic on all $\mathbb{C}$ because $h_{\overline{z}} = (\varphi - \Theta)_{\overline{z}} = 0$. Also $h(0) = h(1) = 0$ and $|h(z)| = \mathcal{O}\big(|z|^2\big)$ when $|z| \to \infty$. Therefore $h \equiv 0$.

By lemma 3.1, $\sigma \in \mathcal{D}_{\epsilon,p}$ for all $2 < p < q$. Therefore $a \in \mathcal{L}_p(\mathbb{C})$ and $b \in \mathcal{L}_p(\mathbb{C})$ for all $2 < p < q$. Let $2 < p < q$, we have that

$$\begin{aligned}
\Theta_z^a(z) &= Ha - Pa(1) \\
\|\Theta_z^a\|_{\frac{1}{\epsilon},p} &\leq \|Ha\|_p + |Pa(1)| \left(\pi \tfrac{1}{\epsilon^2}\right)^{\frac{1}{p}} \\
&\leq C_p \, \|a\|_p + K_p \, \|a\|_p \, \left(\frac{\pi}{\epsilon^2}\right)^{\frac{1}{p}}
\end{aligned}$$

$$\int_{|z|<\epsilon} \left|Ha\left(\tfrac{1}{z}\right)\right|^p = \int_{|w|>\frac{1}{\epsilon}} \frac{1}{|w|^4} \, |Ha(w)|^p \leq \epsilon^4 \left(\|Ha\|_p\right)^p$$

$$\left\|\Theta_z^a\left(\tfrac{1}{z}\right)\right\|_{\epsilon,p} \leq \epsilon^{\frac{4}{p}} \, C_p \, \|a\|_p + K_p \, \|a\|_p \, (\pi \, \epsilon^2)^{\frac{1}{p}}$$

$$\|\Theta_z^a\|_{\mathcal{D}_{\epsilon,p}} \leq \left(1 + \epsilon^{\frac{4}{p}}\right) \left(C_p + K_p \, \left(\frac{\pi}{\epsilon^2}\right)^{\frac{1}{p}}\right) \, \|a\|_p.$$

We write $C(p, q, \epsilon)$ for constants depending only on p, q, ϵ.

$$\Theta_z^b = -2 \, z \, Pb\left(\tfrac{1}{z}\right) + Hb\left(\tfrac{1}{z}\right) + Pb(1)$$

$$\int_{|z|<\frac{1}{\epsilon}} \left|Hb\left(\tfrac{1}{z}\right)\right|^p = \int_{|w|>\epsilon} \frac{1}{|w|^4} \, |Hb(w)|^p \leq \frac{1}{\epsilon^4} \, \left(\|Hb\|_p\right)^p$$

$$\int_{|z|<\frac{1}{\epsilon}} \left|2 \, z \, Pb\left(\tfrac{1}{z}\right)\right|^p \leq \int_{|z|<\frac{1}{\epsilon}} \left(2K_p \, \|b\|_p \, |z|^{\frac{2}{p}}\right)^p \leq \left(C_1(p,\epsilon) \, \|b\|_p\right)^p$$

$$\left\|\Theta_z^b\right\|_{\frac{1}{\epsilon},p} \leq \left(C_1(p,\epsilon) + \frac{C_p}{\epsilon^{\frac{4}{p}}} + K_p \, \left(\frac{\pi}{\epsilon^2}\right)^{\frac{1}{p}}\right) \, \|b\|_p$$

$$\int_{|z|<\epsilon} |Hb(z)|^p \leq \left(C_p \, \|b\|_p\right)^p$$

Since $\left|2 \, \frac{1}{z} \, Pb(z)\right| \leq 2 \, K_q \, \|b\|_q \, |z|^{-\frac{2}{q}}$ and $p < q$, we have

$$\int_{|z|<\epsilon} \left|2 \, \frac{1}{z} \, Pb(z)\right|^p \leq \left(2 \, K_q \, \|b\|_q\right)^p \int_{|z|<\epsilon} |z|^{-\frac{2p}{q}} < +\infty\,,$$

$$\left\|\Theta_z^b\left(\tfrac{1}{z}\right)\right\|_{\epsilon,p} \leq C_2(p,q,\epsilon) \, \|b\|_q + C_p \, \|b\|_p + (\pi \, \epsilon^2)^{\frac{1}{p}} \, K_p \, \|b\|_p.$$

Therefore $\|\Theta_z^b\|_{\mathcal{D}_{\epsilon,p}} < +\infty$ and hence $\|\Theta_z\|_{\mathcal{D}_{\epsilon,p}} < +\infty$ for all $2 < p < q$. Now use lemma 3.2 to get (iii).

For (iv) observe that $\|\sigma\|_{\mathcal{D}_{\epsilon,q}} = \|a\|_q + \|b\|_q$ and that $\|a\|_p \leq A(\epsilon,q) \, \|a\|_q$, $\|b\|_p \leq A(\epsilon,q) \, \|b\|_q$ by lemma 3.1. Now use the above estimates and lemma 3.3. $\square$

The following notation will be useful for the next proposition: For $\epsilon > 0$, $p > 2$, $F : \mathbb{C} \to \mathbb{C}$, let

$$\begin{aligned}
\mathcal{D}(\epsilon,p,F) &:= \left\{ \sigma : \mathbb{C} \to \mathbb{C} \;\middle|\; (\sigma \circ F^{-1}) F_z^{-1} \in \mathcal{D}_{\epsilon,p} \right\} \\
\|\sigma\|_{\mathcal{D}(\epsilon,p,F)} &:= \left\|(\sigma \circ F^{-1}) F_z^{-1}\right\|_{\mathcal{D}_{\epsilon,p}}.
\end{aligned}$$

3.4. Proposition. *If $\mu \in \mathcal{L}_\infty(\mathbb{C})$, $\|\mu\|_\infty < k < 1$, $k\,C_q < 1$, $q > 2$ and $\big(\sigma \circ (f^\mu)^{-1}\big)\,(f^\mu)_z^{-1} = \Big(\frac{\sigma}{f^\mu_z}\Big) \circ (f^\mu)^{-1} \in \mathcal{D}_{\epsilon,q}$ for some $\epsilon > 0$. Then there exists a unique solution of*

$$\omega_{\overline{z}} = \mu\,\omega_z + \sigma$$

such that ω is continuous, $\omega(0) = 0$, $\omega(1) = 0$ and

$$\lim_{|z|\to\infty} \frac{\omega(z)}{|f^\mu(z)|^2} = 0\,.$$

Moreover,

(i) *$\big(\omega_z \circ (f^\mu)^{-1}\big)(f^\mu)_z^{-1} \in \mathcal{D}_{r,p}$ for all $r > 0$ and all $2 < p < q$.*

(ii) *$\omega \in B_{R,p}$ for all $R > 0$ and all $2 < p \le \frac{q^2}{q-2}$.*

(iii) *If $\sigma \in B_{R,p}$, $2 < p \le q$, $0 < r < R$ and $\epsilon > 0$, then there exists $C(R,r) = C(R,r,\epsilon,p,k)$ such that*

$$\|\omega\|_{B_{r,p}} \le C(R,r)\,\Big(\|\sigma\|_{\mathcal{D}(\epsilon,p,f^\mu)} + \|\sigma\|_{R,p}\Big)\,.$$

(iv) *For all $r > 0$ and $2 < p < q$, there exists $A(r,p) = A(r,\epsilon,p,q) > 0$ such that*

$$\|\omega_z\|_{\mathcal{D}(r,p,f^\mu)} \le A(r,p)\;\|\sigma\|_{\mathcal{D}(\epsilon,q,f^\mu)}\,.$$

(v) *If μ, $\sigma \in W^{n,p}(R)$, $2 < p \le q$, $0 < r < R$ and $\epsilon > 0$, then there exists $D(R,r) = D(R,r,n,p,\epsilon,\|\mu\|_{W^{n,p}(R)})$ such that*

$$\|\omega\|_{W^{n+1,p}(r)} \le D(R,r)\,\Big(\|\sigma\|_{\mathcal{D}(\epsilon,p,f^\mu)} + \|\sigma\|_{W^{n,p}(R)}\Big)\,.$$

Proof: We first prove the uniqueness of the solution. Suppose that u is another solution. Then $v := \omega - u$ satisfies $v_{\overline{z}} = \mu\,v_z$, $v(0) = v(1) = 0$ and $|v(z)| = \mathcal{O}\big(|f^\mu(z)|^2\big)$ when $|z| \to \infty$. Let $h := v \circ (f^\mu)^{-1}$. Then $h(0) = h(1) = 0$, by theorem 1.5.(viii) we have that h is holomorphic on all $\mathbb{C}$ and

$$\lim_{z\to\infty} \frac{|h(z)|}{|z|^2} = \lim_{z\to\infty} \frac{\big|v(f^{-1}(z))\big|}{|z|^2} = \lim_{y\to\infty} \frac{|v(y)|}{|f(y)|^2} = 0\,,$$

where $f = f^\mu$. Therefore $h \equiv 0$.

For the existence, write $\omega = \Theta \circ f^\mu$. Using the formulas of lemma 1.6, and that

$$\overline{f}_z = \overline{(f_{\overline{z}})} = \overline{\mu}\,\overline{(f_z)} = \overline{\mu}\,\overline{f}_{\overline{z}}\,,$$

we see that ω is a solution of the problem if and only if, for $f = f^\mu$, we have

$$\Theta_{\overline{z}} = \left(\frac{1}{1-|\mu|^2}\,\frac{\sigma}{\overline{f}_{\overline{z}}}\right) \circ f^{-1} =: \rho\,, \tag{15}$$

$\Theta(0) = 0$, $\Theta(1) = 0$ and $\lim_{z\to\infty} \frac{\Theta(z)}{|z|^2} = 0$. Since $\overline{f}_{\overline{z}} = \overline{f_z}$, we have by hypothesis that $\rho \in \mathcal{D}_{\epsilon,q}$. By lemma 3.3, such Θ exists and is unique.

For (i) observe that, for $f = f^{\mu}$,

$$\omega_z = (\Theta_z \circ f)\, f_z + (\Theta_{\overline{z}} \circ f)\, \overline{f}_z$$
$$\omega_z = (\Theta_z \circ f)\, f_z + \left(\frac{1}{1-|\mu|^2}\, \frac{\sigma}{\overline{f_{\overline{z}}}}\right)\, \overline{\mu}\, \overline{f_{\overline{z}}}$$

$$(\omega_z \circ f^{-1})\, f_z^{-1} = \left(\frac{\omega_z}{f_z}\right) \circ f^{-1} = \Theta_z + \left(\frac{\overline{\mu}}{1-|\mu|^2}\, \frac{\sigma}{f_z}\right) \circ f^{-1}. \tag{16}$$

By the hypothesis on σ and μ and by lemma 3.2, we have that $(\omega_z \circ f^{-1}) f_z^{-1} \in \mathcal{D}_{r,p}$ if and only if $\Theta_z \in \mathcal{D}_{r,p}$; but this is true by lemma 3.3.

(iv) From (15), (16) and lemmas 3.3 and 3.2, we get that

$$\|\omega_z\|_{\mathcal{D}(r,p,f^{\mu})} \leq \left(\frac{B}{1-k^2} + \frac{A\,k}{1-k^2}\right)\, \|\sigma\|_{\mathcal{D}(\epsilon,q,f^{\mu})}\ .$$

This proves (iv).

For (ii) we know that $f \in B_{R,p}$ for all $2 < p \leq q$ and by lemma 3.3, $\Theta \in B_{R,p}$ for all $2 < p \leq q$. Now use lemma 1.6.

We now prove (iii). Let $\lambda : \mathbb{C} \to [0,1]$ be a C^{∞} function such that $\lambda(z) = 1$ on $|z| \leq r$ and $\lambda(z) = 0$ on $|z| \geq R$. We have that

$$(\lambda\omega)_{\overline{z}} = \mu\,(\lambda\omega)_z + (\lambda_{\overline{z}} - \mu\lambda_z)\,\omega + \lambda\sigma\,.$$

By lemma 3.3, there exists $C_1(R) = C_1(R,p,\epsilon)$ such that for $\Theta = \omega \circ f^{-1}$, $f = f^{\mu}$, we have

$$\|\Theta\|_{R,\infty} \leq C_1(R)\ \left\|\left(\frac{1}{1-|\mu|^2}\,\frac{\sigma}{\overline{f_{\overline{z}}}}\right) \circ f^{-1}\right\|_{\mathcal{D}_{\epsilon,p}}.$$

Let $A = A(R,k) > 0$ be such that $f(|z| < R) \subseteq [|z| < A]$. Writing $\Xi := (\sigma \circ f^{-1})\, f_z^{-1}$, we have

$$\|\omega\|_{R,\infty} = \|\Theta \circ f\|_{R,\infty} \leq C_1(A,k)\ \|\Xi\|_{\mathcal{D}_{\epsilon,p}}\,,$$
$$\|\omega\|_{R,p} \leq C_3(R,k,p,\epsilon)\ \|\Xi\|_{\mathcal{D}_{\epsilon,p}}\,,$$
$$\|\lambda\sigma\|_{\mathcal{L}_p} \leq \|\sigma\|_{R,p}\,,$$
$$\|(\lambda_{\overline{z}} - \mu\lambda_z)\,\omega\|_{\mathcal{L}_p} \leq C_4(R,r)\ \|\Xi\|_{\mathcal{D}_{\epsilon,p}}\,.$$

By lemma 1.4, we have that

$$\|\lambda\omega\|_{B_p} \leq K(k,p)\ \Big(C_4\ \|\Xi\|_{\mathcal{D}_{\epsilon,p}} + \|\sigma\|_{R,p}\Big)\,.$$

Therefore

$$\|\omega\|_{B_{r,p}} \leq C(R,r,p,k,\epsilon)\ \Big(\|\Xi\|_{\mathcal{D}_{\epsilon,p}} + \|\sigma\|_{R,p}\Big)\,.$$

(v) The case $n = 0$ is proved in item (iii). Suppose by induction that it holds for $n-1$. Cover the disc $|z| \leq r$ by a finite number of discs of radius δ such that the corresponding discs of radius 2δ are all contained in $|z| < R$. Choose δ small enough so that

$$\Theta\big(2\delta, n, p, \|\mu\|_{W^{n,p}(R)}\big) < 1\,,$$

where Θ is from lemma 2.1. Choose one of these discs, say $|z-a|<\delta$. Let $\lambda:\mathbb{C}\to[0,1]$ be a C^∞ function such that $\lambda(z)\equiv 1$ on $|z-a|\le\delta$ and $\lambda(z)\equiv 0$ on $|z-a|\ge 2\delta$. Let

$$u(z):=\lambda(z)\,\big(\omega(z+a)-\omega(a)\big).$$

Then $u(0)=0$, $u_z\in\mathcal{L}_p(\mathbb{C})$ and

$$u_{\overline{z}}=\mu\,u_z+(\lambda_{\overline{z}}-\widehat{\mu}\lambda_z)\,\big(\widehat{\omega}-\omega(a)\big)+\lambda\,\widehat{\sigma}_z\,,$$

where $\widehat{\mu}(z)=\mu(z+a)$, $\widehat{\omega}(z)=\omega(z+a)$, $\widehat{\sigma}(z)=\sigma(z+a)$. By lemma 2.1 we have that

$$\|u\|_{W^{n+1,p}(2\delta)}\le D_1(a)\,\left(\big\|(\lambda_{\overline{z}}-\mu\lambda_z)\big(\widehat{\omega}-\omega(a)\big)\big\|_{W^{n,p}(R)}+\|\lambda\widehat{\sigma}\|_{W^{n,p}(2\delta)}\right),$$

where D_1 depends on 2δ, p, n, $\|\mu\|_{W^{n,p}(R)}$. We have

$$\big\|(\lambda_{\overline{z}}-\widehat{\mu}\lambda_z)\big(\widehat{\omega}-\omega(a)\big)\big\|_{W^{n,p}(2\delta)}\le 2^n\,\|\lambda_{\overline{z}}-\widehat{\mu}\lambda_z\|_{W^{n,p}(R)}\,\|\widehat{\omega}-\omega(a)\|_{W^{n,p}(2\delta)}$$

$$\begin{aligned}
&\le 2^n\,\left(\|\lambda\|_{C^n}+2^n\,\|\lambda\|_{C^n}\,\|\mu\|_{W^{n,p}(R)}\right)\,\|\widehat{\omega}-\omega(a)\|_{W^{n,p}(R)}\\
&\le D_2(a)\,\|\omega\|_{W^{n,p}(R)}\\
&\le D_2(a)\,D(R,n-1)\,\left(\|\sigma\|_{\mathcal{D}(\epsilon,p,f^\mu)}+\|\sigma\|_{W^{n-1,p}(R)}\right)
\end{aligned}$$

where in the second inequality we used that $\|\ \|_{W^{n,p}(2\delta)}\le\|\ \|_{W^{n,p}(R)}$, on the second inequality we used that

$$\|\widehat{\omega}-\omega(a)\|_{W^{n,p}(2\delta)}\le 2\,\|\omega\|_{W^{n,p}(R)}\,,$$

because $D^k\big(\widehat{\omega}-\omega(a)\big)=D^k\,\omega$ for $k>0$ and and $\|\widehat{\omega}-\omega(a)\|_{2\delta,\infty}\le 2\,\|\omega\|_{R,\infty}$, and on the last inequality we used the induction hypothesis. Also

$$\|\lambda\sigma\|_{W^{n,p}(2\delta)}\le 2^n\,\|\lambda\|_{C^n}\,\|\sigma\|_{W^{n,p}(R)}\,.$$

Combining these inequalities, we get that

$$\|u\|_{W^{n+1,p}(2\delta)}\le D_3\,\left(\|\sigma\|_{\mathcal{D}(\epsilon,p,f^\mu)}+\|\sigma\|_{W^{n,p}(R)}\right).$$

In particular

$$\begin{aligned}
\|\omega\|_{W^{n+1,p}(|z-a|<\delta)}&\le\|u\|_{W^{n+1,p}(2\delta)}+|\omega(a)|\\
&\le\|u\|_{W^{n+1,p}(2\delta)}+\|\omega\|_{R,\infty}\\
&\le D_4\,\left(\|\sigma\|_{\mathcal{D}(\epsilon,p,f^\mu)}+\|\sigma\|_{W^{n,p}(R)}\right).
\end{aligned}$$

Adding the estimates of each ball, we get

$$\left(\left\|D^{n+1}\omega\right\|_{r,p}\right)^p = \sum_a \int_{|z-a|<\delta} \left\|D^{n+1}\omega\right\|^p = \sum_a \left(\left\|D^{n+1}\omega\right\|_{|z-a|<\delta,p}\right)^p$$

$$\left\|D^{n+1}\omega\right\|_{r,p} \leq \sum_a \left\|D^{n+1}\omega\right\|_{|z-a|<\delta,p} \leq \sum_a \|\omega\|_{W^{n+1,p}(|z-a|<\delta)}$$

$$\|\omega\|_{C^n(r)} \leq \sup_a \|\omega\|_{C^n(|z-a|<\delta)} \leq \sum_a \|\omega\|_{W^{n+1,p}(|z-a|<\delta)}$$

$$\|\omega\|_{W^{n+1,p}(r)} \leq D_5 \left(\|\sigma\|_{\mathcal{D}(\epsilon,p,f^\mu)} + \|\sigma\|_{W^{n,p}(R)}\right).$$

□

3.5. Lemma. *Let $\lambda \in \mathcal{L}_\infty(\mathbb{C})$, $\|\lambda\|_\infty < k < 1$ and let $h = f^\lambda$. Let $K > 1$, $0 < \alpha < 1$ and $0 < \epsilon < 1$ be such that*

$$\left|h^{-1}(z)\right| < K\, |z|^\alpha \text{ for all } |z| < \tfrac{1}{\epsilon}.$$

Let $p_o > 2$ be such that $k\, C_{p_o} < 1$. Let $q_o > 2$ and

$$p = \frac{p_o\, q_o}{p_o + q_o - 2}.$$

(i) *If $A \in \mathcal{L}_{q_o}\left(|z| < \frac{K}{\epsilon}\right)$ then $(A \circ h^{-1})\, h_z^{-1} \in \mathcal{L}_p\left(|z| < \frac{1}{\epsilon}\right)$ and*

$$\left\|(A \circ h^{-1})\, h_z^{-1}\right\|_{\frac{1}{\epsilon},p} \leq \frac{1}{(1-k^2)^{\frac{1}{p}-\frac{1}{q_o}}}\ \|A\|_{\frac{K}{\epsilon},q_o} \left(\left\|h_z^{-1}\right\|_{\frac{1}{\epsilon},p_o}\right)^{1-\frac{2}{q_o}}.$$

(ii) *Let $k(z) = 1\Big/\overline{h\left(\frac{1}{\bar z}\right)}$ and suppose that there exists $a > 1$ and $Q > 1$ such that*

$$\begin{aligned} |k(z)| &< a\, |z| && \text{for all } |z| < \epsilon \text{ and} \\ \left|k^{-1}(z)\right| &< Q\, |z|^\alpha && \text{for all } |z| < \epsilon. \end{aligned}$$

If $A\left(\frac{1}{z}\right) \in \mathcal{L}_q\left(|z| < a\,\epsilon\right)$ then $\left((A \circ h^{-1})\, h_z^{-1}\right)\left(\frac{1}{z}\right) \in \mathcal{L}_p\left(|z| < \epsilon\right)$ and

$$\left\|\left((A \circ h^{-1})\, h_z^{-1}\right)\ \left(\tfrac{1}{z}\right)\right\|_{\epsilon,p} \leq \frac{a^2}{(1-k^2)^{\frac{1}{p}-\frac{1}{q_o}}}\ \left\|A\left(\tfrac{1}{z}\right)\right\|_{Q\epsilon^\alpha,q_o} \left(\left\|h_z^{-1}\right\|_{\epsilon,p_o}\right)^{1-\frac{2}{q_o}}.$$

Proof: Let $p = \frac{p_o\, q_o}{p_o+q_o-2}$ and let q, $r > 0$ be such that

$$p\, q = q_o \quad \text{and} \quad (p-2)\, r + 2 = p_o.$$

In particular

$$\frac{1}{q} + \frac{1}{r} = 1, \qquad \frac{p_o}{pr} = 1 - \frac{2}{q_o} \quad \text{and} \quad \frac{1}{pr} = \frac{1}{p} - \frac{1}{q_o}.$$

We prove (i) first. We have

$$I := \int_{|z|<\frac{1}{\epsilon}} \left|A \circ h^{-1}\right|^p\ \left|h_z^{-1}\right|^p = \int_{|z|<\frac{1}{\epsilon}} \left|A\left(h^{-1}(z)\right)\right|^p\ \frac{1}{\left|h_z\left(h^{-1}(z)\right)\right|^p}.$$

Write $w = h^{-1}(z)$. Using the Jacobian

$$\text{Jac } h = |h_z|^2 - |h_{\bar z}|^2 = \left(1 - |\lambda|^2\right)\ |h_z|^2 \leq |h_z|^2,$$

we have that

$$I \leq \int_{h^{-1}\left[|z|<\frac{1}{\epsilon}\right]} |A(w)|^p \, \frac{1}{|h_z(w)|^p} \, |h_z(w)|^2 \, dw$$

$$\leq \left[\int_{h^{-1}\left[|z|<\frac{1}{\epsilon}\right]} |A|^{pq}\right]^{\frac{1}{q}} \left[\int_{h^{-1}\left[|z|<\frac{1}{\epsilon}\right]} \frac{1}{|h_z|^{(p-2)r}}\right]^{\frac{1}{r}}.$$

But

$$\int\limits_{h^{-1}\left[|z|<\frac{1}{\epsilon}\right]} \frac{1}{|h_z|^{(p-2)r}} = \int\limits_{h^{-1}\left[|z|<\frac{1}{\epsilon}\right]} \frac{|h_z|^2}{|h_z|^{(p-2)r+2}}$$

$$\leq \frac{1}{1-k^2} \int_{|z|<\frac{1}{\epsilon}} \left|h_z^{-1}\right|^{(p-2)r+2} = \frac{1}{1-k^2} \int_{|z|<\frac{1}{\epsilon}} \left|h_z^{-1}\right|^{p_o}.$$

Therefore

$$I^{\frac{1}{p}} \leq \left[\int_{h^{-1}\left[|z|<\frac{1}{\epsilon}\right]} |A|^{q_o}\right]^{\frac{1}{q_o}} \left[\frac{1}{1-k^2}\right]^{\frac{1}{pr}} \left[\int_{|z|<\frac{1}{\epsilon}} \left|h_z^{-1}\right|^{p_o}\right]^{\frac{1}{pr}}.$$

Since $h^{-1}\left[|z|<\frac{1}{\epsilon}\right] \subseteq \left[|w|<\frac{K}{\epsilon^\alpha}\right] \subseteq \left[|w|, \frac{K}{\epsilon}\right]$, we have

$$I^{\frac{1}{p}} \leq \left[\frac{1}{1-k^2}\right]^{\frac{1}{p}-\frac{1}{q_o}} \|A\|_{\frac{K}{\epsilon},q_o} \left(\left\|h_z^{-1}\right\|_{\frac{1}{\epsilon},p_o}\right)^{1-\frac{2}{q_o}}.$$

For (ii) consider

$$I := \int_{|z|<\epsilon} \left|A \circ h^{-1}\left(\tfrac{1}{z}\right)\right|^p \left|h_z^{-1}\left(\tfrac{1}{z}\right)\right|^p = \int_{|z|<\epsilon} \left|A \circ h^{-1}\left(\tfrac{1}{\overline{z}}\right)\right|^p \left|h_z^{-1}\left(\tfrac{1}{\overline{z}}\right)\right|^p.$$

Write $\frac{1}{\overline{w}} = h^{-1}\left(\frac{1}{\overline{z}}\right)$, i.e. $z = k(w)$. We have

$$h_z^{-1}\left(\tfrac{1}{\overline{z}}\right) = \frac{1}{h_z\left(h^{-1}\left(\frac{1}{\overline{z}}\right)\right)} = \frac{1}{h_z\left(\frac{1}{\overline{w}}\right)}.$$

By corollary 1.7, we have that $k_{\overline{z}}(w) = \overline{\lambda\left(\frac{1}{\overline{w}}\right)}\frac{w^2}{\overline{w}^2}$, hence

$$\frac{1}{1-k^2} \, |k_z|^2 \leq \operatorname{Jac}(h) \leq |k_z|^2,$$

then

$$I \leq \int_{k^{-1}[|z|<\epsilon]} \left|A\left(\tfrac{1}{\overline{w}}\right)\right|^p \frac{1}{\left|h_z\left(\frac{1}{\overline{w}}\right)\right|^p} \, |k_z(w)|^2 \,,$$

$$\leq \left[\int_{k^{-1}[|z|<\epsilon]} \left|A\left(\tfrac{1}{w}\right)\right|^{pq}\right]^{\frac{1}{q}} \left[\int_{k^{-1}[|z|<\epsilon]} \frac{|k_z(w)|^{2r}}{\left|h_z\left(\frac{1}{\overline{w}}\right)\right|^{pr}}\right]^{\frac{1}{r}}.$$

Since $k(w) = 1\Big/\overline{h\left(\frac{1}{\overline{w}}\right)}$, we have that

$$k_z(w) = \frac{\overline{h}_{\overline{z}}\left(\frac{1}{\overline{w}}\right)\left(-\frac{1}{w^2}\right)}{\overline{h\left(\frac{1}{\overline{w}}\right)}^2} = -\overline{h}_{\overline{z}}\left(\frac{1}{\overline{w}}\right)\,\frac{k(w)^2}{w^2}$$

$$\left|h_z\left(\frac{1}{\overline{w}}\right)\right| = \left|\overline{h}_{\overline{z}}\left(\frac{1}{\overline{w}}\right)\right| = |k_z(w)|\,\frac{|w|^2}{|k(w)|^2}$$

$$\frac{|k_z(w)|^{2r}}{\left|h_z\left(\frac{1}{\overline{w}}\right)\right|^{pr}} = \frac{|k_z(w)|^{2r}}{|k_z(w)|^{pr}}\cdot\frac{|k(w)|^{2pr}}{|w|^{2pr}} \le a^{2pr}\,\frac{1}{|k_z(w)|^{(p-2)r}}$$

$$\begin{aligned}\int_{k^{-1}[|z|<\epsilon]}\frac{|k_z(w)|^{2r}}{\left|h_z\left(\frac{1}{\overline{w}}\right)\right|^{pr}} &\le a^{2pr}\int_{k^{-1}[|z|<\epsilon]}\frac{1}{|k_z(w)|^{(p-2)r}}\\ &\le a^{2pr}\int_{k^{-1}[|z|<\epsilon]}\frac{|k_z(w)|^2}{|k_z(w)|^{(p-2)r+2}}\\ \le \frac{a^{2pr}}{1-k^2}\int_{|z|<\epsilon}\frac{1}{|k_z\circ k^{-1}|^{p_o}} &= \frac{a^{2pr}}{1-k^2}\int_{|z|<\epsilon}\left|k_z^{-1}\right|^{p_o}.\end{aligned}$$

Therefore

$$\begin{aligned}\left\|\left(\left(A\circ h^{-1}\right)h_z^{-1}\right)\left(\tfrac{1}{z}\right)\right\|_{\epsilon,p} &\le \left\|A\left(\tfrac{1}{w}\right)\right\|_{Q\epsilon,q_o}\left(\frac{a^{2pr}}{1-k^2}\right)^{\frac{1}{pr}}\left(\left\|k_z^{-1}\right\|_{\epsilon,p_o}\right)^{\frac{p_o}{pr}}\\ &\le \frac{a^2}{(1-k^2)^{\frac{1}{p}-\frac{1}{q_o}}}\left\|A\left(\tfrac{1}{w}\right)\right\|_{Q\epsilon,q_o}\left(\left\|h_z^{-1}\right\|_{\epsilon,p_o}\right)^{1-\frac{2}{q_o}}.\end{aligned}$$

□

Given $\mu\in\mathcal{L}_p(\mathbb{D})$, $\mathbb{D}=\left\{\, z\in\mathbb{C}\,|\ |z|<1\,\right\}$, extend it to $\mathcal{L}_\infty(\mathbb{D})$ by

$$\mu(z) = \overline{\mu\left(\tfrac{1}{\overline{z}}\right)}\,\frac{z^2}{\overline{z}^2}\,,$$

denote by $\widehat{\mu}$ this extension and consider $\mathcal{L}_\infty(\mathbb{D})$ as a subspace of $\mathcal{L}_\infty(\mathbb{C})$ by these extensions. On $\mathcal{L}_\infty(\mathbb{C})\cap W^{1,p}(\epsilon)$ consider the norm

$$\|\mu\|_{LE} := \|\mu\|_{\mathcal{L}_\infty(\mathbb{C})} + \|\mu\|_{W^{1,p}(\epsilon)}$$

and on $\mathcal{D}(\epsilon,p,F)\cap\mathcal{L}_p(D_R)$ consider the norm

$$\|\sigma\|_{DP} := \|\sigma\|_{\mathcal{D}(\epsilon,p,F)} + \|\sigma\|_{R,p}\,.$$

3.6. Proposition. *Suppose that $\mu_o\in\mathcal{L}_\infty(\mathbb{C})\cap W^{1,p_o}(\epsilon)$, $\|\mu_o\|<k<1$, $k\,C_{p_o}<1$, $p_o>2$ and let $F=f^{\widehat{\mu}_o}$. Then the map*

$$\left(\mathcal{L}_\infty(\mathbb{D})\cap W^{1,p}(\epsilon)\right)\times\left(\mathcal{D}(\epsilon,p,F)\cap\mathcal{L}_p(D_R)\right)\to B_{r,p}$$

given by $(\mu,\sigma)\mapsto\omega^{\widehat{\mu},\sigma}$, is differentiable for μ in a neighbourhood of μ_o, for all $0<r<R$ and any $2<p<p_o$.

Proof: Write $\omega^o := \omega^{\widehat{\mu}_o,\sigma_o}$ and for $(\mu,\sigma) \in \big(\mathcal{L}_\infty(\mathbb{D})\cap W^{1,p}(\epsilon)\big)\times\big(\mathcal{D}(\epsilon_o,p,F)\cap \mathcal{L}_p(D_R)\big)$ write $\omega = \omega^{\widehat{\mu},\sigma}$. For simplicity write $\mu = \widehat{\mu} \in \mathcal{L}_\infty(\mathbb{C})$. Let $\nu := \mu - \mu_o$ and $\rho := \sigma - \sigma_o$. By proposition 3.4(i), $\omega_z^o \in \mathcal{D}(\epsilon_o,p,F)$ for all $2<p<p_o$ and hence there exists a solution of

$$\ell_{\overline{z}} = \mu_o\,\ell_z + \nu\,\omega_z^o + \rho \tag{17}$$

such that $\ell(0)=\ell(1)=0$ and $|\ell(z)| = \mathcal{O}\big(|F(z)|^2\big)$ when $|z|\to\infty$. Moreover, since

$$\begin{aligned} \|\nu\,\omega_z^o + \rho\|_{\mathcal{D}(\epsilon_o,p,F)} &\le \|\nu\|_\infty\ \|\omega_z^o\|_{\mathcal{D}(\epsilon_o,p,F)} + \|\rho\|_{\mathcal{D}(\epsilon_o,p,F)}\,, \\ \|\nu\,\omega_z^o + \rho\|_{R,p} &\le \|\nu\|_\infty\ \|\omega_z^o\|_{R,p} + \|\rho\|_{R,p}\,. \end{aligned} \tag{18}$$

By proposition 3.4.(i), $\|\omega_z^o\|_{\mathcal{D}(\epsilon,p,F)} < +\infty$ and by 3.4.(iii), $\|\omega_z^o\|_{R,p} < +\infty$. Therefore by 3.4.(iii), the linear map $L(\nu,\rho) = \ell \in B_{R,p}$ is continuous. In particular, for all $2<p<p_o$, we have that

$$\lim_{\substack{\mu\to\mu_o\\ \sigma\to\sigma_o}} \|\ell\|_{B_{R,p}} = 0 \qquad , \qquad \lim_{\substack{\mu\to\mu_o\\ \sigma\to\sigma_o}} \|\ell_z\|_{R,p} = 0. \tag{19}$$

By 3.4.(iv) we also have that

$$\lim_{\substack{\mu\to\mu_o\\ \sigma\to\sigma_o}} \|\ell_z\|_{\mathcal{D}(\epsilon,p,F)} \le A \lim_{\substack{\mu\to\mu_o\\ \sigma\to\sigma_o}} \|\nu\,\omega_z^o + \rho\|_{\mathcal{D}(\epsilon_o,p,F)} = 0\,,$$

and by lemma 3.2,

$$\lim_{\substack{\mu\to\mu_o\\ \sigma\to\sigma_o}} \|\ell_z\|_{\mathcal{D}(r,p,F)} = 0 \quad \text{ for all } r>0,\ \ 2<p<p_o. \tag{20}$$

Let $h := \omega^{\mu,\sigma} - \omega^{\mu_o,\sigma_o} - \ell$, then

$$h_{\overline{z}} = \mu\,h_z + \nu\,\ell_z\,. \tag{21}$$

Let $H^\mu = H := f^\lambda$, where

$$\lambda = \lambda^u := \left(\frac{\mu-\mu_o}{1-\mu\,\mu_o}\,\frac{F_z}{\overline{F_{\overline{z}}}}\right)\circ F^{-1},$$

where $F = f^{\mu_o}$. We have that $F^\mu := f^\mu = H^\mu \circ F$.

We now see that we can use H^μ on lemma 3.5. Let $\eta_2 > 0$ be such that $\|\lambda^\mu\|_\infty < k$ for all $\|\mu-\mu_o\| < \eta_2$. From corollary 2.3, we obtain that $\mu \mapsto H^\mu \in C^1(\epsilon_2)$ is continuous for some $0<\epsilon_2<\epsilon_o$. In particular there exists $0<\eta_3<\eta_2$ and $a>1$ such that

$$|H^\mu(z)| < a\ |z| \ \text{ for all }\ |z| < \epsilon_3 := \frac{\epsilon_2}{2} \text{ and all } \ \|\mu-\mu_o\| < \eta_3.$$

From the definition of $\lambda = \lambda^\mu$ we get that $\lambda(z) = \overline{\lambda\left(\frac{1}{\overline{z}}\right)}\,\frac{z^2}{\overline{z}^2}$ for almost every $z\in\mathbb{C}$. Therefore, writing $G^\mu(z) := 1\big/\overline{H^\mu\left(\frac{1}{\overline{z}}\right)}$, we have that $G^\mu = H^\mu$.

Observe that $\lambda^{\mu_o} \equiv 0$ and $H^{\mu_o} = Id$. By corollary 1.7, we have that $(H^\mu)^{-1} = f^{\widehat{\lambda}}$, where $\widehat{\lambda} = -\left(\lambda\,\frac{H_z}{\overline{H_{\overline{z}}}}\right)\circ H^{-1}$. In particular, for any $0<\delta<1$, there exists $0<\eta_4=\eta_4(\delta)<\eta_3$ such that $\big\|\widehat{\lambda}^\mu\big\|_\infty = \|\lambda^\mu\|_\infty < \delta$ for all

$\|\mu - \mu_o\| < \eta_4$. By theorem 1.9.(a), for any $r_o = r_o(\delta) > 0$ with $\delta\, C_{r_o(\delta)} < 1$ and some $K = K(\epsilon_3, \delta, r_o) > 1$, $C = C(\epsilon_3, \delta, r_o) > 1$, we have that

$$\begin{aligned} \left|(H^\mu)^{-1}(z)\right| &< K\; |z|^{1-\frac{2}{r_o(\delta)}} && \text{for all } |z| < \tfrac{1}{\epsilon_3} \text{ and all } \|\mu - \mu_o\| < \eta_4, \\ \left\|(H^\mu)_z^{-1}\right\|_{\frac{1}{\epsilon_3}, r_o(\delta)} &< C(\epsilon_3, k) && \text{for all } |\mu - \mu_o| < \eta_4\,, \end{aligned}$$

with $r_o(\delta) \to \infty$ when $\delta \to 0$ and $\eta_4 \to 0$.

Therefore the conditions on lemma 3.5 are satisfied by H^μ with uniform constants a, K, ϵ_3 for all $\|\mu - \mu_o\| < \eta_4$ and with $\alpha = 1 - \frac{2}{r_o(\delta)}$.

For any g we have that

$$\left(g \circ (F^\mu)^{-1}\right)(F^\mu)_z^{-1} = \left(h \circ (H^\mu)^{-1}\right)(H^\mu)_z^{-1},$$

where $h = (g \circ F^{-1})\, F_z^{-1}$, $F = f^{\mu_o}$.

Given $0 < p < p_o$ choose $0 < \delta < 1$ (hence $\eta_4(\delta) > 0$) and $p < q_o = q_o(p) < p_o$ such that

$$0 < p \leq \frac{q_o\, r_o(\delta)}{q_o + r_o(\delta) - 1}\,.$$

Applying lemma 3.5, we have that $\ell_z \in \mathcal{D}(\epsilon_3, p, f^\mu)$ and

$$\|\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} \leq \frac{a^2}{(1-k^2)^{\frac{1}{p}-\frac{1}{q_o}}} \left(\|\ell_z\|_{\mathcal{D}(\frac{K}{\epsilon_3}, q_o, F)} + \|\ell_z\|_{\mathcal{D}(K\,\epsilon^\alpha, q_o, F)} \right) \cdot C(\epsilon_3, k).$$

In particular, by lemma 3.2 and (20), we have that

$$\lim_{\substack{\mu \to \mu_o \\ \sigma \to \sigma_o}} \|\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} = 0 \text{ for all } 2 < p < p_o. \tag{22}$$

By proposition 3.4 and (21), we have that

$$\begin{aligned} \|h\|_{B_{r,p}} &\leq C(R,r)\; \left(\|\nu\, \ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} + \|\nu\, \ell_z\|_{R,p} \right) \\ &\leq C(R,r)\; \left(\|\nu\|_\infty\; \|\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} + \|\nu\|_\infty\; \|\ell_z\|_{R,p} \right)\,, \\ \frac{\|h\|_{B_{r,p}}}{\|\nu\|_\infty} &\leq C(R,r)\; \left(\|\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} + \|\ell_z\|_{R,p} \right)\,, \end{aligned}$$

for all $R > 0$ and any $2 < p < p_o$. From (19) and (22) we get that

$$\lim_{\substack{\mu \to \mu_o \\ \sigma \to \sigma_o}} \frac{\|h\|_{B_{r,p}}}{\|\nu\| + \|\rho\|_{\mathcal{D}(\epsilon_3,p,F)}} \leq \lim_{\substack{\mu \to \mu_o \\ \sigma \to \sigma_o}} \frac{\|h\|_{B_{r,p}}}{\|\nu\|_\infty} = 0\,.$$

By 3.4.(iv) and 3.2, we also have that for $2 < p < p_1 < p_o$,

$$\|\ell_z\|_{\mathcal{D}(\epsilon,p,F)} \leq A\; \|\nu\, \ell_z\|_{\mathcal{D}(\epsilon,p_1,F)} \leq A\; \|\nu\|_\infty\; \|\ell_z\|_{\mathcal{D}(\epsilon_3,p_1,F)}$$

$$\lim_{\substack{\mu \to \mu_o \\ \sigma \to \sigma_o}} \frac{\|h_z\|_{\mathcal{D}(\epsilon,p,F)}}{\|\nu\| + \|\rho\|_{\mathcal{D}(\epsilon_3,p,F)}} \leq A \lim_{\substack{\mu \to \mu_o \\ \sigma \to \sigma_o}} \|\ell_z\|_{\mathcal{D}(\epsilon_3,p_1,F)} = 0\,. \tag{23}$$

$\square$

On $\mathcal{L}_\infty(\mathbb{C})\cap W^{n,p}(R)$ and on $\mathcal{D}(\epsilon,p,F)\cap W^{n,p}(R)$ consider the norms

$$\|\mu\|_{LW} := \|\mu\|_{\mathcal{L}_\infty(\mathbb{C})} + \|\mu\|_{W^{n,p}(R)} \ ,$$
$$\|\sigma\|_{DW(F)} := \|\sigma\|_{\mathcal{D}(\epsilon,p,F)} + \|\sigma\|_{W^{n,p}(R)} \ .$$

3.7. Proposition. *Suppose that $\mu_o\in\mathcal{L}_\infty(\mathbb{D})\cap W^{n,p_o}(R)$, $\|\mu_o\|_\infty<k<1$, $k\,C_{p_o}<1$, $p_o>2$ and let $F=f^{\mu_o}$. Then the map $(\mu,\rho)\mapsto\omega^{\mu,\rho}$,*

$$\big(\mathcal{L}_\infty(\mathbb{D})\cap W^{n,p_o}(R)\big)\times\big(\mathcal{D}(\epsilon,p_o,F)\cap W^{n,p_o}(R)\big)\to W^{n+1,p}(r)$$

is differentiable for μ in a neighbourhood of μ_o for all $0<r<R$ and any $2<p<p_o$.

Proof: We have the same equations (17), (18), (21), (22) from proposition 3.6. Also,

$$\|\nu\,\omega_z^o+\rho\|_{W^{n,p}(R)}\le 2^n\ \|\nu\|_{W^{n,p}(R)}\ \|\omega_z^o\|_{W^{n,p}(R)}+\|\rho\|_{W^{n,p}(R)}\,.$$

By proposition 3.4.(v), $\|\omega_z^o\|_{W^{n,p}(r)}$ is finite for all $0<r<R$ and $2<p<p_o$. Using (18) and 3.4.(v), we get that

$$\lim_{\substack{\mu\to\mu_o\\ \sigma\to\sigma_o}}\|\ell\|_{W^{n+1,p}(r)}=0 \tag{24}$$

for all $0<r<R$, $2<p<p_o$. In particular, the linear map $L(\nu,\rho)=\ell\in W^{n+1,p}(r)$ is continuous. From equation (21) and 3.4.(v) we have that, for $S:=\frac{R+r}{2}$,

$$\begin{aligned}
\|h\|_{W^{n+1,p}(r)} &\le C(S,r)\ \Big(\|\nu\,\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)}+\|\nu\,\ell_z\|_{W^{n,p}(S)}\Big)\\
&\le C(S,r)\ \Big(\|\nu\|_\infty\ \|\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)}+2^n\ \|\nu\|_{W^{n,p}(R)}\ \|\ell_z\|_{W^{n,p}(S)}\Big)\\
\frac{\|h\|_{W^{n+1,p}(r)}}{\|\nu\|_{W^{n,p}(r)}} &\le C(S,r)\ \Big(\|\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)}+2^n\ \|\ell_z\|_{W^{n,p}(S)}\Big)\ .
\end{aligned}$$

By (22) and (24) we have that

$$\lim_{\substack{\mu\to\mu_o\\ \sigma\to\sigma_o}}\frac{\|h\|_{W^{n+1,p}(r)}}{\|\nu\|_{LW}+\|\rho\|_{DW(F)}}\le\lim_{\substack{\nu\to0\\ \rho\to0}}\frac{\|h\|_{W^{n+1,p}(r)}}{\|\nu\|_{W^{n,p}(r)}}=0$$

for all $2<p<p_o$ and $0<r<R$. This completes the proof.

□

3.8. Corollary. *The maps of propositions 3.6 and 3.7 are C^∞.*

Proof: We prove the corollary for the map in proposition 3.7, the proof for the other map is similar. Define the following Banach spaces: $\mathbf{E}:=\mathcal{L}_\infty(\mathbb{C})\cap W^{n,p}(R)$, $\mathbf{F}:=\mathcal{D}(\epsilon,p,F)\cap W^{n,p}(R)$, $\mathbf{G}:=W^{n+1,p}(r)\cap\mathcal{F}(\epsilon,p,F)$, $\mathcal{F}(\epsilon,p,F):=\big\{\,\ell\,|\,\ell_z\in\mathcal{D}(\epsilon,p,F)\,\big\}$ with $\|\ell\|_{\mathcal{F}}:=\|\ell_z\|_{\mathcal{D}(\epsilon,p,F)}$ and $\mathcal{L}(\mathbf{E}\times\mathbf{F},\mathbf{G}):=\big\{\,L:\mathbf{E}\times\mathbf{F}\to\mathbf{G}\,|\,\mathcal{L}\text{ linear}\,\big\}$.

There is no map $\mathbf{G} \to \mathbf{F}$ given by $\omega \to \omega_z$ because $r < R$. We leave to the reader the technicalities that appear with this problem. Define the maps $F : U \times \mathbf{F} \subseteq \mathbf{E} \times \mathbf{F} \to \mathbf{G}$, $F(\mu, \sigma) = \omega^{\mu,\sigma}$ where $U \subseteq \mathbf{E}$ is the open subset defined in proposition 3.7. Let $\overline{F} : U \subseteq \mathbf{E} \to \mathcal{L}(\mathbf{F}, \mathbf{G})$, $\overline{F}(\mu) \cdot \sigma = \omega^{\mu,\sigma}$; and $D : U \times \mathbf{F} \to \mathcal{L}(\mathbf{E} \times \mathbf{F}, \mathbf{G})$, $D(\mu, \sigma)(\nu, \rho) := \ell$, the derivative on equation (17) of proposition 3.6. Let $B : \mathbf{E} \times \mathbf{G} \to \mathbf{F}$ be the linear map $B(\nu, \omega) = \nu\, \omega_z$. We have that

$$D(\mu, \sigma)(\nu, \rho) = \overline{F}(\mu) \circ B\big(\nu, F(\mu, \sigma)\big) + \overline{F}(\mu)(\rho)$$

$$D(\mu, \sigma) = \overline{F}(\mu) \circ B\big(\pi_1, F(\mu, \sigma)\big) + \overline{F}(\mu) \circ \pi_2 \tag{25}$$

where $\pi_1 : \mathbf{E} \times \mathbf{F} \to \mathbf{E}$ and $\pi_2 : \mathbf{E} \times \mathbf{F} \to \mathbf{F}$ are the projections. We have that

$$\begin{aligned}
\|B(\nu, \omega)\|_{\mathbf{F}} = \|\nu\, \omega_z\|_{DW^n} &\leq \|\nu\|_\infty\ \|\omega_z\|_{\mathcal{D}(\epsilon,p,F)} + 2^n\ \|\nu\|_{W^n}\ \|\omega_z\|_{W^n} \\
&\leq 2^n\ \|\nu\|_{W^n}\ \big(\|\omega\|_{\mathcal{F}} + \|\omega\|_{W^{n+1}}\big) \\
&\leq 2^n\ \|\nu\|_{\mathbf{E}}\ \|\omega\|_{\mathbf{G}}\, .
\end{aligned}$$

Therefore the bilinear map B is C^∞. By proposition 3.7 and the limit (23) in the proof of proposition 3.6, we have that $F : U \times \mathbf{F} \to \mathbf{G}$ is differentiable. Using the notation of proposition 3.6 and 3.7, we have that

$$\begin{aligned}
\big\|\overline{F}(\mu + \nu)(\sigma) - \overline{F}(\mu)(\sigma) - \ell\big\|_{\mathbf{G}} &= \|h\|_{W^{n+1}} + \|h\|_{\mathcal{F}} \\
&\leq A_1(\mu)\ \|\nu\, \ell_z\|_{DW^n} \\
&\leq A_1(\mu)\ \Big(\|\nu\|_\infty\ \|\ell_z\|_{\mathcal{D}(\epsilon,p,F)} + 2^n\ \|\nu\|_{W^n}\ \|\ell_z\|_{W^n}\Big) \\
&\leq A_1(\mu)\ \|\nu\|_{W^n}\ 2^n\ \Big(\|\ell_z\|_{\mathcal{D}(\epsilon,p,F)} + \|\nu\|_{W^n}\Big) \\
&\leq A_1(\mu)\ \|\nu\|_{\mathbf{E}}\ A_2(\mu)\ \|\nu\, \omega_z^o + (\rho \equiv 0)\|_{DW^n} \\
&\leq A_3(\mu)\ \|\nu\|_{\mathbf{E}}\ \Big(\|\nu\|_\infty\ \|\omega_z^o\|_{\mathcal{D}(\epsilon,p,F)} + 2^n\ \|\nu\|_{W^n}\ \|\omega\|_{W^{n+1}}\Big) \\
&\leq A_4(\mu)\ \|\nu\|_{\mathbf{E}}\ \|\nu\|_{\mathbf{E}}\ \|\sigma\|_{\mathbf{F}}
\end{aligned}$$

$$\lim_{\|\nu\|_{\mathbf{E}} \to 0} \frac{1}{\|\nu\|_{\mathbf{E}}}\ \max_\sigma \left\{ \frac{\|h\|_{\mathbf{G}}}{\|\sigma\|_{\mathbf{F}}} \right\} \leq \lim_{\nu \to 0} A_4(\mu)\ \|\nu\|_{\mathbf{E}} = 0.$$

Therefore the map $\overline{F}$ is differentiable and its derivative is given by $(D\overline{F}(\mu) \cdot \nu)(\sigma) = D(\mu, \sigma)(\nu, 0)$, or

$$D\overline{F}(\mu) \cdot \nu = D(\mu, \cdot)(\nu, 0)\, . \tag{26}$$

Suppose that F and $\overline{F}$ are r-times differentiable. Then from formula (25) we have that D is r-times differentiable. But D is the derivative of F so that F is $(r + 1)$-times differentiable. Formula (26) implies that $\overline{F}$ is also $(r + 1)$-times differentiable. We conclude that F is C^∞.

□

3.9. Theorem.

(i) *Let $0 < k < 1$ and $p > 2$ with $k\, C_p < 1$. Then for any $R > 0$, the map*

$$\left\{\, \mu \in \mathcal{L}_\infty(\mathbb{D}) \mid \|\mu\|_\infty < k \,\right\} \longrightarrow B_{R,p}$$

given by $\mu \mapsto f^{\widehat{\mu}}$ is C^∞.

(ii) *Let $0 < k < 1$ and $p > 2$ with $k\, C_p < 1$. Then the map*

$$\mathcal{L}_\infty(\mathbb{D}) \cap W^{n,p}(R) \cap \left\{\, \|\mu\|_\infty < k \,\right\} \longrightarrow W^{n+1,p}(r)$$

given by $\mu \mapsto f^{\widehat{\mu}}$ is C^∞ for any $0 < r < R$.

(iii) *In particular, for any $n \geq 1$ and any $0 < r < S < R$ the map*

$$\mathcal{L}_\infty(\mathbb{D}) \cap C^n(S) \cap \left\{\, \|\mu\|_\infty < k \,\right\} \longrightarrow C^{n+1-\frac{2}{p}}(r) \cap C^{1-\frac{2}{p}}(R)$$

given by $\mu \mapsto f^{\widehat{\mu}}$ is C^∞.

Proof: Define the spaces $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$, $\mathcal{L}(\mathbf{F}, \mathbf{G})$ and the maps $\overline{F}(\mu) \cdot \sigma = \omega^{\mu,\sigma}$ and $B : \mathbf{E} \to \mathcal{L}(\mathbf{F}, \mathbf{G})$, $B(\nu, \omega) = \nu\, \omega_z$ as in the proof of corollary 3.8. We have that $\overline{F}$ and B are C^∞. Define the map $H : \mathbf{E} \to \mathbf{G}$ by $H(\mu) := f^\mu$.
Claim: *H is differentiable and $DH(\mu) \cdot \nu = \omega^{\mu, \nu f^\mu_z}$, i.e.*

$$DH(\mu) = \overline{F}(\mu) \circ B\big(\cdot, H(\mu)\big). \tag{27}$$

Suppose that the claim is true. From formula (27) we have that if H is r-times differentiable, then DH is r-times differentiable and hence H is $(r+1)$-times differentiable. By the claim, the induction starts at $r = 1$ and then H is C^∞.

□

Proof of the Claim: Let μ, $\nu \in \mathbf{E}$, $\omega := \omega^{\mu, \nu f^\mu_z}$, $h := f^{\mu+\nu} - f^\mu - \omega$. Then

$$h_{\overline{z}} = (\mu + \nu)\, h_z + \nu\, \omega_z$$
$$\omega_{\overline{z}} = \mu\, \omega_z + \nu\, f_z$$

with $h(0) = h(1) = \omega(0) = \omega(1) = 0$, $|\omega(z)| = \mathcal{O}\big(|f^\mu(z)|^2\big)$ and $|h(z)| = \mathcal{O}\big(|f^{\mu+\nu}(z)|^2\big)$. We have that

$$\begin{aligned}
\|H(\mu + \nu) - H(\mu) - \omega\|_{\mathbf{G}} &= \|h\|_{W^{n+1}} + \|h\|_{\mathcal{F}} \\
&\leq A\, \left(\|\nu\, \omega_z\|_{W^n} + \|\nu\, \omega_z\|_{\mathcal{D}(\epsilon,p,f^\mu)} \right) \\
&= A\, \|B(\nu, \omega)\|_{\mathbf{F}} \\
&\leq 2^n\, A\, \|\nu\|_{\mathbf{E}}\, \|\omega\|_{\mathbf{G}} \\
&\leq 2^n\, A\, \|\nu\|_{\mathbf{E}}\, A\, \|\nu\, f^\mu_z\|_{\mathbf{F}} \\
&\leq 2^n\, A\, \|\nu\|_{\mathbf{E}}\, A\, 2^n\, \|\nu\|_{\mathbf{E}}\, \|f^\mu\|_{\mathbf{G}}
\end{aligned}$$

with $\|f\|_{\mathbf{G}} = \|f\|_{W^{n+1}} + \|f_z\|_{\mathcal{D}(\epsilon,p,f^\mu)}$. By considering small bump functions λ as in proposition 3.4.(v), one can show that $f \in W^{n+1,p}(R)$, hence $\|f\|_{W^{n+1}} < +\infty$. We have that $\|f_z\|_{\mathcal{D}(\epsilon,p,f^\mu)} = \|1\|_{\mathcal{D}_{\epsilon,p}} < +\infty$. Therefore

$$\lim_{\nu\to 0} \frac{\|H(\mu+\nu) - H(\mu) - \omega\|_{\mathbf{G}}}{\|\nu\|_{\mathbf{E}}} \leq \lim_{\nu\to 0} (2^n\, A)^2\ \|\nu\|_{\mathbf{E}}\ \|f\|_{\mathbf{G}} = 0.$$

And the linear map $DH(\mu) : \nu \mapsto \omega$ is continuous because

$$\|\omega\|_{\mathbf{G}} \leq A\ \|\nu\, f_z^\mu\|_{\mathbf{F}} \leq 2^n\, A\ \|f^\mu\|_{\mathbf{G}}\ \|\nu\|_{\mathbf{E}}\,.$$

This proves the claim.

□

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