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The Hausdorff Dimension of the Harmonic Class on Negatively Curved Surfaces

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ABSTRACT. We study the regularity of the Hausdorff dimension of the harmonic class of a surface M of negative curvature as a function of the riemannian metric. We prove that it is a C^{r-3} function of the metric in the Banach manifold of C^r riemannian metrics on M. We also prove regularity results for some asymptotic quantities associated to the Brownian motion on $\tilde{M}$.

1. Regularity of the harmonic class

1.1. Introduction

In the last years there has been increasing interest in potential theory on simply connected manifolds $\tilde{M}$ of bounded negative curvature. Anderson [3], Anderson and Schoen [4], and Sullivan [28] have proven that the Dirichlet problem on $\tilde{M}$ can be solved for continuous data on the sphere at infinity $S(\infty)$ of $\tilde{M}$. In [17], Kifer gives a probabilistic proof of this result, relating it to the Brownian motion on $\tilde{M}$. When $\tilde{M}$ is the universal cover of a closed manifold of negative curvature M, Ledrappier [21] related some asymptotic quantities associated to the Brownian motion on $\tilde{M}$ with ergodic quantities associated to the geodesic flow of M and obtained rigidity results for the metric on M (see Theorem 1.2). For example, if $(\rho, \theta) \in \mathbb{R}^+ \times \{ v \in T_x\tilde{M} \mid |v| = 1 \}$ are the geodesic polar coordinates about x of a point $z = \exp_x \rho\,\theta \in \tilde{M}$, and $A(x, z)$ is the function defined by $dV(z) = A(x, z)\, d\rho\, d\theta$, where dV is the volume element of $\tilde{M}$, then for almost every Brownian path $\tilde{\omega}(t)$ on $\tilde{M}$ we have the same limit:

$$\lambda = \lim_{t\to\infty} \frac{\log A\,(x, \tilde{\omega}(t))}{d\,(x, \tilde{\omega}(t))} ,$$

where $d(x, y)$ is the distance function on $\tilde{M}$. We restrict ourselves to the case of the universal cover of a closed surface and consider λ as a function of the riemannian metric g. We prove that the map $g \mapsto \lambda(g)$ is C^{r-3} when g varies in the C^r topology.

Solving the Dirichlet problem on $\tilde{M}$ for boundary data on $S(\infty)$ gives rise to harmonic measures ω_x associated to each point x of $\tilde{M}$. All these measures are absolutely continuous with respect to each other and define a measure class on $S(\infty)$. Since, in the case of surfaces, the sphere at infinity

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has a natural C^1 structure, the Hausdorff dimension of the harmonic class $HD(\omega_g)$ is well defined. It gives a measure of the deviation of g from a metric of constant curvature (cf. Katok [15]). We prove that the map $g \mapsto HD(\omega_g)$ is C^{r-3} varying g in the C^r topology.

The actual condition that we need on the riemannian metric g on M is that the geodesic flow of g is Anosov. This allows some sets of positive curvature but not conjugate points (cf. Klingenberg [19] or Mañé [24]). We state the theorems in this setting.

1.2. Notations and statements of results

Let (M, g) be a closed surface of genus $g \geq 2$ endowed with a riemannian metric whose geodesic flow is Anosov, for example, a metric with variable negative curvature $-b^2 \leq K \leq -a^2$. Let $\pi : \tilde{M} \to M$ be its universal cover with the metric induced by π and (M, g). Let $S_g M$ (resp. $S_g \tilde{M}$) be the unit tangent bundle of (M, g) (resp. $(\tilde{M}, \tilde{g})$ the lift of g) with the natural projection $p : S_g M \to M$ (resp. $\tilde{p} : S_g \tilde{M} \to \tilde{M}$). Let $\mathbf{\Gamma} = \pi_1(M)$ be the group of deck transformations of $\tilde{M}$.

The harmonic class

Two geodesics γ and η in $\tilde{M}$ are said to be equivalent if $\sup_{t \geq 0} d\big(\gamma(t), \eta(t)\big) < +\infty$. The space of equivalence classes is called the *sphere at infinity* and is denoted by $S(\infty)$ (see, e.g., [6]). For $\tilde{X}$ in $S\tilde{M}$ let $\gamma_{\tilde{X}}$ be the geodesic in $\tilde{M}$ defined by $\big(\gamma_{\tilde{X}}(0), \gamma'_{\tilde{X}}(0)\big) = \tilde{X}$. Denote by $\tau : S\tilde{M} \to S(\infty)$ the map that associates to each $\tilde{X}$, the class of $\gamma_{\tilde{X}}$. For x in $\tilde{M}$, the restriction τ_x of τ to $S_x \tilde{M} = \tilde{p}^{-1}\{x\}$ is a homeomorphism between $S_x \tilde{M}$ and $S(\infty)$. The *cone topology* on $\tilde{M} \cup S(\infty)$ is obtained by adding to the topology of $\tilde{M}$ and $S(\infty)$ the open sets $C(A, R) := \tau(A) \cap \bigcap_{t > R} \exp_{\tilde{x}}(tA)$, where A is an open subset of $S_{\tilde{x}} \tilde{M}$.

Let $\tilde{\phi} : S\tilde{M} \times \mathbb{R} \to S\tilde{M}$ be the geodesic flow of $(\tilde{M}, \tilde{g})$, $\tilde{\phi}_t(\tilde{X}) = \big(\gamma_{\tilde{X}}(t), \gamma'_{\tilde{X}}(t)\big)$ and $\phi : SM \times \mathbb{R} \to SM$ be the geodesic flow of (M, g). Given $\tilde{X} \in S\tilde{M}$, the *weak stable manifold* of $\tilde{X}$ is defined by

$$\widetilde{W}^s(\tilde{X}) := \left\{ \tilde{Y} \in S\tilde{M} \;\middle|\; \sup_{t \geq 0} d\big(\tilde{\phi}_t(\tilde{X}), \tilde{\phi}_t(\tilde{X})\big) < +\infty \right\}.$$

$\widetilde{W}^s(\tilde{X})$ is a C^1 submanifold of $S\tilde{M}$ homeomorphic to $\mathbb{R}^2$. The *stable foliation* $\widetilde{\mathcal{F}}^s = \{ \widetilde{W}^s(\tilde{X}) \mid \tilde{X} \in S\tilde{M} \}$ is $\mathbf{\Gamma}$-invariant and projects onto the stable foliation $\mathcal{F}^s = \{W^s(X) \mid X \in SM\}$, $W^s(\pi X) := \pi\big(\widetilde{W}^s(\tilde{X})\big)$, for the Anosov flow ϕ_t on SM. Since $\dim M = 2$, the foliations $\widetilde{\mathcal{F}}^s$, $\mathcal{F}^s$ are C^1 (see [13] or [14]). The *strong unstable manifold*

$$\widetilde{W}^{uu}(\tilde{X}) = \left\{ \tilde{Y} \in S\tilde{M} \;\middle|\; \lim_{t \to +\infty} d\big(\tilde{\phi}_{-t}(\tilde{X}), \tilde{\phi}_{-t}(\tilde{Y})\big) = 0 \right\}$$

is the negative horosphere passing through $\tilde{X}$, which is a $\mathbf{\Gamma}$-invariant embedded submanifold of $S\tilde{M}$ homeomorphic to $\mathbb{R}$ and projects onto the strong unstable manifold $W^{uu}(\pi\tilde{X}) = \big(\widetilde{W}^{uu}(\tilde{X})\big)$ for $\pi(\tilde{X})$. They form a foliation $\widetilde{\mathcal{F}}^{uu} = \{ \widetilde{W}^{uu}(\tilde{X}) \mid \tilde{X} \in S\tilde{M} \}$, called the strong unstable foliation or the horospheric foliation which is transversal to the stable foliation. The spheric foliation $\mathcal{S} = \{ S_x\tilde{M} \mid x \in \tilde{M} \}$ is also transversal to $\widetilde{\mathcal{F}}^s$. The restrictions $\tau : \widetilde{W}^{uu}(\tilde{X}) \to S(\infty) - \{\tau(-\tilde{X})\}$ and $\tau_x : S_x\tilde{M} \to S(\infty)$ are homeomorphisms whose transition maps $\tau|_{\widetilde{W}^{uu}(\tilde{X})} \circ (\tau|_{\widetilde{W}^{uu}(\tilde{Y})})^{-1}$, $\tau|_{\widetilde{W}^{uu}(\tilde{X})} \circ \tau_y^{-1}$, $\tau_x \circ \tau_y^{-1}$ are the holonomy maps of the stable foliation, i.e., the diagram

$$\begin{array}{ccc} \widetilde{W}^{uu}(\tilde{X}) & \xrightarrow{\text{holonomy}} & \widetilde{W}^{uu}(\tilde{Y}) \\ & \tau \searrow \quad \swarrow \tau & \\ & S(\infty) & \end{array}$$

commutes. Since the holonomy maps of $\mathcal{F}^s$ are C^1, this gives a natural C^1 structure to $S(\infty)$.

The Laplacian operator on $\tilde{M}$ is the operator $\Delta\varphi = \operatorname{div}(\operatorname{grad}(\varphi))$ on $C^2(\tilde{M}, \mathbb{R})$, where $\langle \operatorname{grad}(\varphi), \tilde{X} \rangle = \tilde{X}(\varphi)$, $\forall \tilde{X} \in T\tilde{M}$ and $\operatorname{div}(F)$ is the trace of $Y \mapsto \nabla_Y F$, the riemannian connection on a vectorfield $F : \tilde{M} \to T\tilde{M}$. The Dirichlet problem $\Delta\varphi = 0$, $\varphi|_{S(\infty)} = f$ can be solved for any $f : S(\infty) \to \mathbb{R}$ continuous (see [3, 4, 28], or [17]). Let $Hf = \varphi$ be the solution to the problem. For $x \in \tilde{M}$, the *harmonic measure at* x is the unique Borel measure ω_x on $S(\infty)$ such that

$$(Hf)(x) = \int_{S(\infty)} f \, d\omega_x$$

for any $f \in C^0(S(\infty), \mathbb{R})$. All these measures are absolutely continuous with respect to each other. Their equivalence class is called the *harmonic class* of $\tilde{M}$.

Given a subset K of a separable metric space (Ω, d), the *Hausdorff dimension* of K is defined to be

$$\begin{aligned} HD(K) &:= \inf \{ \delta > 0 \,|\, m_\delta(K) = 0 \} , \\ m_\delta(K) &:= \liminf_{\delta \to 0} \left\{ \sum_{V \in \mathcal{O}} (\operatorname{diam} V)^\delta \right\} , \end{aligned}$$

where the infimum on $m_\delta(K)$ is taken over all open covers $\mathcal{O}$ of K with $\operatorname{diam} \mathcal{O} < \epsilon$. Given a Borel probability measure μ on (Ω, d), the Hausdorff dimension of μ is defined to be

$$HD(\mu) := \inf \{ HD(\Lambda) \,|\, \mu(\Lambda) = 1 \} .$$

This number is constant in an equivalence class of (absolutely continuous) probabilities.

Since C^1 maps preserve Hausdorff dimension and $HD(\cup_{n=1}^\infty K_n) = \sup_{n \in \mathbb{N}} HD(K_n)$, we can define the *Hausdorff dimension of the harmonic class* to be $HD(\omega) := HD(\omega_x \circ \tau_y^{-1}) = HD(\omega_x \circ (\tau|_{\widetilde{W}^{uu}(\tilde{Z})})^{-1})$ for any $x, y \in \tilde{M}$, $\tilde{Z} \in S\tilde{M}$. We write $HD(\omega_g)$ when we want to make explicit the dependence of $HD(\omega)$ on the riemannian metric g of M.

Kifer and Ledrappier [20] proved that for a simply connected complete riemannian manifold $\tilde{M}$ of bounded negative sectional curvatures $-b^2 \le K \le -a^2$, the Hausdorff dimensions $HD(\omega_x \circ \tau_x^{-1})$ (which *a priori* depend on $x \in \tilde{M}$ because the maps $\tau_x \circ \tau_y^{-1}$ are only Hölder continuous) are all positive. Actually, they are all equal by Remark 1.5.

Let $R^r(M)$ be the Banach manifold of C^r riemannian metrics on M with the C^r topology and let $A^r(M)$ be the open subset of C^r metrics whose geodesic flow is Anosov, in particular, metrics with negative curvature. Here we prove the following:

Theorem 1.1. *The map* $A^r(M) \ni g \mapsto HD(\omega_g) \in \mathbb{R}$ *is* C^{r-3}, $r \ge 3$.

The Brownian motion

Let (M, g) be as above. Denote by $\widetilde{\Omega} = C^0([0, +\infty[, \tilde{M})$ the space of continuous paths on $\tilde{M}$ with the topology given by uniform convergence on compact subsets. For $x \in \tilde{M}$ let $\boldsymbol{P}_x$ be the Borel probability on $\widetilde{\Omega}_x := [\, \tilde{\omega} \in \widetilde{\Omega} \mid \tilde{\omega}(0) = x \,]$ defined by

$$\boldsymbol{P}_x\left[\tilde{\omega} \in \widetilde{\Omega} \mid \tilde{\omega}(0) = x,\ \tilde{\omega}(t) \in A\right] = \int_A p(t, x, y)\, dm_g(y)$$

for any $t > 0$ and any Borel subset $A \subset \tilde{M}$, where m_g is the volume element of $\tilde{M}$ and $P : \mathbb{R} \times \tilde{M} \times \tilde{M} \to \mathbb{R}$ is the fundamental solution of the heat equation on $\tilde{M}$:

$$\begin{aligned} \frac{\partial p}{\partial t} + \nabla p(t, \cdot, y) &= 0\,, \\ \lim_{t \downarrow 0} \int_{\tilde{M}} p(t, x, y)\, f(y)\, dm_g(y) &= f(x)\,, \end{aligned}$$

for any continuous function $f : \tilde{M} \to \mathbb{R}$. Since the *heat kernel* satisfies (see [8, Theorem VIII.4, VIII.5]):

$$\begin{array}{llll} (i) & p(t, y, x) = p(t, x, y) \geq 0, & \forall\, t\ \geq 0, & \forall\, x, y\ \in \tilde{M}\,, \\ (ii) & \int_{\tilde{M}} p(t, x, y)\, dm_g(y) = 1, & \forall g\ \geq 0, & \forall\, x, y\ \in \tilde{M}\,, \\ (iii) & \int_{\tilde{M}} p(s, x, y)\, p(t, y, z)\, dm_g(y) = p(s+t, x, z), & \forall s, t \geq 0, & \forall x, y, z \in \tilde{M}\,; \end{array}$$

we have that the family $\mathcal{P} = \{\, \boldsymbol{P}_x \mid x \in \tilde{M} \,\}$ of probability measures defines a continuous Markov process on $\tilde{M}$ called the *Brownian motion* on $\tilde{M}$. The induced probabilities $P_{\pi(\tilde{X})} = \boldsymbol{P}_{\tilde{x}} \circ \pi$ on $\Omega = C^0\big([0, +\infty[, M)$ define the Brownian motion on M.

Since the geodesic flow ϕ_t is Anosov, $\tilde{M}$ cannot have conjugate points and $\exp_x T_x\tilde{M} \to \tilde{M}$ is a diffeomorphism for every $x \in M$ (see [19] or [24]). For $x \in \tilde{M}$, we consider geodesic polar coordinates about x, i.e., we identify $T_x\tilde{M}$ with $]0, +\infty[\times S_x\tilde{M} \cup \{0\}$ and a point $z \in \tilde{M}$ is described by the polar coordinates of $\exp_x^{-1}(z)$. For $\tilde{\omega} \in \widetilde{\Omega}$, denote by $\big(r(\tilde{\omega}, t), \theta(\tilde{\omega}, t)\big)$ the geodesic polar coordinate about x of the point $\tilde{\omega}(t)$. For $x \in \tilde{M}$, let λ_x be the Lebesgue measure on $S_x\tilde{M}$ and denote by $A_g(x, z)$ the function on $\tilde{M} \times \tilde{M}$ such that

$$dm_g\left(\exp_z t\xi\right) = A\left(x, (t, \xi)\right)\ dt\, d\lambda_x(\xi)$$

for $\xi \in S_x\tilde{M}$. Let $V_g(x, t)$ be the volume of the ball of radius t about x:

$$V_g(x, t) = \int_0^t \left(\int_{S_x\tilde{M}} A_g\left(x, (s, \xi)\right)\, d\lambda_x(\xi) \right) ds\,.$$

The following theorem has been proved by several people:

Theorem 1.2.

[26] *For all $x \in \tilde{M}$, $\boldsymbol{P}_x$-a.e. $\tilde{\omega} \in \widetilde{\Omega}$, $\theta(\tilde{\omega}, t)$ converges as t goes to infinity towards some limit $\theta(\tilde{\omega}, +\infty) \in S_x\tilde{M}$.*
The induced measure on $S(\infty) \approx S_x\tilde{M}$ by $\theta(\cdot, +\infty) : (\Omega, \boldsymbol{P}_x) \to S_x\tilde{M}$ is the harmonic measure ω_x.

[29] *There exists a number $\alpha > 0$ such that for all $x \in \tilde{M}$, $\boldsymbol{P}_x$-a.e. $\tilde{\omega}$, $\lim_{t \to +\infty} \frac{1}{t}\, r(\tilde{\omega}, t) = \alpha(g)$.*

[16] *There exists a number $\beta > 0$ such that for all $x \in \tilde{M}$, $\boldsymbol{P}_x$-a.e. $\tilde{\omega}$,* $\lim_{t\to+\infty} -\frac{1}{t}\log p(t,x,\tilde{\omega}(t)) = \beta(g)$.

[21] *There exists a number $\gamma > 0$ such that for all $x \in \tilde{M}$, $\boldsymbol{P}_x$-a.e. $\tilde{\omega}$,* $\lim_{t\to+\infty} \frac{1}{t}\log A\big(x,\tilde{\omega}(t)\big) = \gamma(g)$.

[21] *In general $\beta \le \gamma$ and $\beta \le \alpha\, h$, where h is the topological entropy of the geodesic flow on SM.*

[21] *Each of the equalities $\beta = \gamma$ or $\beta = \alpha h$ hold if and only if the surface M has constant curvature.*

We prove the following slightly more general result than Theorem 1.1:

Theorem 1.3.

(i) *The map $A^r(M) \ni g \mapsto \frac{\beta(g)}{\alpha(g)} \in \mathbb{R}$ is C^{r-2}.*

(ii) *The map $A^r(M) \ni g \mapsto \frac{\gamma(g)}{\alpha(g)} \in \mathbb{R}$ is C^{r-3}.*

(iii) *The Hausdorff dimension of the harmonic measure is $HD(\omega_g) = \frac{\beta(g)}{\gamma(g)}$.*

Since for surfaces the harmonic measure on $S_x\tilde{M} \approx S(\infty)$ is absolutely continuous with respect to the Lebesgue measure only in the case of constant (negative) curvature and in this case $HD(\omega) = 1$, then $HD(\omega_g)$ can be seen as a measure of the deviation of g from a metric of constant curvature (cf. [15]).

1.3. Equilibrium states

Given a Hölder continuous function $F : SM \to \mathbb{R}$, there exists a unique ϕ-invariant probability measure μ_F on SM, called the *equilibrium state* of (ϕ, F) such that it maximizes the functional

$$\nu \mapsto h_\nu(\phi_1) + \int F\, d\mu$$

over all the ϕ-invariant Borel probability measures on SM, where $h_\nu(\phi_1)$ is the entropy of ϕ_1 with respect to ν (see [7]).

For $X \in SM$ define the local stable and strong unstable manifolds of X by

$$\begin{aligned} W^s_\epsilon(X) &= \Big\{ Y \in SM \,\Big|\, d\big(\phi_t(X), \phi_t(Y)\big) \le \epsilon, \quad \forall t \ge 0 \Big\} \\ W^{uu}_\epsilon(X) &= \Big\{ Y \in SM \,\Big|\, d(X,Y) \le \epsilon \text{ and } \lim_{t\to+\infty} d\big(\phi_{-t}(X), \phi_{-t}(Y)\big) = 0 \Big\}. \end{aligned}$$

If $\epsilon > 0$ is sufficiently small, then they are transversal embedded discs in SM with $\dim W^s_\epsilon(X) = 2$, $\dim W^{uu}_\epsilon(X) = 1$. For $\epsilon > 0$ small there exists a partition ξ of SM with $\operatorname{diam} \xi < \epsilon$ such that it is *subordinate* to $\mathcal{F}^{uu}$, i.e., $\xi(X) \subset W^{uu}_\epsilon(X)$ for all $X \in SM$ (see [22]) and such that it is a *measurable partition*, i.e., the quotient space SM/ξ is separated by a countable number of measurable sets (see [27]). Then (cf. [27]) there exists a system of *conditional measures* associated to it, i.e., for μ-a.e. $X \in SM$ there exists a probability measure $\mu_X = \mu_{\xi(X)}$ on $\xi(X)$ such that for any Borel set A on SM, the function $X \mapsto \mu_{\xi(X)}\big(A \cap \xi(X)\big)$ is measurable and $\mu(A) = \int_{SM} \mu_{\xi(X)}\big(A \cap \xi(X)\big)\, d\mu(X)$.

If μ^F is an equilibrium state and $\mathcal{L}$ is the holonomy map of the stable foliation $\mathcal{F}^s$ from (a subset of) $\xi(X)$ to $\xi\big(\phi_t(X)\big)$: $\mathcal{L}(Y) = W^s(Y) \cap \xi\big(\phi_t(X)\big)$, then the measures $\mu^F_{\xi(X)}$ and $\mu^F_{\xi(\phi_t(X))} \circ \mathcal{L}^{-1}$

are equivalent on $\xi(X)\cap\mathcal{L}^{-1}\big(\xi(\phi_t(X))\big)$. It follows that the measure ν on $\xi(X)$ defined by $\nu(A)=\mu^F\big(\cup_{Y\in A}W^s_\epsilon(Y)\big)$ is equivalent to $\mu^F_{\xi(X)}$.

Observe that if $Hf=\varphi$ is the solution of $\Delta\varphi=0$, $\varphi|_{S(\infty)}=f$ on $\tilde M$ and $\Gamma\in\mathbf{\Gamma}$, then $H(f\circ\Gamma)=(Ff)\circ\Gamma$ so that the harmonic measures satisfy $\omega_{\Gamma(x)}=\omega_x\circ\tilde\Gamma^{-1}$, where $\tilde\Gamma$ is the extension of Γ to $S(\infty)$. Since for $\tilde X\in S\tilde M$, $\tau|_{W^{uu}(D\Gamma\cdot\tilde X)}\circ D\Gamma=\tilde\Gamma\circ\tau|_{W^{uu}(\tilde X)}$, we have that the measures $\nu_{\tilde X}:=\omega_{p(\tilde X)}\circ\tau_{W^{uu}(\tilde X)}$ satisfy $\tilde\nu_{D\Gamma\cdot\tilde X}=\tilde\nu_{\tilde X}\circ D\Gamma$ and hence the system $\{\,\tilde\nu_{\tilde X}\mid\tilde X\in S\tilde M\,\}$ projects to a family of measures $\{\nu_X\mid X\in SM,\}$, $\nu_{\pi\tilde X}\circ D\pi=\tilde\nu_{\tilde X}$, that we call the *horospheric harmonic measure* on SM.

Theorem 1.4 [21].

1. *The horospheric harmonic measures are equivalent to the conditional measures on local strong unstable manifolds of the equilibrium state μ^F of the function*

$$F(\pi\tilde X)=\log K\big(\gamma_{\tilde X}(0),\gamma_{\tilde X}(1),\tau(\tilde X)\big)\,, \tag{1.4.1}$$

where $K:\tilde M\times\tilde M\times S(\infty)\to\mathbb{R}$ is the Poisson kernel of $\tilde M$ (see Section 1.5): $\mu^F_{\xi(X)}\approx\nu_X$ for all $X\in SM$.

2. *We have, for the Brownian motion in $\tilde M$, that $\gamma=\alpha\int J^u\,d\mu^F$, where*

$$J^u(X)=\frac{d}{dt}\big[\log\big|\det D\phi_t|_{T_XW^{uu}(X)}\big|\big]_{t=0}\,.$$

In particular $\frac{\gamma}{\alpha}$ is the positive Lyapunov exponent of $(SM,\{\phi_t,t\in\mathbb{R}\},\mu^F)$.

3. *We have $\beta=\alpha\,h_\mu(\phi)$.*

Since $D\pi$ is a C^1 map and the Hausdorff dimension $HD\big(\mu^F_{\xi(X)}\big)$ is constant for μ-a.e. $X\in SM$ (cf. [23]), then we have that $HD(\omega_g)=HD(\nu_X)=HD\big(\mu^F_{\xi(X)}\big)$, μ-a.e. $X\in SM$. Ledrappier, Manning, and Young (cf. [22, 23, 30]) proved that $HD\big(\mu^F_{\xi(X)}\big)=h_\mu(\phi)/\lambda(\mu)$, where $\lambda(\mu)$ is the positive Lyapunov exponent of $(\phi,\,\mu)$. In particular, we have that $HD(\omega_g)=\beta(g)/\gamma(g)$.

Remark 1.5. Ledrappier and Young [22] proved that in higher dimensions, $\dim SM\geq 4$, the Hausdorff dimension of conditional measures on W^{uu}, $HD\big(\mu_{\xi(X)}\big)$, of invariant probabilities μ, are the same μ-a.e. $X\in SM$. This implies that in $\dim M>2$, even when the holonomy maps of the stable foliation $\mathcal{F}^s$ are only Hölder continuous and hence the sphere at infinity has only a Hölder structure, the Hausdorff dimension of the harmonic class is well defined (and positive). □

We are going to use the following:

Theorem 1.6 [9].

Let X be a C^r vectorfield on a compact manifold N whose flow is Anosov. Let $\mathfrak{X}^r(N)$ be the Banach space of C^r vectorfields on N and $C^\alpha(N,\mathbb{R})$ be the Banach space of α-Hölder continuous functions on N. Let $\psi:\mathcal{V}\subset\mathfrak{X}^r(N)\to C^0(N,\mathbb{R})$ be a continuous map from a neighborhood $\mathcal{V}$ of X of vectorfields whose flows are Anosov. For $Y\in\mathcal{V}$ let u_Y be the topological equivalence of Proposition 1.13, and suppose that the map $F(Y):=\psi(Y)\circ u_Y$ is such that $F:\mathcal{V}\subset\mathfrak{X}^r(N)\to C^\alpha(N,\mathbb{R})$ is C^{s-1}, $s\leq r$. For $Y\in\mathcal{V}$, let μ_Y be the equilibrium state for $\big(Y,\psi(Y)\big)$ and $h(\mu_Y)$ the metric entropy of Y with respect to μ_Y. Then there exists a neighborhood $\mathcal{U}\subset\mathcal{V}$ of X in $\mathfrak{X}^r(N)$ such that the maps

(i) $\mathcal{U} \ni Y \mapsto h(\mu_Y) \in \mathbb{R}$ *is* C^{s-1} ,
(ii) $\mathcal{U} \ni Y \mapsto \mu_Y \in \big(C^\alpha(N,\mathbb{R})\big)^*$ *is* C^{s-1} ,
(iii) $\mathcal{U} \ni Y \mapsto \lambda(\mu_Y) := \int \frac{d}{dt}\big[\log\big|\det D\big(\phi_t(Y)\big)|_{E_Y^{uu}(p)}\big|\big]_{t=0} d\mu_Y(p) \in \mathbb{R}$ *is* C^t *with* $t := \min\{s-1, r-2\}$,

where $(p,t) \mapsto \big(\phi_t(Y)\big)(p)$ *is the flow of* $Y \in \mathcal{U}$ *and* $E_Y^{uu}(p) = T_p W_Y^{uu}(p) \subset T_p N$ *is the unstable subspace for* Y *at* p.

Moreover, if $F : \mathcal{V} \subset \mathfrak{X}^r(N) \to C^\alpha(N,\mathbb{R})$ *is* C^{s-1} *and* $F : \mathcal{V} \subset \mathfrak{X}^r(N) \to C^\alpha(N,\mathbb{R})$ *is* C^s, *then the map* $\mathcal{U} \ni Y \mapsto P\big(F(Y)\big) \in \mathbb{R}$ *is* C^s, *where* $P\big(F(Y)\big)$ *is the pressure function* $F(Y)$ *for the flow of* Y.

Sketch of the proof of Theorems 1.1 and 1.3

We will apply Theorem 1.6 to our case: let $R^r(M)$ be the Banach manifold of C^r riemannian metrics on M. Given $g \in R^r(M)$, the geodesic flow of (M, g) is generated by a C^{r-1} vectorfield $X(g)$. Fix a riemannian metric $g_o \in \mathcal{A}^r(M)$ and a small neighborhood $g_o \in \mathcal{V} \subset \mathcal{A}^r(M)$. Let $\Sigma M = S_{g_o}M$ be the g_o-unit tangent bundle. For $g \in \mathcal{V}$, using the orthogonal projection $S_g M \to \Sigma M$, conjugate the geodesic flow for g to a flow on ΣM with vectorfield $Y(g)$. Since this projection is differentiable, entropies and Lyapunov exponents for $Y(g)$ are the same as the corresponding ones for $X(g)$. We will prove (cf. Lemma 1.12) that the map $\mathcal{R}^r(M) \ni g \mapsto Y(g) \in \mathfrak{X}^{r-1}(\Sigma M)$ is C^∞. Let F_g be the function defined in Theorem 1.4. In Section 1.7 we will prove that the map $\mathcal{R}^r(M) \ni g \mapsto F_g \circ u_g \in C^\alpha(\Sigma M, \mathbb{R})$ is C^{r-2} for some $0 < \alpha < 1$ and the map $\mathcal{R}^r(M) \ni g \mapsto F_g \circ u_g \in C^0(\Sigma M, \mathbb{R})$ is C^{r-1}. Then using Theorem 1.4 and Theorem 1.6, we obtain Theorems 1.1 and 1.3. □

1.4. Conformal equivalence

Given an initial riemannian metric g_o on M, the existence of isothermal coordinates (see below) implies that we can find an oriented atlas on M in which locally we can write $g_o = f(x,y)\,(dx \otimes dx + dy \otimes dy)$, where f is a smooth scalar function. Writing $z = x + iy$ we obtain an analytic atlas. Indeed, for other isothermal charts (u, v), writing $w = u + iv$ and $g_o = h(u,v)\,(du \otimes du + dv \otimes dv)$ we have that the derivatives of the transition maps $w \circ z^{-1}$ must satisfy $\left[\frac{\partial(u,v)}{\partial(x,y)}\right]\left[\frac{\partial(u,v)}{\partial(x,y)}\right]^T = \frac{f(x,y)}{h(u,v)}\, Id$, which gives the Cauchy-Riemann equations for $\frac{dw}{dx}$.

This gives to M and $\tilde{M}$ the structure of riemann surfaces. The uniformization theorem [11] implies that $\tilde{M}$ is conformally equivalent to $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ with the euclidean metric. We identify $\mathbb{D} \approx \tilde{M}$ so that the covering map $\pi : \mathbb{D} \to (M, g_o)$ is conformal, the deck transformations are holomorphic and the lifted metric $\tilde{g_o}$ can be written as $\tilde{g_o} = \rho_o(dx \otimes dx + dy \otimes dy)$, where $\rho_o : \mathbb{D} \to \mathbb{R}$ is a positive smooth function. Denote by $z = x + iy : \tilde{M} \to \mathbb{D}$ this coordinate system.

Consider the lift $\tilde{g}$ to $\tilde{M}$ of another riemannian metric g on M. We look for coordinates $w : \tilde{M} \to \mathbb{D}$ such that $\tilde{g} = \rho\,|dz + \mu\, d\bar{z}|^2$ where $\rho : \tilde{M} \to \mathbb{R}^+$ and $\mu : \tilde{M} \to \mathbb{C}$ is a smooth function such that $|\mu(p)| < 1$ for all $p \in \tilde{M}$. Writing $\tilde{g}$ in the coordinates $z = x + iy$ as

$$\tilde{g} = \mathrm{A}\, dx \otimes dx + 2\,\mathrm{B}\, dx \otimes dy + \mathrm{D}\, dy \otimes dy\,,$$

we have

$$|dz + \mu\, d\bar{z}|^2 = \left(1 + |\mu|^2\right)\,|dz|^2 + 2\ \mathrm{Re}\left(\overline{\mu}\, dz^2\right) = \lambda\, \tilde{g}$$

with $\rho = \frac{1}{\lambda}$. Form this we get that

$$\alpha = \frac{\text{A} - \text{D}}{4}\lambda \quad , \quad \beta = \frac{\text{B}}{2}\lambda \quad , \quad 1 + |\mu|^2 = \frac{\text{A} + \text{D}}{\text{B}}\beta \,.$$

We choose the solution

$$\begin{aligned} \beta &= 2\,\text{B}\,\frac{(\text{A}+\text{D}) - 2\sqrt{\text{AD}-\text{B}^2}}{(\text{A}+\text{D})^2 - 4\left(\text{AD}-\text{B}^2\right)} = 2\,\text{B}\,\frac{p-q}{p^2-q^2} \\ \beta &= \frac{2\,\text{B}}{p+q} \quad , \quad \alpha = \frac{\text{A}-\text{D}}{p+q} \end{aligned} \tag{1.7}$$

where $p := \text{A} + \text{D} > 0$ and $q := 2\sqrt{\text{AD} - \text{B}^2} > 0$ because the matrix $A = \begin{bmatrix} \text{A} & \text{B} \\ \text{B} & \text{D} \end{bmatrix}$ is positive definite. Observe that we get that

$$1 + |\mu|^2 = \frac{2\,p}{p+q} = \frac{2}{1+\frac{q}{p}} < 2$$

because $\frac{q}{p} > 0$, and then $|\mu| < 1$.

Let $C^k(r)$ be the Banach space of C^k functions $f : \{z \in \mathbb{C} \mid |z| < r\} \to \mathbb{D}$ with the C^k norm and let $C^0(\mathbb{D}, \mathbb{D})$ be the space of continuous functions of the open disc to itself with the C^0 norm.

Lemma 1.8. *For all $0 < r < 1$ and all $k \geq 0$, the map $\mu : \mathcal{R}^k(M) \to C^k(r) \cap C^0(\mathbb{D}, \mathbb{D})$ given by $g \mapsto \mu(g) \circ z^{-1}$ is C^∞.*

Proof. From formula (1.7) it is clear that the map $\mu : \mathcal{R}^k(M) \to C^k(r)$ is C^∞. Observe that the equation $\tilde{g} = \rho\ |dz + \mu\, d\bar{z}|^2$ has exactly two solutions for μ at each point, one with $|\mu| > 1$ and one with $|\mu| < 1$. We choose the one with $|\mu| < 1$. Let h be a deck transformation and write $w = h(z)$. The map h is holomorphic, so that $h_{\bar{z}} = 0$. Since h is a g-isometry, we have that

$$\begin{aligned} \rho\ |dz + \mu\, d\bar{z}| &= \rho(w)\ |dw + \mu(w)\, d\bar{w}| \\ &= (\rho \circ h)\ |h_z|\ \left| dz + (\mu \circ h)\,\frac{\overline{h_{\bar{z}}}}{h_z}\, d\bar{z} \right| . \end{aligned}$$

Therefore,

$$\mu \circ h = \mu\,\frac{h_z}{\overline{h_{\bar{z}}}} \cdot$$

Let $\mathcal{D}$ be a fundamental domain and choose $r_0 > 0$ such that $\mathcal{D} \subset [\,|z| < r_0\,]$. Choose any point $\tilde{p} \in \mathbb{D} = \tilde{M}$, then

$$\begin{aligned} |\mu(g_1)(\tilde{p}) - \mu(g_2)(\tilde{p})| &= |\mu(g_1)(q) - \mu(g_2)(q)|\ \left|\frac{h_z(q)}{\overline{h_{\bar{z}}(q)}}\right| \\ &= |\mu(g_1)(q) - \mu(g_2)(q)|\,, \end{aligned}$$

where h is a deck transformation such that $\tilde{p} \in h(\mathcal{D})$ and $q \in \mathcal{D}$ is such that $h(q) = \tilde{p}$. Therefore,

$$\|\mu(g_1) - \mu(g_2)\|_{C^0(\mathbb{D},\mathbb{D})} \leq \|\mu(g_1) - \mu(g_2)\|_{C^0(r)} \ .$$

This implies that $\mathcal{R}^k(M) \to C^0(\mathbb{D}, \mathbb{D})$ is C^∞.

1.5. The Poisson kernel

Given a riemannian metric g on M, the Laplace–Beltrami operator can be written in local coordinates as

$$\Delta = \sum_{i,j} g^{ij} \left(\frac{\partial^2}{\partial x^i \, \partial x^j} - \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k} \right)$$

where

$$\Gamma_{ij}^k = \tfrac{1}{2} \sum_m \left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right) g^{mk}$$

are the Christoffel's symbols of g and $[g^{ij}] = [g_{ij}]^{-1}$ is the inverse matrix of the local representation of $g = \sum g_{ij}(dx^i \otimes dx^j)$. If we multiply a metric g on $\tilde{M}$ by a smooth function $\lambda : \tilde{M} \to \mathbb{R}^+$, the Laplace–Beltrami operator for λg takes the form:

$$\begin{aligned} \Delta_{\lambda g} &= \sum_{i,j} (\lambda g)^{ij} \left(\frac{\partial^2}{\partial x^i \, \partial x^j} - \sum_k \Gamma_{ij}^k(\lambda g) \frac{\partial}{\partial x^k} \right) \\ &= \frac{1}{\lambda} \Delta_g - \left(\frac{2 - \dim M}{2} \right) \frac{1}{\lambda^2} \sum_{m,k} \frac{\partial \lambda}{\partial x^m} g^{mk} \frac{\partial}{\partial x^k} \\ &= \frac{1}{\lambda} \Delta_g \, . \end{aligned}$$

so that the set of harmonic functions for g coincides with the set of harmonic functions for λg.

The Poisson kernel on $\tilde{M}$, $K : \tilde{M} \times \tilde{M} \times S(\infty) \to \mathbb{R}$ is defined as the Radon–Nikodym derivative of the harmonic measures:

$$K(x, y, \theta) := \frac{d\omega_y}{d\omega_x}(\theta) \, .$$

Fix a riemannian metric g_o on M and its lift $\tilde{g}_o$ to $\tilde{M}$. Suppose that its geodesic flow is Anosov. Fix an isothermal chart $z : (\tilde{M}, \tilde{g}_o) \to (\mathbb{D}, e)$, where e is the euclidean metric (actually its conformal type) on $\mathbb{D}$.

Lemma 1.9.

There exists a neighborhood $\mathcal{U}$ of g_o in the C^3-topology such that for all $g \in \mathcal{U}$ the chart z induces a homeomorphism $z : S_g(\infty) \to S^1 = \partial\mathbb{D} \subset \mathbb{C}$ of the sphere at infinity of g and S^1 by $z[\gamma_g] := \lim_{t \to +\infty} z \circ \gamma_g(t)$.

Moreover,

(i) The extension $z : \tilde{M} \cup S_g(\infty) \to \overline{\mathbb{D}}$ is a homeomorphism.
(ii) The map $z : S_{g_o}(\infty) \to S^1$ is Hölder continuous.

Proof.

(i) In [12] it is proved that for any metric on M whose flow is Anosov, the map $z : M \cup S_g(\infty) \to \overline{\mathbb{D}}$ is a homeomorphism. It is also proved that any two Anosov geodesic flows for M are topologically equivalent.

(ii) Let $\phi : S_g\tilde{M} \times \mathbb{R} \to S_g\tilde{M}$ be the lift of the geodesic flow for g and let ρ be the g-distance on $S_g\tilde{M}$. Let $\psi : \Sigma\tilde{M} \times \mathbb{R} \to \Sigma\tilde{M}$ be the lift of the geodesic flow for the metric g_1 with constant curvature $K \equiv -1$ and let d be the hyperbolic distance on $\Sigma\tilde{M} = S_{g_1}\tilde{M}$. Let $h : S_gM \to \Sigma M$ be a topological equivalence of the geodesic flows for g and g_1, and let $\tilde{h} : S_g\tilde{M} \to \Sigma\tilde{M}$ be its lift. Since $S_g(\infty)$ is compact, it is enough to prove that for any $w \in S_g\tilde{M}$, the map $H : \widetilde{W}^{uu}(w, \phi) \approx S_g(\infty) \xrightarrow{z} S^1 \approx \widetilde{W}^{uu}(\tilde{h}(w), \psi)$ is Hölder continuous on a neighborhood of w. We use local strong unstable manifolds:

$$\widetilde{W}^{uu}_\beta(p, \psi) := \left\{ q \in \Sigma\tilde{M} \,\middle|\, d(p, q) < \beta \text{ and } \lim_{t\to-\infty} d\left(\psi_t(p), \psi_t(q)\right) = 0 \right\}.$$

We have that $H = P\circ\tilde{h}$ where $P : D_\theta \subseteq \Sigma\tilde{M} \to \widetilde{W}^{uu}_\beta(\tilde{h}(w), \psi)$ is the projection along the flow lines of ψ, $D_\theta = \{ z \in \widetilde{W}^u(\tilde{h}(w), \psi) \,|\, d(z, \tilde{h}(w)) < \theta \}$, and $\widetilde{W}^u(\tilde{h}(w), \psi) = \cup_{t\in\mathbb{R}} \psi_t(\widetilde{W}^{uu}(\tilde{h}(w), \psi))$ is the weak unstable manifold of $\tilde{h}(w)$. Fix $\epsilon > 0$ small and such that if $p, q \in \widetilde{W}^u(\tilde{h}(w), \psi)$, $d(p, \tilde{h}(w)) < \epsilon$ and $d(q, \tilde{h}(w)) < \epsilon$, then there exists exactly one point in the intersection

$$\{w\} = \widetilde{W}^{uu}_{4\epsilon}(q, \psi) \cap \{ \psi_t(p) \,|\, -4\epsilon \le t \le 4\epsilon \} \ne \emptyset .$$

Let $\theta := 12\,\epsilon$. For the hyperbolic metric g_1 we know that P is C^1. Let $B > 0$ be such that $d(P(x), P(y)) < B\,\rho(x, y)$ for all $x, y \in D_\theta$. We need the following:

Claim.

(a) *There exists $0 < a < A$ such that if $x \in S_g = \tilde{M}$ and $s(x) > 0$ is such that $\tilde{h}(\phi(x, 1)) = \psi(\tilde{h}(x), s(x))$, then $2a < s(x) < \frac{A}{2}$.*

(b) *There exists $0 < a < A$ such that if $x \in S_g\tilde{M}$, $T > 2$ and $s(x, T) > 0$ is such that $\tilde{h}(\phi(x, T)) = \psi(\tilde{h}(x), s(x, T))$, then $a\,T < s(x, T) < A\,T$.*

□

Proof. For (a) use the continuity of h and the compactness of S_gM. For (b), suppose that $T = n+\delta$, with $n \in \mathbb{Z}^+$ and $\delta \in [0, 1[$. From item (a) we get that $2a\,\frac{t}{2} \le 2a\,n \le s(x, T) \le \frac{A}{2}\,n + \frac{A}{2} \le \frac{A}{2}\,2T$.
□

Let $\lambda > 0$ be such that $\rho(\phi_t(x), \phi_t(y)) \le e^{\lambda t}\,\rho(x, y)$ for all $t \ge 0$ and all $x, y \in S_g\tilde{M}$. Let $\eta > 0$ be such that $\rho(x, y) < 2\eta$ implies $d(\tilde{h}(x), \tilde{h}(y)) < \epsilon$. If $x, y \in \widetilde{W}^{uu}_\delta(w, \phi)$ with $\delta < e^{-3\lambda}\eta$, let $T := \min\{ s > 0 \,|\, \rho(\phi_s(x), \phi_s(y)) = \eta \}$. This number exists by the expansivity of ϕ if η is small enough (in fact for any η using the conjugacy $\tilde{h}$). We have that $T > 3$ and $\rho(x, y) \ge \eta\, e^{-\lambda T}$.

There exist continuous functions $\sigma, \tau : \mathbb{R} \to \mathbb{R}$ such that $\sigma(0) = 0 = \tau(0)$ and $\tilde{h}(\phi(x, s)) = \psi(\tilde{h}(x), \sigma(x))$, $\tilde{h}(\phi(y), t) = \psi(\tilde{h}(y), \tau(t))$. By the claim, $aT \le \tau(T) \le AT$ and $aT \le \sigma(T) \le AT$. Since $x, y \in W^{uu}_\delta(w, \phi)$, then $\tilde{h}(x)$ and $\tilde{h}(y)$ are in the weak unstable manifold $W^u(\tilde{h}(w), \psi)$ of $\tilde{h}(w)$. Write $p := \tilde{h}(x)$, $q := \tilde{h}(y)$ and let

$$m := W^{uu}_{4\epsilon}(\tilde{h}(y)) \cap \left\{ \psi_t\tilde{h}(x) \,\middle|\, -4\epsilon \le t \le 4\epsilon \right\}.$$

Then

$$\begin{aligned} d\left(\psi_{\tau(T)}(q), \psi_{\tau(T)}(m)\right) &\ge e^{\tau(T)}\, d(q, m) \ge e^{aT}\, d(q, m) \\ d\left(\psi_{s(T)}(p), \psi_{\tau(T)}(q)\right) &= d\left(\tilde{h}\phi_T(x), \tilde{h}\phi_T(y)\right) \le \epsilon \end{aligned}$$

$$\begin{aligned} e^{aT} d(q, m) &\leq d\left(\psi_{\sigma(T)}(p), \psi_{\tau(T)}(q)\right) + d\left(\psi_{\sigma(T)}(p), \psi_{\tau(T)}(m)\right) \\ &\leq \epsilon + |\sigma(T) - \tau(T)| + d(p, m) \\ &\leq \epsilon + 2AT + 4\epsilon \\ d(q, m) &\leq (5\epsilon + 2AT)\, e^{-aT} \leq e^{-\frac{a}{2}T} \end{aligned}$$

if $T > T_0 := T_0(\epsilon, A, a) > 0$. If we choose $0 < \delta < e^{-\lambda T_0 \eta}$, then $x, y \in \widetilde{W}^{uu}_\delta(w, \phi)$ implies that $T > T_0$. In particular

$$d(q, m) \leq e^{-\frac{a}{2}T} \leq \left(\eta\, e^{-\lambda T}\right)^{\frac{a}{2\lambda}} \eta^{-\frac{a}{2\lambda}} \leq \eta^{-\alpha} \rho(x, y)^\alpha$$

for $\alpha = \frac{a}{2\lambda}$. We have that $d(\tilde{h}(w), m) \leq d(\tilde{h}(w), p) + d(p, m) \leq \epsilon + 4\epsilon < 12\epsilon = \theta$. Then

$$\begin{aligned} d\left(H(x), H(y)\right) &= d\left(P(p), P(q)\right) = d\left(P(m), P(q)\right) \\ &\leq B\, d(q, m) \leq B\, \eta^{-\alpha} \rho(x, y)^\alpha . \end{aligned}$$

This proves Lemma 1.9. □

Let $g \in \mathcal{U}$ be another metric on M and $\tilde{g}$ its lift to $\tilde{M}$. Suppose that $f : (\overline{\mathbb{D}}, S^1) \to (\overline{\mathbb{D}}, S^1)$ is a homeomorphism which is differentiable on $\mathbb{D}$ and satisfies

$$f_{\bar{z}} = \mu(g)\, f_z ,$$

where $\mu(g)$ is from Section 1.4. Then for $w = f \circ z$, the metric $\tilde{g}$ is written as $\tilde{g} = \lambda\, |dw|^2$. By the remark above, the $\tilde{g}$-harmonic functions on $\mathbb{D}$ in the coordinates w are the harmonic functions for the euclidean Laplacian on $\mathbb{D}$.

From now on we identify $\tilde{M} \approx \mathbb{D}$ and $S_g(\infty) \approx S^1$ for any metric g on $\mathcal{U}$, using z.

Lemma 1.10. *The Poisson kernel for $\tilde{g}$ on $\mathbb{D} \cup S^1 \approx \tilde{M} \cup S_g(\infty)$ is given by*

$$k(x, y, \theta) = \mathbb{P}\left(f(x), f(y), f(\theta)\right)$$

for any $x, y \in \mathbb{D}$, $\theta \in S^1$, where $\mathbb{P}$ is the Poisson kernel for the euclidean Laplacian

$$\mathbb{P}(z, w, \theta) = \mathrm{Re}\left(\frac{e^{i\theta} + w}{e^{i\theta} - w} \cdot \frac{e^{i\theta} - z}{e^{i\theta} + z}\right) .$$

Proof. For $z \in \mathbb{D}$, let ω_z be the $\tilde{g}$-harmonic measure at z and λ_z be the euclidean harmonic measure at z. Let $\varphi : S^1 \to \mathbb{R}$ be a continuous function. By Lemma 1.9, it corresponds to a continuous function $S_g(\infty) \to \mathbb{R}$. Let $\varphi(z)$ be its $\tilde{g}$-harmonic extension to $\overline{\mathbb{D}}$, $\Delta_{\tilde{g}}(\varphi) = 0$. Let $\phi(w) = \varphi(f^{-1}(w))$ be the function φ, written in the coordinates $w = f(z)$. Let Δ be the euclidean Laplacian on $\mathbb{D}$. Then $\Delta\phi = 0$ and hence

$$\begin{aligned} \int \varphi\, d\omega_y &= \varphi(y) = \phi(f(y)) = \int_{S^1} \phi(\theta)\, d\lambda_{f(y)}(\theta) \\ &= \int_{S^1} \varphi \circ f^{-1}\, d\lambda_{f(y)} \\ &= \int_{S^1} \varphi \circ f^{-1}(\theta)\;\; \mathbb{P}(f(x), f(y), \theta)\;\; d\lambda_{f(x)}(\theta) \\ &= \int_{S^1} \left(\varphi \circ f^{-1}\right)\left(k \circ f^{-1}\right)\; d\lambda_{f(x)} = \int_{S^1} \varphi \cdot k\; d\omega_x \end{aligned}$$

where $k(\theta) := \mathbb{P}\big(f(x), f(y), f(\theta)\big)$. Therefore,

$$k(x, y, \theta) := \frac{d\omega_y}{d\omega_x}(\theta) = \mathbb{P}\left(f(x), f(y), f(\theta)\right) . \qquad \square$$

1.6. Stability of the geodesic flow

Fix a C^r riemannian metric $g_o \in \mathcal{A}^r(M) \subset \mathcal{R}^r(M)$ such that the geodesic flow of g_o is Anosov. Let ΣM be the g_o-unit tangent bundle $\Sigma M := \{\, v \in TM \mid g_o(v, v) = 1 \,\}$. Given another riemannian metric $g \in \mathcal{R}^r(M)$ and its unit tangent bundle $S_g M = \{\, v \in TM \mid g(v, v) = 1 \,\}$, define the map $F : S_g M \to \Sigma M$ by $F(v) = v\,\big(g_o(v, v)\big)^{-\frac{1}{2}}$. Let ψ_t be the geodesic flow for g and define $\varphi_t := F \circ \psi_t \circ F^{-1}$. Then F is a C^r conjugacy between ψ_t and ϕ_t.

Given a chart $x : U \subseteq M \to \mathbb{R}^2$, consider the chart $(\overline{x}, \overline{y}) = (x, dx) : TU \to \mathbb{R}^2 \times \mathbb{R}^2$, with $\overline{y}(v) = (y_1, y_2)$ if $v = \sum y_i \frac{\partial}{\partial x^i}$. In this chart, the geodesic flow for g satisfies

$$\frac{dx^k}{dt} = y_k \quad , \quad \frac{dy_k}{dt} = -\sum_{ij} \Gamma^k_{ij}\, y_i\, y_j \quad , \quad k = 1, 2 ;$$

where

$$\Gamma^k_{ij} = \tfrac{1}{2} \sum_{\ell} \left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial_{\ell i}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right) g^{\ell k} ,$$

are the Christoffel symbols for $g = \sum g_{ij}(dx^i \otimes dx^j)$ and $[g^{k\ell}] = [g_{ij}]^{-1}$.

Let $\lambda(s) = \big(p(s), \vec{v}(s)\big) \in T_{p(s)}M$ be an orbit of ψ_s. Then $F(\lambda(s)) = \big(p(s), \vec{v}(s)(g_o)^{-\frac{1}{2}}\big)$, $g_o := g_o\big(\vec{v}(s), \vec{v}(s)\big)$, and

$$\begin{aligned}
\frac{d(F \circ \lambda)}{ds} &= \left(\frac{dp}{ds} , \frac{1}{\sqrt{g_o}} \frac{d\vec{v}}{ds} - \frac{\vec{v}}{2\,(g_o)^{\frac{3}{2}}} \frac{d}{ds} g_o \right) \\
&= \left(\vec{v} \, , -\frac{1}{\sqrt{g_o}} \sum_{ij} \Gamma^k_{ij} v^i v^j \frac{\partial}{\partial x^k} - \frac{\vec{v}}{2\,(g_o)^{\frac{3}{2}}} \frac{d}{ds} g_o \right) \\
\frac{d}{ds} g_o\left(\vec{v}(s), \vec{v}(s)\right) &= \frac{d}{ds} \sum_{ij} g^o_{ij}(p(s))\, v^i(s)\, v^j(s) \\
&= \sum_{ijk} \frac{\partial g^o_{ij}}{\partial x^k} v^k v^i v^j - 2 \sum_{ijk\ell} g^o_{ij} \Gamma^i_{k\ell} v^k v^\ell v^j .
\end{aligned}$$

If $\vec{w}(s) = F\big(\vec{v}(s)\big)$, then $\vec{v}(s) = \frac{\vec{w}}{\sqrt{g(w,w)}}$ and $\sqrt{g_o(\vec{v}, \vec{v})} = \frac{\sqrt{g_o(w,w)}}{\sqrt{g(w,w)}}$. We have

$$\frac{dw}{ds} = \left(\frac{w}{\sqrt{g}} , -\frac{1}{\sqrt{g}\sqrt{g_o}} \sum_{ij} \Gamma^k_{ij} w^i w^j \frac{\partial}{\partial x^k} - \frac{\vec{w}}{2\sqrt{g}\,(g_o)^{\frac{3}{2}}} \sum_{ijk} \left(\frac{\partial g^o_{ij}}{\partial x_k} - 2 \sum_{\ell} g^o_{\ell k} \Gamma^k_{ij} \right) w^i w^j w^k \right)$$

This is the vectorfield of ψ_s, denote it by $X(g)$. Let $\mathfrak{X}^{r-1}(\Sigma M)$ be the Banach space of C^{r-1} vectorfields on ΣM with the C^r norm. The formula above proves the following lemma.

Lemma 1.12. *The map $\mathcal{R}^r(M) \to \mathfrak{X}^{r-1}(\Sigma M)$ is C^∞.*

We will need the following version of the structural stability theorem:

Proposition 1.13 [9].

Let $X \in \mathfrak{X}^{r-1}(\Sigma M)$ be an Anosov flow, then there exists a neighborhood $\mathcal{V} \subset \mathfrak{X}^{r-1}(\Sigma M)$, $0 < \beta < 1$ and C^{r-2} maps $\mathcal{V} \to C^{\beta}_{\phi}(\Sigma M, \Sigma M) : Y \mapsto u_Y$ and $\mathcal{V} \to C^{\beta}_{\phi}(\Sigma M, [\frac{1}{2}, +\infty[) : Y \mapsto \gamma_Y$ such that $Y \circ u_Y = \gamma_Y\ D_{\phi} u_Y$.

Moreover, the corresponding maps $Y \mapsto u_Y$ and $Y \mapsto \gamma_Y$ for $\beta = 0$ are C^{r-1}.

Where $C^{\beta}_{\phi}(\Sigma M, \Sigma M)$ is the space of β-Hölder continuous functions $u : \Sigma M \to \Sigma M$ such that $\frac{d}{dt} u(\phi_t(p))|_{t=0}$ exists and it is β-Hölder continuous endowed with the norm $[\![u]\!]_{\beta} = \|u\|_{\beta} + \left\|\frac{d}{dt}(u \circ \phi_t)\right\|_{\beta}$ where $\|\ \ \|_{\beta}$ is the β-Hölder norm for a fixed C^r riemannian metric and $C^0_{\phi}(\Sigma M, \Sigma M)$ is the space of continuous functions $u : \Sigma M \to \Sigma M$ such that $\frac{d}{dt}(u \circ \phi_t)$ exists, with the norm $[\![u]\!]_0 = \|u\|_{\sup} + \left\|\frac{d}{dt}(u \circ \phi_t)\right\|_{\sup}$.

Corollary 1.14. *For $Y \in \mathcal{V}$ consider the map $\sigma_Y : M \to \mathbb{R}^+$ defined by $\psi_Y(u_Y(p), 1) = u_Y \circ \phi(p, \sigma_Y(p))$, where ψ_Y is the flow of Y. Then*

(i) The map $\mathcal{U} \to C^{\beta}(\Sigma M, \mathbb{R}^+) : Y \mapsto \sigma_Y$ is C^{r-2}.
(ii) In particular the maps $\mathcal{U} \to C^{\beta}(\Sigma M, \Sigma M) : Y \mapsto \psi_Y(u_Y(p), 1)$ is C^{r-2}.
(iii) The corresponding maps for $\beta = 0$ are C^{r-1}.

Proof. From the equation $\psi(u(p), t) = u(\phi(p, s(t)))$we get that $\frac{ds}{dt} = \gamma(\phi_s(p))$. Consider the map $F : \mathcal{U} \times C^{\beta}(\Sigma M, \mathbb{R}^+) \to C^{\beta}(\Sigma M, \mathbb{R}^+)$ given by $F(Y, \sigma)(p) = \int_0^{\sigma} \frac{1}{\gamma_Y(\phi_s(p))}\, ds$. Then the function σ_Y is characterized by $F(Y, \sigma_Y) \equiv 1$. Observe that $\left(\frac{\partial F}{\partial \sigma} \cdot \tau\right)(p) = \left(\gamma_Y(\phi_s(p))\right)^{-1} \tau(p)$ is invertible because $\gamma_Y(\phi_s(p)) > 0$. Since F is C^r, the implicit function theorem implies that $Y \mapsto \sigma_Y \in C^{\beta}(\Sigma M, \mathbb{R}^+)$ is C^{r-2}. The case $\beta = 0$ is similar.

For (ii) use the fact that $\mathfrak{X}^r(\Sigma M) \to C^{\gamma}(\Sigma M \times \mathbb{R}, \Sigma M) : Y \mapsto \psi_Y$ is C^{r-2} for $0 < \gamma < 1$ and $\mathfrak{X}^r(\Sigma M) \to C^0(\Sigma M \times \mathbb{R}, \Sigma M) : Y \mapsto \psi_Y$ is C^{r-1}. □

1.7. Proof of Theorems 1.1 and 1.3

We will need the following generalization of a theorem by Ahlfors and Bers which will be proved in Section 2.

Theorem 1.15. *Given $\mu : \mathbb{D} \to \mathbb{D}$ measurable with $\|\mu\|_{\infty} < k < 1$ there exists a unique homeomorphism of the closed disk $f^{\mu} = f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ satisfying $f_{\bar{z}} = \mu\ f_z$, with generalized derivatives $f_z, f_{\bar{z}}$, such that $f(0) = 0$, $f(1) = 1$, $f(S^1) = S^1$. Moreover*

(i) The map f is Hölder continuous on $\overline{\mathbb{D}}$ and if $0 < r < 1$ and $\mu \in C^n(|z| < r, \mathbb{D})$ then $f \in C^{n+\alpha}(|z| < r, \mathbb{D})$ for some $0 < \alpha = \alpha(k) < 1$.
(ii) For any $n \geq 1$ and any $0 < r < R < 1$, the map $\mathcal{L}_{\infty}(\mathbb{D}) \cap C^n(|z| < R, \mathbb{C}) \cap \{\|\mu\| < k\} \to C^{n+\alpha}(r) \cap C^{\alpha}(\mathbb{D}, \mathbb{D})$ given by $\mu \mapsto f^{\mu}$ is C^{∞}.

We now prove Theorems 1.1 and 1.3. Let g be a riemannian metric in a small C^r neighborhood of g_o. Then the map $F : S_g M \to \Sigma M$ of Section 1.6 is a C^r conjugacy between the geodesic flow of g and the flow of $X(g)$. In particular, F maps strong stable and strong unstable manifolds of the geodesic flows to strong stable and strong unstable manifolds of $X(g)$.

Let $\psi(g)$ be the geodesic flow of g and $\varphi(g) := F \circ \psi(g) \circ F^{-1}$. Let $\pi : TM \to M$ be the projection. Let $P_g : S_gM \to \mathbb{R}$ be $P_g(X) = \log K_g\big(\pi\tilde{X}, \pi\psi(g)(\tilde{X}, 1), \tau\tilde{M}\big)$ where $\tilde{X}$ is a lift of X under $p : T\tilde{M} \to TM$. Let μ_g be the equilibrium state of P_g for $\psi(g)$. Consider the measure $\nu_g := F^*(\mu_g)$, $\nu_g(A) = \mu_g(F^{-1}(A))$. We have for the metric entropies that $h_{\nu_g}(\varphi(g)) = h_{\mu_g}(\psi(g))$. Since the conjugacy F is differentiable, we have that the Lyapunov exponents of ν_g and μ_g coincide $\lambda^+(\nu_g) = \lambda^+(\mu_g)$.

In particular, the Hausdorff dimension of the conditional measures on local strong manifolds are equal:

$$HD^u\left(\nu_g\right) = HD^u\left(\mu_g\right) = \frac{h_{\mu_g}\left(\psi_g\right)}{\lambda^+\left(\mu_g\right)} = \frac{h_{\nu_g}\left(\varphi_g\right)}{\lambda^+(\nu_g)} \,.$$

For any $\varphi(g)$-invariant measure ν, we have that

$$h_\nu\left(\varphi_g\right) + \int_{\Sigma M} p_g \circ F^{-1}\, d\nu = h_\mu\left(\psi_g\right) + \int_{S_gM} P_g\, d\mu\,,$$

where $\nu = F^*(\mu)$, i.e., $\mu\big(F^{-1}(A)\big) := \nu(A)$. Therefore, the maximum of these numbers is attained at $\nu = \nu_g = F^*(\mu_g)$. Hence, ν_g is the equilibrium state of $G_g = P_g \circ F^{-1}$ for φ_g. We have that

$$\begin{aligned}
G_g(X) &= P_g \circ F^{-1}(X) = P_g\left(\frac{X}{\|X\|_g}\right) \\
&= \log K_g\left(\pi\left(\frac{\tilde{X}}{\|X\|_g}\right), \pi\,\tilde{\psi}_h\left(\frac{\tilde{X}}{\|X\|_g}, 1\right),\ \tau_g\tilde{X}\right) \\
&= \log K_g\left(\pi\left(\tilde{X}\right), \pi\tilde{\varphi}_g\left(\tilde{X}, 1\right), \tau_g\tilde{X}\right) \\
&= \log \mathbb{P}\left(f_g\left(\pi\tilde{X}\right), f_g\left(\pi\tilde{\varphi}_g\left(\tilde{X}, 1\right)\right), f_g\tau_g\tilde{X}\right),
\end{aligned}$$

where $\mathbb{P}$ is the euclidean Poisson kernel on $\mathbb{D}$ and we consider $\pi : T\tilde{M} \to \tilde{M} \approx \mathbb{D}$. In order to apply Proposition 1.13 we need to see that $g \mapsto G_g \circ u_{X(g)} \in C^\beta(\Sigma M, \mathbb{R})$ is C^{r-2} for some $\beta > 0$, where $u_{X(g)}$ is the topological equivalence of Proposition 1.13.

Fix a fundamental domain of $p : \mathbb{D} \approx \tilde{M} \to M$ and its corresponding lift $q : M \to \mathbb{D}$. Since the C^r or C^α, $0 < \alpha < 1$ norms of maps are equivalent to sums of C^r or C^α norms of local restrictions of the maps, we do not bother with the discontinuities of this lift q. We have that

$$G_g \circ u_{X(g)}(V) = \log \mathbb{P}\left(f_g\, \pi\, q\, u_g(V),\ f_g\, \pi\, \tilde{\varphi}_g\left(q\, u_g(V), 1\right), f_g\, \tau_g\, q\, u_g(V)\right)\,. \tag{1.16}$$

By the structural stability theorem, the $\tilde{g}_o$-geodesic of $q(V)$ and the $\tilde{g}$-geodesic of $q\big(u_g(V)\big)$ remain at bounded distance of each other. In particular, their limit on $\overline{\mathbb{D}}$ as $t \to +\infty$ is the same:

$$\Theta(V) := \tau_g\, q\, u_g(V) = \tau_{g_o}\, q(V) \quad \text{for all } g \text{ near } g_o\,.$$

By Lemma 1.9 (ii), the map $\Theta : \Sigma M \to S^1$ is Hölder continuous. By Theorem 1.15, for some $0 < \alpha < 1$, the map $g \mapsto f_g \in C^\alpha(S^1, S^1)$ is C^∞. Therefore, for some $0 < \beta < 1$ the map $g \mapsto f_g \circ \Theta \in C^\beta(S^1, S^1)$ is C^∞.

By Proposition 1.13 and Lemma 1.12, for some $0 < \gamma < 1$, the map $g \mapsto u_g \in C^\gamma(\Sigma M, \Sigma M)$ is C^{r-2}. The maps $\pi : T\tilde{M} \to T\tilde{M}$ and $q : M \to \mathbb{D}$ are C^r and by Theorem 1.15 and Lemma 1.8, the map $g \mapsto f_g \in C^{r-1+\alpha}(|z| < R, \mathbb{D})$ is C^∞ for some $0 < R < 1$ such that

$$\left\{ w \mid \tilde{d}_g\,(w, q(M)) \le 4,\ \text{for some } g \in \mathcal{V} \right\} \subseteq [|z| < R]\ ,$$

where $\mathcal{V}$ is a neighborhood of g_o and $\tilde{d}_g$ is the $\tilde{g}$-distance in $\tilde{M} \approx \mathbb{D}$. Therefore, the map $g \mapsto f_g \circ \pi \circ q \circ u_g \in C^\delta(\Sigma M, \mathbb{D})$ is C^{r-2} for some $0 < \delta < 1$. Observe that we used here the derivatives of f_g. For $\delta = 0$, this map is C^{r-1}.

By Corollary 1.14, the map $g \mapsto \tilde{\psi}_g\big(q\, u_g(\cdot), 1\big) = q \circ \varphi_g\big(u_g(\cdot), 1\big) \in C^\beta(\Sigma M, \mathbb{D})$ is C^{r-2} for some $0 < \beta < 1$ and it is C^{r-1} for $\beta = 0$. Since $g \mapsto f_g \in C^{r-1+\alpha}\big(|z| < R, \mathbb{D}\big)$ is C^∞, we have that the second component of (1.16): $g \mapsto f_g \circ \pi \circ q \circ \varphi_g\big(u_g(\cdot), 1\big) \in C^\delta\big(\Sigma M, \mathbb{D}\big)$ is C^{r-2} for some $0 < \delta < 1$ and it is C^{r-1} for $\delta = 0$.

Since $\mathbb{P}$ is C^∞, from Equation (1.16) we get that the map $\mathcal{A}^r(M) \supseteq \mathcal{U} \ni g \mapsto G_g \circ u_{X(g)} \in C^\alpha(\Sigma M, \mathbb{R})$ is C^{r-2} for some $0 < \alpha < 1$ and it is C^{r-1} for $\alpha = 0$. Applying Theorem 1.6, we have that $g \mapsto h(\nu_g) = h(\mu_g)$ is C^{r-2}, $g \mapsto \lambda^+(\nu_g) = \lambda^+(\mu_g)$ is C^{r-3} and also that $g \mapsto P(\varphi_g)$ is C^{r-1}, where $P(\varphi_g)$ is the pressure of G_g for ψ_g. □

2. Regularity of quasiconformal mappings

Our aim here is to prove (cf. Theorem 1.15) that if $f^\mu : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a μ-quasiconformal map normalized by $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$, then the map $\mu \mapsto f^\mu$ is C^∞; where μ varies in the space of C^k maps and f^μ in the space of $C^{k+\alpha}$ maps. We obtain similar results for solutions of non-homogeneous Beltrami equations (cf. Corollary 2.38).

Bers [5] proved that f^μ is $C^{k+1+\alpha}$ if μ is $C^{k+\alpha}$. Ahlfors and Bers [2] proved that the map $\mu \mapsto f^\mu$ is C^1 when μ is in $\mathcal{L}_\infty$ and f^μ is Hölder continuous. In order to get the second derivative, we are forced to deal with derivatives of non-homogeneous Beltrami equations.

The proof that the map $\mu \mapsto f^\mu$ is C^∞ relies in the fact that its derivative satisfies a non-homogeneous Beltrami equation and that the derivatives of such equations can be expressed again in terms of non-homogeneous Beltrami equations. In fact, if $f^\mu_{\bar{z}} = \mu\, f^\mu_z$, the derivative $\frac{d}{d\mu} f^\mu \cdot h = \omega$ satisfies [2] $\omega_{\bar{z}} = \mu\, \omega_z + h\, f_z$. Consider the map $F(\mu, \sigma) = \omega^{\mu,\sigma}$, where $\omega = \omega^{\mu,\sigma}$ satisfies $\omega_{\bar{z}} = \mu\, \omega_z + \sigma$. Since F is linear on σ we will have that

$$\frac{\partial F}{\partial \sigma} \cdot h = F(\mu, h)\ .$$

A formal computation shows that the derivative $\lambda = \frac{\partial F}{\partial \mu} \cdot h$ should satisfy $\lambda_{\bar{z}} = \mu\, \lambda_z + h\, \omega_z$. So that

$$\frac{\partial F}{\partial \mu} \cdot h = F(\mu, h \cdot F(\mu, \sigma))\ .$$

We will prove that F is C^1. Then a recursive argument will give that F is C^∞ and then $\mu \mapsto f^\mu$ is C^∞.

2.1. Preliminaries

Given a C^1 function $f(x, y)$ defined on a region $\Omega \subseteq \mathbb{R}^2$ with values on $\mathbb{C}$, define the derivatives

$$f_z := \tfrac{1}{2}\left(f_x - i\, f_y\right) \quad , \quad f_{\bar{z}} := \tfrac{1}{2}\left(f_x + i\, f_y\right)\ . \tag{2.1}$$

If $f : \Omega \to \mathbb{C}$ is locally integrable, then we say that f_z and $f_{\bar z}$ are the generalized derivatives of f if they are locally integrable and satisfy

$$\begin{aligned} \iint_\Omega f_z \varphi \, dx\, dy &= -\iint_\Omega f \varphi_z \, dx\, dy \\ \iint_\Omega f_{\bar z} \varphi \, dx\, dy &= -\iint_\Omega f \varphi_{\bar z} \, dx\, dy \end{aligned} \tag{2.2}$$

for all $\varphi \in C^1$ with compact support in Ω. The following lemma is well known:

Lemma 2.3. *If $f_{\bar z} \equiv 0$, then f is holomorphic.*

More precisely, there exists a holomorphic function which is almost everywhere equal to f. Define the following operators

$$\begin{aligned} (Ph)(w) &= -\frac{1}{\pi}\iint_{\mathbb{C}} h(z)\left[\frac{1}{z-w} - \frac{1}{z}\right] dx\, dy \quad , \quad z = x + iy\,, \\ (Hh)(w) &= -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{h(z) - h(w)}{(z-w)^2}\, dx\, dy \quad , \quad z = x + iy\,, \\ &= -\frac{1}{\pi}\lim_{\epsilon\to 0}\iint_{|z|>\epsilon} \frac{h(z)-h(w)}{(z-w)^2}\, dx\, dy\,. \end{aligned}$$

Lemma 2.4. *Suppose that $g \in \mathcal{L}_p(\mathbb{C})$, $p > 2$. Then Pg exists everywhere as an absolutely convergent integral and Hg exists almost everywhere as a Cauchy principal limit. The following relations hold:*

$$(Pg)_{\bar z} = g \quad , \quad (Pg)_z = Hg\,. \tag{2.4.1}$$

$$|Pg(z_1) - Pg(z_2)| \le K_p \, \|g\|_p \, |z_1 - z_2|^{1-\frac{2}{p}}\,. \tag{2.4.2}$$

$$\|Hg\|_g \le C_p \, \|g\|_p\,. \tag{2.4.3}$$

Actually, (2.4.3) holds for $p > 1$ and for $p = 2$ and it can be replaced by

$$\|Hg\|_p = \|g\|_2\,. \tag{2.4.4}$$

$$\lim_{p\to 2} C_p = 1\,. \tag{2.4.5}$$

Write $\partial g = g_z$, $\bar\partial g = g_{\bar z}$; *then the operators* ∂, $\bar\partial$, *and* H *commute*. (2.4.6)

The relation (2.4.3) is a deep result called Calderon–Zygmund's inequality. The proof of this lemma can be found in [1].

We now see the behavior of the operator H on small discs. For $0 < R < 1$, define the operator

$$\begin{aligned} H_R h(w) &= -\frac{1}{\pi}\iint_{|z|<R} \frac{h(z)-h(w)}{(z-w)^2}\, dx\, dy \quad , \quad z = x + iy \\ &= -\frac{1}{\pi}\lim_{\epsilon\to 0}\iint_{\epsilon<|x|<R} \frac{h(z)-h(w)}{(z-w)^2}\, dx\, dy\,. \\ P_R h(w) &= -\frac{1}{\pi}\iint_{|z|<R} h(z)\left[\frac{1}{z-w} - \frac{1}{z}\right] dx\, dy\,. \end{aligned}$$

Define the norm

$$\|h\|_{R,p} = \left(\iint_{|z|<R} |h(z)|^p \; dx\, dy\right)^{\frac{1}{p}} .$$

Let $C^\alpha(D_R, \mathbb{C}) = C^\alpha(R)$ be a Banach space of α-Hölder continuous functions on the disc $D_R := \{ z \in \mathbb{C} \mid |z| < R \}$, provided with the norm

$$\begin{aligned} [[h]]_{R,p} &:= \|h\|_{R,\infty} + [h]_{R,\alpha} \;, \\ \|h\|_{R,\infty} &:= \sup_{|z|<R} |h(z)| \;, \\ [h]_{R,\alpha} &:= \sup_{|z-w|<1} \frac{|h(z)-h(w)|}{|z-w|^\alpha} \;. \end{aligned}$$

Observe that

$$[[g \cdot h]]_{R,\alpha} \leq 2\, [[g]]_{R,\alpha}\, [[h]]_{R,\alpha} \,.$$

For $R > 0$, $p > 2$, $n \geq 0$ define

$$\begin{aligned} W^{n,p}(R,0) &:= \Big\{ h : \mathbb{C} \to \mathbb{C} \mid h \in C^{n-1}(\mathbb{C},\mathbb{C}),\ \left\|D^n h\right\| \in \mathcal{L}_p\,(D_R) \text{ and} \\ h(z) &= 0 \text{ for } |z| > R \Big\} \;, \\ W^{n,p}(R) &:= \Big\{ h : D_R \to \mathbb{C} \mid h \in C^{n-1}\,(D_R, \mathbb{C})\,,\ \left\|D^n h\right\| \in \mathcal{L}_p\,(D_R) \Big\} \,. \\ \|h\|_{W^{n,p}(R)} &:= \|h\|_{C^{n-1}(R)} + \left\|D^n h\right\|_{R,p} \;, \\ \|h\|_{C^{n-1}(R)} &:= \textstyle\sum_{k=0}^{n-1} \left\|D^k h\right\|_{R,\infty} \;, \\ \left\|D^k h\right\|_{R,\infty} &:= \sum_{i+j=k} \left\|\partial^i \bar{\partial}^j h\right\|_{R,\infty} \,. \end{aligned}$$

On both $W^{n,p}(R,0)$ and $W^{n,p}(R)$ consider the norm $\| \; \|_{W^{n,p}(R)}$. Observe that for $0 < R < 1$, we have

$$[[h]]_{R,\alpha} \leq \|h\|_{R,\infty} + \|Dh\|_{R,\infty} = \|h\|_{R,\infty} + \|\partial h\|_{R,\infty} + \left\|\bar{\partial} h\right\|_{R,\infty} \,.$$

Lemma 2.5.

(a) *For all* $0 < \alpha < 1$ *there exists* $C(\alpha) > 0$ *such that*

$$[[H_R h]]_{R,\alpha} \leq C(\alpha)\, [[h]]_{R,\alpha}\, R^\alpha \quad \textit{for all} \;\; 0 < R < 1 \,.$$

Moreover, if $h \in C^\alpha(D_R, \mathbb{C})$, *then* $P_R h$ *is* $C^{1+\alpha}$ *and*

$$(P_R\, h)_{\bar{z}} = h \quad , \quad (P_R\, h)_z = Hh \;\; \textit{on} \;\; 0 < R < 1 \,.$$

(b) *For all* $0 < \alpha < 1$ *there exists* $D(\alpha) > 0$ *such that*

$$[[P_R\, h]]_{R,\alpha} \leq D(\alpha)\, [[h]]_{R,\alpha}\, R^{1-\alpha} \,.$$

(c) *For all* $p > 2$ *there exists* $A(p) > 1$ *such that for all* $0 < R < 1$ *and* $h \in W^{n,p}(R,0)$,

$$\begin{aligned} \|H_R\, h\|_{C^{n-1}(R)} &\leq A(p)\, R^{1-\frac{2}{p}}\, \|h\|_{W^{n,p}(R)} \;, \\ \left\|D^n H_R h\right\|_{R,p} &\leq C_p \left\|D^n h\right\|_{R,p} \,. \end{aligned}$$

In particular, the operator $H_R : W^{n,p}(R,0) \to W^{n,p}(R)$ *is continuous and has norm* $\|H_R\|_{W^{n,p}(R)} \leq A(p)$.

(d) *For all $p > 2$ there exists $B(p, R) > 0$ such that for $h \in W^{n,p}(R, 0)$*

$$\|P_R\, h\|_{W^{n+1,p}(R)} \le B(p, R)\ \|h\|_{W^{n,p}(R)}\ .$$

in particular, the operator $P_R : W^{n,p}(R, 0) \to W^{n+1,p}(R)$ is continuous.

The proof of part (a) of this lemma can be found in [5].

Proof. (b) Let $p > 2$ be such that $\alpha = 1 - \frac{2}{p}$. By Lemma 2.4 we have that

$$\begin{aligned}
[P_R\, h]_{R,\alpha} &\le K_p\ \|h\|_{R,p} \le K_p\ \|h\|_{R,\infty}\ \|1\|_{R,p} \\
&\le K_p\, [[h]]_{R,\alpha}\, \pi^{\frac{1}{p}}\ R^{\frac{2}{p}} \\
\|P_R\, h\|_{R,\infty} &\le |P_R\, h(0)| + [P_R\, h]_{R,\alpha}\ R^{\alpha} \\
&\le K_p\, [[h]]_{R,\alpha}\, \pi^{\frac{1}{p}}\ R^{\frac{2}{p}}\ R^{\alpha}\,,
\end{aligned}$$

because $P_R h(0) = 0$. Now observe that $\frac{2}{p} = 1 - \alpha$ and $R^{\alpha} < 1$ to get (b).

(c) Given $0 \le k < n$, let $\delta^k h$ be a kth partial derivative of h, $\delta^k h = \partial^i \bar{\partial}^j h$, $i + j = k$. Then

$$\delta^k h = P\left(\delta^k h_{\bar{z}}\right) + F\ , \tag{2.6}$$

where $F : \mathbb{C} \to \mathbb{C}$ is holomorphic. Since $\delta^k h \in \mathcal{L}_p(\mathbb{C})$ and by Lemma 2.4. (2.4.2), $P(\delta^k h_{\bar{z}})$ is $\mathcal{O}(|z|^2)$ when $|z| \to \infty$, then F is constant. In particular, for $\alpha = 1 - \frac{2}{p}$,

$$\begin{aligned}
\left[\delta^k h\right]_{R,\alpha} &= \left[P\left(\delta^k h_{\bar{z}}\right)\right]_{R,\alpha} \le K_p\ \left\|\delta^k h_{\bar{z}}\right\|_p = K_p\ \left\|\delta^k h_{\bar{z}}\right\|_{R,p} \\
\left[\left[\delta^k h\right]\right]_{R,\alpha} &\le \left\|\delta^k h\right\|_{R,\infty} + K_p\ \left\|\delta^k h_{\bar{z}}\right\|_{R,p}\ .
\end{aligned}$$

By Lemma 2.4. (2.4.6), $\delta^k H h = H \delta^k h$ and

$$\begin{aligned}
\left\|\delta^k H h\right\|_{R,\infty} &= \left\|H_R\, \delta^k h\right\|_{R,\infty} \le \left[\left[H_R\, \delta^k h\right]\right]_{R,\alpha} \\
&\le C(\alpha)\, R^{\alpha}\ \left[\left[\delta^k h\right]\right]_{R,\alpha} \\
&\le C(\alpha)\, R^{\alpha}\ \left(\left\|\delta^k h\right\|_{R,\infty} + \left\|\partial \delta^k h\right\|_{R,\infty} + \left\|\bar{\partial} \delta^k h\right\|_{R,\infty}\right) \quad \text{if } k \le n-2\,, \\
&\le C(\alpha)\, R^{\alpha}\ \left(\left\|\delta^{n-1} h\right\|_{R,\infty} + K_p\ \left\|\delta^{n-1} h_{\bar{z}}\right\|_{R,p}\right) \quad \text{if } k = n-1\,.
\end{aligned}$$

Adding over all kth partial derivatives, we get

$$\|Hh\|_{C^{n-1}(R)} \le C(\alpha)\ \left(3 + K_p\right)\ R^{\alpha}\ \|h\|_{W^{n,p}(R)}\ .$$

For $k = n$, we have that

$$\begin{aligned}
\left\|\delta^n H_R h\right\|_{R,p} &= \left\|H_R\, \delta^n h\right\|_{R,p} \le C_p\ \left\|\delta^n h\right\|_{R,p} \\
\left\|D^n H_R\, h\right\|_{R,p} &\le C_p\ \left\|D^n h\right\|_{R,p} \\
\|H_R h\|_{W^{n,p}(R)} &\le \left(C_p + C(\alpha)\ \left(K_p + 3\right)\ R^{\alpha}\right)\ \|h\|_{W^{n,p}(R)}\ .
\end{aligned}$$

(d) Let $\delta = \partial^i \overline{\partial}^j$, then

$$\begin{aligned} \partial^i \overline{\partial}^j P_R h &= \partial^i \overline{\partial}^{j-1} h \quad \text{if } j \geq 1 \,, \\ &= \partial^{i-1} \overline{\partial}^j H h \quad \text{if } i \geq 1 \,. \end{aligned}$$

Therefore,

$$\begin{aligned} \|P_R h\|_{W^{n+1,p}(R)} &\leq \max\left\{ \|h\|_{W^{n,p}(R)} \,,\, \|Hh\|_{W^{n,p}(R)} \,,\, \|P_R h\|_{R,p} \,,\, \|P_R h\|_{R,\infty} \right\} \\ &\leq B(p,R)\ \|h\|_{W^{n,p}(R)} \end{aligned}$$

for $B(p, R) := \sum\{ 1\,,\ A(p)\,,\ K_p \pi^{\frac{1}{p}} R\,,\ K_p R^{1-\frac{2}{p}}\,,\ \pi^{\frac{1}{p}} R^{\frac{2}{p}} \}$. □

For $\omega : \mathbb{C} \to \mathbb{C}$ and $p > 2$, let $\alpha = 1 - \frac{2}{p}$ and

$$\|\omega\|_{B_p} = \sup_{z_1 \neq z_2} \frac{|\omega(z_1) - \omega(z_2)|}{|z_1 - z_2|^{\alpha}} + \left(\iint_{\mathbb{C}} |\omega_z|^p \right)^{\frac{1}{p}} + \left(\iint_{\mathbb{C}} |\omega_{\overline{z}}|^p \right)^{\frac{1}{p}}$$

and define B_p as the space of maps $\omega : \mathbb{C} \to \mathbb{C}$ with $\omega(0) = 0$ and $\|\omega\|_{B_p} < \infty$, endowed with the norm $\|\ \|_{B_p}$.

Lemma 2.7. *Given $\mu \in \mathcal{L}_\infty(\mathbb{C})$, $\|\mu\|_\infty < k < 1$, $\sigma \in \mathcal{L}_p(\mathbb{C})$ with $k\,C_p < 1$. Then there exists a unique solution $\omega^{\mu,\sigma}$ of $\omega_{\overline{z}} = \mu\,\omega_z + \sigma$ with $\omega(0) = 0$ and $\omega_z \in \mathcal{L}_p(\mathbb{C})$. Moreover,*

- *(i) There exists $K = K(k, p)$ such that $\|\omega\|_{B_p} \leq K(k, p)\ \|\sigma\|_{\mathcal{L}_p}$.*
- *(ii) If $\mu_n \to \mu$ almost everywhere, $\|\mu_n\|_\infty < k$ and $\sigma_n \to \sigma$ in $\mathcal{L}_p$, then $\omega^{\mu_n,\sigma_n} \to \omega^{\mu,\sigma}$ in B_p.*
- *(iii) The unique solution of $\omega_{\overline{z}} = \mu\,\omega_z + \sigma$ such that $\omega(0) = a \in \mathbb{C}$ and $\omega_z \in \mathcal{L}_p$ is $\omega(z) = a + \omega^{\mu,\sigma}(z)$.*

The proof of all of this lemma except item (iii) can be found in [2]. Uniqueness in item (iii) is proved by substracting two such solutions and obtaining a solution of the homogeneous problem which is zero by (i).

Theorem 2.8 (Ahlfors–Bers) [2].

Given $\mu : \mathbb{C} \to \mathbb{C}$ measurable with $\|\mu\|_\infty < k < 1$ and $p > 2$ with $k\,C_p < 1$. Then there exists a unique homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that $f_{\overline{z}} = \mu\, f_z$, $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$. Moreover,

- *(i) f is $\alpha = 1 - \frac{2}{p}$ Hölder continuous on $S^2 = \mathbb{C} \cup \{\infty\}$.*
- *(ii) f_z is locally of class $\mathcal{L}_p$.*
- *(iii) $f_z \neq 0$ almost everywhere.*
- *(iv) f^{-1} is $\alpha = 1 - \frac{2}{p}$ Hölder continuous and has generalized derivatives which are locally of class $\mathcal{L}_p$.*
- *(v) $(f^{-1})_z$ and $(f^{-1})_{\overline{z}}$ are determined by the classical formulas.*
- *(vi) f and f^{-1} transform measurable sets into measurable sets.*
- *(vii) Integrals are transformed according to the classical rule.*
- *(viii) If $\varphi_{\overline{z}} = \mu\,\varphi_z$ on a region $\Omega \subseteq \mathbb{C}$, then $\varphi \circ f^{-1}$ is holomorphic on $f(\Omega)$.*

The solution f of Theorem 2.8 will be denoted by f^μ through the rest of the paper.

Lemma 2.9 [2].

Let $f = f^\mu$, $\Omega \subseteq \mathbb{C}$ bounded and suppose that $h_z, h_{\overline{z}} \in \mathcal{L}_q(f(\Omega))$, $q > 2$. Then $h \circ f$ has generalized derivatives given by

$$\begin{aligned}(h \circ f)_z &= (h_z \circ f)\ f_z + (h_{\overline{z}} \circ f)\ \overline{f}_z \\ (h \circ f)_{\overline{z}} &= (h_z \circ f)\ f_{\overline{z}} + (h_{\overline{z}} \circ f)\ \overline{f}_{\overline{z}}\end{aligned}$$

and

$$\|(h \circ f)_z\|_r \leq M\ \left(\|h_z\|_q + \|h_{\overline{z}}\|_q\right), \quad r = \frac{pq}{p+q-2},$$

where the norms are over the corresponding bounded regions Ω, $f(\Omega)$ and M is independent of h.

Corollary 2.10.

Let $f = f^\mu$ and suppose that $h_{\overline{z}} = \nu\, h_z$, then

(i) $(f^{-1})_{\overline{z}} = \lambda\, (f^{-1})_z$ *with* $\lambda = -\left(\frac{f_z}{\overline{f}_{\overline{z}}}\, \mu\right) \circ f^{-1}$, $\overline{f}_{\overline{z}} = \overline{(f_z)}$.

(ii) *If* $(h \circ f)_{\overline{z}} = \eta\, (h \circ f)_z$, *then* $\nu = \left(\frac{\eta - \nu}{1 - \eta\,\overline{\mu}}\ \frac{f_z}{\overline{f}_{\overline{z}}}\right) \circ f^{-1}$.

(iii) *If* $\overline{g(z)} = \frac{1}{f(1/\overline{z})}$, *then* $g_{\overline{z}} = \lambda\, g_z$, *with* $\overline{\lambda(z)} = \mu\, \left(\frac{1}{\overline{z}}\right) \frac{\overline{z}^2}{z^2}$.

Write

$$\mathbb{D} := \left\{ z \in \mathbb{C} \,\middle|\, |z| < 1 \right\} \quad , \quad S^1 := \left\{ x \in \mathbb{C} \,\middle|\, |z| = 1 \right\}.$$

Corollary 2.11. *If $\overline{\mu(z)} = \mu\left(\frac{1}{\overline{z}}\right) \frac{\overline{z}^2}{z^2}$, then $F := f^\mu$ restricted to $\mathbb{D}$ is the unique solution of $F_{\overline{z}} = \mu F_z$ on $\mathbb{D}$ such that $F(0) = 0$, $F(1) = 1$ and $F(\mathbb{D}) = \mathbb{D}$. We have that $f = f^\mu$ satisfies $\overline{f(z)} = \frac{1}{f(1/\overline{z})}$. In particular, F is an $\alpha = 1 - \frac{2}{p}$ Hölder continuous homeomorphism of $\mathbb{D}$.*

Proof. By Corollary 2.10. (iii) and the uniqueness of the solution in Theorem 2.8, we have that $\overline{f\left(\frac{1}{\overline{z}}\right)} = \frac{1}{f(z)}$ and $f(0) = 0$, therefore $f(\mathbb{D}) \subset \mathbb{D}$ and it is a solution for F. If there exists another solution G on $\mathbb{D}$, then $H = G \circ F^{-1}$ is analytic on $\mathbb{D}$ and $H(0) = 0$, $H(1) = 1$. By Schwartz's lemma, $H(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi[$. Since $H(1) = 1$, then $\theta = 0$, $H(z) = z$ and hence $G = F$. □

Given $0 < R < \infty$, let $B_{R,p}$ be the Banach space of functions $\omega : \mathbb{C} \to \mathbb{C}$ with $\omega(0) = 0$ and finite norm $\|\ \|_{B_{R,p}}$:

$$\|\omega\|_{B_{R,p}} = \sup_{|z_1|, |z_2| \leq R} \frac{|\omega(z_1) - \omega(z_2)|}{|z_1 - z_2|^{1 - \frac{2}{p}}} + \left(\iint_{|z| \leq R} |\omega_z|^p\ dx\, dy\right)^{\frac{1}{p}}.$$

The following theorems are due to Ahlfors and Bers:

Theorem 2.12 [2]. *Suppose that $\|\mu\|_\infty \leq k$, $\|\mu\|_\infty \leq k$, $\|\nu\|_\infty \leq k$, $k\, C_p < 1$, $p > 2$. Then for all $R > 0$,*

(a) $\|f^\mu - f^\nu\|_{B_{R,p}} \le c(R)\ \|\mu - \nu\|_\infty$, *with* $c(R)$ *depending only on* R, k, p.
(b) *If* $\mu_n \to \mu$ *almost everywhere, then* $\|f^{\mu_n} - f^\mu\|_{B_{R,p}} \to 0$.

Theorem 2.13 [2]. *Let* $t = (t_1, \ldots, t_n)$ *and* $s = (s_1, \ldots, s_n)$ *be real vectors in* $\mathbb{R}^n$. *Suppose that for all* t *in some open set* Δ *we have*

$$\mu(s+t) = \mu(t) + \sum_{i=1}^{n} a_i(t)\, s_i + |s|\ \alpha(t,s)$$

with $\|\mu(t)\|_\infty \le k < 1$, $\|\alpha(t,s)\|_\infty \le c$ *and* $\alpha(t,s) \to 0$ *almost everywhere as* $s \to 0$. *Suppose further that the norms* $\|a_i(t+s)\|_\infty$ *are bounded and that* $a_i(t+s) \to a_i(t)$ *almost everywhere for* $s \to 0$. *Then* $\omega^{\mu(t)}$ *has a development*

$$f^{\mu(s+t)} = f^{\mu(t)} + \sum_{i=1}^{n} \omega_i(t)\, s_i + |s|\ \gamma(t,s)$$

with $\|\gamma(t,s)\|_{B_{R,p}} \to 0$ *for* $s \to 0$. *Where* $\omega_i(t)$ *is the solution of*

$$W_{\bar z} = \mu(t)\ W_z + a_i(t)\ f_z^{\mu(t)}$$

such that $W(0) = 0$, $W(1) = 0$ *and* $|W(z)| = \mathcal{O}\left(\left|f^{\mu(t)}\right|^2\right)$ *as* $z \to \infty$.

2.2. The local non-homogeneous Beltrami equation

From now on the functions μ are assumed to be measurable and with $\|\mu\|_\infty \le k < 1$ for some fixed k and p is assumed to be $p > 2$ and such that $k\, C_p < 1$ unless otherwise stated.

Lemma 2.14. *Let* $0 < R < 1$ *and* $p > 2$. *Let* $\mu, \sigma \in W^{n,p}(R)$ *be such that* $\mu(z) = \sigma(z) = 0$ *for all* $|z| \ge R$. *Let* ω *be the solution of* $\omega_{\bar z} = \mu\, \omega_z + \sigma$ *such that* $\omega(0) = 0$ *and* $\omega_z \in \mathcal{L}_p(\mathbb{C})$. *Suppose that* $k := \|\mu\|_\infty < 1$ *and*

$$\Theta := \Theta\left(R, n, p, \|\mu\|_{W^{n,p}(R)}, k\right) = k\, C_p + 2^n\ \|\mu\|_{W^{n,p}(R)}\ A(p)\, R^{1-\frac{2}{p}} < 1$$

with $A(p)$ *from Lemma 2.5. Then* $\omega \in W^{n+1,p}(R)$ *and there exists* $D(R,n) = D(R, n, p, \|\mu\|_{W^{n,p}(R)}, k) > 0$ *such that*

$$\|\omega\|_{W^{n+1,p}(R)} \le D(R,n)\ \|\sigma\|_{W^{n,p}(R)}\ .$$

Proof. Let q be a solution of

$$q = \mu\, Hq + \sigma \tag{2.15}$$

in $\mathcal{L}_p(\mathbb{C})$. This is possible because the norm of the operator μH in $\mathcal{L}_p(\mathbb{C})$ is $\le k\, C_p < 1$ and hence $(I - \mu H)$ is invertible in $\mathcal{L}_p(\mathbb{C})$. Let

$$\omega = Pq = P(I - \mu H)^{-1}\sigma\,. \tag{2.16}$$

Then we have that $\omega_z = Hq$, $\omega_{\bar z} = q = \mu\, Hq + \sigma$. Therefore, ω is the unique solution of $\omega_{\bar z} = \mu\, \omega_z + \sigma$ with $\omega(0) = 0$, $\omega_z \in \mathcal{L}_p(\mathbb{C})$ of Lemma 2.7.

Observe that we only need to use P_R and H_R in (2.16) because $q(z) \equiv 0$ on $|z| \ge R$ by (2.15) and μH sends $W^{n,p}(R, 0)$ into itself.

Now we estimate the norm of the operator $(I-\mu H)^{-1}$ on $W^{n,p}(R,0)$:

$$\begin{aligned}\|\mu\, H(\sigma)\|_{W^{n,p}(R)} &\le \|\mu\|_{R,\infty}\, \left\|D^n H(\sigma)\right\|_{R,p} + 2^n\, \|\mu\|_{W^{n,p}(R)}\, \|H\sigma\|_{C^{n-1}(R)} \\ &\le \|\mu\|_{R,\infty}\, C_p\, \left\|D^n\sigma\right\|_{R,p} + 2^n\, \|\mu\|_{W^{n,p}(R)}\, A(p)\, R^{1-\frac{2}{p}}\, \|\sigma\|_{W^{n,p}(R)} \\ &\le \Theta\, \|\sigma\|_{W^{n,p}(R)}\,,\end{aligned}$$

where $\Theta := k\, C_p + 2^n\, \|\mu\|_{W^{n,p}(R)}\, A(p)\, R^{1-\frac{2}{p}} < 1$.

$$\begin{aligned}\|\omega\|_{W^{n+1,p}(R)} &= \left\|P_R(I-\mu H)^{-1}\sigma\right\|_{W^{n+1,p}(R)} \\ &= \left\|P_R\left(\textstyle\sum_{k=0}^{\infty}(\mu H)^k\right)\sigma\right\|_{W^{n+1,p}(R)} \\ &\le B(p,R)\,\left(\textstyle\sum_{k=0}^{\infty}\Theta^k\right)\|\sigma\|_{W^{n,p}(R)} \\ &\le D(R,p)\, \|\sigma\|_{W^{n,p}(R)}\end{aligned}$$

where $D(R,n,p,\|\mu\|_{W^{n,p}(R)}) = \frac{B(p,R)}{1-\Theta}$. □

Lemma 2.17. *Let* $\mathcal{W} := W^{n,p}(R,0)\cap\left[\,\|\mu\|_{W^{n,p}(R)} < a\,,\ \|\mu\|_\infty < k < 1\right]$ *with R small enough such that* $\Theta(R,n,p,a,k) < 1$, *where Θ is from Lemma 2.14. Then the map* $\mathcal{W}\times W^{n,p}(R,0)\to W^{n+1,p}(R)$, *given by* $(\mu,\sigma)\mapsto\omega^{\mu,\sigma}$, *is continuous.*

Proof. Let (μ,σ), $(\mu_o,\sigma_o)\in\mathcal{W}\times W^{n,p}(R,0)$. Let $\omega^o=\omega^{\mu_o,\sigma_o}$ and $\omega=\omega^{\mu,\sigma}$, i.e., $\omega^o_{\bar z} = \mu\,\omega^o_z+\sigma_o$ and $\omega_{\bar z}=\mu\,\omega_z+\sigma$ in $\mathcal{L}_p(\mathbb{C})$. We have that

$$\left(\omega-\omega^o\right)_{\bar z} = \mu\,\left(\omega-\omega^o\right)_z + \left(\mu-\mu_o\right)\omega^o_z + \left(\sigma-\sigma_o\right)$$

with

$$\left\|(\mu-\mu_o)\,\omega^o_z+(\sigma-\sigma_o)\right\|_{W^{n,p}(R)} \le 2^n\,\left\|\mu-\mu_o\right\|_{W^{n,p}(R)}\,\left\|\omega^o_z\right\|_{W^{n,p}(R)} + \left\|\sigma-\sigma_o\right\|_{W^{n,p}(R)}\,.$$

From Lemma 2.14, we get that

$$\left\|\omega-\omega^o\right\|_{W^{n+1,p}(R)} \le D(\Theta,k)\,\left(2^n\,\left\|\mu-\mu_o\right\|_{W^{n,p}(R)}\,\left\|\omega^o_z\right\|_{W^{n,p}(R)} + \left\|\sigma-\sigma_o\right\|_{W^{n,p}(R)}\right)\,. \quad \square$$

On $\mathcal{L}_\infty(\mathbb{C})\cap W^{1,p}(R)$ consider the topology given by $\langle\mu_n\rangle\to\mu$ if $\mu_n\to\mu$ almost everywhere in $\mathbb{C}$ and $\|\mu_n-\mu\|_{W^{1,p}(R)}\to 0$.

Corollary 2.19. *Given* $0<R<1$, $0<k<1$, $L>0$ *with* $k\,C_p<1$, *there exists* $0<r<r(k,L)<R$ *such that the map* $\mu\mapsto\omega^\mu$*:*

$$\mathcal{L}_\infty(\mathbb{C})\cap W^{1,p}(R)\cap\left\{\mu\;\middle|\;\|\mu\|_\infty<k,\ \|\mu\|_{W^{1,p}(R)}<L\right\}\to C^1\left(|z|<r,\mathbb{C}\right)$$

is continuous.

Proof. Let $\langle\mu_n\rangle$ be a sequence converging to μ_o in $\mathcal{L}_\infty(\mathbb{C})\cap W^{1,p}(R)$ with $\|\mu_n\|_\infty<k$, $\|\mu_n\|_{W^{1,p}(R)}<L$. Let $\omega^n := f^{\mu_n}-f^{\mu_o}$, $f^o := f^{\mu_o}$. Since $\mu_n\to\mu_o$ a.e, then by Theorem 2.8, $\|\omega^n\|_{B_{R,p}}\to 0$. In particular $\|\omega^n\|_{W^{1,p}(R)}\to 0$. We have that

$$\omega^n_{\bar z} = \mu_n\,\omega^n_z + (\mu_n-\mu_o)\,f^o_z\,.$$

Let $\lambda : \mathbb{C} \to [0, 1]$ be a C^∞ function such that $\lambda(z) \equiv 1$ for $|z| \leq r$ and $f(z) \equiv 0$ for $|z| \leq 2r$. Choose r such that $\Theta(2r, n = 1, p, L, k) < 1$ where Θ is from Lemma 2.17 and $0 < 2r < R$. We have

$$(\lambda\,\omega^n)_{\bar{z}} = \widehat{\mu}_n\,(\lambda\,\omega^n)_z + (\lambda_{\bar{z}} - \mu_n\,\lambda_z)\,\omega^n + \lambda\,(\mu_n - \mu_o)\,f^o_z\,,$$

where $\widehat{\mu}_n(z) = \mu_n(z)$ for $|z| < 2r$ and $\widehat{\mu}_n(z) = 0$ for $\|z\| \geq 2r$. Since the sequence $\|\mu_n\|_{W^{1,p}(2r)}$ is bounded and $\|\omega^n\|_{W^{1,p}(R)} \to 0$, then $\|(\lambda_{\bar{z}} - \mu_n\,\lambda_z)\,\omega^n\|_{W^{1,p}(R)} \to 0$. Also $\left\|\lambda\,(\mu_n - \mu_o)\,f^o_z\right\|_{W^{1,p}(2r)} \to 0$. In particular

$$\left\|\omega^n\right\|_{C^1(D_r,\mathbb{C})} \leq \left\|\lambda\,\omega^n\right\|_{C^1(D_{2r},\mathbb{C})} \leq \left\|\lambda\,\omega^n\right\|_{W^{2,p}(2r)} \to 0\,. \qquad \square$$

2.3. Global non-homogeneous Beltrami equations

Lemma 2.21. *If $0 < R < +\infty$ and $h \in \mathcal{L}_q(|z| < R)$ for some $q > 2$, then $h \in \mathcal{L}_p(|z| < R)$ for all $2 < p < q$, and*

$$\|h\|_{R,p} \leq A\;\|h\|_{R,q}\;,$$

where $A = A(R, q) = \max\left\{1,\ \sqrt{\pi}^{\,1-\frac{2}{q}}\,R^{1-\frac{2}{q}}\right\}$.

Proof. Let $\alpha = \frac{q}{p} > 1$ and $\frac{1}{\beta} + \frac{1}{\alpha} = 1$, then

$$\begin{aligned}
\int_{|z|<R} |h|^p = \int_{|z|<R} |h|^{\frac{q}{\alpha}} \cdot 1 \ &\leq\ \left[\int\!\!\int_{|z|<R} |h|^q\right]^{\frac{1}{\alpha}} \left[\int\!\!\int_{|z|<R} 1\right]^{\frac{1}{\beta}}\,,\\
\left(\|h\|_{R,p}\right)^p \ &\leq\ \left(\|h\|_{R,q}\right)^{\frac{q}{\alpha}} \left(\pi\,R^2\right)^{\frac{1}{\beta}}\\
\|h\|_{R,p} \ &\leq\ \|h\|_{R,q}\,\left(\pi\,R^2\right)^{\frac{1}{\beta p}}\,.
\end{aligned}$$

From this we get the lemma because $\frac{1}{\beta p} = \frac{1}{p} - \frac{1}{q} \in \left]0, \frac{1}{2}\left(1 - \frac{2}{q}\right)\right[$. $\square$

Given $\epsilon > 0$ and $q > 2$, define

$$\mathcal{D}_{\epsilon,q} := \left\{\sigma : \mathbb{C} \to \mathbb{C} \,\middle|\, \sigma(z) \in \mathcal{L}_q\left(|z| < \frac{1}{\epsilon}\right) \text{ and } \sigma\left(\frac{1}{z}\right) \in \mathcal{L}_q(|z| < \epsilon)\right\},$$

with the norm

$$\|\sigma\|_{\mathcal{D}_{\epsilon,q}} := \left(\int\!\!\int_{|z|<\frac{1}{\epsilon}} |\sigma(z)|^q\ dx\,dy\right)^{\frac{1}{q}} + \left(\int\!\!\int_{|z|<\epsilon} \left|\sigma\left(\frac{1}{z}\right)\right|^q\ dx\,dy\right)^{\frac{1}{q}}.$$

Lemma 2.22. *For all $0 < \epsilon < +\infty$ and $q > 2$, we have that*

$$\mathcal{D}_{\epsilon,q} = \bigcap_{0<r<\infty}\ \bigcap_{2<p\leq q} \mathcal{D}_{r,p}\,.$$

Moreover, given $0 < r < \infty$ and $2 < p \leq q$, there exists $A = A(p, q, r, \epsilon) > 0$ such that

$$\|h\|_{\mathcal{D}_{r,p}} \leq A\;\|h\|_{\mathcal{D}_{\epsilon,q}}$$

for all $h \in \mathcal{D}_{\epsilon,q}$.

Proof. One inclusion is trivial. For the other one, let $r > \epsilon$, then $\left[|z| < \frac{1}{\epsilon}\right] \subset \left[|z| < \frac{1}{r}\right]$ and hence $\sigma \in \mathcal{L}_q(|z| < \frac{1}{\epsilon}) \subseteq \mathcal{L}_q(|z| < \frac{1}{r})$. We have that

$$\begin{aligned}
\iint\limits_{\epsilon<|z|<r} \left|\sigma\left(\frac{1}{z}\right)\right|^q &= \iint\limits_{\frac{1}{r}<|w|<\frac{1}{\epsilon}} \frac{1}{|w|^4}\,|\sigma(w)|^q \le \frac{1}{\epsilon^4} \iint\limits_{\frac{1}{r}<|w|<\frac{1}{\epsilon}} |\sigma(w)|^q \\
&\le \frac{1}{\epsilon^4}\,\|\sigma\|_{\mathcal{D}_{\epsilon,q}} < +\infty\,, \\
\int\limits_{|z|<r} \left|\sigma\left(\frac{1}{z}\right)\right|^q &= \int\limits_{|z|<\epsilon} \left|\sigma\left(\frac{1}{z}\right)\right|^q + \int\limits_{\epsilon<|z|<r} \left|\sigma\left(\frac{1}{z}\right)\right|^q < \left(1+\frac{1}{\epsilon^4}\right)\|\sigma\|_{\mathcal{D}_{\epsilon,q}}\,.
\end{aligned}$$

Therefore, $\sigma \in \mathcal{D}_{r,q}$ and $\|\sigma\|_{\mathcal{D}_{r,q}} \le A_1\,\|\sigma\|_{\mathcal{D}_{\epsilon,q}}$. By Lemma 2.21, we have that $\sigma \in \mathcal{D}_{r,p}$ and $\|\sigma\|_{\mathcal{D}_{r,q}} \le A_2(p,q,r,\epsilon)\,\|\sigma\|_{\mathcal{D}_{\epsilon,q}}$ for all $r > \epsilon$ and all $2 < p \le q$. The case $0 < r < \epsilon$ is similar to this case. □

Lemma 2.23. *Let $\epsilon > 0$, $q > 2$, $\sigma \in \mathcal{D}_{\epsilon,q}$. Then there exists a unique solution of $\Theta_{\bar z} = \sigma$, such that Θ is continuous, $\Theta(0) = 0$, $\Theta(1) = 0$ and*

$$\lim_{|z|\to\infty} \frac{\Theta(z)}{|z|^2} = 0\,.$$

The solution satisfies

- *(i)* $|\Theta(z)| \le 2\,K_q\,\|\sigma\|_{\mathcal{D}_{\epsilon,q}}\,\max\left\{|z|^{1-\frac{2}{q}}, |z|^{1+\frac{2}{q}}\right\}$, *for all $z \in \mathbb{C}$, where K_q is from Lemma 2.4. (2.4.2).*
- *(ii)* $\Theta \in B_{R,p}$ *for all $R > 0$ and all $2 < p \le q$.*
- *(iii)* $\Theta_z \in \mathcal{D}_{r,p}$ *for all $2 < p < q$ and all $r > 0$.*
- *(iv)* *For all $r > 0$ and $2 < p < q$ there exists $B = B(r,\epsilon,p,q) > 0$, such that $\|\Theta_z\|_{\mathcal{D}_{r,p}} < B\,\|\sigma\|_{\mathcal{D}_{\epsilon,q}}$.*

Proof. Let $a(z) = \sigma(z)$ for $|z| < \frac{1}{\epsilon}$ and $a(z) = 0$ for $|z| > \frac{1}{\epsilon}$, and let $b(z) = 0$ for $|z| > \epsilon$ and $b(z) = \sigma\left(\frac{1}{z}\right)\frac{z^2}{\bar z^2}$ for $|z| < \epsilon$. Since $\sigma \in \mathcal{D}_{\epsilon,p}$ we have that $a \in \mathcal{L}_q(\mathbb{C})$ and $b \in \mathcal{L}_q(\mathbb{C})$. Define

$$\begin{aligned}
\Theta^a(z) &:= Pa(z) - z\,Pa(1) \\
\Theta^b(z) &:= -z^2\,Pb\left(\frac{1}{z}\right) + z\,Pb(1) \\
\Theta(z) &:= \Theta^a(z) + \Theta^b(z)\,.
\end{aligned}$$

We have that $\Theta^a_{\bar z} = a$, $\Theta^b_{\bar z} = \sigma$ on $|z| > \frac{1}{\epsilon}$ and $\Theta^b_{\bar z} = 0$ on $|z| < \frac{1}{\epsilon}$. Therefore, $\Theta_{\bar z} = \sigma$. Also $\Theta(0) = \Theta(1) = 0$. Moreover,

$$\begin{aligned}
|\Theta(z)| &\le K_q\,\|a\|_q\left(|z|^{1-\frac{2}{q}} + |z|\right) + K_q\,\|b\|_q\,|z|^2\left|\frac{1}{z}\right|^{1-\frac{2}{q}} + K_q\,\|b\|_q\,|z| \\
&\le 2\,K_q\,\|\sigma\|_{\mathcal{D}_{\epsilon,q}}\,\max\left\{|z|^{1-\frac{2}{q}},\,|z|^{1+\frac{2}{q}}\right\}\,.
\end{aligned}$$

Suppose that φ is another solution. Let $h = \varphi - \Theta$. Then h is analytic on all $\mathbb{C}$ because $h_{\bar z} = (\varphi - \Theta)_{\bar z} = 0$. Also $h(0) = h(1) = 0$ and $|h(z)| = \mathcal{O}(|z|^2)$ when $|z| \to \infty$. Therefore, $h \equiv 0$.

By Lemma 2.21, $\sigma \in \mathcal{D}_{\epsilon,p}$ for all $2 < p < q$. Therefore, $a \in \mathcal{L}_p(\mathbb{C})$ and $b \in \mathcal{L}_p(\mathbb{C})$ for all $2 < p < q$. Let $2 < p < q$, we have that

$$\begin{aligned}
\Theta_z^a(z) &= Ha - Pa(1) \\
\left\|\Theta_z^a\right\|_{\frac{1}{\epsilon},p} &\le \|Ha\|_p + |Pa(1)| \left(\pi \frac{1}{\epsilon^2}\right)^{\frac{1}{p}} \\
&\le C_p \|a\|_p + K_p \|a\|_p \left(\frac{\pi}{\epsilon^2}\right)^{\frac{1}{p}} \\
\int_{|z|<\epsilon} \left|Ha\left(\frac{1}{z}\right)\right|^p &= \int_{|w|>\frac{1}{\epsilon}} \frac{1}{|w|^4} |Ha(w)|^p \le \epsilon^4 \left(\|Ha\|_p\right)^p \\
\left\|\Theta_z^a\left(\frac{1}{z}\right)\right\|_{\epsilon,p} &\le \epsilon^{\frac{4}{p}} C_p \|a\|_p + K_p \|a\|_p \left(\pi\,\epsilon^2\right)^{\frac{1}{p}} \\
\|\Theta_z^a\|_{\mathcal{D}_{\epsilon,p}} &\le \left(1+\epsilon^{\frac{4}{p}}\right) \left(C_p + K_p \left(\frac{\pi}{\epsilon^2}\right)^{\frac{1}{p}}\right) \|a\|_p .
\end{aligned}$$

We write $C(p, q, \epsilon)$ for constants depending only on p, q, ϵ.

$$\begin{aligned}
\Theta_z^b &= -2\,z\,Pb\left(\frac{1}{z}\right) + Hb\left(\frac{1}{z}\right) + Pb(1) \\
\int_{|z|<\frac{1}{\epsilon}} \left|Hb\left(\frac{1}{z}\right)\right|^p &= \int_{|w|>\epsilon} \frac{1}{|w|^4} |Hb(w)|^p \le \frac{1}{\epsilon^4} \left(\|Hb\|_p\right)^p \\
\int_{|z|<\frac{1}{\epsilon}} \left|2\,z\,Pb\left(\frac{1}{z}\right)\right|^p &\le \int_{|z|<\frac{1}{\epsilon}} \left(2K_p \|b\|_p\, |z|^{\frac{2}{p}}\right)^p \le \left(C_1(p,\epsilon)\,\|b\|_p\right)^p \\
\left\|\Theta_z^b\right\|_{\frac{1}{\epsilon},p} &\le \left(C_1(p,\epsilon) + \frac{C_p}{\epsilon^{\frac{4}{p}}} + K_p \left(\frac{\pi}{\epsilon^2}\right)^{\frac{1}{p}}\right) \|b\|_p \\
\int_{|z|<\epsilon} |Hb(z)|^p &\le \left(C_p \|b\|_p\right)^p
\end{aligned}$$

Since $\left|2\,\frac{1}{z}\,Pb(z)\right| \le 2\,K_q\,\|b\|_q\,|z|^{-\frac{2}{q}}$ and $p < q$, we have

$$\begin{aligned}
\int_{|z|<\epsilon} \left|2\,\frac{1}{z}\,Pb(z)\right|^p &\le \left(2\,K_q\,\|b\|_q\right)^p \int_{|z|<\epsilon} |z|^{-\frac{2p}{q}} < +\infty , \\
\left\|\Theta_z^b\left(\frac{1}{z}\right)\right\|_{\epsilon,p} &\le C_2(p,q,\epsilon)\,\|b\|_q + C_p\,\|b\|_p + \left(\pi\,\epsilon^2\right)^{\frac{1}{p}} K_p\,\|b\|_p .
\end{aligned}$$

Therefore, $\|\Theta_z^b\|_{\mathcal{D}_{\epsilon,p}} < +\infty$ and hence $\|\Theta_z\|_{\mathcal{D}_{\epsilon,p}} < +\infty$ for all $2 < p < q$. Now use Lemma 2.22 to get (iii).

For (iv) observe that $\|\sigma\|_{\mathcal{D}_{\epsilon,q}} = \|a\|_q + \|b\|_q$ and that $\|a\|_p \le A(\epsilon, q)\,\|a\|_q$, $\|b\|_p \le A(\epsilon, q)\,\|b\|_q$ by Lemma 2.21. Now use the above estimates and Lemma 2.23. □

The following notation will be useful for the next proposition: For $\epsilon > 0$, $p > 2$, $F : \mathbb{C} \to \mathbb{C}$, let

$$\begin{aligned}
\mathcal{D}(\epsilon, p, F) &:= \left\{\sigma : \mathbb{C} \to \mathbb{C} \,\middle|\, \left(\sigma \circ F^{-1}\right) F_z^{-1} \in \mathcal{D}_{\epsilon,p}\right\} \\
\|\sigma\|_{\mathcal{D}(\epsilon,p,F)} &:= \left\|\left(\sigma \circ F^{-1}\right) F_z^{-1}\right\|_{\mathcal{D}_{\epsilon,p}} .
\end{aligned}$$

Proposition 2.24. *If $\mu \in \mathcal{L}_\infty(\mathbb{C})$, $\|\mu\|_\infty < k < 1$, $k\, C_q < 1$, $q > 2$ and $\big(\sigma \circ (f^\mu)^{-1}\big)\, (f^\mu)_z^{-1} = \left(\frac{\sigma}{f_z^\mu}\right) \circ (f^\mu)^{-1} \in \mathcal{D}_{\epsilon,q}$ for some $\epsilon > 0$. Then there exists a unique solution of*

$$\omega_{\overline{z}} = \mu\, \omega_z + \sigma$$

such that ω is continuous, $\omega(0) = 0$, $\omega(1) = 0$ and

$$\lim_{|z|\to\infty} \frac{\omega(z)}{|f^\mu(z)|^2} = 0\,.$$

Moreover,

(i) $\big(\omega_z \circ (f^\mu)^{-1}\big)(f^\mu)_z^{-1} \in \mathcal{D}_{r,p}$ *for all $r > 0$ and all $2 < p < q$.*

(ii) $\omega \in B_{R,p}$ *for all $R > 0$ and all $2 < p \le \frac{q^2}{q-2}$.*

(iii) If $\sigma \in B_{R,p}$, $2 < p \le q$, $0 < r < R$ and $\epsilon > 0$, then there exists $C(R,r) = C(R,r,\epsilon,p,k)$ such that

$$\|\omega\|_{B_{r,p}} \le C(R,r)\, \left(\|\sigma\|_{\mathcal{D}(\epsilon,p,f^\mu)} + \|\sigma\|_{R,p}\right)\,.$$

(iv) For all $r > 0$ and $2 < p < q$, there exists $A(r,p) = A(r,\epsilon,p,q) > 0$ such that

$$\|\omega_z\|_{\mathcal{D}(r,p,f^\mu)} \le A(r,p)\ \|\sigma\|_{\mathcal{D}(\epsilon,q,f^\mu)}\,.$$

(v) If $\mu, \sigma \in W^{n,p}(R)$, $2 < p \le q$, $0 < r < R$ and $\epsilon > 0$, then there exists $D(R,r) = D(R,r,n,p,\epsilon,\|\mu\|_{W^{n,p}(R)})$ such that

$$\|\omega\|_{W^{n+1,p}(r)} \le D(R,r)\, \left(\|\sigma\|_{\mathcal{D}(\epsilon,p,f^\mu)} + \|\sigma\|_{W^{n,p}(R)}\right)\,.$$

Proof. We first prove the uniqueness of the solution. Suppose that u is another solution. Then $v := \omega - u$ satisfies $v_{\overline{z}} = \mu\, v_z$, $v(0) = v(1) = 0$ and $|v(z)| = \mathcal{O}\big(|f^\mu(z)|^2\big)$ when $|z| \to \infty$. Let $h := v \circ (f^\mu)^{-1}$. Then $h(0) = h(1) = 0$, by Theorem 2.8 (viii) we have that h is holomorphic on all $\mathbb{C}$ and

$$\lim_{z\to\infty} \frac{|h(z)|}{|z|^2} = \lim_{z\to\infty} \frac{\left|v(f^{-1}(z))\right|}{|z|^2} = \lim_{y\to\infty} \frac{|v(y)|}{|f(y)|^2} = 0\,,$$

where $f = f^\mu$. Therefore, $h \equiv 0$.

For the existence, write $\omega = \Theta \circ f^\mu$. Using the formulas of Lemma 2.9, and that

$$\overline{f}_z = \overline{(f_{\overline{z}})} = \overline{\mu}\,\overline{(f_z)} = \overline{\mu}\,\overline{f}_{\overline{z}}\,,$$

we see that ω is a solution of the problem if and only if, for $f = f^\mu$, we have

$$\Theta_{\overline{z}} = \left(\frac{1}{1-|\mu|^2}\,\frac{\sigma}{\overline{f}_{\overline{z}}}\right) \circ f^{-1} =: \rho\,, \tag{2.25}$$

$\Theta(0) = 0$, $\Theta(1) = 0$ and $\lim_{z\to\infty} \frac{\Theta(z)}{|z|^2} = 0$. Since $\overline{f}_{\overline{z}} = \overline{f_z}$, we have by hypothesis that $\rho \in \mathcal{D}_{\epsilon,q}$. By Lemma 2.23, such Θ exists and is unique.

For (i) observe that, for $f = f^\mu$,

$$\begin{aligned}
\omega_z &= (\Theta_z \circ f)\ f_z + (\Theta_{\overline{z}} \circ f)\ \overline{f}_z \\
\omega_z &= (\Theta_z \circ f)\ f_z + \Big(\frac{1}{1-|\mu|^2}\,\frac{\sigma}{\overline{f}_{\overline{z}}}\Big)\,\overline{\mu}\,\overline{f}_{\overline{z}} \\
\left(\omega_z \circ f^{-1}\right)\, f_z^{-1} &= \Big(\frac{\omega_z}{f_z}\Big) \circ f^{-1} = \Theta_z + \Big(\frac{\overline{\mu}}{1-|\mu|^2}\,\frac{\sigma}{f_z}\Big) \circ f^{-1}\,.
\end{aligned} \tag{2.26}$$

By the hypothesis on σ and μ and by Lemma 2.22, we have that $(\omega_z \circ f^{-1}) f_z^{-1} \in \mathcal{D}_{r,p}$ if and only if $\Theta_z \in \mathcal{D}_{r,p}$; but this is true by Lemma 2.23.

(iv) From (2.25), (2.26), and Lemmas 2.23 and 2.22, we get that

$$\|\omega_z\|_{\mathcal{D}(r,p,f^\mu)} \le \left(\frac{B}{1-k^2} + \frac{A\,k}{1-k^2}\right) \|\sigma\|_{\mathcal{D}(\epsilon,q,f^\mu)} .$$

This proves (iv).

For (ii) we know that $f \in B_{R,p}$ for all $2 < p \le q$ and by Lemma 2.23, $\Theta \in B_{R,p}$ for all $2 < p \le q$. Now use Lemma 2.9.

We now prove (iii). Let $\lambda : \mathbb{C} \to [0,1]$ be a C^∞ function such that $\lambda(z) = 1$ on $|z| \le r$ and $\lambda(z) = 0$ on $|z| \ge R$. We have that

$$(\lambda\omega)_{\bar z} = \mu\,(\lambda\omega)_z + (\lambda_{\bar z} - \mu\lambda_z)\;\omega + \lambda\sigma .$$

By Lemma 2.23, there exists $C_1(R) = C_1(R,p,\epsilon)$ such that for $\Theta = \omega \circ f^{-1}$, $f = f^\mu$, we have

$$\|\Theta\|_{R,\infty} \le C_1(R) \left\| \left(\frac{1}{1-|\mu|^2}\,\frac{\sigma}{\overline{f_{\bar z}}}\right) \circ f^{-1} \right\|_{\mathcal{D}_{\epsilon,p}} .$$

Let $A = A(R,k) > 0$ be such that $f(|z| < R) \subseteq [|z| < A]$. Writing $\Xi := (\sigma \circ f^{-1})\, f_z^{-1}$, we have

$$\begin{aligned}
\|\omega\|_{R,\infty} &= \|\Theta \circ f\|_{R,\infty} \le C_1(A,k)\;\|\Xi\|_{\mathcal{D}_{\epsilon,p}} , \\
\|\omega\|_{R,p} &\le C_3(R,k,p,\epsilon)\;\|\Xi\|_{\mathcal{D}_{\epsilon,p}} , \\
\|\lambda\sigma\|_{\mathcal{L}_p} &\le \|\sigma\|_{R,p} , \\
\|(\lambda_{\bar z} - \mu\lambda_z)\;\omega\|_{\mathcal{L}_p} &\le C_4(R,r)\;\|\Xi\|_{\mathcal{D}_{\epsilon,p}} .
\end{aligned}$$

By Lemma 2.7, we have that

$$\|\lambda\omega\|_{B_p} \le K(k,p)\;\left(C_4\;\|\Xi\|_{\mathcal{D}_{\epsilon,p}} + \|\sigma\|_{R,p}\right) .$$

Therefore,

$$\|\omega\|_{B_{r,p}} \le C(R,r,p,k,\epsilon)\;\left(\|\Xi\|_{\mathcal{D}_{\epsilon,p}} + \|\sigma\|_{R,p}\right) .$$

(v) The case $n = 0$ is proved in item (iii). Suppose by induction that it holds for $n-1$. Cover the disc $|z| \le r$ by a finite number of discs of radius δ such that the corresponding discs of radius 2δ are all contained in $|z| < R$. Choose δ small enough so that

$$\Theta\left(2\delta, n, p, \|\mu\|_{W^{n,p}(R)}\right) < 1 ,$$

where Θ is from Lemma 2.14. Choose one of these discs, say $|z-a| < \delta$. Let $\lambda : \mathbb{C} \to [0,1]$ be a C^∞ function such that $\lambda(z) \equiv 1$ on $|z-a| \le \delta$ and $\lambda(z) \equiv 0$ on $|z-a| \ge 2\delta$. Let

$$u(z) := \lambda(z)\;(\omega(z+a) - \omega(a)) .$$

Then $u(0) = 0$, $u_z \in \mathcal{L}_p(\mathbb{C})$ and

$$u_{\bar z} = \mu\,u_z + (\lambda_{\bar z} - \widehat{\mu}\lambda_z)\;(\widehat{\omega} - \omega(a)) + \lambda\,\widehat{\sigma}_z ,$$

where $\widehat{\mu}(z) = \mu(z+a)$, $\widehat{\omega}(z) = \omega(z+a)$, $\widehat{\sigma}(z) = \sigma(z+a)$. By Lemma 2.14 we have that

$$\|u\|_{W^{n+1,p}(2\delta)} \le D_1(a)\;\left(\|(\lambda_{\bar z} - \mu\lambda_z)\,(\widehat{\omega} - \omega(a))\|_{W^{n,p}(R)} + \|\lambda\widehat{\sigma}\|_{W^{n,p}(2\delta)}\right) ,$$

where D_1 depends on 2δ, p, n, $\|\mu\|_{W^{n,p}(R)}$. We have

$$\begin{aligned}
\|(\lambda_{\bar z}-\widehat{\mu}\lambda_z)\,(\widehat{\omega}-\omega(a))\|_{W^{n,p}(2\delta)} &\le 2^n\ \|\lambda_{\bar z}-\widehat{\mu}\lambda_z\|_{W^{n,p}(R)}\ \|\widehat{\omega}-\omega(a)\|_{W^{n,p}(2\delta)}\\
&\le 2^n\ \big(\|\lambda\|_{C^n}+2^n\ \|\lambda\|_{C^n}\ \|\mu\|_{W^{n,p}(R)}\big)\\
&\quad\ \|\widehat{\omega}-\omega(a)\|_{W^{n,p}(R)}\\
&\le D_2(a)\ \|\omega\|_{W^{n,p}(R)}\\
&\le D_2(a)\,D(R,n-1)\ \big(\|\sigma\|_{\mathcal{D}(\epsilon,p,f^\mu)}+\|\sigma\|_{W^{n-1,p}(R)}\big)
\end{aligned}$$

where in the first inequality we used that $\|\ \|_{W^{n,p}(2\delta)}\le\|\ \|_{W^{n,p}(R)}$, on the second inequality we used that

$$\|\widehat{\omega}-\omega(a)\|_{W^{n,p}(2\delta)}\le 2\ \|\omega\|_{W^{n,p}(R)}\,,$$

because $D^k\big(\widehat{\omega}-\omega(a)\big)=D^k\,\omega$ for $k>0$ and $\|\widehat{\omega}-\omega(a)\|_{2\delta,\infty}\le 2\ \|\omega\|_{R,\infty}$, and on the last inequality we used the induction hypothesis. Also

$$\|\lambda\sigma\|_{W^{n,p}(2\delta)}\le 2^n\ \|\lambda\|_{C^n}\ \|\sigma\|_{W^{n,p}(R)}\ .$$

Combining these inequalities, we get that

$$\|u\|_{W^{n+1,p}(2\delta)}\le D_3\ \big(\|\sigma\|_{\mathcal{D}(\epsilon,p,f^\mu)}+\|\sigma\|_{W^{n,p}(R)}\big)\ .$$

In particular

$$\begin{aligned}
\|\omega\|_{W^{n+1,p}(|z-a|<\delta)} &\le \|u\|_{W^{n+1,p}(2\delta)}+|\omega(a)|\\
&\le \|u\|_{W^{n+1,p}(2\delta)}+\|\omega\|_{R,\infty}\\
&\le D_4\ \big(\|\sigma\|_{\mathcal{D}(\epsilon,p,f^\mu)}+\|\sigma\|_{W^{n,p}(R)}\big)\ .
\end{aligned}$$

Adding the estimates of each ball, we get

$$\begin{aligned}
\Big(\big\|D^{n+1}\omega\big\|_{r,p}\Big)^p &= \sum_a\int_{|z-a|<\delta}\big\|D^{n+1}\omega\big\|^p=\textstyle\sum_a\Big(\big\|D^{n+1}\omega\big\|_{|z-a|<\delta,p}\Big)^p\\
\big\|D^{n+1}\omega\big\|_{r,p} &\le \sum_a\big\|D^{n+1}\omega\big\|_{|z-a|<\delta,p}\le\sum_a\ \|\omega\|_{W^{n+1,p}(|z-a|<\delta)}\\
\|\omega\|_{C^n(r)} &\le \sup_a\ \|\omega\|_{C^n(|z-a|<\delta)}\le\sum_a\|\omega\|_{W^{n+1,p}(|z-a|<\delta)}\\
\|\omega\|_{W^{n+1,p}(r)} &\le D_5\ \big(\|\sigma\|_{\mathcal{D}(\epsilon,p,f^\mu)}+\|\sigma\|_{W^{n,p}(R)}\big)\ .
\end{aligned}$$

□

Lemma 2.27. *Let $\lambda\in\mathcal{L}_\infty(\mathbb{C})$, $\|\lambda\|_\infty<k<1$ and let $h=f^\lambda$. Let $K>1$, $0<\alpha<1$ and $0<\epsilon<1$ be such that*

$$\left|h^{-1}(z)\right|<K\ |z|^\alpha\ \text{ for all } |z|<\frac{1}{\epsilon}\,.$$

Let $p_o>2$ be such that $k\,C_{p_o}<1$. Let $q_o>2$ and

$$p=\frac{p_o\,q_o}{p_o+q_o-2}\,.$$

(i) *If $A\in\mathcal{L}_{q_o}\big(|z|<\frac{K}{\epsilon}\big)$, then $(A\circ h^{-1})\,h_z^{-1}\in\mathcal{L}_p\big(|z|<\frac{1}{\epsilon}\big)$ and*

$$\left\|\left(A\circ h^{-1}\right)\,h_z^{-1}\right\|_{\frac{1}{\epsilon},p}\le\frac{1}{\left(1-k^2\right)^{\frac{1}{p}-\frac{1}{q_o}}}\ \|A\|_{\frac{K}{\epsilon},q_o}\left(\left\|h_z^{-1}\right\|_{\frac{1}{\epsilon},p_o}\right)^{1-\frac{2}{q_o}}\,.$$

(ii) *Let* $k(z) = 1\Big/\overline{h\left(\frac{1}{z}\right)}$ *and suppose that there exists* $a > 1$ *and* $Q > 1$ *such that*

$$\begin{aligned} |k(z)| &< a\,|z| && \text{for all } |z| < \epsilon \text{ and} \\ \left|k^{-1}(z)\right| &< Q\,|z|^{\alpha} && \text{for all } |z| < \epsilon\,. \end{aligned}$$

If $A\left(\frac{1}{z}\right) \in \mathcal{L}_q\big(|z| < a\,\epsilon\big)$, *then* $\left(\left(A \circ h^{-1}\right) h_z^{-1}\right)\left(\frac{1}{z}\right) \in \mathcal{L}_p\big(|z| < \epsilon\big)$ *and*

$$\left\|\left(\left(A \circ h^{-1}\right) h_z^{-1}\right)\left(\frac{1}{z}\right)\right\|_{\epsilon,p} \le \frac{a^2}{\left(1-k^2\right)^{\frac{1}{p}-\frac{1}{q_o}}} \left\|A\left(\frac{1}{z}\right)\right\|_{Q\epsilon^{\alpha},q_o} \left(\left\|h_z^{-1}\right\|_{\epsilon,p_o}\right)^{1-\frac{2}{q_o}}.$$

Proof. Let $p = \frac{p_o\,q_o}{p_o+q_o-2}$ and let $q, r > 0$ be such that

$$p\,q = q_o \quad \text{and} \quad (p-2)\,r + 2 = p_o.$$

In particular

$$\frac{1}{q} + \frac{1}{r} = 1, \qquad \frac{p_o}{pr} = 1 - \frac{2}{q_o} \quad \text{and} \quad \frac{1}{pr} = \frac{1}{p} - \frac{1}{q_o}\,.$$

We prove (i) first. We have

$$I := \int_{|z|<\frac{1}{\epsilon}} \left|A \circ h^{-1}\right|^{p} \left|h_z^{-1}\right|^{p} = \int_{|z|<\frac{1}{\epsilon}} \left|A\left(h^{-1}(z)\right)\right|^{p} \frac{1}{\left|h_z\left(h^{-1}(z)\right)\right|^{p}}\,.$$

Write $w = h^{-1}(z)$. Using the Jacobian

$$\text{Jac}\,h = |h_z|^2 - |h_{\overline{z}}|^2 = \left(1 - |\lambda|^2\right)\,|h_z|^2 \le |h_z|^2\,,$$

we have that

$$\begin{aligned} I &\le \int_{h^{-1}\left[|z|<\frac{1}{\epsilon}\right]} |A(w)|^{p} \frac{1}{|h_z(w)|^{p}}\,|h_z(w)|^2\,dw \\ &\le \left[\int_{h^{-1}\left[|z|<\frac{1}{\epsilon}\right]} |A|^{pq}\right]^{\frac{1}{q}} \left[\int_{h^{-1}\left[|z|<\frac{1}{\epsilon}\right]} \frac{1}{|h_z|^{(p-2)r}}\right]^{\frac{1}{r}}. \end{aligned}$$

But

$$\begin{aligned} \int\limits_{h^{-1}\left[|z|<\frac{1}{\epsilon}\right]} \frac{1}{|h_z|^{(p-2)r}} &= \int\limits_{h^{-1}\left[|z|<\frac{1}{\epsilon}\right]} \frac{\|h_z|^2}{|h_z|^{(p-2)r+2}} \\ &\le \frac{1}{1-k^2} \int_{|z|<\frac{1}{\epsilon}} \left|h_z^{-1}\right|^{(p-2)r+2} = \frac{1}{1-k^2} \int_{|z|<\frac{1}{\epsilon}} \left|h_z^{-1}\right\|^{p_o}. \end{aligned}$$

Therefore,

$$I^{\frac{1}{p}} \le \left[\int_{h^{-1}\left[|z|<\frac{1}{\epsilon}\right]} |A|^{q_o}\right]^{\frac{1}{q_o}} \left[\frac{1}{1-k^2}\right]^{\frac{1}{pr}} \left[\int_{|z|<\frac{1}{\epsilon}} \left|h_z^{-1}\right|^{p_o}\right]^{\frac{1}{pr}}.$$

Since $h^{-1}\left[|z|<\frac{1}{\epsilon}\right]\subseteq\left[|w|<\frac{K}{\epsilon^{\alpha}}\right]\subseteq\left[|w|\,,\frac{K}{\epsilon}\right]$, we have

$$I^{\frac{1}{p}}\leq\left[\frac{1}{1-k^2}\right]^{\frac{1}{p}-\frac{1}{q_o}}\|A\|_{\frac{K}{\epsilon},q_o}\left(\left\|h_z^{-1}\right\|_{\frac{1}{\epsilon},p_o}\right)^{1-\frac{2}{q_o}}\ .$$

For (ii) consider

$$\mathbf{I}:=\int_{|z|<\epsilon}\left|A\circ h^{-1}\left(\frac{1}{z}\right)\right|^p\left|h_z^{-1}\left(\frac{1}{z}\right)\right|^p=\int_{|z|<\epsilon}\left|A\circ h^{-1}\left(\frac{1}{\overline{z}}\right)\right|^p\left|h_z^{-1}\left(\frac{1}{\overline{z}}\right)\right|^p\ .$$

Write $\frac{1}{w}=h^{-1}\left(\frac{1}{\overline{z}}\right)$, i.e., $z=k(w)$. We have

$$h_z^{-1}\left(\frac{1}{\overline{z}}\right)=\frac{1}{h_z\left(h^{-1}\left(\frac{1}{\overline{z}}\right)\right)}=\frac{1}{h_z\left(\frac{1}{w}\right)}\cdot$$

By Corollary 2.10, we have that $k_{\overline{z}}(w)=\overline{\lambda\left(\frac{1}{w}\right)}\frac{w^2}{\overline{w}^2}$, hence

$$\frac{1}{1-k^2}\ |k_z|^2\leq\ \mathrm{Jac}(h)\leq|k_z|^2\ ,$$

then

$$\begin{aligned}\mathbf{I}\ &\leq\ \int_{k^{-1}[|z|<\epsilon]}\left|A\left(\frac{1}{\overline{w}}\right)\right|^p\frac{1}{\left|h_z\left(\frac{1}{\overline{w}}\right)\right|^p}\ |k_z(w)|^2\ ,\\ &\leq\ \left[\int_{k^{-1}[|z|<\epsilon]}\left|A\left(\frac{1}{w}\right)\right|^{pq}\right]^{\frac{1}{q}}\left[\int_{k^{-1}[|z|<\epsilon]}\frac{|k_z(w)|^{2r}}{\left|h_z\left(\frac{1}{\overline{w}}\right)\right|^{pr}}\right]^{\frac{1}{r}}\ .\end{aligned}$$

Since $k(w)=1\Big/\overline{h\left(\frac{1}{\overline{w}}\right)}$, we have that

$$\begin{aligned}k_z(w)\ &=\ \frac{\overline{h}_{\overline{z}}\left(\frac{1}{\overline{w}}\right)\left(-\frac{1}{w^2}\right)}{\overline{h\left(\frac{1}{\overline{w}}\right)}^{\,2}}=-\overline{h}_{\overline{z}}\left(\frac{1}{\overline{w}}\right)\frac{k(w)^2}{w^2}\\ \left|h_z\left(\frac{1}{\overline{w}}\right)\right|\ &=\ \left|\overline{h}_{\overline{z}}\left(\frac{1}{\overline{w}}\right)\right|=|k_z(w)|\ \frac{|w|^2}{|k(w)|^2}\\ \frac{|k_z(w)|^{2r}}{\left|h_z\left(\frac{1}{\overline{w}}\right)\right|^{pr}}\ &=\ \frac{|k_z(w)|^{2r}}{|k_z(w)|^{pr}}\cdot\frac{|k(w)|^{2pr}}{|w|^{2pr}}\leq a^{2pr}\ \frac{1}{|k_z(w)|^{(p-2)r}}\\ \int_{k^{-1}[|z|<\epsilon]}\frac{|k_z(w)|^{2r}}{\left|h_z\left(\frac{1}{\overline{w}}\right)\right|^{pr}}\ &\leq\ a^{2pr}\int_{k^{-1}[|z|<\epsilon]}\frac{1}{|k_z(w)|^{(p-2)r}}\\ &\leq\ a^{2pr}\int_{k^{-1}[|z|<\epsilon]}\frac{|k_z(w)|^2}{|k_z(w)|^{(p-2)r+2}}\\ &\leq\ \frac{a^{2pr}}{1-k^2}\int_{|z|<\epsilon}\frac{1}{|k_z\circ k^{-1}|^{p_o}}=\frac{a^{2pr}}{1-k^2}\int_{|z|<\epsilon}\left|k_z^{-1}\right|^{p_o}\ .\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\left(\left(A\circ h^{-1}\right)h_z^{-1}\right)\left(\frac{1}{z}\right)\right\|_{\epsilon,p} &\le \left\|A\left(\frac{1}{w}\right)\right\|_{Q\epsilon,q_o}\left(\frac{a^{2pr}}{1-k^2}\right)^{\frac{1}{pr}}\left(\left\|k_z^{-1}\right\|_{\epsilon,p_o}\right)^{\frac{p_o}{pr}}\\ &\le \frac{a^2}{\left(1-k^2\right)^{\frac{1}{p}-\frac{1}{q_o}}}\left\|A\left(\frac{1}{w}\right)\right\|_{Q\epsilon,q_o}\left(\left\|h_z^{-1}\right\|_{\epsilon,p_o}\right)^{1-\frac{2}{q_o}}\ . \quad \square\end{aligned}$$

Given $\mu\in\mathcal{L}_p(\mathbb{D})$, $\mathbb{D}=\{z\in\mathbb{C}\mid |z|<1\}$, extend it to $\mathcal{L}_\infty(\mathbb{D})$ by

$$\mu(z)=\overline{\mu\left(\frac{1}{\bar z}\right)}\,\frac{z^2}{\bar z^2}\,,$$

denote by $\widehat{\mu}$ this extension and consider $\mathcal{L}_\infty(\mathbb{D})$ as a subspace of $\mathcal{L}_\infty(\mathbb{C})$ by these extensions. On $\mathcal{L}_\infty(\mathbb{C})\cap W^{1,p}(\epsilon)$ consider the norm

$$\|\mu\|_{LE}:=\|\mu\|_{\mathcal{L}_\infty(\mathbb{C})}+\|\mu\|_{W^{1,p}(\epsilon)}$$

and on $\mathcal{D}(\epsilon,p,F)\cap\mathcal{L}_p(D_R)$ consider the norm

$$\|\sigma\|_{DP}:=\|\sigma\|_{\mathcal{D}(\epsilon,p,F)}+\|\sigma\|_{R,p}\ .$$

Proposition 2.28. *Suppose that $\mu_o\in\mathcal{L}_\infty(\mathbb{C})\cap W^{1,p_o}(\epsilon)$, $\|\mu_o\|<k<1$, $kC_{p_o}<1$, $p_o>2$ and let $F=f^{\widehat{\mu}_o}$. Then the map*

$$\left(\mathcal{L}_\infty(\mathbb{D})\cap W^{1,p}(\epsilon)\right)\times\left(\mathcal{D}(\epsilon,p,F)\cap\mathcal{L}_p(D_R)\right)\to B_{r,p}$$

given by $(\mu,\sigma)\mapsto\omega^{\widehat{\mu},\sigma}$, is differentiable for μ in a neighborhood of μ_o, for all $0<r<R$ and any $2<p<p_o$.

Proof. Write $\omega^o:=\omega^{\widehat{\mu}_o,\sigma_o}$ and for $(\mu,\sigma)\in\left(\mathcal{L}_\infty(\mathbb{D})\cap W^{1,p}(\epsilon)\right)\times\left(\mathcal{D}(\epsilon_o,p,F)\cap\mathcal{L}_p(D_R)\right)$ write $\omega=\omega^{\widehat{\mu},\sigma}$. For simplicity write $\mu=\widehat{\mu}\in\mathcal{L}_\infty(\mathbb{C})$. Let $\nu:=\mu-\mu_o$ and $\rho:=\sigma-\sigma_o$. By Proposition 2.24(i), $\omega_z^o\in\mathcal{D}(\epsilon_o,p,F)$ for all $2<p<p_o$ and hence there exists a solution of

$$\ell_{\bar z}=\mu_o\,\ell_z+\nu\,\omega_z^o+\rho \tag{2.29}$$

such that $\ell(0)=\ell(1)=0$ and $|\ell(z)|=\mathcal{O}(|F(z)|^2)$ when $|z|\to\infty$. Moreover, since

$$\begin{aligned}\left\|\nu\,\omega_z^o+\rho\right\|_{\mathcal{D}(\epsilon_o,p,F)} &\le \|\nu\|_\infty\left\|\omega_z^o\right\|_{\mathcal{D}(\epsilon_o,p,F)}+\|\rho\|_{\mathcal{D}(\epsilon_o,p,F)}\ ,\\ \left\|\nu\,\omega_z^o+\rho\right\|_{R,p} &\le \|\nu\|_\infty\left\|\omega_z^o\right\|_{R,p}+\|\rho\|_{R,p}\ .\end{aligned} \tag{2.30}$$

By Proposition 2.24 (i), $\left\|\omega_z^o\right\|_{\mathcal{D}(\epsilon,p,F)}<+\infty$ and by Proposition 2.24 (iii), $\left\|\omega_z^o\right\|_{R,p}<+\infty$. Therefore, by Proposition 2.24 (iii), the linear map $L(\nu,\rho)=\ell\in B_{R,p}$ is continuous. In particular, for all $2<p<p_o$, we have that

$$\lim_{\substack{\mu\to\mu_o\\ \sigma\to\sigma_o}}\|\ell\|_{B_{R,p}}=0\quad,\quad\lim_{\substack{\mu\to\mu_o\\ \sigma\to\sigma_o}}\|\ell_z\|_{R,p}=0\,. \tag{2.31}$$

By Proposition 2.24 (iv) we also have that

$$\lim_{\substack{\mu\to\mu_o\\ \sigma\to\sigma_o}}\|\ell_z\|_{\mathcal{D}(\epsilon,p,F)}\le A\lim_{\substack{\mu\to\mu_o\\ \sigma\to\sigma_o}}\left\|\nu\,\omega_z^o+\rho\right\|_{\mathcal{D}(\epsilon_o,p,F)}=0\,,$$

and by Lemma 2.22,

$$\lim_{\substack{\mu\to\mu_o\\ \sigma\to\sigma_o}} \|\ell_z\|_{\mathcal{D}(r,p,F)} = 0 \quad \text{for all } r>0,\ \ 2<p<p_o\,. \tag{2.32}$$

Let $h := \omega^{\mu,\sigma} - \omega^{\mu_o,\sigma_o} - \ell$, then

$$h_{\bar z} = \mu\, h_z + \nu\, \ell_z\,. \tag{2.33}$$

Let $H^\mu = H := f^\lambda$, where

$$\lambda = \lambda^u := \left(\frac{\mu-\mu_o}{1-\mu\,\mu_o}\,\frac{F_z}{\overline{F_{\bar z}}}\right)\circ F^{-1},$$

where $F = f^{\mu_o}$. We have that $F^\mu := f^\mu = H^\mu \circ F$.

We now see that we can use H^μ on Lemma 2.27. Let $\eta_2 > 0$ be such that $\|\lambda^\mu\|_\infty < k$ for all $\|\mu-\mu_o\| < \eta_2$. From Corollary 2.19, we obtain that $\mu \mapsto H^\mu \in C^1(\epsilon_2)$ is continuous for some $0<\epsilon_2<\epsilon_o$. In particular, there exists $0<\eta_3<\eta_2$ and $a>1$ such that

$$\left|H^\mu(z)\right| < a\ |z| \ \text{ for all } |z| < \epsilon_3 := \frac{\epsilon_2}{2} \text{ and all } \|\mu-\mu_o\| < \eta_3\,.$$

From the definition of $\lambda = \lambda^\mu$ we get that $\lambda(z) = \overline{\lambda\left(\frac{1}{\bar z}\right)}\,\frac{z^2}{\bar z^2}$ for almost every $z\in\mathbb{C}$. Therefore, writing $G^\mu(z) := 1\big/\overline{H^\mu\left(\frac{1}{\bar z}\right)}$, we have that $G^\mu = H^\mu$.

Observe that $\lambda^{\mu_o}\equiv 0$ and $H^{\mu_o} = Id$. By Corollary 2.10, we have that $(H^\mu)^{-1} = f^{\widehat\lambda}$, where $\widehat\lambda = -\left(\lambda\,\frac{H_z}{\overline{H_{\bar z}}}\right)\circ H^{-1}$. In particular, for any $0<\delta<1$, there exists $0<\eta_4=\eta_4(\delta)<\eta_3$ such that $\left\|\widehat\lambda^\mu\right\|_\infty = \|\lambda^\mu\|_\infty < \delta$ for all $\|\mu-\mu_o\|<\eta_4$. By Theorem 2.12 (a), for any $r_o = r_o(\delta)>0$ with $\delta\, C_{r_o(\delta)} < 1$ and some $K = K(\epsilon_3,\delta,r_o)>1$, $C = C(\epsilon_3,\delta,r_o)>1$, we have that

$$\begin{aligned}
\left|\left(H^\mu\right)^{-1}(z)\right| &< K\ |z|^{1-\frac{2}{r_o(\delta)}} && \text{for all } |z| < \frac{1}{\epsilon_3} \text{ and all } \|\mu-\mu_o\| < \eta_4\,,\\
\left\|\left(H^\mu\right)^{-1}_z\right\|_{\frac{1}{\epsilon_3},r_o(\delta)} &< C\,(\epsilon_3,k) && \text{for all } |\mu-\mu_o| < \eta_4\,,
\end{aligned}$$

with $r_o(\delta)\to\infty$ when $\delta\to 0$ and $\eta_4\to 0$.

Therefore, the conditions on Lemma 2.27 are satisfied by H^μ with uniform constants a, K, ϵ_3 for all $\|\mu-\mu_o\|<\eta_4$ and with $\alpha = 1-\frac{2}{r_o(\delta)}$.

For any g we have that

$$\left(g\circ\left(F^\mu\right)^{-1}\right)\left(F^\mu\right)^{-1}_z = \left(h\circ\left(H^\mu\right)^{-1}\right)\left(H^\mu\right)^{-1}_z\,,$$

where $h = (g\circ F^{-1})\,F_z^{-1}$, $F = f^{\mu_o}$.

Given $0<p<p_o$ choose $0<\delta<1$ (hence $\eta_4(\delta)>0$) and $p<q_o=q_o(p)<p_o$ such that

$$0 < p \le \frac{q_o\, r_o(\delta)}{q_o + r_o(\delta) - 1}\,.$$

Applying Lemma 2.27, we have that $\ell_z \in \mathcal{D}(\epsilon_3, p, f^\mu)$ and

$$\|\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} \leq \frac{a^2}{(1-k^2)^{\frac{1}{p}-\frac{1}{q_o}}} \left(\|\ell_z\|_{\mathcal{D}(\frac{K}{\epsilon_3},q_o,F)} + \|\ell_z\|_{\mathcal{D}(K\,\epsilon^\alpha,q_o,F)}\right) \cdot C\,(\epsilon_3, k) \ .$$

In particular, by Lemma 2.22 and (2.32), we have that

$$\lim_{\substack{\mu\to\mu_o \\ \sigma\to\sigma_o}} \|\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} = 0 \text{ for all } 2 < p < p_o \ . \tag{2.34}$$

By Proposition 2.24 and (2.33), we have that

$$\begin{aligned} \|h\|_{B_{r,p}} &\leq C(R,r)\ \left(\|\nu\,\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} + \|\nu\,\ell_z\|_{R,p}\right) \\ &\leq C(R,r)\ \left(\|\nu\|_\infty\ \|\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} + \|\nu\|_\infty\ \|\ell_z\|_{R,p}\right) , \\ \frac{\|h\|_{B_{r,p}}}{\|\nu\|_\infty} &\leq C(R,r)\ \left(\|\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} + \|\ell_z\|_{R,p}\right) , \end{aligned}$$

for all $R > 0$ and any $2 < p < p_o$. From (2.31) and (2.34) we get that

$$\lim_{\substack{\mu\to\mu_o \\ \sigma\to\sigma_o}} \frac{\|h\|_{B_{r,p}}}{\|\nu\| + \|\rho\|_{\mathcal{D}(\epsilon_3,p,F)}} \leq \lim_{\substack{\mu\to\mu_o \\ \sigma\to\sigma_o}} \frac{\|h\|_{B_{r,p}}}{\|\nu\|_\infty} = 0 \,.$$

By Proposition 2.24 (iv) and Lemma 2.22, we also have that for $2 < p < p_1 < p_o$,

$$\begin{aligned} \|\ell_z\|_{\mathcal{D}(\epsilon,p,F)} &\leq A\ \|\nu\,\ell_z\|_{\mathcal{D}(\epsilon,p_1,F)} \leq A\ \|\nu\|_\infty\ \|\ell_z\|_{\mathcal{D}(\epsilon_3,p_1,F)} \\ \lim_{\substack{\mu\to\mu_o \\ \sigma\to\sigma_o}} \frac{\|h_z\|_{\mathcal{D}(\epsilon,p,F)}}{\|\nu\| + \|\rho\|_{\mathcal{D}(\epsilon_3,p,F)}} &\leq A\ \lim_{\substack{\mu\to\mu_o \\ \sigma\to\sigma_o}} \|\ell_z\|_{\mathcal{D}(\epsilon_3,p_1,F)} = 0 \,. \end{aligned} \tag{2.35}$$

On $\mathcal{L}_\infty(\mathbb{C}) \cap W^{n,p}(R)$ and on $\mathcal{D}(\epsilon, p, F) \cap W^{n,p}(R)$ consider the norms

$$\begin{aligned} \|\mu\|_{LW} &:= \|\mu\|_{\mathcal{L}_\infty(\mathbb{C})} + \|\mu\|_{W^{n,p}(R)} \ , \\ \|\sigma\|_{DW(F)} &:= \|\sigma\|_{\mathcal{D}(\epsilon,p,F)} + \|\sigma\|_{W^{n,p}(R)} \ . \end{aligned}$$

Proposition 2.36. *Suppose that $\mu_o \in \mathcal{L}_\infty(\mathbb{D}) \cap W^{n,p_o}(R)$, $\|\mu_o\|_\infty < k < 1$, $k\,C_{p_o} < 1$, $p_o > 2$ and let $F = f^{\mu_o}$. Then the map $(\mu, \rho) \mapsto \omega^{\mu,\rho}$,*

$$\left(\mathcal{L}_\infty(\mathbb{D}) \cap W^{n,p_o}(R)\right) \times \left(\mathcal{D}(\epsilon, p_o, F) \cap W^{n,p_o}(R)\right) \to W^{n+1,p}(r)$$

is differentiable for μ in a neighborhood of μ_o for all $0 < r < R$ and any $2 < p < p_o$.

Proof. We have the same Equations (2.29), (2.30), (2.33), and (2.34) from Proposition 2.28. Also,

$$\left\|\nu\,\omega_z^o + \rho\right\|_{W^{n,p}(R)} \leq 2^n\ \|\nu\|_{W^{n,p}(R)}\ \left\|\omega_z^o\right\|_{W^{n,p}(R)} + \|\rho\|_{W^{n,p}(R)} \ .$$

By Proposition 2.24 (v), $\left\|\omega_z^o\right\|_{W^{n,p}(r)}$ is finite for all $0 < r < R$ and $2 < p < p_o$. Using (2.30) and Proposition 2.24 (v), we get that

$$\lim_{\substack{\mu\to\mu_o \\ \sigma\to\sigma_o}} \|\ell\|_{W^{n+1,p}(r)} = 0 \tag{2.37}$$

for all $0 < r < R$, $2 < p < p_o$. In particular, the linear map $L(\nu, \rho) = \ell \in W^{n+1,p}(r)$ is continuous. From Equation (2.33) and Proposition 2.24 (v) we have that, for $S := \frac{R+r}{2}$,

$$\begin{aligned}
\|h\|_{W^{n+1,p}(r)} &\leq C(S,r)\ \left(\|\nu\,\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} + \|\nu\,\ell_z\|_{W^{n,p}(S)}\right) \\
&\leq C(S,r)\ \left(\|\nu\|_\infty\ \|\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} + 2^n\ \|\nu\|_{W^{n,p}(R)}\ \|\ell_z\|_{W^{n,p}(S)}\right) \\
\frac{\|h\|_{W^{n+1,p}(r)}}{\|\nu\|_{W^{n,p}(r)}} &\leq C(S,r)\ \left(\|\ell_z\|_{\mathcal{D}(\epsilon_3,p,f^\mu)} + 2^n\ \|\ell_z\|_{W^{n,p}(S)}\right).
\end{aligned}$$

By (2.34) and (2.37) we have that

$$\lim_{\substack{\mu\to\mu_o \\ \sigma\to\sigma_o}} \frac{\|h\|_{W^{n+1,p}(r)}}{\|\nu\|_{LW} + \|\rho\|_{DW(F)}} \leq \lim_{\substack{\nu\to 0 \\ \rho\to 0}} \frac{\|h\|_{W^{n+1,p}(r)}}{\|\nu\|_{W^{n,p}(r)}} = 0$$

for all $2 < p < p_o$ and $0 < r < R$. This completes the proof. □

Corollary 2.38. *The maps of Propositions 2.28 and 2.36 are C^∞.*

Proof. We prove the corollary for the map in Proposition 2.36, the proof for the other map is similar. Define the following Banach spaces: $\mathbf{E} := \mathcal{L}_\infty(\mathbb{C}) \cap W^{n,p}(R)$, $\mathbf{F} := \mathcal{D}(\epsilon, p, F) \cap W^{n,p}(R)$, $\mathbf{G} := W^{n+1,p}(r) \cap \mathcal{F}(\epsilon, p, F)$, $\mathcal{F}(\epsilon, p, F) := \{\, \ell \mid \ell_z \in \mathcal{D}(\epsilon, p, F) \,\}$ with $\|\ell\|_{\mathcal{F}} := \|\ell_z\|_{\mathcal{D}(\epsilon,p,F)}$ and $\mathcal{L}(\mathbf{E}\times\mathbf{F}, \mathbf{G}) := \{\, L : \mathbf{E}\times\mathbf{F} \to \mathbf{G} \mid \mathcal{L} \text{ linear} \,\}$.

There is no map $\mathbf{G} \to \mathbf{F}$ given by $\omega \to \omega_z$ because $r < R$. We leave to the reader the technicalities that appear with this problem. Define the maps $F : U \times \mathbf{F} \subseteq \mathbf{E}\times\mathbf{F} \to \mathbf{G}$, $F(\mu,\sigma) = \omega^{\mu,\sigma}$ where $U \subseteq \mathbf{E}$ is the open subset defined in Proposition 2.36. Let $\overline{F} : U \subseteq \mathbf{E} \to \mathcal{L}(\mathbf{F}, \mathbf{G})$, $\overline{F}(\mu)\cdot\sigma = \omega^{\mu,\sigma}$; and $D : U \times \mathbf{F} \to \mathcal{L}(\mathbf{E}\times\mathbf{F}, \mathbf{G})$, $D(\mu,\sigma)(\nu,\rho) := \ell$, the derivative on Equation (2.29) of Proposition 2.28. Let $B : \mathbf{E}\times\mathbf{G} \to \mathbf{F}$ be the linear map $B(\nu,\omega) = \nu\,\omega_z$. We have that

$$\begin{aligned}
D(\mu,\sigma)(\nu,\rho) &= \overline{F}(\mu)\circ B\,(\nu, F(\mu,\sigma)) + \overline{F}(\mu)(\rho) \\
D(\mu,\sigma) &= \overline{F}(\mu)\circ B\,(\pi_1, F(\mu,\sigma)) + \overline{F}(\mu)\circ\pi_2
\end{aligned} \tag{2.39}$$

where $\pi_1 : \mathbf{E}\times\mathbf{F} \to \mathbf{E}$ and $\pi_2 : \mathbf{E}\times\mathbf{F} \to \mathbf{F}$ are the projections. We have that

$$\begin{aligned}
\|B(\nu,\omega)\|_{\mathbf{F}} &= \|\nu\,\omega_z\|_{DW^n} \leq \|\nu\|_\infty\ \|\omega_z\|_{\mathcal{D}(\epsilon,p,F)} + 2^n\ \|\nu\|_{W^n}\ \|\omega_z\|_{W^n} \\
&\leq 2^n\ \|\nu\|_{W^n}\ \left(\|\omega\|_{\mathcal{F}} + \|\omega\|_{W^{n+1}}\right) \\
&\leq 2^n\ \|\nu\|_{\mathbf{E}}\ \|\omega\|_{\mathbf{G}}\,.
\end{aligned}$$

Therefore, the bilinear map B is C^∞. By Proposition 2.36 and the limit (2.35) in the proof of Proposition 2.28, we have that $F : U \times \mathbf{F} \to \mathbf{G}$ is differentiable. Using the notation of Propositions 2.28 and 2.36, we have that

$$\begin{aligned}
\left\|\overline{F}(\mu+\nu)(\sigma) - \overline{F}(\mu)(\sigma) - \ell\right\|_{\mathbf{G}} &= \|h\|_{W^{n+1}} + \|h\|_{\mathcal{F}} \\
&\leq A_1(\mu)\ \|\nu\,\ell_z\|_{DW^n} \\
&\leq A_1(\mu)\ \left(\|\nu\|_\infty\ \|\ell_z\|_{\mathcal{D}(\epsilon,p,F)} + 2^n\ \|\nu\|_{W^n}\ \|\ell_z\|_{W^n}\right) \\
&\leq A_1(\mu)\ \|\nu\|_{W^n}\ 2^n\ \left(\|\ell_z\|_{\mathcal{D}(\epsilon,p,F)} + \|\nu\|_{W^n}\right) \\
&\leq A_1(\mu)\ \|\nu\|_{\mathbf{E}}\ A_2(\mu)\ \left\|\nu\,\omega_z^o + (\rho\equiv 0)\right\|_{DW^n}
\end{aligned}$$

$$\begin{aligned} &\leq A_3(\mu)\,\|\nu\|_{\mathbf{E}}\left(\|\nu\|_\infty\,\|\omega_z^\rho\|_{\mathcal{D}(\epsilon,p,F)}+2^n\,\|\nu\|_{W^n}\,\|\omega\|_{W^{n+1}}\right)\\ &\leq A_4(\mu)\,\|\nu\|_{\mathbf{E}}\,\|\nu\|_{\mathbf{E}}\,\|\sigma\|_{\mathbf{F}} \end{aligned}$$

$$\lim_{\|\nu\|_{\mathbf{E}}\to 0}\frac{1}{\|\nu\|_{\mathbf{E}}}\max_\sigma\left\{\frac{\|h\|_{\mathbf{G}}}{\|\sigma\|_{\mathbf{F}}}\right\} \leq \lim_{\nu\to 0} A_4(\mu)\,\|\nu\|_{\mathbf{E}}=0\,.$$

Therefore, the map $\overline{F}$ is differentiable and its derivative is given by $(D\overline{F}(\mu)\cdot\nu)(\sigma) = D(\mu,\sigma)(\nu,0)$, or

$$D\overline{F}(\mu)\cdot\nu = D(\mu,\cdot)(\nu,0)\,. \tag{2.40}$$

Suppose that F and $\overline{F}$ are r-times differentiable. Then from formula (2.39) we have that D is r-times differentiable. But D is the derivative of F so that F is $(r+1)$-times differentiable. Formula (2.40) implies that $\overline{F}$ is also $(r+1)$-times differentiable. We conclude that F is C^∞. □

Theorem 2.41.

(i) *Let $0<k<1$ and $p>2$ with $k\,C_p<1$. Then for any $R>0$, the map*

$$\left\{\mu\in\mathcal{L}_\infty(\mathbb{D})\mid\|\mu\|_\infty<k\right\}\longrightarrow B_{R,p}$$

given by $\mu\mapsto f^{\widehat{\mu}}$ is C^∞.

(ii) *Let $0<k<1$ and $p>2$ with $k\,C_p<1$. Then the map*

$$\mathcal{L}_\infty(\mathbb{D})\cap W^{n,p}(R)\cap\left\{\|\mu\|_\infty<k\right\}\longrightarrow W^{n+1,p}(r)$$

given by $\mu\mapsto f^{\widehat{\mu}}$ is C^∞ for any $0<r<R$.

(iii) *In particular, for any $n\geq 1$ and any $0<r<S<R$ the map*

$$\mathcal{L}_\infty(\mathbb{D})\cap C^n(S)\cap\left\{\|\mu\|_\infty<k\right\}\longrightarrow C^{n+1-\frac{2}{p}}(r)\cap C^{1-\frac{2}{p}}(R)$$

given by $\mu\mapsto f^{\widehat{\mu}}$ is C^∞.

Proof. Define the spaces $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$, $\mathcal{L}(\mathbf{F},\mathbf{G})$ and the maps $\overline{F}(\mu)\cdot\sigma=\omega^{\mu,\sigma}$ and $B:\mathbf{E}\to\mathcal{L}(\mathbf{F},\mathbf{G})$, $B(\nu,\omega)=\nu\,\omega_z$ as in the proof of Corollary 2.38. We have that $\overline{F}$ and B are C^∞. Define the map $H:\mathbf{E}\to\mathbf{G}$ by $H(\mu):=f^\mu$.

Claim. *H is differentiable and $DH(\mu)\cdot\nu=\omega^{\mu,\nu}f_z^\mu$, i.e.,*

$$DH(\mu)=\overline{F}(\mu)\circ B\left(\cdot,H(\mu)\right)\,. \tag{2.42}$$

□

Suppose that the claim is true. From formula (2.42) we have that if H is r-times differentiable, then DH is r-times differentiable and hence H is $(r+1)$-times differentiable. By the claim, the induction starts at $r=1$ and then H is C^∞. □

Proof of the Claim. Let $\mu,\nu\in\mathbf{E}$, $\omega:=\omega^{\mu,\nu f_z^\mu}$, $h:=f^{\mu+\nu}-f^\mu-\omega$. Then

$$\begin{aligned} h_{\bar z} &= (\mu+\nu)\,h_z+\nu\,\omega_z\\ \omega_{\bar z} &= \mu\,\omega_z+\nu\,f_z \end{aligned}$$

with $h(0) = h(1) = \omega(0) = \omega(1) = 0$, $|\omega(z)| = \mathcal{O}\big(|f^{\mu}(z)|^2\big)$ and $|h(z)| = \mathcal{O}\big(\big|f^{\mu+\nu}(z)\big|^2\big)$. We have that

$$\begin{aligned}
\|H(\mu+\nu) - H(\mu) - \omega\|_{\mathbf{G}} &= \|h\|_{W^{n+1}} + \|h\|_{\mathcal{F}} \\
&\le A\, \big(\|\nu\, \omega_z\|_{W^n} + \|\nu\, \omega_z\|_{\mathcal{D}(\epsilon,p,f^{\mu})}\big) \\
&= A\, \|B(\nu,\omega)\|_{\mathbf{F}} \\
&\le 2^n\, A\, \|\nu\|_{\mathbf{E}}\, \|\omega\|_{\mathbf{G}} \\
&\le 2^n\, A\, \|\nu\|_{\mathbf{E}}\, A\, \big\|\nu\, f_z^{\mu}\big\|_{\mathbf{F}} \\
&\le 2^n\, A\, \|\nu\|_{\mathbf{E}}\, A\, 2^n\, \|\nu\|_{\mathbf{E}}\, \big\|f^{\mu}\big\|_{\mathbf{G}}
\end{aligned}$$

with $\|f\|_{\mathbf{G}} = \|f\|_{W^{n+1}} + \|f_z\|_{\mathcal{D}(\epsilon,p,f^{\mu})}$. By considering small bump functions λ as in Proposition 2.24 (v), one can show that $f \in W^{n+1,p}(R)$, hence $\|f\|_{W^{n+1}} < +\infty$. We have that $\|f_z\|_{\mathcal{D}(\epsilon,p,f^{\mu})} = \|1\|_{\mathcal{D}_{\epsilon,p}} < +\infty$. Therefore,

$$\lim_{\nu\to 0} \frac{\|H(\mu+\nu) - H(\mu) - \omega\|_{\mathbf{G}}}{\|\nu\|_{\mathbf{E}}} \le \lim_{\nu\to 0} \big(2^n\, A\big)^2\, \|\nu\|_{\mathbf{E}}\, \|f\|_{\mathbf{G}} = 0\,.$$

And the linear map $DH(\mu) : \nu \mapsto \omega$ is continuous because

$$\|\omega\|_{\mathbf{G}} \le A\, \big\|\nu\, f_z^{\mu}\big\|_{\mathbf{F}} \le 2^n\, A\, \big\|f^{\mu}\big\|_{\mathbf{G}}\, \|\nu\|_{\mathbf{E}}\,.$$

This proves the claim. □

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