HOMOGENIZATION ON ARBITRARY MANIFOLDS

GONZALO CONTRERAS, RENATO ITURRIAGA, AND ANTONIO SICONOLFI

Abstract. We describe a setting for homogenization of convex hamiltonians on abelian covers of any compact manifold. In this context we also provide a new simple variational proof of standard homogenization results.

1. Introduction

An homogenization result refers to the convergence of solutions u_ϵ of a problem P_ϵ with an increasingly fast variation,parametrized by ϵ, to a function u_0 solution of an "averaged" problem P_0. In this paper we propose a setting in which homogenization results for the Hamilton-Jacobi equation which have only been obtained in the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, or equivalently in $\mathbb{R}^n$ with $\mathbb{Z}^n$-periodic conditions, can be carried out to arbitrary compact manifolds in a natural way. Moreover we show a new and very simple proof of the homogenization result.

We choose to present the simplest model of homogenization introduced in the celebrated paper by Lions, Papanicolaou and Varadhan [8], leaving more sophisticated versions for future work. We believe that this setting will allow to translate many classical homogenization results in $\mathbb{T}^n$ to more general manifolds.

A *Tonelli Hamiltonian* on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a C^2 function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which is

(a) $\mathbb{Z}^n$*-periodic*, i.e. $H(x + \mathbf{z}, p) = H(x, p)$, for all $\mathbf{z} \in \mathbb{Z}^n$.

(b) *Convex:* The Hessian $\dfrac{\partial^2 H}{\partial p^2}(x, p)$ is positive definite for all (x, p).

(c) *Superlinear:* $\displaystyle\lim_{p\to\infty} \frac{H(x, p)}{|p|} = +\infty$, uniformly on x.

In the setting of [8] one considers a small parameter $\varepsilon > 0$ and the initial value problem for the Hamilton-Jacobi equation

$$\partial_t u^\varepsilon + H(\tfrac{x}{\varepsilon}, \partial_x u^\varepsilon) = 0, \tag{1}$$

$$u^\varepsilon(x, 0) = f_\varepsilon(x); \tag{2}$$

If f_ε is Lipschitz on $\mathbb{R}^n$, from [3], [7], [4] we know that there is a unique viscosity solution of the problem (1)-(2). Lions, Papanicolaou and Varadhan prove in [8] that if $f_\varepsilon \to f$

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uniformly when $\varepsilon \to 0$ and f is Lipschitz, the solutions u^ε converge uniformly to the unique viscosity solution of the problem (3)-(4):

$$\partial_t u + \overline{H}(\partial_x u) = 0, \tag{3}$$

$$u(x,0) = f(x); \tag{4}$$

where $\overline{H}$ is a convex hamiltonian which does not depend on x and is called the *effective hamiltonian.*

Equation (1) is seen as the Hamilton-Jacobi equation for a modified hamiltonian

$$H_\varepsilon(x,p) := H(\tfrac{x}{\varepsilon}, p). \tag{5}$$

And it is said that $H_\varepsilon \to \overline{H}$ in the sense that the solutions of their Hamilton-Jacobi equations converge as stated by the homogenization result. The homogenization is interpreted as the convergence of solutions of Hamilton-Jacobi equations and of its action minimizing characteristics when the space is *"seen from far away"*. Alternatively, one says that the limiting problems have a slow variable p and a fast oscillating variable $\frac{x}{\varepsilon}$ which is *"averaged"* by the homogenization limit.

The effective hamiltonian $\overline{H}$ is usually highly non-differentiable, but the solutions of the problem (3)-(4) are easily written because the characteristic curves for the equation (3) are the straight lines, and $p = d_x u$ is constant along them. Thus

$$u(y,t) = \min_{x\in\mathbb{R}^n} \left\{ f(x) + t\,\overline{L}\left(\tfrac{y-x}{t}\right)\right\}, \tag{6}$$

where

$$\overline{L}(v) := \max_{p\in\mathbb{R}^n} \left[\, p(v) - \overline{H}(p)\,\right] \tag{7}$$

is the *effective lagrangian.* The simplicity of this (limit) solution and the possibility of using coarse grids in numerical analysis are the main advantage of the homogenization in applications.

It turns out that the effective lagrangian $\overline{L}$ is Mather's minimal action functional β and $\overline{H}$ is Mather's α function. Indeed knowing that : 1) The convergence holds. 2) The fact from Fathi's weak KAM theory [6], [5], that there is only one constant, $\alpha(P)$, such that there are viscosity solutions of the Hamilton Jacobi equation

$$H(x, P + d_x v) = \alpha(P).$$

And 3) Using the special case of affine initial conditions, $a \in \mathbb{R}^n$, $P \in (\mathbb{R}^n)^*$,

$$f(x) = u(x,0) = a + P\cdot x. \tag{8}$$

It is easy to show that the the effective hamiltonian $\overline{H}$ is Mather's alpha function.

From this we obtain several interpretations for the effective hamiltonian, for example as Mañé's critical value [9], $\overline{H}(P) = c(L-P)$, $L = H^*$, when the free time minimal action becomes bounded; or the energy of Mather's minimizing measures for $L - P$, see [1]; or the min-max formula

$$\overline{H}(P) = \min_{u\in C^1(\mathbb{T}^n,\mathbb{R})} \max_{x\in\mathbb{T}^n} H(x, P + d_x u),$$

with its symplectic interpretation [2].

Let M be a compact riemannian manifold without boundary. A *Tonelli hamiltonian* on M is a C^2 function $H : T^*M \to \mathbb{R}$ on the cotangent bundle T^*M which is convex and superlinear as in (b), (c) above. We want to generalize Lions, Papanicolau, Varadhan Theorem to this setting. The generalization of the previous setting to other compact manifolds has three problems, namely

(1) It is not clear how to choose the generalization of $\frac{x}{\varepsilon}$.
(2) In the modified hamiltonian H_ε in (5) the base point changes to $\frac{x}{\varepsilon}$ but the moment p *"remains the same"*. It is not clear how to do this in non-parallelizable manifolds.
(3) Mather's alpha function, the candidate for effective hamiltonian, is defined in the first cohomology group $\alpha : H^1(M,\mathbb{R}) \approx \mathbb{R}^k \to \mathbb{R}$, which may not be a cover of the manifold. Thus the (limiting) effective hamiltonian and the Hamilton-Jacobi equations for H_ε would be defined in very different spaces. In particular, these spaces usually have different dimensions.

To solve the last problem we will use an ad hoc definition of convergence of spaces very much inspired in the Gromov Hausdorff convergence. For the second problem a change of variables in the torus allows to change the parameter ϵ in the space variables $\frac{x}{\varepsilon}$ to the momentum variables. Indeed

Write

$$u^\varepsilon(x,t) = v^\varepsilon\big(\tfrac{x}{\varepsilon}, t\big). \tag{9}$$

Then the problem (1)-(2) for $v^\varepsilon(y,t)$ becomes

$$\partial_t v^\varepsilon + H(y, \tfrac{1}{\varepsilon}\partial_y v^\varepsilon) = 0. \tag{10}$$

$$v^\varepsilon(y,0) = f_\varepsilon(\varepsilon y). \tag{11}$$

Observe that now equation (10) makes sense in any manifold, but equation (11) does not. We will take care of that afterwards.

Given a metric space (M,d), a family of metric spaces (M_n,d_n) and continuous maps $F_n : (M_n,d_n) \to (M,d)$, we say that $\lim_n(M_n,d_n,F_n) = (M,d)$ iff

(1) There are $K_n > 0$ and $A_n > 0$ such that $\lim_n A_n = 0$, K_n, K_n^{-1} are bounded and $\forall x,y \in M_n$,

$$\forall x,y \in M_n, \qquad K_n^{-1}\, d_n(x,y) - A_n \le d\big(F_n(x),F_n(y)\big) \le K_n\, d_n(x,y). \tag{12}$$

(2) For any $x \in M$ there is a sequence $x_n \in M_n$ such that $\lim_n F_n(x_n) = x$.

Observe that condition (1) implies that $\lim_n \operatorname{diam} F_n^{-1}\{y\} = 0$ for all $y \in M$. If $\lim_n(M_n, d_n, F_n) = (M, d)$ and $f_n : M_n \to \mathbb{R}$, $f : M \to \mathbb{R}$, we say that $\lim_n f_n = f$ *uniformly* iff

$$\lim_n \sup_{x \in M_n} |f_n(x) - f(F_n(x))| = 0.$$

We say that $\lim_n f_n = f$ *uniformly on compact sets* if for any compact subset $K \subset M$

$$\lim_n \sup_{x \in F_n^{-1}(K)} |f_n(x) - f(F_n(x))| = 0.$$

We say that the family f_n is *equicontinuous* if $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$x, y \in M_n, \quad d_n(x, y) < \delta \quad \Longrightarrow \quad |f_n(x) - f_n(y)| < \varepsilon.$$

The initial Hamiltonian will be the lift of H to the maximal free abelian cover of M defined as follows. Let $\tilde{M}$ be the covering space of M defined by $\pi_1(\tilde{M}) = \ker \mathfrak{h}$, where $\mathfrak{h} : \pi_1(M) \to H_1(M, \mathbb{R})$ is the Hurewicz homomorphism. Its group of deck tranformations is $\mathbb{G} = \operatorname{im}[\pi_1(M) \to H_1(M, \mathbb{R})]$, which is a free abelian group, $\mathbb{G} \approx \mathbb{Z}^k \subset H_1(M, \mathbb{R}) \approx \mathbb{R}^k$. Observe that the large-scale structure of the covering space $\tilde{M}$ is given by $\mathbb{G} = \mathbb{Z}^k$.

Let $\tilde{d}$ be the metric on $\tilde{M}$ induced by the lift of the riemannian metric on M. Let $\tilde{M}_\varepsilon$ be the metric space $(\tilde{M}, \tilde{d}_\varepsilon)$, where $\tilde{d}_\varepsilon := \varepsilon \tilde{d}$. Then $\tilde{M}_\varepsilon$ converges to $H_1(M, \mathbb{R})$ in the same way as $\varepsilon \mathbb{Z}^k$ converges to $\mathbb{R}^k$ or $\varepsilon\, \mathbb{G} \xrightarrow{\varepsilon} H_1(M, \mathbb{R})$.

To be precise, by the universal coefficient theorem $H_1(M, \mathbb{R}) = H^1(M, \mathbb{R})^*$. Let $G : \tilde{M} \to H_1(M, \mathbb{R})$ be given by

$$G(x) \cdot c = \oint_{x_0}^{x} \widetilde{\omega}, \tag{13}$$

where $c \in H^1(M, \mathbb{R})$, ω is a 1-form on M with cohomology class c and $\widetilde{\omega}$ is the lift of ω to $\tilde{M}$. In section §2 we prove that $\lim_\varepsilon(\tilde{M}, d_\varepsilon, \varepsilon G) = H_1(M, \mathbb{R})$.

In our homogenization result, the hamiltonian is the lift $\tilde{H}$ of H to the cover $\tilde{M}$, or equivalently, we start with a hamiltonian H on the cover $\tilde{M}$ which is $\mathbb{G}$-invariant. The effective hamiltonian is also Mather's alpha function $\overline{H} = \alpha : H^1(M, \mathbb{R}) \to \mathbb{R}$. In the homogenized problem the position space, or configuration space, is the homology group $x \in H_1(M, \mathbb{R}) \approx \mathbb{R}^k$, and the momenta, and the derivatives $d_x u$, are in its dual, the cohomology group $\{p,\, d_x u\} \subset H^1(M, \mathbb{R}) \approx \mathbb{R}^k$.

We finally address the problem of initial conditions. The functions f_ε will be defined in the limit space $f_\varepsilon : H_1(M, \mathbb{R}) \to \mathbb{R}$. Equation (10) will be on the manifold $(\tilde{M}, \tilde{d})$[1], and equation (11) will be replaced by

$$v^\varepsilon(y, 0) = g_\varepsilon(y) := f_\varepsilon(\varepsilon\, G(y)), \quad y \in \tilde{M},$$

[1] Or on the manifold $(\tilde{M}, \tilde{d}_\varepsilon)$. Both manifolds $(\tilde{M}, \tilde{d})$, $(\tilde{M}, \tilde{d}_\varepsilon)$ have the same differential structure.

which is interpreted as f_ε "seen" on $\tilde M_\varepsilon$.

Our main theorem is

1.1. **Theorem.**

Let $H : T^*M \to \mathbb{R}$ be a Tonelli hamiltonian on a closed manifold M. Let $f_\varepsilon : H_1(M,\mathbb{R}) \to \mathbb{R}$ be uniformly continuous and such that $f_\varepsilon \to f$ uniformly on $H_1(M,\mathbb{R})$, with f with linear growth. Let $\tilde H : T^*\tilde M \to \mathbb{R}$ be the lift of H to $\tilde M$. Let v^ε be the variational solution to the problem

$$\partial_t v^\varepsilon + \tilde H(x, \tfrac{1}{\varepsilon}\partial_x v^\varepsilon) = 0; \qquad x \in \tilde M,\ t > 0. \tag{14}$$

$$v^\varepsilon(x,0) = f_\varepsilon(\varepsilon\, G(x)). \tag{15}$$

Then the family of functions $v^\varepsilon : \tilde M_\varepsilon \times [0,+\infty[\to \mathbb{R}$ is equicontinuous and converges uniformly on compact sets to the solution $u : H^1(M,\mathbb{R}) \times [0,+\infty[\to \mathbb{R}$ of the problem

$$\partial_t u + \overline{H}(\partial_x u) = 0, \tag{16}$$

$$u(x,0) = f(x); \tag{17}$$

where the effective hamiltonian $\overline{H}$ is Mather's alpha function $\overline{H} = \alpha : H^1(M,\mathbb{R}) \to \mathbb{R}$.

Several comments are still in order:

1) The convergence destroys the differential structure of the spaces. Nevertheless we obtain convergence of solutions u^ε to a solution of a partial differential equation on the limit space because the Hamilton-Jacobi equation is an encoding of a variational principle. Namely, its solutions are the minimal cost functions under the Lagrangian. This variational principle is preserved under the limit of spaces.

2) Once all this setting is provided, the usual proof, follows. This is very good news since we can expect to generalize a lot of homogenization results to other manifolds. However, using a result of Mather, we will provide another proof, which is essentially a change of variables in the Lax formula.

3) Motivated by possible applications we extend the result to other Abelian coverings.

1.1. **Subcovers.**

For general abelian covers, i.e. covering spaces whose group of deck transformations $\mathbb{D}$ is abelian, the torsion part of $\mathbb{D}$ is killed under the limit of $\hat M_\varepsilon = (\hat M, \varepsilon \hat d)$. Thus the limit is the same as in a free abelian cover, where $\mathbb{D}$ is free abelian. These coverings are subcovers of the maximal free abelian cover $\tilde M$. In this case we have similar results as in Theorem 1.1.

Let $L : TM \to \mathbb{R}$ be the lagrangian of H i.e.

$$L(x,v) := \max_{p \in T_x^*M} \big[\, p(v) - H(x,p)\, \big]. \tag{18}$$

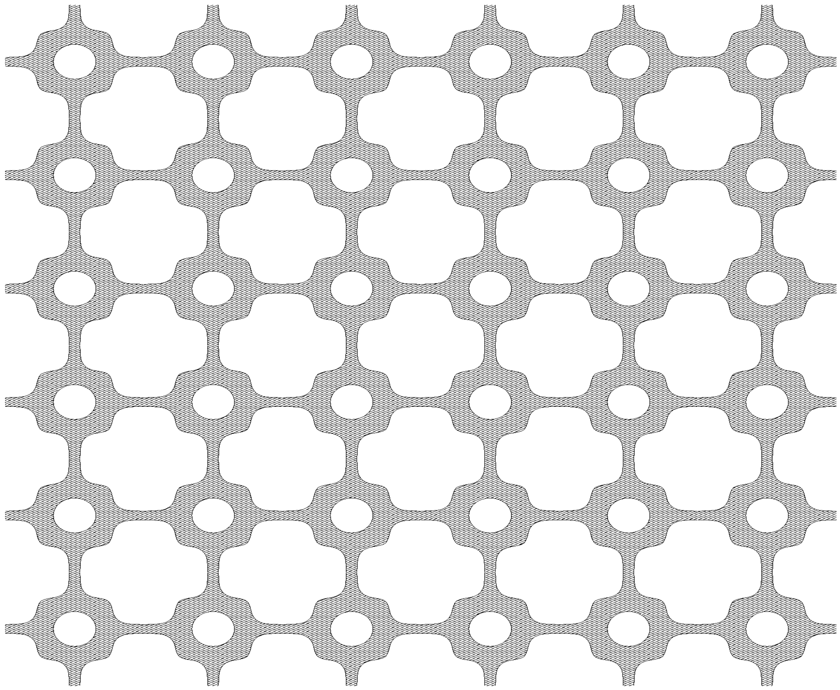

FIGURE 1. A free abelian cover of the compact orientable surface of genus 3 with group of covering transformationns $\mathbb{Z}^2$.

The Euler-Lagrange equation for L is

$$\tfrac{d}{dt}\partial_v L = \partial_x L,$$

it determines a complete flow φ on TM by $\varphi_t(x,v) = (\gamma(t), \dot\gamma(t))$ where γ is the solution of (18) with $(\gamma(0), \dot\gamma(0)) = (x, v)$. Given an invariant Borel probability μ for φ_t with compact support define its *homology class* $\rho(\mu) \in H_1(M,\mathbb{R}) = H^1(M,\mathbb{R})^*$ by

$$\rho(\mu)\cdot c = \int_{TM} \omega \, d\mu,$$

where ω is any closed 1-form on M with cohomology class c. Mather's minimal action function is $\beta : H_1(M,\mathbb{R}) \to \mathbb{R}$,

$$\beta(h) := \inf_{\rho(\mu)=h} \int L \, d\mu, \tag{19}$$

where the infimum is along all φ_t-invariant probabilities with homology $\rho(\mu) = h$.

Free abelian covers $\hat M$ are obtained from epimorphisms $\mathfrak{g} : \pi_1(M) \to \mathbb{Z}^\ell$, with $\pi_1(\hat M) = \ker \mathfrak{g}$ and group of deck transformations $\mathbb{Z}^\ell = \operatorname{im} \mathfrak{g}$. Since $H_1(M,\mathbb{Z})$ is the abelianization of $\pi_1(M)$ such epimorphisms $\mathfrak{g}$ factor as $\mathfrak{g} = \mathfrak{f} \circ \mathfrak{h}$ with $\mathfrak{g} : \pi_1(M) \xrightarrow{\mathfrak{h}} H_1(M,\mathbb{Z}) \xrightarrow{\mathfrak{f}} \mathbb{Z}^\ell$. The linearization of $\mathfrak{f}$ gives a linear epimorphism

$\mathfrak{f}: H_1(M,\mathbb{R}) \to \mathbb{R}^\ell$. The minimal action functional for the cover $\hat{M}$ is $\widehat{\beta}: \mathbb{R}^\ell = \operatorname{im}\mathfrak{f} \to \mathbb{R}$,

$$\widehat{\beta}(z) = \inf\{\, \beta(h) \,|\, \mathfrak{f}(h) = z \,\}.$$

This can also be interpreted as the average action of minimizing Euler-Lagrange orbits on $\hat{M}$ with asymptotic direction z. The effective hamiltonian for $\hat{M}$ is $\overline{H} = \widehat{\beta}^*$, the convex dual of $\widehat{\beta}$:

$$\begin{aligned} \overline{H}(p) = \widehat{\beta}^*(p) &= \max_{z\in\mathbb{R}^\ell} p(z) - \widehat{\beta}(z) \\ &= \max_{h\in H_1(M,\mathbb{R})} p(\mathfrak{f}(h)) - \beta(h) \\ &= \alpha(\mathfrak{f}^*(p)), \end{aligned} \tag{20}$$

where $\mathfrak{f}^*$ is the homomorphism $\mathfrak{f}^*: (\mathbb{R}^\ell)^* \to H_1(M,\mathbb{R})^* = H^1(M,\mathbb{R})$ induced by $\mathfrak{f}$.

Let $\pi: \hat{M} \to M$ be the projection. Define $F: \hat{M} \to \mathbb{R}^\ell$ by

$$\langle F(x), y\rangle = \oint_{x_0}^{x} \pi^*\mathfrak{f}^* y,$$

where $\mathfrak{f}^* y$ is interpreted as any closed 1-form in M in the cohomology class $\mathfrak{f}^* y$.

This integral does no depend on the chosen path in $\hat{M}$ from x_0 to x. Indeed, the projection π induces an isomorphism $\pi_*: \pi_1(\hat{M}, x_0) \to \mathbb{S}(x_0) \subset \pi_1(M, b_0)$, $b_0 = \pi(x_0)$, onto the stabilizer $\mathbb{S}(x_0)$ of x_0 in the fiber over b_0 among the Deck transformations of the covering $\hat{M}$:

$$\mathbb{S}(x_0) = \{\, T \in \pi_1(M, b_0) \mid T(x_0) = x_0 \,\}.$$

Therefore $\ker(\mathfrak{f}\circ\mathfrak{h}) = \ker(\mathfrak{g}) = \pi_*(\pi_1(\hat{M}, x_0)) = \mathbb{S}(x_0)$. If γ is a loop in $\hat{M}$ with endpoint x_0, we have that

$$\oint_\gamma \pi^*\mathfrak{f}^* y = \oint_{\pi\circ\gamma} \mathfrak{f}^* y = \langle \mathfrak{f}([\pi\circ\gamma]), y\rangle = \langle 0, y\rangle = 0,$$

because

$$[\pi\circ\gamma] = \pi_*([\gamma]) \subset \pi_*(H_1(\hat{M},\mathbb{R})) \subset \ker\mathfrak{f}.$$

1.2. **Theorem.**

Let $\hat{M}$ be a free abelian cover of M obtained from the epimorphism $\mathfrak{f}: H_1(M,\mathbb{Z}) \to \mathbb{Z}^\ell$ with $\pi_1(\hat{M}) = \ker\mathfrak{h}\circ\mathfrak{f}$ and group of deck transformations $\mathbb{F} \approx \mathbb{Z}^\ell = \operatorname{im}\mathfrak{f}$.

Let $F: \hat{M} \to \mathbb{R}^\ell$ be given by

$$\langle F(x), y\rangle = \oint_{x_0}^{x} \pi^*\mathfrak{f}^* y\,, \qquad \forall y \in \mathbb{R}^\ell.$$

Let $f_\varepsilon : \mathbb{R}^\ell \to \mathbb{R}$ be uniformly continuous and such that $f_\varepsilon \to f$ uniformly on $\mathbb{R}^\ell$, where f has linear growth. Let H be a Tonelli hamiltonian on M. Let $\hat{H} : T^*\hat{M} \to \mathbb{R}$ be the lift of H to $\hat{M}$. Let v^ε be the variational solution to the problem

$$\partial_t v^\varepsilon + \hat{H}(x, \tfrac{1}{\varepsilon}\partial_x v^\varepsilon) = 0; \qquad x \in \hat{M},\ t > 0. \tag{21}$$

$$v^\varepsilon(x, 0) = f_\varepsilon(\varepsilon\, F(x)). \tag{22}$$

Writing $\hat{M}_\varepsilon = (\hat{M}, \varepsilon\hat{d})$, where $\hat{d}$ is the metric on $\hat{M}$, we have that

$$\lim_\varepsilon (\hat{M}, \varepsilon\hat{d}, \varepsilon F) = \mathbb{R}^\ell.$$

Then the family of functions $v^\varepsilon : \hat{M}_\varepsilon \times [0, +\infty[\to \mathbb{R}$, is equicontinuous and converges uniformly on compact sets to the solution $u : \mathbb{R}^\ell \times [0, +\infty[\to \mathbb{R}$ of the problem

$$\partial_t u + \overline{H}(\partial_x u) = 0, \tag{23}$$

$$u(x, 0) = f(x); \tag{24}$$

where the effective hamiltonian $\overline{H} : (\mathbb{R}^\ell)^* \to \mathbb{R}$, is $\overline{H} = \mathfrak{f}_*\alpha$ given by (20).

2. Convergence of Spaces.

2.1. **Proposition.** *For the maximal free abelian cover we have that*

$$\lim_\varepsilon (\tilde{M}, d_\varepsilon, F_\varepsilon) = H_1(M, \mathbb{R}).$$

Proof: Observe that for the finite dimensional space $H_1(M, \mathbb{R})$ we can use any norm.

If ω is a closed 1-form in M and $\|\omega\| := \sup_{x \in M} |\omega(x)|$,

$$\big|\,[G(x) - G(y)] \cdot \omega\big| = \left| \oint_x^y \widehat{\omega}\ \right| \le \|\omega\|\ d(x, y)$$

Then there is $K_0 > 0$ such that

$$|G(x) - G(y)| \le K_0\ d(x, y), \tag{25}$$

and using that $F_\varepsilon = \varepsilon\, G$ and $d_\varepsilon = \varepsilon\, d$, we have that

$$|F_\varepsilon(x) - F_\varepsilon(y)| \le K_0\ d_\varepsilon(x, y). \tag{26}$$

Write $\mathbb{G} = \operatorname{im}[\pi_1(M) \to H_1(M, \mathbb{R})]$, the group of covering transformations for $\tilde{M}$. Fix $x_0 \in \tilde{M}$. Let $e_1, \ldots, e_k$ be a basis for $\mathbb{G}$ and let $\widehat{\gamma}_i$ be a minimal geodesic from x_0 to $e_i(x_0) =: x_0 + e_i$. If $\pi : \tilde{M} \to M$ is the projection, the concatenation $(\pi\circ\widehat{\gamma}_1)^{n_1} * \cdots * (\pi\circ\widehat{\gamma}_k)^{n_k}$ lifts to a curve from x_0 to $x_0 + \overline{n}$, where $\overline{n} = \sum n_i e_i$. Let ℓ_i be the length of γ_i. Then

$$d(x_0, x_0 + \overline{n}) \le \textstyle\sum_i n_i\, \ell_i \le (\max_i \ell_i)\, k\, |\overline{n}| = K_2\, |\overline{n}|.$$

For any $\overline{n}, \overline{m} \in \mathbb{G} \approx \mathbb{Z}^k$, we have

$$d(x_0 + \overline{n}, x_0 + \overline{m}) = d(x_0, x_0 + (\overline{n} - \overline{m})) \leq K_2\, |\overline{n} - \overline{m}|.$$

If $x, y \in \tilde{M}$ there are two elements $x_0 + \overline{n}$, $x_0 + \overline{m}$ of the orbit of x_0 such that $d(x, x_0 + \overline{n}) \leq D$ and $d(y, x_0 + \overline{m}) \leq D$, where $D := \operatorname{diam} M$. We have that

$$\begin{aligned} d(x, y) &\leq d(x, x_0 + \overline{n}) + d(x_0 + \overline{n}, x_0 + \overline{m}) + d(x_0 + \overline{m}, y) \\ &\leq K_2\, |\overline{n} - \overline{m}| + 2D. \end{aligned}$$

Observe that

$$G(x_0 + \overline{m}) - G(x_0 + \overline{n}) = \overline{m} - \overline{n} \in H_1(M, \mathbb{R}). \tag{27}$$

Using the Lipschitz property (25) for G,

$$\begin{aligned} |G(x) - G(y)| &\geq |\overline{m} - \overline{n}| - |G(x) - G(x_0 + \overline{n})| - |G(y) - G(x_0 + \overline{m})| \\ &\geq |\overline{m} - \overline{n}| - 2\, K_0\, D. \end{aligned}$$

Therefore

$$d(x, y) \leq K_2\, |G(x) - G(y)| + 2\, K_0\, K_2\, D + 2D.$$

For $A := 2K_0 D + 2K_2^{-1} D$,

$$\forall x, y \in \tilde{M}, \qquad K_2^{-1}\, d(x, y) - A \leq |G(x) - G(y)|.$$

Multiplying the inequality by ε, we get

$$\forall x, y \in \tilde{M}, \qquad K_2^{-1}\, d_\varepsilon(x, y) - \varepsilon\, A \leq |F_\varepsilon(x) - F_\varepsilon(y)|. \tag{28}$$

Inequalities (26) and (28) prove condition (1) of the convergence.

Condition (2) follows from the fact that the image of the $\mathbb{G}$-orbit of x_0,

$$F_\varepsilon(x_0 + \mathbb{G}) = \varepsilon \mathbb{G} = \varepsilon \mathbb{Z}^k \subset \mathbb{R}^k = H_1(M, \mathbb{R}),$$

is ε-dense in $H_1(M, \mathbb{R})$.

□

If $\lim_n (M_n, d_n, F_n) = (M, d)$, we say that a family of functions $f_n : (M_n, d_n) \to \mathbb{R}$ converges pointwise to $f : (M, d) \to \mathbb{R}$ if for every $x \in M$ there are sequences $x_n \in M_n$ with $\lim_n F_n(x_n) = x$ and $\lim_n f(x_n) = f(x)$.

2.2. **Proposition.**

If $\lim_n (M_n, d_n, F_n) = (M, d)$ *and* $f_n : (M_n, d_n) \to \mathbb{R}$ *is an equicontinuous family which converges pointwise to* $f : (M, d) \to \mathbb{R}$*, then the convergence is uniform on compact sets and* f *is uniformly continuous.*

If $\limsup_n K_n = 1$ *in* (12)*, then* f *has the same modulus of continuity as the* f_n*.*

Proof: We first prove that f is uniformly continuous. Given $\varepsilon > 0$ let $\delta = \delta(\varepsilon) > 0$ be such that

$$\forall n, \quad \forall a, b \in M_n, \quad d_n(a,b) < \delta \Longrightarrow |f_n(a) - f_n(b)| < \varepsilon.$$

Let $Q := \limsup_n K_n$. Let $x, y \in M$ with $d(x,y) < \frac{\delta}{Q}$. Let $x_n, y_n \in M_n$ with $\lim_n F_n(x_n) = x$, $\lim_n F_n(y_n) = y$ and $\lim_n f_n(x_n) = f(x)$, $\lim_n f_n(y_n) = f(y)$. Then

$$d_n(x_n, y_n) \le K_n\, d(F_n(x_n), F_n(y_n)) + A_n\, K_n \xrightarrow{n} (\limsup_n K_n)\, d(x,y) < \delta.$$

Thus

$$|f_n(x_n) - f_n(y_n)| < \varepsilon.$$

Taking $\limsup_n$ on inequality

$$|f(x) - f(y)| \le |f(x) - f_n(x_n)| + |f_n(x_n) - f_n(y_n)| + |f_n(y_n) - f(y)|,$$

we obtain that

$$|f(x) - f(y)| < \varepsilon.$$

Thus, f is uniformly continuous. If $Q = 1$, f has the same modulus of continuity as the f_n.

We now prove that the convergence is uniform on compact sets. Let $\mathbb{K} \subset M$ be compact. Let $Q := \sup_n (1 + K_n) > 1$. Given $\varepsilon > 0$ let $\delta > 0$ be such that

$$\begin{cases} d_n(x,y) < \delta & \Longrightarrow \quad |f_n(x) - f_n(y)| < \delta\,, \\ d(x,y) < \delta & \Longrightarrow \quad |f(x) - f(y)| < \delta. \end{cases}$$

Let $\{U_\alpha\}_{\alpha=1}^m$ be a finite open cover of $\mathbb{K}$ such that $\operatorname{diam} U_\alpha < \frac{\delta}{2Q}$. Let $N_0 > 0$ be such that $n > N_0 \Longrightarrow A_n < \frac{\delta}{2}$. Then

$$F_n(x),\ F_n(y) \in U_\alpha \quad \Longrightarrow \quad d_n(x,y) < K_n\, d(F_n(x), F_n(y)) + A_n < \frac{\delta}{2} + A_n < \delta.$$

Thus

$$n > N_0 \quad \Longrightarrow \quad \begin{cases} x, y \in U_\alpha & \Longrightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{3}, \\ F_n(x),\ F_n(y) \in U_\alpha & \Longrightarrow \quad |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}. \end{cases}$$

Let $N_1 > N_0$ be such that $F_n^{-1}(U_\alpha) \neq \emptyset$ for all $n > N_1$. For each $n > N_1$ and $1 \le \alpha \le m$ choose one $x_{n,\alpha} \in M_n$ with $F_n(x_{n,\alpha}) \in U_\alpha$.

Since $f_n \to f$ point wise, for each α there is $M_\alpha > 0$ such that

$$\forall n > M_\alpha \quad : \qquad |f_n(x_{n,\alpha}) - f(F_n(x_{n,\alpha}))| < \tfrac{\varepsilon}{3}.$$

Let $N_2 := \max\{ N_0, N_1, M_1, \dots, M_m \}$. Then if $n > N_2$, $y \in F_n^{-1}(\mathbb{K})$, take α such that $F_n(y) \in U_a$, then

$$\begin{aligned} |f_n(y) - f(F_n(y))| &\le |f_n(y) - f_n(x_{n,\alpha})| + |f_n(x_{n,\alpha}) - f(F_n(x_{n,\alpha}))| + |f(F_n(x_{n,\alpha})) - f(F_n(y))| \\ &\le \qquad \frac{\varepsilon}{3} \qquad + \qquad \frac{\varepsilon}{3} \qquad + \qquad \frac{\varepsilon}{3}. \end{aligned}$$

□

We now deal with the sub-covering case. Let $\mathfrak{f} : H_1(M,\mathbb{Z}) \to \mathbb{Z}^\ell$ be an epimorphism and $\mathfrak{h} : \pi_1(M) \to H_1(M,\mathbb{Z})$ the Hurewicz map. Let $\pi : \hat{M} \to M$ be the covering of M with $\pi_1(\hat{M}) = \ker(\mathfrak{h} \circ f)$ and group of Deck transformations $\operatorname{im} \mathfrak{f} = \mathbb{Z}^\ell$. Let $F : \hat{M} \to \mathbb{R}^\ell$ be defined by

$$\langle F(x), y \rangle = \oint_{x_0}^{x} \pi^* \mathfrak{f}^* y \,, \qquad \forall y \in \mathbb{R}^\ell.$$

2.3. **Proposition.**

$$\lim_{\varepsilon} (M, \varepsilon d, \varepsilon F) = \mathbb{R}^\ell.$$

Proof: We first prove that F is Lipschitz. Observe that

$$\langle F(x) - F(y), z \rangle = \oint \pi^* \mathfrak{f}^* z,$$

$$|\langle F(x) - F(y), z \rangle| \le |z| \; \|\mathfrak{f}^*\| \, \|\pi^*\| \; d(x,y).$$

Applying the inequality to $z = F(x) - F(y)$ we obtain

$$\|F(x) - F(y)\| \le \|\mathfrak{f}^*\| \; \|\pi^*\| \; d(x,y).$$

Multiplying the inequality by ε, for $K = \|\mathfrak{f}^*\| \; \|\pi^*\|$, we get

$$\|F_\varepsilon(x) - F_\varepsilon(y)\| \le K \, d_\varepsilon(x,y).$$

The proof of the other inequality for F is the same as in proposition 2.1, we only need to prove the analogous of equation (27). But if $\overline{n}, \overline{m} \in \operatorname{im} \mathfrak{f} = \mathbb{Z}^\ell$ are Deck transformations of $\hat{M}$,

$$\begin{aligned} \langle F(x_0 + \overline{m}) - F(x_0 + \overline{n}), y \rangle &= \oint_{x_0+\overline{n}}^{x_0+\overline{m}} \pi^* \mathfrak{f}^* y = \oint_{\pi \circ \gamma} f^* y \\ &= \langle f([\pi \circ \gamma]), y \rangle = \langle \overline{m} - \overline{n}, y \rangle. \end{aligned}$$

Thus

$$F(x_0 + \overline{n}) - F(x_0 + \overline{m}) = \overline{m} - \overline{n}.$$

With this formula we obtain condition (2) of the convergence because the $\mathbb{Z}^\ell$-orbit of $x_0 \in \hat{M}$,

$$F_\varepsilon(x_0 + \mathbb{Z}^\ell) = \varepsilon\mathbb{Z}^\ell \subset \mathbb{R}^\ell$$

which is ε-dense in $\mathbb{R}^\ell$.

□

3. Homogenization in the maximal free abelian cover.

Proof of theorem 1.1:

Write

$$\begin{aligned} H_\varepsilon(x,p) &:= \tilde{H}\big(x, \tfrac{1}{\varepsilon}p\big). \\ L_\varepsilon(x,v) &:= \max_{p\in T_x^*M} \{\, p\cdot v - H_\varepsilon(x,p)\,\} \\ &= \max_{p\in T_x^*M} \left\{ \frac{p}{\varepsilon}\cdot(v\varepsilon) - H\big(x,\tfrac{p}{\varepsilon}\big)\right\} \\ &= L(x,\varepsilon v). \end{aligned}$$

The solution to the problem (14)–(15) is given by the Lax-Oleinik formula

$$\begin{aligned} v^\varepsilon(x,T) &= \inf_{\gamma(T)=x} \left\{ v^\varepsilon(\gamma(0),0) + \int_0^T L_\varepsilon(\gamma,\dot\gamma)\,dt \;\middle|\; \gamma\in C^1([0,T],\tilde{M}),\ \gamma(T)=x \right\}, \\ &= \inf_{\gamma(T)=x} \left\{ v^\varepsilon(\gamma(0),0) + \int_0^T L(\gamma,\varepsilon\,\dot\gamma) \right\}. \end{aligned}$$

Write $\eta:[0,\frac{T}{\varepsilon}]\to\tilde{M}$, $\eta(s):=\gamma(\varepsilon s)$, then

(29)
$$\int_0^T L(\gamma,\varepsilon\dot\gamma)\,dt = \int_0^{\frac{T}{\varepsilon}} L(\eta(s),\dot\eta(s))\cdot\varepsilon\,ds.$$

$$v^\varepsilon(x,T) = \inf_{\eta(\frac{T}{\varepsilon})=x} \left\{ v^\varepsilon(\eta(0),0) + \varepsilon\int_0^{\frac{T}{\varepsilon}} L(\eta,\dot\eta)\,ds \;\middle|\; \eta\in C^1([0,\tfrac{T}{\varepsilon}],\tilde{M}),\ \ \eta\big(\tfrac{T}{\varepsilon}\big)=x \right\}.$$

$$= \inf_{y\in\tilde{M}} \left\{ v^\varepsilon(y,0) + \varepsilon\,\tilde\phi\big(y,x,\tfrac{T}{\varepsilon}\big)\right\}, \tag{30}$$

where

$$\tilde\phi(y,x,S) := \inf\left\{ \int_0^S L(\eta,\dot\eta)\,ds \;\middle|\; \eta\in C^1\big([0,S],\tilde{M}\big),\ \ \eta(0)=y,\ \eta(S)=x \right\}.$$

The solution to the limit problem (16) - (17) is

$$u(y,T) = \inf_{z\in H_1(M,\mathbb{R})} \left\{ f(z) + T\,\beta\left(\tfrac{y-z}{T}\right)\right\}, \tag{31}$$

where β is Mather's minimal action functional (19).

The proof is just to show that formula (30) converges to formula (31), using Mather's proposition 3.4 below on the uniform convergence of mean minimal actions to the beta function. It is done at the end of the section. But the statement of theorem 1.1 requires to prove that the family v^ε is equicontinuous and that the limit is uniform. For this we shall need some slightly more technical work.

We say that a curve $\gamma : [0,T] \to \tilde{M}$ is a *Tonelli minimizer* iff

$$\int_0^T L(\gamma,\dot\gamma)\,dt = \tilde\phi(\gamma(0),\gamma(T),T).$$

3.1. **Lemma.** *The map $(x,y) \mapsto \tilde\phi(x,y,T)$ is Lipschitz on $T > d(x,y)$.*

Proof: There is $A > 1$ such that if $\gamma : [0,T] \to \tilde{M}$ is a Tonelli minimizer and $d(\gamma(0),\gamma(T)) > T$ then $|\dot\gamma(t)| < A$ for all $t \in [0,T]$. Let

$$Q_1 := \sup_{|v|\le 2A} |L(x,v)|, \qquad Q_2 := \sup_{|v|\le 2A} |\partial_v L(x,v)\cdot v|.$$

If $a \in [\frac12, 2]$ and $|v| \le A$, we have that

$$\begin{aligned}\left|L(x,av)\cdot \tfrac1a - L(x,v)\right| &\le \tfrac1a\ |L(x,av)-L(x,v)| + \left|\tfrac{a-1}{a}\right|\ |L(x,v)|\\ &\le \frac1a\left(\int_1^a \partial_v L(x,sv)\cdot v\,ds\right) + \left|\frac{a-1}{a}\right|\ |L(x,v)|\\ &\le \left|\tfrac{a-1}{a}\right|\ Q_2 + \left|\tfrac{a-1}{a}\right|\ Q_1.\end{aligned}$$

Let $z \in \tilde{M}$, $d := d(y,z) \ll T$. Let $\gamma \in C^1([0,T],\tilde{M})$, $\gamma(0) = x$, $\gamma(T) = z$ be such that

$$\tilde\phi(x,z,T) = \int_0^T L(\gamma,\dot\gamma)\,dt.$$

Define $\eta : [0,T-d] \to \tilde{M}$ by $\eta(s) = \gamma\big(s\,\frac{T}{T-d}\big)$, we have that

$$\begin{aligned}\tilde\phi(x,z,T-d) &\le \int_0^{T-d} L(\eta,\dot\eta)\,ds = \int_0^T L(\gamma, \tfrac{T}{T-d}\,\dot\gamma)\,\big(\tfrac{T-d}{T}\big)\ dt\\ &\le \int_0^T L(\gamma,\dot\gamma)\,dt + \left|\tfrac{d}{T-d}\right|\ (Q_1+Q_2)\\ &\le \tilde\phi(x,z,T) + d(y,z)\,\tfrac{2}{T}\,(Q_1+Q_2).\end{aligned}$$

$$\tilde\phi(z,y,d) \le Q_1\ d(y,z).$$

By the triangle inequality,

$$\begin{aligned}\tilde\phi(x,y,T) &\le \tilde\phi(x,z,T-d)+\tilde\phi(z,y,d).\\ &\le \tilde\phi(x,z,T)+\big(Q_1+\tfrac{2}{T}(Q_1+Q_2)\big)\,d(y,z).\end{aligned}$$

Similarly

$$\tilde\phi(x,z,T)\le\tilde\phi(x,y,T)+K_1\,d(y,z),$$

with $K_1:=Q_1+\frac{2}{T}\,(Q_1+Q_2)$. □

3.2. **Lemma.** *There is $K_1>0$ such that if $\frac{T}{\varepsilon}>2$ then $v^\varepsilon(x,T)$ is εK_1-Lipschitz on $x\in\tilde M$.*

Proof: Let $x,z\in\tilde M$ with $d(x,z)<2<\frac{T}{\varepsilon}$. Let K_1 be the Lipschitz constant from lemma 3.1 Let $y_n\in\tilde M$ be such that

$$\begin{aligned}v^\varepsilon(x,T)&=\lim_n\big\{v^\varepsilon(y_n,0)+\varepsilon\,\tilde\phi\big(y_n,x,\tfrac{T}{\varepsilon}\big)\big\}\\ &\ge\lim_n\big\{v^\varepsilon(y_n,0)+\varepsilon\,\tilde\phi\big(y_n,z,\tfrac{T}{\varepsilon}\big)-\varepsilon K_1\,d(x,z)\big\}\\ &\ge v^\varepsilon(z,T)-\varepsilon K_1\,d(x,z).\end{aligned}$$

The other inequality is similar.

□

3.3. **Corollary.** *If $\frac{T}{\varepsilon}>2$ the function $v^\varepsilon(\,\cdot\,,T):\tilde M_\varepsilon\to\mathbb{R}$ is K_1-Lipschitz.*
In particular the family $v^\varepsilon(\,\cdot\,,T)$ is equicontinuous on $\tilde M_\varepsilon$.

The solution to the limit problem (16)–(17) is given by (6) where the effective lagrangian (7) is Mather's beta function (19), thus

$$u(y,T)=\inf_{x\in H_1(M,\mathbb{R})}\big\{f(x)+T\,\beta\left(\tfrac{y-x}{T}\right)\big\} \tag{32}$$

Following Mather [10, page 180], given $x,y\in\tilde M$ define the *difference vector* $y-x\in H_1(M,\mathbb{R})$ by

$$\langle[\omega],y-x\rangle=\oint_x^y\widetilde\omega, \tag{33}$$

where ω is a closed 1-form on M and $\widetilde\omega=\pi^*\omega$ is its lift to $\tilde M$.

We shall need the following result by Mather [10, Corollary on page 181]:

3.4. **Proposition.**

For every $A>0$, $\delta>0$ there is $T_0>0$ such that if $x,y\in\tilde M$, $T\ge T_0$, $\left\|\frac{y-x}{T}\right\|\le A$, then

$$\left|\tfrac{1}{T}\,\tilde\phi(x,y,T)-\beta\left(\tfrac{y-x}{T}\right)\right|<\delta.$$

From (33) we obtain

$$y - x = G(y) - G(x).$$

Applying proposition 3.4 with time $\frac{T}{\varepsilon}$, and recalling that $F_\varepsilon = \varepsilon G$, we have that

$$\left\| \tfrac{F_\varepsilon(y)-F_\varepsilon(x)}{T} \right\| < A \quad \& \quad \tfrac{T}{\varepsilon} > T_0(A,\delta) \quad \Longrightarrow \quad \left| \varepsilon\, \tilde{\phi}\left(x, y, \tfrac{T}{\varepsilon}\right) - T\,\beta\left(\tfrac{F_\varepsilon(y)-F_\varepsilon(x)}{T}\right) \right| < \delta\, T. \tag{34}$$

By hypothesis f has linear growth. Since β is superlinear, given $y \in H_1(M,\mathbb{R})$ there is $z_0 = z_0(y) \in H_1(M,\mathbb{R})$ such that

$$\begin{aligned} u(y,T) &= \inf_{z\in H_1(M,\mathbb{R})} \left\{ f(z) + T\,\beta\left(\tfrac{y-z}{T}\right) \right\} \\ &= f(z_0) + T\,\beta\left(\tfrac{y-z_0}{T}\right). \end{aligned}$$

Let $z_\varepsilon, y_\varepsilon \in \tilde{M}$ be such that $F_\varepsilon(z_\varepsilon) \to z_0$ and $F_\varepsilon(y_\varepsilon) \to y$. Then using (34) and $\delta := 2\max\{d(z_0, F_\varepsilon(z_\varepsilon)), d(y, F_\varepsilon(y_\varepsilon))\}$,

$$\begin{aligned} u(y,T) &\geq f_\varepsilon(F_\varepsilon(z_\varepsilon)) - \|f - f_\varepsilon\|_0 - \operatorname{osc}\left(f|_{z_0+[-\delta,\delta]^k} \right) \\ &\qquad + \varepsilon\,\tilde{\phi}\left(z_\varepsilon, y_\varepsilon, \tfrac{T}{\varepsilon}\right) - \delta\,T - \left| \beta\left(\tfrac{F_\varepsilon(y_\varepsilon)-F_\varepsilon(z_\varepsilon)}{T}\right) - \beta\left(\tfrac{y-z_0}{T}\right) \right| \\ &\geq v^\varepsilon(y_\varepsilon, T) \quad - \left[\|f_\varepsilon - f\| + \operatorname{osc}(f_\varepsilon, z_0, \delta) + \delta T + \operatorname{osc}\left(\beta, \tfrac{y-z_0}{T}, \tfrac{\delta}{T}\right) \right]. \end{aligned} \tag{35}$$

Similarly, f_ε has linear growth and L is superlinear, there is $x \in \tilde{M}$ such that

$$\begin{aligned} v^\varepsilon(y_\varepsilon, T) &= f_\varepsilon(F_\varepsilon(x)) + \varepsilon\,\tilde{\phi}\big(x, y_\varepsilon, \tfrac{T}{\varepsilon}\big) \\ &\geq f(F_\varepsilon(x)) - \|f_\varepsilon - f\|_0 + T\,\beta\left(\tfrac{F_\varepsilon(y_\varepsilon)-F_\varepsilon(x)}{T}\right) - \delta\,T \\ &\geq u(F_\varepsilon(y_\varepsilon), T) - \big[\, \|f_\varepsilon - f\|_0 + \delta\,T \big]. \\ &\geq u(y,T) - \Big[\, \|f_\varepsilon - f\|_0 + \delta\,T + \operatorname{osc}\big(u(\,\cdot\,,T), y, \delta\big) \Big]. \end{aligned} \tag{36}$$

Since β and u are uniformly continuous on compact subsets, from (35), (36) we obtain that $\lim_n v^\varepsilon = u$ uniformly on compact subsets.

□

4. SUBCOVERS

The convergence of the spaces is proven in Proposition 2.3.

Proof of Theorem 1.2:

Write the projections of the coverings as $\tilde M \xrightarrow{p} \hat M \xrightarrow{\pi} M$. Consider the lifts to $\tilde M$ of the solutions v^ε and the lift to $H_1(M,\mathbb{R})$ of the initial conditions f_ε:

$$\tilde v^\varepsilon(x,t) := v^\varepsilon(p(x),t), \qquad \tilde f_\varepsilon(x) = f_\varepsilon(\mathfrak{f}(x)), \qquad \tilde f(x) = f(\mathfrak{f}(x)).$$

By Theorem 1.1 we have that the family $\tilde v^\varepsilon$ is equicontinuous and converges uniformly on compact subsets to the solution $\tilde u$ of the problem (16)–(17). Since the solutions v^ε are invariant under the deck transformations for the cover p so is $\tilde u$, and induces a function on $\hat M \times [0,+\infty[$. The equicontinuity and the uniform convergence on compacts also descend to the cover $\hat M$.

Now we check the form of the effective lagrangian and here of the effective hamiltonian. We have that

$$\begin{aligned} u(y,T) &= \inf_{x\in\mathbb{R}^\ell} \Big\{ \tilde f(x) + T\beta\left(\tfrac{y-x}{T}\right) \Big\}, \\ &= \inf_{x\in\mathbb{R}^\ell} \Big\{ \tilde f(x+z) + T\beta\left(\tfrac{y-x-z}{T}\right) \Big\}, \qquad \forall z \in \ker \mathfrak{f}, \\ &= \inf_{x\in\mathbb{R}^\ell} \Big\{ \tilde f(x) + T\beta\left(\tfrac{y-x}{T} - \tfrac{z}{T}\right) \Big\}, \qquad \forall z \in \ker \mathfrak{f}, \\ &= \inf_{x\in\mathbb{R}^\ell} \Big\{ \tilde f(x) + T \inf_{w\in\ker\mathfrak{f}} \beta\left(\tfrac{y-x}{T} + w\right) \Big\}, \\ &= \inf_{x\in\mathbb{R}^\ell} \Big\{ \tilde f(x) + T\hat\beta\left(\tfrac{p(y)-x}{T}\right) \Big\}, \\ &= \hat u(p(y),T), \end{aligned}$$

where $\hat u$ is the solution to the limit problem (23)–(24) in Theorem 1.2.

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CIMAT, A.P. 402, 36.000, Guanajuato. GTO, México.
E-mail address: gonzalo@cimat.mx

CIMAT, A.P. 402, 36.000, Guanajuato. GTO, México.
E-mail address: renato@cimat.mx

Dipartimento di Matematica, Università degli Studi di Roma "La Sapienza", 00185 Roma, Italy.
E-mail address: siconolf@mat.uniroma1.it