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Average linking numbers

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To the memory of Ricardo Mañé

Abstract. We prove the existence of the average linking number, also called the Hopf invariant, for any invariant measure under a differentiable flow on S^3 without singularities, which has no periodic orbit of positive measure.

1. *Introduction*

Let γ_1, γ_2 be two disjoint oriented knots in the three-dimensional sphere S^3. The linking number $\ell(\gamma_1, \gamma_2)$ is defined as the algebraic intersection number of γ_1 with an orientable surface N transversal to γ_1, with oriented boundary γ_2. For a better understanding of our results we need the two following interpretations of the linking number (see [**5**]):

(1) for any regular planar projection it is the algebraic crossing number of γ_1 *over* γ_2;

(2) it is also the degree of the map

$$\begin{aligned} G : T^2 &\to S^2 \\ (t_1, t_2) &\mapsto \frac{\gamma_1(t_1) - \gamma_2(t_2)}{|\gamma_1(t_1) - \gamma_2(t_2)|}, \end{aligned}$$

where T^2 is the two-dimensional torus and S^2 is the two-dimensional sphere.

Hence

$$\begin{aligned} \ell(\gamma_1, \gamma_2) &= \frac{1}{\mathrm{vol}(S^2)} \int_{S^1 \times S^1} \det(DG)\, ds\, dt \\ &= \frac{1}{4\pi} \int_0^{t_1} \int_0^{t_2} \frac{(\gamma_1' \times \gamma_2')(\gamma_1 - \gamma_2)}{|\gamma_1 - \gamma_2|^3}, \end{aligned}$$

where $\gamma_i : [0, t_i] \to S^3 = \mathbb{R}^3 \cup \{\infty\}$ is a parametrization of γ_i; $i = 1, 2$. This formula is known as the Gauss formula.

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Given a flow φ_t on S^3 generated by a vector field F, and two closed orbits γ_1, γ_2, we define the average linking number of γ_1 and γ_2 as $(1/T_1T_2)\ell(\gamma_1, \gamma_2)$, where T_i is the period of γ_i. In order to define the average linking number for general orbits we need a convenient set of short curves.

Define a good set of short curves as a system of piecewise differentiable paths joining points $x, y \in S^3$, depending in a measurable way on x and y such that the Gauss integrals of every pair of non-intersecting pairs of the system, and the Gauss integrals of every pair of non-intersecting {paths of the system, segments of orbits $\varphi_t(p)$, $0 \le t \le 1$}, are bounded independently of the paths by a constant k (see [**1**]). In §3 we will prove the existence of such a set.

For any $x \in S^3$ and $T \in \mathbb{R}$ let $\hat{x}_T$ denote the knot formed by the orbit from x to $\varphi_T(x)$ and a path $\alpha_{\varphi_T(x),x}$, in a chosen system of short curves, joining $\varphi_T(x)$ to x. For any two points in different orbits p, q and times T_1, T_2 define, when possible, their average linking number as

$$\ell(x, y) := \lim_{T_1, T_2 \to \infty} \frac{1}{T_1T_2} \ell(\hat{x}_T, \hat{y}_T).$$

The main purpose of this paper is to decide when this limit exists and does not depend on the set of short curves. Our main theorem is as follows.

THEOREM 1.1. *Let μ, ν be two invariant measures on S^3. Then the function $J : S^3 \times S^3 \to \mathbb{R}$ defined by*

$$J(p, q) = \frac{F(p) \times F(q) \cdot (p - q)}{|p - q|^3}$$

is in $L^1(\mu \times \nu)$.

From Birkhoff's ergodic theorem we obtain the following.

THEOREM 1.2.

(a) *For $(\mu \times \nu)$-almost every pair of points the limit*

$$\hat{\ell}(x, y) = \lim_{T_1, T_2 \to \infty} \frac{1}{T_1T_2} \int_0^{T_1} \int_0^{T_2} \frac{(\gamma_1' \times \gamma_2', \gamma_1 - \gamma_2)}{|\gamma_1 - \gamma_2|^3}$$

exists.

(b) *If there is no periodic orbit with positive μ and ν measures this limit is for $(\mu \times \nu)$-almost every pair the same as $\ell(x, y)$, in particular the asymptotic linking number does not depend on the choice of the set of short curves. If the measures μ and ν are ergodic, the asymptotic linking $\ell(x, y)$ is given by the integral $\iint J(p, q)\, d\mu(p)\, d\nu(q)$.*

The same assertion of this theorem was proven by Arnold in [**1**] for smooth measures and in [**2**] for Gibbs measures. The proof in [**1**] relies on the fact that the singularity of the formula has order two which is less than the codimension of the diagonal, the singular set of J. The proof in [**2**] relies on the observation that the singularity of the Gauss formula has actually order one and that the Hausdorff dimension of a Gibbs measure is greater than one.

The idea of our proof is that the singularity of the Gauss formula disappears when we integrate along the orbits in a small flow box. This is because the integral of the Gauss formula is the integral of the Jacobian of G, therefore in small time intervals the area of the image of G should be contained in one half of the sphere. However, since we actually want to bound the integral of the absolute value of the Jacobian of G, it was easier to carry on an analytical proof, rather than to worry about of regions of injectivity of G and the sign of its Jacobian.

As an application, in [**6**] Verjovsky and Vila, following the same philosophy, extended other topological invariants, such as Witten's invariant for links, to an average value for an invariant smooth measure of a flow. In their paper, the only obstruction to defining the average Witten's invariant for singular invariant measures is the existence of the average linking number for such a measure, which is proven here in Theorem 1.1. We also quote the work of Freedman and He [**4**], where they relate the asymptotic crossing number (the integral of $|J(x, y)|$) with the energy of a vector field with a smooth invariant measure.

Finally let us mention that Gambaudo and Ghys [**3**] have proved that the average linking number is a topological invariant for suspensions of diffeomorphisms of the two discs.

2. *Proof of the theorems*

LEMMA 2.1. *For any compact subset $K \subset \mathbb{R}^3$, there exist $\varepsilon > 0$ and $A > 0$, such that if $p \in K$, $|x - p| < \varepsilon$, $|y - p| < \varepsilon$, then*

$$J(x, y) = \frac{H^T \mathbf{B} H}{|H|^3} + \phi(x, y),$$

where $|\phi(x, y)| < A$ is bounded, $H = y - x$ and $\mathbf{B}$ is the matrix defined by $\mathbf{B} \cdot H = F(p) \times (DF(p) \cdot H)$ for all $H \in \mathbb{R}^3$.

Proof. We have

$$J(x, y) = -\frac{F(x) \times F(y)}{|H|} \cdot \frac{H}{|H|^2},$$

where $y = x + H$. Since $F(y) = F(x) + DF(x) \cdot H + \psi(x, H)$, with $|\psi(x, H)| \le a|H|^2$ uniformly on all $p \in K$, $|x - p| < \varepsilon$, then

$$\begin{aligned} J(x, y)|H|^3 &= H \cdot [F(x) \times (F(x) + DF(x) \cdot H + \psi(x, H))] \\ &= H \cdot F \times DF(x) \cdot H + H \cdot F(x) \times \psi(x, H). \end{aligned}$$

But if $A := a \sup_{d(x,K) \le \varepsilon} |F(x)|$, then

$$\frac{H \cdot F(x) \times \psi(x, H)}{|H|^3} \le a|F(x)| \le A. \qquad \square$$

Given $p \in \mathbb{R}^3$ and $\varepsilon > 0$, let $\mathbf{E}(p)$ and $\mathbf{N}(p)$ be defined by

$$\begin{aligned} \mathbf{E}(p) &= \mathbf{E}(p, \varepsilon) = \{q \in \mathbb{R}^3 \mid (q - p) \cdot F(p) = 0, |q - p| < \varepsilon\}, \\ \mathbf{N}(p) &= \mathbf{N}(p, \varepsilon) = \bigcup_{t \in [-\varepsilon, \varepsilon]} \varphi_t(\mathbf{E}(p)). \end{aligned}$$

For $s, t \in \mathbb{R}$ and $q \in \mathbf{E}(p)$, write $p_s = \varphi_s(p)$, $q_t = \varphi_t(q)$. Given $q \in \mathbf{E}(p)$, let $\tau : [-\varepsilon, \varepsilon] \to \mathbb{R}$ be such that $q_{\tau(s)} \in \mathbf{E}(p_s, 1)$, i.e. $(q_{\tau(s)} - p_s) \cdot F(p_s) = 0$.

LEMMA 2.2. *For any compact subset $K \subset \mathbb{R}^3$, there exist $\varepsilon > 0$ and $M > 0$, such that if $p \in K$, $q \in \mathbf{E}(p, \varepsilon)$ and $|s|, |t| < \varepsilon$, then*

$$J(p_s, q_t) = \frac{Q_s \mathbf{B}_s Q_s + (t - \tau(s)) Q_s (F_s \times \mathbf{A}_s F_s)}{|H_{st}|^3} + \psi(p, q, s, t)$$

where $|\psi(p, q, s, t)| \le M$, $F_s = F(p_s)$, $\mathbf{A}_s = DF(p_s)$, $H_{st} = q_t - p_s$, $Q_s = q_{\tau(s)} - p_s$ and $\mathbf{B}_s$ is the matrix defined by $\mathbf{B}_s H = F_s \times (\mathbf{A}_s H)$ for all $H \in \mathbb{R}^3$.

Proof. We have $q_{\tau(s)} = p_s + Q_s$, and

$$\begin{aligned}
F(q_{\tau(s)}) &= F(p_s) + DF(p_s) Q_s \\
&\qquad + G_1(p, q, s)|Q_s|^2, \; |G_1(p, q, s)| < a_1 \quad \text{for some } a_1 > 0, \\
q_t - p_s &= Q_s + (q_t - q_{\tau(s)}) \\
&= Q_s + (t - \tau(s)) F(q_{\tau(s)}) + G_2 |t - \tau(s)|^2, \quad |G_2| < a_2, \\
H_{st} &= Q_s + (t - \tau(s)) F(p_s) + G_3(|t - \tau(s)|^2 + |t - \tau(s)||Q_s|), \quad |G_3| < a_3.
\end{aligned}$$

Observe that, since $Q_s \perp F(p_s)$, writing $\Delta t := t - \tau(s)$,

$$\begin{aligned}
|H_{st}|^2 &\ge |Q_{st}|^2 + |\Delta t|^2 |F_s|^2 - a_3^2 (|\Delta t|^2 + |\Delta t||Q_s|)^2 \\
&\ge |Q_s|^2 + \tfrac{1}{2} |\Delta t|^2 |F_s|^2.
\end{aligned}$$

So

$$|H_{st}|^2 \ge \tfrac{1}{2} [|Q_s|^2 + |\Delta t|^2 |F_s|^2], \tag{1}$$

if $\varepsilon > 0$ is small enough. Therefore $|Q_s| \le |H_{st}|$ and $|\Delta t| \le \alpha |H_{st}|$ for some $\alpha = \alpha(K, \varepsilon) > 0$. Now

$$\begin{aligned}
H_{st} \mathbf{B}_s H_{st} &= (Q_s + \Delta t F_s + a_3(|\Delta t|^2 + |\Delta t||Q_s|)) \cdot (F_s \times \mathbf{A}_s H_{st}) \\
&= Q_s F_s \times \mathbf{A}_s H_{st} + a_3 \mathcal{O}(|H_{st}|^2) \mathcal{O}(|H_{st}|) \\
&= Q_s F_s \times \mathbf{A}_s (Q_s + (\Delta t) F_s + \mathcal{O}(|H_{st}|^2)) + \mathcal{O}(|H_{st}|^3) \\
&= Q_s \mathbf{B}_s Q_s + Q_s (F_s \times \mathbf{A}_s F_s) \Delta t + \mathcal{O}(|H_{st}|^3).
\end{aligned}$$

By Lemma 2.1, we have

$$J(p_s, q_t) = \frac{H_{st} \mathbf{B}_s H_{st}}{|H_{st}|^3} + \phi(p_s, q_t),$$

with $|\phi(p_s, s, t)| < A$. This completes the proof of the lemma. □

LEMMA 2.3. *For any compact subset $K \subset \mathbb{R}^3$, there exist $\epsilon > 0$ and $a, b > 0$, such that if $q \in E(p, \epsilon)$ and $|s|, |t| < \epsilon$, then*

$$a|Q_0| \le |Q_s| \le b|Q_0|$$

where Q_s is defined as in Lemma 2.2.

Proof. Define $\tau : \cup_{p\in K}(\{p\}\times \mathbf{E}(p,\epsilon)\times[-\epsilon,\epsilon]) \to \mathbb{R}$ by $\tau(p,q,s)$ such that $q_{\tau(p,q,s)} \in \mathbf{E}(p_s, 1)$. We first show that τ is differentiable on q. By the definition of τ, we have

$$G(p,q,\tau(p,q,s)) := F(p_s)\cdot(\varphi(q,\tau(p,q,s)) - p_s) = 0.$$

Since

$$\frac{\partial G}{\partial \tau} = F(p_s)\cdot \left.\frac{\partial G}{\partial \tau}\right|_{(q,\tau)} = F(p_s)\cdot F(q_\tau) > 0,$$

then by the implicit function theorem we have that τ is differentiable with respect to q and

$$\frac{\partial \tau}{\partial q} = \frac{-F(p_s)\cdot \left.\frac{\partial\varphi}{\partial q}\right|_{(q,\tau)}}{F(p_s)\cdot F(q_\tau)}.$$

Moreover, there exists $C = C(K,\epsilon) > 0$ such that $|\partial\tau/\partial q| < C$. We have

$$Q_s = q_{\tau(s)} - p_s = (q_{\tau(s)} - p_s) + (q_s - p_s).$$

Since $q_{\tau(s)} - q_s = (\tau(s) - s)\; F(q_\sigma)$ for some σ between s and $\tau(s)$, we have

$$|q_{\tau(s)} - q_s| \le C\;|q-p| \max_q |F(q)| \le B|Q_0|,$$

for $B := C\max_q |F(q)|$. By Gronwall's inequality

$$|q_s - p_s| \le D|q-p| = D|Q_0|,$$

for some uniform $D = D(K,\epsilon)$. Therefore if $b = C + D$, we have

$$|Q_s| \le (C+D)|Q_0| = b|Q_0|.$$

The other inequality is obtained from this one by changing the roles of p and p_s and reversing the time (observe that $\varphi(q_s, \tau(p_s, q_s, -s) = q)$. Therefore

$$Q_0 \le bQ_s.$$

Now take $a = 1/b$. □

LEMMA 2.4. *For any compact subset $K \subset \mathbb{R}^3$ there exist $\varepsilon > 0$ and $M > 0$, such that if $p \in K$ and $q \in \mathbf{E}(p,\varepsilon)$, then*

$$\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} J(p_s,q_t)\,ds\,dt < M\varepsilon.$$

Proof. We first prove the case when $p \ne q$. By Lemma 2.2, we have

$$J(p_s,q_t) = \frac{Q_s\mathbf{B}_sQ_s}{|H_{st}|^3} + \frac{\Delta t\,Q_s(F_s\times \mathbf{A}_sF_s)}{|H_{st}|^3} + \psi(p,q,s,t),$$

with $\psi(p,q,s,t) \le M_1$. We bound the integral of each term.

From Lemma 2.3 and equation (1), we have

$$a|Q_0| \le |Q_s| \le b|Q_0|,$$
$$|H_{st}|^2 \ge \tfrac{1}{2}(|Q_s|^2 + |\Delta t|^2|F_s|^2) \ge \tfrac{1}{2}(|Q_s|^2 + |\Delta t|^2 D),$$

where $D = \min_{|q-p|<\varepsilon} |F(q)| > 0$. Writing $\alpha = \frac{1}{4}a$ and $\beta = \frac{1}{2}\sqrt{D}$, we have

$$|H_{st}|^2 \geq \tfrac{1}{2}(a^2|Q_0|^2 + |\Delta t|^2 D),$$
$$|H_{st}| \geq (\alpha^2|Q_0|^2 + \beta^2|\Delta t|^2)^{1/2}.$$

Since $\mathbf{B}_s H = F(p_s) \times (DF(p_s)\cdot H)$, let $B = \max_{p_s} \|\mathbf{B}_s\|$, then

$$\frac{|Q_s B_s Q_s|}{|H_{st}|^3} \leq \frac{B|Q_s|^2}{|H_{st}|^3} \leq \frac{Bb^2|Q_0|^2}{(\alpha^2|Q_0|^2 + \beta^2|\Delta t|^2)^{3/2}} = \frac{Bb^2|Q_0|^2}{\beta((\alpha^2/\beta^2)|Q_0|^2 + |\Delta t|^2)^{3/2}}.$$

Observe that, since $\Delta t = t - \tau(s)$, then

$$\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} |J|\, dt\, ds = \int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon-\tau(s)}^{\varepsilon-\tau(s)} |J|\, d(\Delta t)\, ds.$$

Since

$$\int \frac{dx}{r^3} = \frac{1}{a^2}\frac{x}{r}, \quad r = \sqrt{x^2+a^2},$$

we have

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{Q_s \mathbf{B}_s Q_s}{|H_{st}|^3}\, dt\, ds &\leq \int_{-\varepsilon}^{\varepsilon} 2\int_0^{2\varepsilon} \frac{Bb^2|Q_0|^2}{\beta\sqrt{(\alpha^2/\beta^2)|Q_0|^2 + |\Delta t|^2}}\, d(\Delta t)\, ds \\ &\leq \frac{2\varepsilon 2Bb^2|Q_0|^2}{(\alpha^2/\beta)|Q_0|^2}\, \frac{\Delta t}{\sqrt{(\alpha^2/\beta^2)|Q_0|^2 + |\Delta t|^2}}\Bigg|_0^{2\varepsilon} \\ &\leq \frac{4\varepsilon Bb^2\beta}{\alpha^2}(1+1) = \frac{8\varepsilon Bb^2\beta}{\alpha^2} = \varepsilon M_2.\end{aligned}$$

Now we bound the integral of the second term. Let

$$A := \max_{p_s} |F(p_s) \times (DF(p_s)\cdot F(p_s))|.$$

Since

$$\int \frac{x}{r^3}\, dx = -\frac{1}{r}, \quad r = \sqrt{x^2+a^2},$$

we have

$$\begin{aligned}&\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{(t-\tau(s))Q_s(F_s \times \mathbf{A}_s F_s)}{|H_{st}|^3}\, dt\, ds \\ &\leq \int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{|\Delta t| b |Q_0| A}{(\sqrt{\alpha^2|Q_0|^2 + \beta^2|\Delta t|^2})^3}\, dt\, ds \\ &\leq \int_{-\varepsilon}^{\varepsilon} 2\int_0^{2\varepsilon} \frac{b|Q_0|A|\Delta t|}{(\beta\sqrt{(\alpha^2/\beta^2)|Q_0|^2 + |\Delta t|^2})^3}\, d(\Delta t)\, ds \\ &\leq \frac{4\varepsilon bA}{\beta}|Q_0| \int_0^{2\varepsilon} \frac{|\Delta t|\, d(\Delta t)}{(\sqrt{(\alpha^2/\beta^2)|Q_0|^2 + |\Delta t|^2})^3} \\ &\leq \frac{4\varepsilon bA}{\beta^3}|Q_0|\, \frac{1}{\sqrt{(\alpha^2/\beta^2)|Q_0|^2 + |\Delta t|^2}}\Bigg|_{2\varepsilon}^{0} \\ &\leq \frac{4\varepsilon bA}{\beta^3}\frac{|Q_0|}{(\alpha/\beta)|Q_0|} \leq \frac{4bA}{\alpha\beta^2}\varepsilon = \varepsilon M_3.\end{aligned}$$

From Lemma 2.2, $|\psi(p, q, s, t)| \leq M_1$, therefore, if $\varepsilon < 1$,

$$\int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} |J(p_s, q_t)| \, ds \, dt \leq \varepsilon M_2 + \varepsilon M_3 + 4\varepsilon^2 M_1 \leq \varepsilon M_4,$$

where $M_4 := M_1 + M_2 + M_3$.

Now suppose that $p = q$. We have

$$F(q_s) = F(q) + s\mathbf{A}F(q) + s^2 K(q, s)$$
$$q_s = q + sF(q) + \tfrac{1}{2}s^2\mathbf{A}F(q) + s^3 G(q, s)$$

where $\mathbf{A} = DF(q)$, $K(q, s)$, and $G(q, s)$ are bounded. Then

$$\begin{aligned}|q_s - q|^3 J(q, q_s) &= F(q) \times (F(q) + s\mathbf{A}F(q) + s^2 K) \cdot (sF(q) + \tfrac{1}{2}s^2\mathbf{A}F(q) + s^3 G) \\ &= (sF \times \mathbf{A}F + s^2 F \times K) \cdot (sF + \tfrac{1}{2}s^2\mathbf{A}F + s^3 G) \\ &= s^4 F \times \mathbf{A}F \cdot G + \tfrac{1}{2}s^4 F \times K \cdot \mathbf{A}F + s^5 F \times K \cdot G \\ |q_s - q|^3 |J(q, q_s)| &\leq s^4 D,\end{aligned}$$

for some $D = D(K, \varepsilon) > 0$, for all $q \in K$, $|s| < \varepsilon$. Furthermore,

$$|q_s - q| = |sF(q) + \tfrac{1}{2}s^2\mathbf{A}F(q) + s^3 G| \geq as,$$

for $a = \frac{1}{2}\min_{q \in K} |F(q)|$ if $\varepsilon = \varepsilon(K) > 0$ is small enough. Therefore

$$|J(q, q_s)| \leq \frac{s^4 D}{a^3 s^3} \leq \frac{\varepsilon D}{a^3},$$

and

$$\int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} J(p_s, p_t) \, ds \, dt \leq \frac{\varepsilon^3 D}{a^3} \leq \varepsilon M_5 \quad \text{for all } p \in K,$$

for $M_5 := D/a^3$. □

COROLLARY 2.5. *For any compact subset $K \subset \mathbb{R}^3$ there exist δ, $\varepsilon > 0$ and $M > 0$ such that if p, q are in K,*

$$\int_{-\delta}^{\delta} \int_{-\delta}^{\delta} J(p_s, q_t) \, ds \, dt < M\varepsilon.$$

Proof. Take ϵ as in Lemma 2.4. Since the vector field never vanishes, then $\mathbf{N}(p, \delta)$ is a neighbourhood of p. Hence it contains a ball of radius r. Moreover, this r may be taken independent of the point p. Given any two points p and q, if δ is small enough then either there exists s_0, t_0 with $|s_0|$, $|t_0| < 2\delta$ such that q_{s_0} is in $\mathbf{E}(p_{t_0})$ and in this case

$$\int_{-\delta}^{\delta} \int_{-\delta}^{\delta} J(p_s, q_t) \, ds \, dt < \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} J(p_s, q_t) \, ds \, dt < M\varepsilon,$$

or the distance between q_s and p_t with $|s_0|$ $|t_0| < \delta$ is bounded by r and hence the integral is bounded. □

Proof of Theorem 1.1. Since J is bounded outside of a neighbourhood of the diagonal $\Delta := \{(p, p) \mid p \in S^3\}$, it is enough to prove that J is in $L^1(\mu \times \nu)$ on a neighbourhood of Δ. The compact set Δ may be covered by a finite number of open sets of the form

$$\mathbf{V}(p, \delta) := \{(q_s^1, q_t^2) \mid (q^1, q^2) \in \mathbf{E}(p, \delta) \times \mathbf{E}(p, \delta); |s|, |t| < \delta\},$$

with $p \in S^3$ and $\delta > 0$ from Corollary 2.5. So, it is enough to prove that the restriction $J|_{\mathbf{V}(p,\delta)}$ is in $L^1(\mu \times \nu)$. Let $\hat{\mu}$, $\hat{\nu}$ be the transversal measures on $E(p, \varepsilon)$ defined by $\hat{\mu}(A) := \mu(\cup_{|t|<\varepsilon}\phi_t(A))$, $\hat{\nu}(A) := \nu(\cup_{|t|<\varepsilon}\phi_t(A))$. Then by Lemma 2.4 we have

$$\begin{aligned}\int_{V(q,\delta)} |J(x, y)|\, d\mu(x)\, d\nu(y) &= \int_{E(q,\delta)} \int_{E(q,\delta)} \left(\int_{-\delta}^{\delta} \int_{-\delta}^{\delta} |J(p_s, y_t)| \right) d\hat{\mu}(p)\, d\hat{\nu}(y) \\ &\le \int_{E(q,\delta)} \int_{E(q,\delta)} M\delta\, d\hat{\mu}(p)\, d\hat{\nu}(y) \le M\delta. \qquad \square\end{aligned}$$

Proof of Theorem 1.2. Item (a) is a direct consequence of Birkhoff's theorem and Theorem 1.1. Item (b) is a consequence of the existence of a good set of short curves because

$$\begin{aligned}\left| \int_0^{T_1+1} \int_0^{T_2+1} - \int_0^{T_1} \int_0^{T_2} \right| &\le \left| \int_0^{T_1} \int_{T_2}^{T_2+1} \right| + \left| \int_{T_1}^{T_1+1} \int_0^{T_2} \right| + \left| \int_0^1 \int_0^1 \right| \\ &\le kT_1 + kT_2 + k\end{aligned}$$

and hence

$$\lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \int_0^{T_1+1} \int_0^{T_2+1} = \lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \int_0^{T_1} \int_0^{T_2}. \qquad \square$$

3. *Existence of a good set of short curves*

The following lemmas prove the existence of a good set of short curves if they are polygonal curves made of small straight line segments which are always transversal to the direction of the flow.

LEMMA 3.1. *Let $L(t) = a + tb$ be a small line segment in $\mathbb{R}^3$ and let $\gamma(s)$ be a small differentiable curve such that:*

(i) *for any $s, t \in [-\varepsilon, \varepsilon]$, the vectors $L'(t) = b$, $\gamma'(s)$ and $\gamma(s) - L(t)$ are not coplanar;*

(ii) *the vectors $(\gamma(s) - L(t))/\|\gamma(s) - L(t)\|$ are always in a hemisphere of S^2.*

Then the Gauss (absolute) integral

$$\int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} |I(s, t)|\, ds\, dt < \tfrac{1}{2} \operatorname{vol}(S^2),$$

where

$$I(s, t) := \frac{(\gamma'(s) \times L'(t)) \cdot (\gamma(s) - L(t))}{|\gamma(s) - L(t)|}.$$

Proof. The integral $I(s,t)$ is the Jacobian of the function $T(s,t) = (\gamma(s) - L(t))/|\gamma(s) - L(t)|\cdot$ The condition (i) implies that $I(s,t)$ always has the same sign. We will see that $T : [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon] \to S^2$ is injective and hence

$$\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} |I(s,t)|\, ds\, dt = \text{area}(\Gamma([-\varepsilon,\varepsilon]\times[-\varepsilon,\varepsilon])) < \tfrac{1}{2}\,\text{vol}(S^2).$$

We show that T is injective. By the transversality condition (i), if $s_1 \neq s_2$, then the planes generated by ($\gamma(s_1)$ and L) and ($\gamma(s_2)$ and L) are different. Therefore if $\gamma(s_1) - L(t_1)$ is parallel to $\gamma(s_2) - L(t_2)$ then $s_1 = s_2$. If $\gamma(\hat{s}) - L(t_1) = \lambda(\gamma(\hat{s}) - L(t_2))$ then $L(t_1) = \lambda L(t_2)$ and $t_1 = t_2$. □

LEMMA 3.2. *Let F be a Lipschitz nonsingular vectorfield on $\mathbb{R}^3$ and let $A_1, A_2, A_3 > 0$ be such that*

$$|F(x) - F(y)| \le A_1|x - y|, \quad A_2 < |F(x)| < A_3 \quad \textit{for all } x, y \in \mathbb{R}^3.$$

Then there exists $\varepsilon = \varepsilon(A_1, A_2, A_3) > 0$ such that if $L(t) = a + t\mathbf{b}$, $\|\mathbf{b}\| = 1$, $|t| < 1$ is a small line segment and $\gamma(s)$, $|s| < \varepsilon$ is a small orbit segment of F such that:

(i) *the angles $\sphericalangle(\gamma'(s), \mathbf{b}) > \pi/3$, $\sphericalangle(\gamma'(s), -\mathbf{b}) > \pi/3$ for all $|s| < \varepsilon$;*

(ii) *the vectors $L'(0) = b$, $\gamma'(0)$ and $\gamma(0) - L(0)$ are not coplanar.*

Then the Gauss (absolute) integral

$$\int_{-1}^{1}\int_{-\varepsilon}^{\varepsilon} |I(s,t)|\, ds\, dt < 2\,\text{vol}(S^2).$$

Proof. It is enough to show that $\int_{-1}^{1}\int_{0}^{\varepsilon} |I| < \text{vol}(S^2)$. Let $g(s) := (\gamma'(s) \times L'(t)) \cdot (\gamma(s) - L(t)) = (\gamma'(s) \times \mathbf{b}) \cdot (\gamma(s) - \mathbf{a})$, and let

$$\begin{aligned} s_0 &:= \min\{s > 0 \mid g(s) = 0\} \\ t_0 &:= \max\{s > s_0 \mid g(s) = 0 \text{ for all } s_0 \le s \le t_0\} \\ s_{n+1} &:= \min\{s > t_n \mid g(s) = 0\} \\ t_{n+1} &:= \max\{s > s_{n+1} \mid g(s) = 0 \text{ for all } s_{n+1} \le s \le t_{n+1}\}. \end{aligned}$$

By Lemma 3.1,

$$\int_{-1}^{1}\int_{0}^{s_0} |I(s,t)|\, ds\, dt < \tfrac{1}{2}\,\text{vol}(S^2).$$

Let $\mathbf{N} := (\mathbf{b} \times \gamma'(0))/|\mathbf{b} \times \gamma'(0)|$, $\mathbf{c} := \mathbf{N} \times \mathbf{b}$ and consider the orthonormal basis $\{\mathbf{b}, \mathbf{N}, \mathbf{c}\}$ of $\mathbb{R}^3$. Given $u_1, u_2 \in \mathbf{T} := \{w \in S^3 \mid \sphericalangle(w, \mathbf{b}) > \pi/3, \sphericalangle(w, -\mathbf{b}) > \pi/3\}$, consider their cylindrical coordinates

$$u_i = \lambda_i \mathbf{b} + r_i \cos\theta_i \mathbf{c} + r_i \sin\theta_i \mathbf{N} \quad \text{some } \lambda_i, r_i, \theta_i.$$

Let $A_4 > 0$ be such that if $u_i \in \mathbf{T}$, $|u_i| > A_2$, $i = 1, 2$, then

$$|u_1 - u_2| > A_4|\theta_1 - \theta_2|. \tag{2}$$

Let θ_n, φ_n be defined by

$$\gamma'(s_n) = \lambda_n \mathbf{b} + r_n \cos\theta_n \mathbf{c} + r_n \sin\theta_n \mathbf{N},$$
$$\gamma'(t_n) = \ell_n \mathbf{b} + k_n \cos\varphi_n \mathbf{c} + k_n \sin\varphi_n \mathbf{N}.$$

Observe that since $\gamma(s_n) - L(t)$ is coplanar with $\mathbf{b}$ and $\gamma'(s_n)$ then $\gamma(s_n) - L(t) = \mu_n(t)\mathbf{b} + \rho_n(t)\cos\theta_n\mathbf{c} + \rho_n(t)\mu\sin\theta_n\mathbf{N}$ with the same θ_n. Similarly $\gamma(t_n) - L(t)$ has the same angle φ_n as $\gamma'(t_n)$.

For $|t| < 1$, $t_n < s < s_{n+1}$, the point $G(s,t) := (\gamma(s) - L(t))/|\gamma(s) - L(t)|$ remains on the sector

$$H_n := \{v \in S^2 \mid v = \lambda\mathbf{b} + r\cos\theta\mathbf{c} + r\sin\theta\mathbf{N}, r > 0, \lambda \in \mathbb{R}, \theta \in [\varphi_n, \theta_{n+1}]\}.$$

Moreover, by the argument of Lemma 3.1, since $I(s,t) \neq 0$, for $t_n < s < s_{n+1}$, the map $G|_{[t_n, s_{n+1}]\times[-1,1]}$ is injective. Hence

$$\int_{-1}^{1}\int_{t_n}^{s_{n+1}} |I| < \text{ Area of sector } H_n = \frac{1}{2\pi}\operatorname{vol}(S^2)|\theta_{n+1} - \varphi_n|.$$

Also $\int_{-1}^{1}\int_{s_n}^{t_n} |I| = 0$, because $I(s,t) = 0$ on $s_n \le s \le t_n$. By (2) we have

$$\begin{aligned} A_4|\theta_{n+1} - \varphi_n| &< |\gamma'(s_{n+1}) - \gamma'(t_n)| = |F(\gamma(s_{n+1})) - F(\gamma(t_n))| \\ &< A_1|\gamma(s_{n+1}) - \gamma(t_n)| < A_1A_3|s_{n+1} - t_n|, \\ |\theta_{n+1} - \varphi_n| &< A_5|s_{n+1} - t_n|, \end{aligned}$$

where $A_5 := A_1A_3/A_4 > 0$. Therefore,

$$\begin{aligned} \int_{-1}^{1}\int_{0}^{\varepsilon} |I(s,t)|\,ds\,dt &= \int_{-1}^{1}\int_{0}^{s_0} |I| + \sum_{n=0}^{\infty}\int_{-1}^{1}\int_{t_n}^{s_{n+1}} |I| \\ &\le \frac{1}{2}\operatorname{vol}(S^2) + \sum_{n=0}^{\infty}\frac{1}{2\pi}\operatorname{vol}(S^2)A_5|s_{n+1} - t_n| \\ &\le \frac{1}{2}\operatorname{vol}(S^2) + \frac{1}{2\pi}\operatorname{vol}(S^2)A_5\varepsilon \\ &\le \operatorname{vol}(S^2), \end{aligned}$$

if we take $\varepsilon < \pi/A_5$. □

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