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Generic dynamics of geodesic flows

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Abstract. We present some perturbation methods which help to describe the generic dynamical behaviour of geodesic flows.

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The difficulties in studying generic properties of geodesic flows with respect to other classes of dynamical systems are twofold. The obvious difficulty of making perturbations for the geodesic equations and the fact that perturbations of geodesic flows are never local.

Indeed, let (M, g) be a compact riemannian manifold and write $g = \sum_{ij} g_{ij}(x)\, dx_i \otimes dx_j$. The phase space of the geodesic flow is the unit tangent bundle SM. A perturbation of the coefficients of the riemannian metric $g_{ij}(x)$ with support $A \subset M$ changes the geodesic vector field along the whole interior of the fiber $SA = \pi^{-1}(A)$, where $\pi : SM \to M$ is the projection. All the known proofs of the closing lemma (cf. [24]) use local perturbations and hence they can not be applied to geodesic flows.

Similar difficulties arise when one tries to change the Euler-Lagrange flow of a lagrangian $L : TM \to \mathbb{R}$ on a given energy level with perturbations by a potential i.e. $L'(x, v) = L(x, v) + \psi(x)$, where $\psi : M \to \mathbb{R}$ is a function on M. For a mechanical lagrangian, this corresponds to perturbing the conditions of the problem without changing Newton's law.

In [6] we prove Theorem 1 below, here we describe the ingredients its proof.

Theorem 1.
On any closed manifold M with $\dim M \geq 2$ the set of C^∞ riemannian metrics whose geodesic flow contains a non-trivial hyperbolic basic set is open and dense in the C^2 topology.

That basic set is a horseshoe obtained from a homoclinic point. Using symbolic dynamics, the existence of a horseshoe implies that such geodesic flows have positive topological entropy and that the number of closed geodesics grow exponentially with their length.

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1. Bumpy metrics.

The simplest invariant set in a geodesic flow is a periodic orbit $\Gamma=(\gamma,\dot\gamma)$ arising from a closed geodesic γ on M. Given a small transversal section Σ to Γ at $\Gamma(0)$ in SM define the *Poincaré map* $\mathcal{P}=\mathcal{P}(\Sigma,\gamma):\Sigma\hookleftarrow$ as the first return map under the geodesic flow. Its derivative $P=d_{\Gamma(0)}\mathcal{P}$ is called the *linearized Poincaré map*. The geodesic γ is said *non-degenerate* if 1 is not an eigenvalue of P. This is the necessary condition in the implicit function theorem to obtain a continuation of Γ under perturbations of the metric. It is also equivalent to Γ being a non-degenerate critical point of the action functional $A=\int\|v\|^2$ with appropriate normalizations[1]. A metric is said *bumpy* if all its periodic orbits are non-degenerate.

Let $\mathcal{R}^r(M)$, be the set of C^r riemannian metrics on M endowed with the C^r topology. The *Bumpy Metric Theorem* states that the set of bumpy metrics contains a residual subset in $\mathcal{R}^r(M)$, $2\le r\le\infty$. It was first announced by Abraham [1] but the first complete proof was given by Anosov [2]. Klingenberg and Takens [18] made a useful improvement:

Write $n+1=\dim M$. Given an integer $k\ge 1$ let $J^k_s(n)$ be the set of k-jets of smooth symplectic maps of $(\mathbb{R}^{2n},0)\hookleftarrow$. A set $Q\subset J^k_s(n)$ is said *invariant* if for all $\sigma\in J^k_s(n)$, $\sigma\,Q\,\sigma^{-1}=Q$. The theorem in [18] proves that if Q is a residual and invariant subset of $J^k(n)$ then the set of metrics such that the Poincaré map of every closed geodesic is in Q contains a residual set in $\mathcal{R}^r(M)$. See [29], [28] for analogous theorems on hypersurfaces of $\mathbb{R}^{n+2}$.

The proof in [18] is based on a local perturbation theorem which says that if γ is a closed geodesic for $g\in\mathcal{R}^r(M)$ there is $g'\in\mathcal{R}^r(M)$ arbitrarily close to g such that γ is a closed geodesic for g' and its Poincaré map belongs to Q. This implies the theorem above provided that the set of closed geodesics is countable. This condition is ensured by the case $k=1$ proved by Anosov [2] together with the Bumpy Metric Theorem.

2. Twist maps.

We say that a closed geodesic is *hyperbolic* if its linearized Poincaré map has no eigenvalues of modulus 1 (in a transversal section inside SM). We say that it is *elliptic* if it is non-degenerate and non-hyperbolic.

The existence of a generic elliptic periodic orbit gives dynamical information about the geodesic flow. If it is partially elliptic, i.e. if not all eigenvalues have modulus 1, using [14] one can obtain an invariant central manifold N where the Poincaré map $\mathcal{P}|_N$ is totally elliptic and N is normally hyperbolic.

Imposing generic conditions specifying only the jets of the Poincaré maps at the periodic points [6, §3] it is possible apply Klingenberg and Takens Theorem [18] to obtain coordinates in which a restriction of the Poincaré map $\mathcal{P}|_N$ becomes a weakly monotonous exact twist map on $\mathbb{T}^q\times\mathbb{R}^q$ which is C^1 near a totally

[1]i.e. on the space of closed curves in M with fixed parametrization interval $[0,\ell]$ and initial point in the transversal section Σ.

integrable twist map. In this conditions we have the Birkhoff-Lewis theorem (see Moser [17, appendix 3.3]) which says that any punctured neighbourhood of the elliptic point contains a periodic point.

Indeed, the condition to write the Birkhoff normal form are that the elliptic points are *4-elementary*, this is that the eigenvalues of modulus one $\rho_1, \ldots, \rho_q$; $\overline{\rho_1}, \ldots, \overline{\rho_q}$ satisfy

$$\textstyle\prod_{i=1}^q \rho_i^{\nu_i} \neq 1 \quad \text{whenever} \quad 1 \leq \sum_{i=1}^q |\nu_i| \leq 4. \tag{1}$$

Then the normal form is $\mathcal{P}(x,y) = (X,Y)$, where

$$\begin{aligned} Z_k &= e^{2\pi i\,\phi_k}\; z_k + g_k(z), \\ \phi_k(z) &= a_k + \textstyle\sum_{\ell=1}^q \beta_{k\ell}\,|z_\ell|^2 \end{aligned}$$

$z = x + iy$, $Z = X + iY$, $\rho_i = e^{2\pi\, i\, a_k}$ and $g(z) = g(x,y)$ has vanishing derivatives up to order 3 at the origin. We say that the normal form is *weakly monotonous* if the matrix $\beta_{k\ell}$ is non-singular. The property $\det \beta_{k\ell} \neq 0$ is independent of the particular choice of normal form. In these coordinates, the matrix $\beta_{k\ell}$ can be detected from the 3-jet of $\mathcal{P}$ at $\theta = (0,0)$ and it can be seen that the property $\{\,(1) \text{ and } \det\beta_{k\ell} \neq 0\,\}$ is open and dense in the jet space $J^3_s(q)$. Changing the coordinates to $(\theta, r) \in \mathbb{T}^q \times \mathbb{R}^q$, where $z_j = \sqrt{\varepsilon\, r_j}\, e^{2\pi i \theta_j}$ on $r_j > 0$, $\forall j$, the Poincaré map becomes a weakly monotonous exact twist map of $\mathbb{T}^q \times \mathbb{R}^q$. We restrict our discussion to the generic set of riemannian metrics all of whose closed geodesics are 4-elementary and have weakly monotonous normal forms.

Moreover, using techniques developed by Arnaud [3] we prove in [6] that $\mathcal{P}|_N$ has a 1-elliptic periodic point. This is a periodic point whose linearized Poincare map on a transversal Σ inside SM has exactly two eigenvalues of modulus 1. Such a periodic point has a normally hyperbolic central manifold where the Poincaré map is an exact twist map of the 2-dimensional annulus $\mathbb{S}^1 \times \mathbb{R}$.

Such a generic twist map contains periodic points for all rational rotation numbers in an interval. In fact for any such rational rotation number there are elliptic and hyperbolic periodic points which have homoclinic intersections [19].

3. The Kupka-Smale Theorem.

A single homoclinic intersection in a geodesic flow can be made transversal by a perturbation argument by Donnay [10] in dimension 2 and Petroll [22] in higher dimensions. But perhaps this is not enough to make transversal two invariant manifolds.

Another argument that can be used to change invariant manifolds or single orbits in geodesic flows and also in lagrangian systems with perturbations by a potential can be made along the following lines. Weak stable and weak unstable manifolds are lagrangian submanifolds for the canonical symplectic form. A lagrangian submanifold contained in a level set of an autonomous hamiltonian is invariant under the hamiltonian flow. Then it is enough to deform the stable manifold W^s to a lagrangian submanifold Λ which is transversal to W^u and then perturb the metric so that the geodesic hamiltonian $H|_\Lambda$ is constant. We will have that $W^s = \Lambda$ for the new geodesic flow. The details appear in [8, Theorem 2.5 and Appendix A].

This argument together with Anosov-Klingenberg-Takens theorem gives

Theorem 2.
Let $Q \subset J_s^{k-1}(n)$ be residual and invariant. There is a residual subset $\mathcal{G} \subset \mathcal{R}^k(M)$ such that if $g \in \mathcal{G}$ then

- *The $(k-1)$-jet of the Poincaré map of every closed geodesic of g is in Q.*
- *All heteroclinic intersections of hyperbolic orbits of g are transversal.*

Choosing Q in the previous theorem as the condition $\{\,(1)$ and $\det \beta_{k\ell} \neq 0\,\}$ as above, we have that for a Kupka-Smale geodesic flow, if it contains an elliptic closed geodesic then it has a transversal homoclinic orbit and hence a hyperbolic subset. It remains to study the case in which all closed geodesics are hyperbolic.

4. Many closed geodesics.

Hingston [13] and Rademacher [25], [27] prove that a C^k generic riemannian metric, $2 \leq k \leq \infty$, contains infinitely many closed geodesics.

If the geodesic flow contains a generic elliptic closed geodesic, this is implied by the Birkhoff-Lewis theorem (Moser [17, appendix 3.3]). But in this case Rademacher [27] obtains infinitely many closed geodesics by imposing only conditions on the 1-jet of the Poincaré map, which is easier to perturb as in Anosov [2]. If there are finitely many closed geodesics Rademacher obtains a resonance condition on the average indices of the geodesics. If there is one elliptic closed geodesic, its average index can be perturbed to break the resonance and hence obtain infinitely many closed geodesics. If all closed geodesics are hyperbolic, then Hingston [13, §6.1] and Rademacher [25, Theorem 1] prove that there are infinitely many.

It is not known if a simply connected manifold can have all its closed geodesics hyperbolic. In [26] Rademacher proves that in the examples of ergodic geodesic flows in S^2 of Donnay [9] and Burns-Gerber [5], all the homologically visible closed geodesics are hyperbolic.

5. Stable hyperbolicity.

In order to prove the generic existence of a homoclinic orbit and hence a hyperbolic set when all closed geodesics are hyperbolic, we use the theory of stable hyperbolicity developed by Mañé [21].

Recall that a linear map $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is *hyperbolic* if it has no eigenvalue of modulus 1. Equivalently, if there is a splitting $\mathbb{R}^{2n} = E^s \oplus E^u$ and $M \in \mathbb{Z}^+$ such that $T(E^s) = E^s$, $T(E^u) = E^u$, $\|T^M|_{E^s}\| < \frac{1}{2}$, $\|T^{-M}|_{E^u}\| < \frac{1}{2}$.

Let $Sp(n)$ be the group of symplectic linear isomorphisms of $\mathbb{R}^{2n}$. We say that a sequence $\xi : \mathbb{Z} \to Sp(n)$ is *periodic* if there is $m \geq 1$ such that $\xi_{m+i} = \xi_i$ for all $i \in \mathbb{Z}$. A periodic sequence is said *hyperbolic* if the linear map $\prod_{i=1}^m \xi_i$ is hyperbolic. In this case the stable and unstable subspaces of $\prod_{i=1}^m \xi_{j+i}$ are denoted by $E_j^s(\xi)$ and $E_j^u(\xi)$ respectively.

A family $\xi = \{\xi^\alpha\}_{\alpha\in\mathcal{A}}$ of sequences in $Sp(n)$ is *bounded* if there exists $Q > 0$ such that $\|\xi_i^\alpha\| < Q$ for every $\alpha \in \mathcal{A}$ and $i \in \mathbb{Z}$. Given two families of periodic sequences in $Sp(n)$, $\xi = \{\xi^\alpha\}_{\alpha\in\mathcal{A}}$ and $\eta = \{\eta^\alpha\}_{\alpha\in\mathcal{A}}$, we say that they are *periodically equivalent* if they have the same indexing set $\mathcal{A}$ and for all $\alpha \in \mathcal{A}$ the periods of ξ^α and η^α coincide. Given two periodically equivalent sequences in $Sp(n)$, $\xi = \{\xi^\alpha\}_{\alpha\in\mathcal{A}}$ and $\eta = \{\eta^\alpha\}_{\alpha\in\mathcal{A}}$ define

$$d(\xi, \eta) = \sup\{\, \|\xi_i^\alpha - \eta_i^\alpha\| \,:\, \alpha \in \mathcal{A},\ i \in \mathbb{Z}\,\}.$$

We say that a family ξ is *hyperbolic* if for all $\alpha \in \mathcal{A}$ the periodic sequence ξ^α is hyperbolic. We say that a hyperbolic periodic family ξ is *stably hyperbolic* if there is $\varepsilon > 0$ such that any periodically equivalent family η satisfying $d(\xi, \eta) < \varepsilon$ is also hyperbolic.

Finally, we say that a family of periodic sequences ξ is *uniformly hyperbolic* if there exist $K > 0$, $0 < \lambda < 1$ and subspaces $E_i^s(\xi^\alpha)$, $E_i^u(\xi^\alpha)$, $\alpha \in \mathcal{A}$, $i \in \mathbb{Z}$ such that $\xi_j(E_j^\tau(\xi^\alpha)) = E_{j+1}^\tau(\xi^\alpha)$ for all $\alpha \in \mathcal{A}$, $j \in \mathbb{Z}$, $\tau \in \{s, u\}$ and

$$\Big\| \prod_{i=1}^m \xi_{j+i}^\alpha \Big|_{E_j^s(\xi^\alpha)} \Big\| < K\,\lambda^m \quad \text{and} \quad \Big\| \Big(\prod_{i=1}^m \xi_{j+i}^\alpha \Big|_{E_j^u(\xi^\alpha)} \Big)^{-1} \Big\| < K\,\lambda^m$$

for all $m \in \mathbb{Z}^+$, $\alpha \in \mathcal{A}$, $j \in \mathbb{Z}$.

In [6] we prove

Theorem 3.

If ξ^α is a bounded stably hyperbolic family of periodic sequences of symplectic linear maps then it is uniformly hyperbolic.

6. The perturbation lemma.

Let Γ be a set of closed geodesics. Construct a family ξ of periodic sequences in $Sp(n)$ given by the linearized time 1 maps of the geodesic flow restricted to the normal bundle $\mathcal{N}$ in SM to the geodesic vector field in $\Gamma \subset SM$.

Suppose that there are infinitely many closed geodesics in Γ and that the family ξ is uniformly hyperbolic. Then the subspaces E^s, E^u are continuous in Γ and hence they can be extended continuously to the closure $\overline{\Gamma}$. The closure would be a uniformly hyperbolic set. By the Spectral Decomposition Theorem it contains a non-trivial hyperbolic basic set because it is not a union of isolated periodic orbits.

We say that a set Γ of closed geodesics for a metric g_0 is *stably hyperbolic* if there is a neighbourhood $\mathcal{U} \subset \mathcal{R}^2(M)$ of g_0 in the C^2-topology such that for every $g \in \mathcal{U}$, the analytic continuation $\Gamma(g)$ of Γ exists and all the orbits in $\Gamma(g)$ are hyperbolic. In [8] we prove a perturbation lemma which implies that if Γ is a stably hyperbolic set of closed geodesics then the corresponding family ξ of symplectic linear maps is stably hyperbolic.

Using that result we obtain that a geodesic flow can either be perturbed to contain a generic elliptic closed geodesic, and hence a twist maps and homoclinics, or the set of its periodic orbits is stably hyperbolic and then contains a hyperbolic basic set.

We need a lemma in which one can perturb the linearized Poincaré map of the time one map in single geodesic in Γ by a fixed amount independently of the length, position, self-intersection or self-accumulation of the geodesic. Since there is no "transversal space" to mitigate the perturbation, such a lemma can only hold in the C^2 topology. The lemma is written for a small geodesic segment.

For simplicity we assume that all our riemannian metrics have injectivity radius larger than 2. Due to an algebraic obstruction in our proof of the lemma we have to assume that the initial metric g_0 is in the set $\mathcal{G}_1$ of metrics with the property that every geodesic segment of length $\frac{1}{2}$ has one point where the sectional curvatures are all different. In [6, §6 and appendix A] we prove that $\mathcal{G}_1$ is C^2 open and C^∞ dense.

For the sequel we need to characterize $\mathcal{G}_1$ precisely. The orthogonal group $\mathcal{O}(n)$ acts on the set of symmetric matrices $\mathcal{S}(n) \subset \mathbb{R}^{n\times n}$ by conjugation: $K \mapsto Q\,K\,Q^*$, $K \in \mathcal{S}(n)$, $Q \in \mathcal{O}(n)$. Given $g \in \mathcal{R}^2(M)$, define the map $K_g : SM \to \mathcal{S}(n)/\mathcal{O}(n)$ as $K_g(\theta) := [K]$, where $K_{ij} = \langle R_g(\theta, e_i)\,\theta,\, e_j\rangle$, R_g is the curvature tensor of g and $\{\,\theta, e_1, \ldots, e_n\,\}$ is a g-orthonormal basis for $T_{\pi(\theta)}M$. Let $h : \mathcal{S}(n)/\mathcal{O}(n) \to \mathbb{R}$ be the function

$$h([K]) = \prod_{1\le i<j\le n} (\lambda_i - \lambda_j)^2,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of K. Let $H : \mathcal{R}^2(M) \to [0, +\infty[$ be

$$H(g) = \min_{\theta\in SM} \max_{t\in[0,\frac{1}{2}]} h(K_g(\theta)).$$

Then $\mathcal{G}_1 = \{\, g \in \mathcal{R}^2(M) \,:\, H(g) > 0\,\}$.

Fix a C^∞ riemannian metric g_0 on M. Let $\gamma : [0,1] \to M$ be a geodesic segment for g_0. Let W be any neighbourhood of $\gamma([0,1])$ in M. Let $\mathcal{F} = \{\,\eta_1, \ldots, \eta_m\,\}$ be any finite set of geodesic segments defined on $[0,1]$ with the following properties

- The endpoints of η_i are not contained in W.
- The segment $\gamma([0,1])$ intersects each η_i transversally.

Let U be a neighbourhood of $\cup\mathcal{F} := \cup_{i=1}^m \eta_i([0,1])$. Denote by $\mathcal{R}^\infty(g_0, \gamma, \mathcal{F}, W, U)$ the set of C^∞ riemannian metrics g on M for which γ is a geodesic segment, $g = g_0$ on $\gamma([0,1])$ and $g = g_0$ on $U \cup (M \setminus W)$.

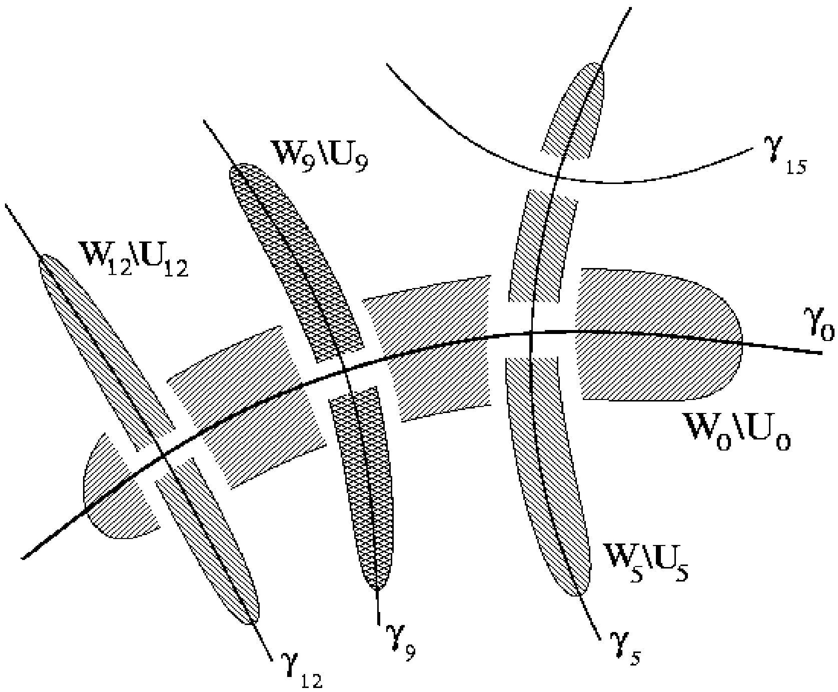

Figure 1. Avoiding self-intersections.

Let $\mathcal{N}_t = \{\, \zeta \in T_{\dot\gamma(t)}SM \mid \langle d\pi(\zeta), \dot\gamma(t)\rangle = 1 \,\}$ be the subspace transversal to the geodesic vector field given by the kernel of the Liouville 1-form $\lambda_{(x,v)}(\zeta) = \langle \zeta, v\rangle_x$. This subspace is the same for all metrics g with $g = g_0$ on $\gamma([0,1])$. Fix symplectic orthonormal basis for $\mathcal{N}_0$ and $\mathcal{N}_1$. Identify these subspaces with $\mathbb{R}^{2n}$ and the symplectic linear maps $\mathcal{N}_0 \to \mathcal{N}_1$ with $Sp(n)$. Let ϕ_t^g be the geodesic flow of a metric g and $S : \mathcal{R}^\infty(g_0, \gamma, \mathcal{F}, W, U) \to Sp(n)$ be the linearized Poincaré map $S(g) := d_{\dot\gamma(0)}\phi_1^g|_{\mathcal{N}_0}$. Write

$$B(g,\gamma) := \max\Big\{\ \|g|_\gamma\|_{C^4}\,,\ \Big[\max_{t\in[\frac14,\frac34]} h\big(K_g(\dot\gamma(t))\big)\Big]^{-1}\Big\}.$$

Theorem 4. *Let $g_0 \in \mathcal{G}_1 \cap \mathcal{R}^r(M)$, $4 \le r \le \infty$. Given a neighbourhood $\mathcal{U} \subset \mathcal{R}^2(M)$ of g_0 there is $\delta = \delta(B(g_0,\gamma),\mathcal{U}) > 0$ such that given any γ, W and $\mathcal{F}$ as above there is a neighbourhood $U = U(B(g_0,\gamma),\mathcal{U},\gamma,W,\mathcal{F})$ of $\cup\mathcal{F}$ in M such that the image of $\mathcal{G}_1 \cap \mathcal{U} \cap \mathcal{R}^\infty(g_0,\gamma,\mathcal{F},W,U)$ under the map S contains the ball of radius δ centered at $S(g_0)$.*

The actual perturbation is made in a small neighbourhood of one point in $\gamma([\frac14,\frac34])$, so that the theorem can be applied independently to adjacent segments. In order to perturb the linearized Poincaré map on a periodic orbit we use Theorem 4 sequentially on segments of the orbit, taking care that the support of the perturbations are disjoint, as suggested in figure 1. The radius δ can remain constant $\delta = \delta(\max\{\|g_0\|_{C^4}, H(g_0)^{-1}\}, \mathcal{U})$ in this process. Because despite the C^4 norm of the perturbed metric grows and $H(g)$ changes, the estimate of δ only depends on the bounds along the segment γ and subsequent perturbations have disjoint supports. Each perturbation is C^2 small and all perturbations remain in $\mathcal{U}$.

The statement of the lemma for a closed orbit is as follows. Given a closed geodesic γ for g_0 and a neighbourhood W of γ let $\mathcal{R}^r(g_0, \gamma, W)$ be the set of C^r riemannian metrics g for which γ is a geodesic, and such that $g = g_0$ on $\gamma \cup (M \setminus W)$. Let T be the minimal period of γ and let $m \in \mathbb{N}$ and $\tau \in [\frac{1}{2}, 1]$ be such that $m\tau = T$. Let $\gamma_k(t) = \gamma(t + k\tau)$, $t \in [0, \tau]$ and for $g \in \mathcal{R}^r(g_0, \gamma, W)$ let $S_k(g) = d_{\gamma_k(0)}\phi^g_\tau \in Sp(n)$, identifying symplectic linear maps $\mathcal{N}(\dot\gamma_k(0)) \to \mathcal{N}(\dot\gamma_k(\tau))$ with $Sp(n)$, $\mathcal{N}(\theta) = \ker \lambda_\theta|_{T_\theta SM}$.

Corollary.
Let $g_0 \in \mathcal{G}_1 \cap \mathcal{R}^r(M)$, $4 \le r \le \infty$. Given a neighbourhood $\mathcal{U}$ of g_0 in $\mathcal{R}^2(M)$, there exists $\delta = \delta(g_0, \mathcal{U}) > 0$ such that if $g \in \mathcal{U}$, γ is a cosed geodesic for g_0 and W is a tubular neighbourhood of γ, then the image of $\mathcal{U} \cap \mathcal{G}_1 \cap \mathcal{R}^r(\gamma, g_0, W) \to \Pi_{k=0}^{m-1} Sp(n)$, under the map $(S_0, \dots, S_{m-1})$, contains the product of balls of radius δ centered at $S_k(g_0)$ for $0 \le k < m$.

The derivative of the geodesic flow is represented by Jacobi fields. To prove Theorem 4 one has to perturb the solutions of the Jacobi equation. The Jacobi equation is difficult to solve but the perturbation of the Jacobi equation giving the derivative $d_g S$ can be solved by variation of parameters in terms of the original solution $S(g)$. This allows to estimate the expansion of $d_g S$ and then the radius δ.

7. Elliptic geodesics in the sphere.

Another problem in which these methods have been used [7] is to prove that there is a C^2 open and dense set of riemannian metrics in $\mathbb{S}^2$ which contain an elliptic closed geodesic.

Henri Poincaré [23] claimed that every convex surface in $\mathbb{R}^3$ contains an elliptic closed geodesic. But Grjuntal [12] showed a counterexample. Pinching conditions to obtain an elliptic closed geodesic on spheres have been given in Grjuntal [11], Thorbergsson [30] and Ballmann, Thorbergsson, Ziller [4].

If the metric can not be perturbed to a metric with an elliptic closed geodesic, then its set of closed orbits is stably hyperbolic and then its closure is uniformly hyperbolic. The geodesic flow of $\mathbb{S}^2$ can not be Anosov, because Anosov geodesic flows do not have conjugate points [16], [20].

The geodesic flow is the Reeb flow of the Liouville contact form on SM. The unit tangent bundle of $\mathbb{S}^2$ is $\mathbb{RP}^3$. Its double cover is $\mathbb{S}^3$ and the geodesic flow of $\mathbb{S}^2$ lifts to the Reeb flow of a tight contact form on $\mathbb{S}^3$. If the metric is bumpy one can apply the theory of Hofer, Wysocki, Zehnder [15].

In the dynamically convex case, there is a surface of section which is a disk transversal to all but one orbit of the Reeb flow which is the boundary of the disk. The return map to the disk preserves the finite area form which is the differential of the contact form. This leads to a contradiction because it can be proved that a homoclinic class of an area preserving map which is not Anosov can not be uniformly hyperbolic. In the non-dynamically convex case we use geometric arguments on the finite energy foliation of [15] to get a contradiction.

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