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# The asymptotic Maslov index and its applications

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*Abstract*. Let $\mathcal{N}$ be a $2n$-dimensional manifold equipped with a symplectic structure $\omega$ and $\Lambda(\mathcal{N})$ be the Lagrangian Grassmann bundle over $\mathcal{N}$. Consider a flow $\phi^t$ on $\mathcal{N}$ that preserves the symplectic structure and a $\phi^t$-invariant connected submanifold $\Sigma$. Given a continuous section $\Sigma \to \Lambda(\mathcal{N})$, we can associate to any finite $\phi^t$-invariant measure with support in $\Sigma$, a quantity, *The asymptotic Maslov index*, which describes the way Lagrangian planes are asymptotically wrapped in average around the Lagrangian Grassmann bundle. We pay particular attention to the case when the flow is derived from an optical Hamiltonian and when the invariant measure is the Liouville measure on compact energy levels. The situation when the energy levels are not compact is discussed.

## 1. *Introduction*

### 1.1. *The Maslov Cocycle.*

In his book *Théorie des perturbations et méthodes asymptotiques* [**15**], Maslov introduced an index of curves relevant in quantum mechanics. Arnold [**2**], in an appendix to Maslov's book, set down the main geometric features of this index introducing a characteristic class. This introductive section is very much guided by Arnold's appendix.

Consider the standard $2n$-dimensional vector space $\mathbb{R}^{2n}$ and its canonical decomposition $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ and denote a point in $\mathbb{R}^{2n}$ by $x = (p, q)$, where $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$. The space $\mathbb{R}^{2n}$ can be endowed with three structures:

- *a Euclidean structure*, i.e. a positive definite quadratic form on $\mathbb{R}^{2n}$:

$$\langle x, x\rangle = \sum_{i=1}^{n}(p_i^2 + q_i^2);$$

- *a complex structure*, i.e. an endomorphism $J$ on $\mathbb{R}^{2n}$ satisfying $J^2 = -Id$:

$$J(p, q) = (-q, p);$$

- *a symplectic structure*, i.e. an antisymmetric non-degenerate bilinear form:

$$\omega(x, y) = \langle J(x), y \rangle = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

The group of automorphisms of $\mathbb{R}^{2n}$ that preserve these structures† is called the *unitary group* and is denoted by $U(n)$. This group is isomorphic to the group of linear isometries of $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n \approx (\mathbb{R}^n \times \{0\}) \oplus J(\mathbb{R}^n \times \{0\}) = \mathbb{R}^{2n}$.

A subspace $W$ in $\mathbb{R}^n$ is said to be *isotropic* if the symplectic form $\omega$ vanishes on $W$. A *Lagrangian plane* is an isotropic subspace of maximal dimension $n$. The subspaces $p = 0$, $q = 0$ and $p = q$ are examples of Lagrangian planes.

The elements of the unitary group map Lagrangian planes onto Lagrangian planes. Actually, the unitary group acts transitively on the set of Lagrangian planes and its stationary group is isomorphic to the orthogonal group $O(n)$. This provides the set of Lagrangian planes $\Lambda(n)$ with the structure of a compact manifold, $\Lambda(n) = U(n)/O(n)$, called the *Lagrangian Grassmann manifold*.

This last identification can be seen as follows. Given an orthonormal basis $\{e_1, \ldots, e_n\}$ of the Lagrangian plane $\mathbb{E}_0 := \{q = 0\}$, consider another Lagrangian plane $\lambda$ and an orthonormal basis $\{\ell_1, \ldots, \ell_n\}$ for $\lambda$. The automorphism of $\mathbb{R}^{2n}$ that maps, for $i = 1, \ldots, n$, each vector $e_i$ to $\ell_i$ and each vector $Je_i$ to $J\ell_i$ is clearly a unitary automorphism. Any unitary automorphism that leaves the Lagrangian plane $\lambda$ invariant, transforms the orthogonal basis $\{e_1, \ldots, e_n\}$ into another orthogonal basis $\{e'_1, \ldots, e'_n\}$ of $\mathbb{E}_0$ and the corresponding orthogonal basis $\{Je_1, \ldots, Je_n\}$ of $J\mathbb{E}_0$ into the orthogonal basis $\{Je'_1, \ldots, Je'_n\}$ of $J\mathbb{E}_0$. Consequently a unitary isomorphism $U$ that fixes $\mathbb{E}_0$ also fixes $J\mathbb{E}_0$. Furthermore, the matrix of $U$ restricted to $\mathbb{E}_0$ written in the basis $\{e_1, \ldots, e_n\}$ coincides with the matrix of $U$ restricted to $J\mathbb{E}_0$ written in the basis $\{Je_1, \ldots, Je_n\}$ and is an element of the orthogonal group $O(n)$. Thus, given two Lagrangian planes $\mathbb{E}_0$ and $\lambda$, there exists a unitary automorphism $u(\lambda)$ mapping the plane $\mathbb{E}_0$ onto the plane $\lambda$. This automorphism is unique up to orthogonal self transformations of the plane $\mathbb{E}_0$, hence its complex determinant is a complex number with modulus one which is unique up to multiplication by $-1$. This defines a map

$$\det{}^2 : \Lambda(n) \to \mathbb{S}^1,$$

which associates, the square of the determinant of the automorphism $u(\lambda)$ to each Lagrangian plane.

The *Maslov cocycle* $\mathfrak{M}$ (see [**2**]) is the element of the first cohomology group $H^1(\Lambda(n), \mathbf{Z})$ which associates to any oriented closed curve $\gamma$ in $\Lambda(n)$ the degree of the map:

$$\mathbb{S}^1 \xrightarrow{\gamma} \Lambda(n) \xrightarrow{\det^2} \mathbb{S}^1.$$

† In fact, an automorphism that preserves two of these structures, necessarily preserves the third one.

The *Maslov cycle* of the Lagrangian plane $\mathbb{E}_0$ (also called the *train* of the Lagrangian plane) is the subset $\Lambda^{\mathbb{E}_0}(n)$ of $\Lambda(n)$, which consists of all the Lagrangian planes whose intersection with $\mathbb{E}_0$ is non-trivial. Arnold [**2**] proved that the Maslov cycle of a Lagrangian plane defines a codimension 1, co-oriented cycle in the Lagrangian Grassmann manifold $\Lambda(n)$†. Writing $\Lambda(n) = U(n)/O(n)$, the transversal orientation of $\Lambda^{\mathbb{E}_0}(n)$ coincides with the orientation of $\theta \mapsto e^{i\theta}\mathbb{E}_0$.

Using the homotopy exact sequence of the fibration

$$O(n) \longrightarrow U(n) \longrightarrow \Lambda(n),$$

we get that $\pi_1(\Lambda(n)) = \mathbb{Z}$. Moreover, the following commutative diagram of fiber bundles

$$\begin{tikzcd}
SO(n) \arrow[r] \arrow[d] & O(n) \arrow[r, "\det"] \arrow[d] & \mathbb{S}^0 \arrow[d] \\
SU(n) \arrow[r] \arrow[d] & U(n) \arrow[r, "\det"] \arrow[d] & \mathbb{S}^1 \arrow[d, "z^2"] \\
S\Lambda(n) \arrow[r] & \Lambda(n) \arrow[r, "\det^2"] & \mathbb{S}^1
\end{tikzcd}$$

where $S\Lambda(n) = \{\lambda \in \Lambda(n) \mid \det^2\lambda = 1\}$, implies that the generator of $\pi_1(\Lambda(n))$ is sent by $\det^2$ to the generator of $\pi_1(\mathbb{S}^1)$. This implies that the Maslov cocycle is a generator of $H^1(\Lambda(n), \mathbb{Z})$ (cf. [**2**]).

The *Maslov index* $\mathfrak{M}(h) \in \mathbb{Z}$ of a homology class $h \in H_1(\Lambda(n), \mathbb{Z})$ is the value that the cocycle $\mathfrak{M}$ takes on the cycle $h$. The Maslov cycle $\Lambda^{\mathbb{E}_0}(n)$ is the Poincaré dual of $\mathfrak{M}$, so that $\mathfrak{M}(h)$ is also equal to the oriented intersection number of $h$ and $\Lambda^{\mathbb{E}_0}(n)$.

*Remark 1.1.* The Maslov cocycle $\mathfrak{M}$ in $H^1(\Lambda(n), \mathbb{Z})$ is induced by a 1-form $\eta$ in $H^1(\Lambda(n), \mathbb{R})$, which is defined for every $\lambda$ in $\Lambda(n)$ by $\eta_\lambda = d_\lambda u$, where $u : \Lambda(n) \to \mathbb{R}$ is locally determined by $\det^2(\lambda) = \exp(2i\pi u(\lambda))$. Thus, for any piecewise differentiable oriented closed curve $\gamma : [0, 1] \to \Lambda(n)$, which is in the homology class $h$, we have

$$\mathfrak{M}(h) = \int_0^1 \eta_{\gamma(t)}(\dot{\gamma}(t))\, dt.$$

In the particular case when the dimension $n$ is 1, Lagrangian planes are lines in $\mathbb{R}^2$ and the Lagrangian Grassmann manifold $\Lambda(1)$ is the projective line $\mathbb{RP}(1) = \mathbb{S}^1$. The unitary group $U(1)$ is the group $\mathbb{S}^1$ of complex numbers with modulus 1. The orthogonal group $O(1)$ is the group with two elements $\pm Id$ and the identification $\Lambda(1) = U(1)/O(1)$ reflects the fact that each line in $\mathbb{R}^2$ is the image of a given line by a multiplication by a complex number with modulus 1, $\exp 2i\pi\theta$, where $\theta$ is defined up to translation by $\frac{1}{2}$ in $\mathbb{R}/\mathbb{Z}$. The Maslov cocycle is simply the morphism that associates to any curve in $\Lambda(1)$ its degree and the Maslov cycle of a Lagrangian plane (i.e. a line) is reduced to the

† In fact, for $k \geq 1$, the set $\Lambda^k(n) = \{\lambda \in \Lambda \mid \dim \lambda \cap \mathbb{E}_0 = k\}$ is an open submanifold of $\Lambda(n)$ with codimension $\frac{1}{2}k(k+1)$, (see Arnold [**2**]). In particular, $\Lambda^{\mathbb{E}_0}(n) = \overline{\Lambda^1(n)} = \bigcup_{k\geq 1} \Lambda^k(n)$ is a cycle in $\Lambda(n)$ of codimension 1 (actually it is an algebraic variety with singular set $\overline{\Lambda^2(n)} = \bigcup_{k\geq 2} \Lambda^k(n)$ of codimension greater than or equal to 3 and thus its boundary chain is null).

line itself. Thus, it is straightforward to notice that any two lines (i.e. Lagrangian planes) are joined by an arc in $\Lambda(1)$ whose intersection number with (the Maslov cycle of) any given line is bounded in norm by 1.

When the dimension $n$ is greater than one, even if more complex, the situation keeps part of this rigidity. More precisely, consider two Lagrangian planes $\lambda_0$ and $\lambda_1$ with trivial intersection, a basis $\{e_1, \dots, e_n\}$ for $\lambda_0$ and a basis $\{e'_1, \dots, e'_n\}$ for $\lambda_1$ such that

$$\omega = \sum_{i=1}^{n} {e_i}^* \wedge {e'_i}^*$$

where $\{{e_i}^*, {e'_i}^*\}$ is the dual basis of $\{e_i, e'_i\}$. Any Lagrangian plane $\lambda$ whose intersection with $\lambda_1$ is reduced to zero can be seen as a graph of a map from $\lambda_0$ to $\lambda_1$. Actually, when written in coordinates with the basis $\{e_1, \dots, e_n\}$ for $\lambda_0$ and $\{e'_1, \dots, e'_n\}$ for $\lambda_1$, a straightforward calculation shows that this graph is given by a $n \times n$-symmetric matrix $A$

$$y = Ax.$$

Furthermore, if the intersection of $\lambda$ with $\lambda_0$ is reduced to zero, the matrix $A$ is invertible. The path $\Gamma_{\lambda_0,\lambda_1,\lambda}$ defined for every $t \in [0, 1]$ by

$$\Gamma_{\lambda_0,\lambda_1,\lambda}(t) = \{(x, y) \in \lambda_0 \oplus \lambda_1 \approx \mathbb{R}^n \times \mathbb{R}^n \mid y = tAx\}$$

is a path from $[0, 1]$ to $\Lambda(n)$ joining $\Gamma_{\lambda_0,\lambda_1,\lambda}(0) = \lambda_0$ to $\Gamma_{\lambda_0,\lambda_1,\lambda}(1) = \lambda$.

The following lemma has its roots in the history of symplectic geometry (see [**3**]). However, for the sake of completeness, and since it will be useful in the following, we include a simple proof of it.

LEMMA 1.2. *Let $\lambda$ be a Lagrangian plane whose intersection with $\lambda_0$ and $\lambda_1$ is reduced to zero and $\mathbb{E}_0$ be a Lagrangian plane whose intersection with $\lambda_1$ is reduced to zero, then the intersection number of the path $\Gamma_{\lambda_0,\lambda_1,\lambda}$ with the Maslov cycle of the Lagrangian plane $\mathbb{E}_0$ is bounded in norm by $n$.*

*Proof.* Written in coordinates in the basis $\{e_1, \dots, e_n\}$ for $\lambda_0$ and $\{e'_1, \dots, e'_n\}$ for $\lambda_1$, the equation of $\mathbb{E}_0$ reads

$$y = Bx,$$

where $B$ is a $n \times n$-symmetric matrix. Assume that for some $t = t_1$ the two planes $\mathbb{E}_0$ and $\Gamma_{\lambda_0,\lambda_1,\lambda}(t_1)$ have a non-trivial intersection. This means that the kernel of $t_1 A - B$ that we denote by $W_{t_1}$, is a subspace of $\mathbb{R}^n$ not reduced to zero. Let $t_1, \dots, t_k$ be a strictly increasing sequence of values of $t$ such that $W_{t_i}$ is not reduced to zero for $i = 1, \dots, k$. Since $A$ is invertible, all these subspaces are linearly independent and thus

$$\sum_{i=1}^{k} \dim W_{t_i} \leq n.$$

It follows that for $\varepsilon > 0$ small, the path $\Gamma_{\lambda_0,\lambda_1,\lambda}|_{t \in [t_i - \varepsilon, t_i + \varepsilon]}$ is homotopic, with fixed endpoints, to a path $h$ which intersects $\mathbb{E}_0$ at most $\dim W_{t_i}$-times and such that, for each $t \in [0, 1]$, $\dim h(t) \cap \mathbb{E}_0 \leq 1$. □

*Remark 1.3.* Consider a linear transformation $T$ of $\mathbb{R}^{2n}$ that preserves the symplectic structure. It is clear that the path $\Gamma_{T\lambda_0,T\lambda_1,T\lambda}$ constructed with the bases $\{Te_1, \dots, Te_n\}$ for $T\lambda_0$ and $\{Te'_1, \dots, Te'_n\}$ for $T\lambda_1$ is the image under $T$ of the path $\Gamma_{\lambda_0,\lambda_1,\lambda}$ and is again given for every $t \in [0, 1]$ by the equation

$$\Gamma_{T\lambda_0,T\lambda_1,T\lambda}(t) = T \circ \Gamma_{\lambda_0,\lambda_1,\lambda}(t) = \{(x, y) \in T\lambda_0 \oplus T\lambda_1 \approx \mathbb{R}^n \times \mathbb{R}^n \mid y = tA\,x\}.$$

1.2. *Symplectic manifolds and the Lagrangian Grassmann bundle.* Let $\mathcal{N}$ be a $2n$-dimensional manifold equipped with a symplectic structure, i.e. a non-degenerate closed 2-form $\omega$.

Given $x$ in $\mathcal{N}$, a subspace $W$ in the tangent space $T_x\mathcal{N}$ is *isotropic* if the 2-form $\omega$ vanishes on $W$. Isotropic subspaces with maximal dimension are called *Lagrangian planes* and have dimension $n$. The *Lagrangian Grassmann bundle* $\Lambda(\mathcal{N})$ is the bundle over $\mathcal{N}$ whose fibers consist of all the Lagrangian planes. We denote by

$$\Pi : \Lambda(\mathcal{N}) \to \mathcal{N},$$

the standard projection.

Given $x$ in $\mathcal{N}$, an *almost complex structure* in the tangent space $T_x\mathcal{N}$ is an endomorphism $J_x$ of $T_x\mathcal{N}$, such that $J_x^2 = -Id$. We say that an almost complex structure $J_x$ and a symplectic structure $\omega_x$ are *compatible* if $u \mapsto \omega_x(J_x u, u)$ is a positive quadratic form on $T_x\mathcal{N}$. This quadratic form induces an Euclidean structure associated to the compatible pair of symplectic and almost complex structures.

Let $\mathcal{J}(\mathcal{N})$ be the bundle over $\mathcal{N}$ whose fiber over any point $x$ in $\mathcal{N}$ consists of all compatible almost complex structures in $T_x\mathcal{N}$. Since the fibers of this bundle are contractible, there exists a continuous section

$$J : \mathcal{N} \to \mathcal{J}(\mathcal{N}),$$

which is unique up to homotopy. The almost complex structure $J$ can be chosen to be smooth.

Consider a connected submanifold $\Sigma$ of $\mathcal{N}$ (neither necessarily compact nor with finite first homology group) and the corresponding bundle $\Lambda(\Sigma) = \Pi^{-1}(\Sigma)$. Assume that there exists a continuous section

$$\mathbb{E} : \Sigma \to \Lambda(\Sigma).$$

This section induces, over each point $x$ in $\Sigma$, a continuous splitting of the tangent space

$$T_x\mathcal{N} = \mathbb{E}(x) \oplus J_x\mathbb{E}(x).$$

Since the unitary group acts transitively on the set of Lagrangian planes of $T_x\mathcal{N}$ and its stationary group is isomorphic to the orthogonal group $O(n)$ (see §1.1), this section yields a trivialization of the bundle $\Lambda(\Sigma)$:

$$I_{\mathbb{E}} : \Lambda(\Sigma) \to \Sigma \times \Lambda(n) = \Sigma \times U(n)/O(n).$$

The *Maslov cocycle of the section* $\mathbb{E}$, $\mathfrak{M}_{\mathbb{E}}$, is the element in $H^1(\Lambda(\Sigma), \mathbb{Z})$ which associates to any oriented closed curve $\gamma$ in $\Lambda(\Sigma)$ the degree of the map

$$\mathbb{S}^1 \xrightarrow{\gamma} \Lambda(\Sigma) \approx \Sigma \times \Lambda(n) \xrightarrow{\tau} \Lambda(n) \overset{\det^2}{\to} \mathbb{S}^1,$$

where $\tau : \Sigma \times \Lambda(n) \to \Lambda(n)$ stands for the projection onto the second factor.

The *Maslov cycle of the section* $\mathbb{E}$ is the sub-bundle $\Lambda^{\mathbb{E}}(\Sigma)$ of $\Lambda(\Sigma)$, whose fiber $\mathcal{F}_x$ over any point $x$ in $\Sigma$ consists of all the Lagrangian planes in $T_x\mathcal{N}$ whose intersection with $\mathbb{E}(x)$ is non-trivial and is given by $\Lambda^{\mathbb{E}}_x = (\tau \circ I_{\mathbb{E}})^{-1}(\Lambda^{\{q=0\}}(n))$. This sub-bundle defines a codimension 1, co-oriented cycle in the bundle $\Lambda(\Sigma)$, the co-orientation of this cycle being induced by the co-orientation of $\Lambda^{\{q=0\}}(n)$.

The *Maslov index* of a homology class $h \in H_1(\Lambda(\Sigma), \mathbb{Z})$ is the value that the cocycle $\mathfrak{M}_{\mathbb{E}}$ takes on the cycle $h$. The Maslov cycle $\Lambda^{\mathbb{E}}(\Sigma)$ is the Poincaré dual of $\mathfrak{M}_{\mathbb{E}}$, so that $\mathfrak{M}_{\mathbb{E}}(h)$ is also equal to the oriented intersection number of $h$ and $\Lambda^{\mathbb{E}}(\Sigma)$.

If $\mathbb{E}$ is a $C^1$ section and $\gamma$ is a piecewise differentiable closed oriented curve in $\Lambda(\Sigma)$ with homology class $h$, Remark 1.1 gives an integral version of the Maslov index of $h$:

$$\mathfrak{M}_{\mathbb{E}}(h) = \int_0^1 \eta_{\tau\circ I_{\mathbb{E}}\circ\gamma(t)}\left(\frac{d}{dt}\tau \circ I_{\mathbb{E}}\gamma(t)\right) dt = \int_0^1 \eta_{\mathbb{E}\gamma(t)}\left(\frac{d}{dt}\gamma(t)\right) dt,$$

where $\eta_{\mathbb{E}}$ is the pullback by $\tau \circ I_{\mathbb{E}}$ of the form $\eta$. When $\mathbb{E}$ is only a continuous section, we can approximate $\mathbb{E}$ by a $C^\infty$ section $\hat{\mathbb{E}}$ such that $\mathfrak{M}_{\mathbb{E}}(h) = \mathfrak{M}_{\hat{\mathbb{E}}}(h)$.

From its definition, the Maslov cocycle depends on the choice of the section $\mathbb{E}$; however, this dependence can be made completely explicit. Consider another section

$$\mathbb{F} : \Sigma \to \Lambda(\Sigma)$$

and the trivialization obtained from this section

$$I_{\mathbb{F}} : \Lambda(\Sigma) \to \Sigma \times \Lambda(n) = \Sigma \times U(n)/O(n).$$

Let us estimate the difference cocycle $\mathfrak{M}_{\mathbb{E}} - \mathfrak{M}_{\mathbb{F}}$ evaluated on a homology class $h$ in $H_1(\Lambda(\Sigma), \mathbb{Z})$. For this purpose consider a closed curve $\gamma : \mathbb{S}^1 \to \Lambda(\Sigma)$ whose homology class is $h$. The curve $\gamma$ induces, through both trivializations, two closed curves $\gamma_{\mathbb{E}}, \gamma_{\mathbb{F}} : \mathbb{S}^1 \to U(n)/O(n)$. On the other hand, the curve $\gamma$ induces a third curve $\gamma_{\mathbb{F},\mathbb{E}} : \mathbb{S}^1 \to U(n)/O(n)$ which is given by the section $\mathbb{F}$ above the projected curve $\Pi \circ \gamma : \mathbb{S}^1 \to \Sigma$, seen through the trivialization $I_{\mathbb{E}}$. For each $\theta \in \mathbb{S}^1$, these three curves are related as follows

$$\det{}^2(\gamma_{\mathbb{E}}(\theta)) = \det{}^2(\gamma_{\mathbb{F}}(\theta)) \cdot \det{}^2(\gamma_{\mathbb{F},\mathbb{E}}(\theta)).$$

Consequently,

$$\mathfrak{M}_{\mathbb{E}}(h) - \mathfrak{M}_{\mathbb{F}}(h) = \mathfrak{M}_{\mathbb{E}}(\mathbb{F}_\star \circ \Pi_\star(h)),$$

where $\mathbb{F}_\star$ and $\Pi_\star$ are, respectively, the maps induced by $\mathbb{F}$ and $\Pi$ on the first homology groups.

In terms of 1-forms we have

$$\eta_{\mathbb{E}} - \eta_{\mathbb{F}} = \Pi^\star \circ \mathbb{F}^\star(\eta_{\mathbb{E}}). \tag{1}$$

It is worth noting that the difference of the two 1-forms $\eta_{\mathbb{E}} - \eta_{\mathbb{F}}$ contains in its kernel the tangent space to the fibers of $\Lambda(\Sigma)$.

This work is organized as follows. In §2, we consider a flow $\phi^t$ defined in a neighborhood of a submanifold $\Sigma$ of $\mathcal{N}$, which leaves $\Sigma$ invariant and preserves the symplectic 2-form. When there exists a continuous section $\Sigma \to \Lambda(\mathcal{N})$, we associate to any finite $\phi^t$-invariant measure with support in $\Sigma$, a quantity: *the asymptotic Maslov index*.

The dependence of this asymptotic Maslov index on the section and the invariant measure is discussed. In particular, we prove that there is no dependence on the section if the Schwartzman asymptotic cycle of the measure vanishes.

In §3 we focus on Hamiltonian flows. We show that the asymptotic cycle of the Liouville measure for a compact energy level is zero if the form $\omega^{n-1}$ is exact, which is always true for Hamiltonian flows on the cotangent bundle of a manifold equipped with its canonical symplectic structure.

In §4 we prove that for Hamiltonians that are optical with respect to a given section (see the definition below or [**5**]), the asymptotic Maslov index of the Liouville measure with respect to this section is always non-negative and it is strictly positive if and only if there are conjugate points. As an application, we prove that an optical Hamiltonian on a compact energy level which possesses a $\phi^t$-invariant Lagrangian section (this is the case, in particular, when the flow on the energy level is Anosov) does not have conjugate points—a result already proved by Klingenberg [**12**] for geodesic flows on compact manifolds, by Mañé [**13**] for geodesic flows with dense non-wandering sets and by Paternain and Paternain [**16**] for convex Hamiltonians.

The cotangent bundle of a manifold is equipped with a canonical symplectic form and a canonical section (the vertical section). Convex Hamiltonian flows—which are optical with respect to the vertical section—are studied in Appendix A.

Finally, in Appendix B, we give an example of a convex Hamiltonian with an invariant Lagrangian section and conjugate points on a non-compact regular energy level with finite volume.

## 2. *The asymptotic Maslov index*

Consider a flow $\phi^t$ defined on a neighborhood of the submanifold $\Sigma$ of $\mathcal{N}$, which leaves $\Sigma$ invariant and preserves the symplectic 2-form:

$$\phi^{t*}\omega = \omega \tag{2}$$

and denote by $X$ the vector field induced by the flow $\phi^t$.

We denote by $n_{\mathbb{E}}(\Gamma)$ the algebraic intersection number† of an oriented curve $\Gamma$ in $\Lambda(\Sigma)$ with $\Lambda^{\mathbb{E}}(\Sigma)$. To a Lagrangian plane $\lambda_x$ in the fiber $\Pi^{-1}(x)$ of $\Lambda(\Sigma)$, we associate the path $\Gamma_{\lambda_x,T} : [0, T] \to \Lambda(\Sigma)$ defined for any $t \in [0, T]$ by

$$\Gamma_{\lambda_x,T}(t) = d\phi^t{}_x(\lambda_x).$$

The following lemma (or some very similar version) is, like Lemma 1.2, a well-known fact in symplectic geometry and again for the sake of completeness we give here a very simple proof.

LEMMA 2.1. *Let $x$ be a point in $\mathcal{N}$ and let $\lambda_x$ and $\lambda'_x$ be two Lagrangian planes in the fiber $\Pi^{-1}(x)$ of $\Lambda(\Sigma)$. Then, for any $T \geq 0$,*

$$|n_{\mathbb{E}}(\Gamma_{\lambda_x,T}) - n_{\mathbb{E}}(\Gamma_{\lambda'_x,T})| \leq 8n.$$

† This intersection number is well defined if its endpoints do not intersect $\Lambda^{\mathbb{E}}(\Sigma)$. If its endpoints intersect $\Lambda^{\mathbb{E}}(\Sigma)$ there is an ambiguity by adding or subtracting at most $n$. Since we shall be interested only in the growth rate of $n_{\mathbb{E}}$, the ambiguity will not matter.

*Proof.* Let $\lambda_{0,x}$ and $\lambda_{1,x}$ be two Lagrangian planes in $\Pi^{-1}(x)$ such that:

- $\lambda_{0,x} \cap \lambda_{1,x} = 0$;
- $\lambda_{0,x} \cap \lambda_x = \lambda_{0,x} \cap \lambda'_x = 0$;
- $\lambda_{1,x} \cap \lambda_x = \lambda_{1,x} \cap \lambda'_x = 0$;
- $\lambda_{1,x} \cap \mathbb{E}(x) = 0$;
- $d\phi_x^T(\lambda_{1,x}) \cap \mathbb{E}_{\phi^T(x)} = 0$.

Consider the two paths $\Gamma_{\lambda_{0,x},\lambda_{1,x},\lambda_x}$ and $\Gamma_{\lambda_{0,x},\lambda_{1,x},\lambda'_x} : [0,1] \to \Pi^{-1}(x)$ joining $\lambda_{0,x}$ to $\lambda_x$ and $\lambda_{0,x}$ to $\lambda'_x$, respectively. Let us denote by $\Gamma_x(\lambda_x, \lambda'_x)$, the path connecting $\lambda_x$ to $\lambda'_x$ obtained by concatenating $\Gamma^{-1}_{\lambda_{0,x},\lambda_{1,x},\lambda_x}$ and $\Gamma_{\lambda_{0,x},\lambda_{1,x},\lambda'_x}$.

The path $\Gamma_x(\lambda_x, \lambda'_x)$ possesses the following two properties.

(i) From Lemma 1.2, its intersection number with the Maslov cycle $\Lambda^{\mathbb{E}}(\Sigma)$ of the section $\mathbb{E}$ is bounded in norm by $2n$.

(ii) From Remark 1.3, it is transported by the flow $\phi^t$. More precisely,

$$d\phi_x^t(\Gamma_x(\lambda_x, \lambda'_x)) = \Gamma_{\phi^t(x)}(d\phi_x^t\lambda_x, d\phi_x^t\lambda'_x)$$

and, consequently, the intersection number of $d\phi_x^t(\Gamma_x(\lambda_x, \lambda'_x))$ with the Maslov cycle $\Lambda^{\mathbb{E}}(\Sigma)$ is also bounded in norm by $2n$.

The 2-chain $A : [0,1] \times [0,T] \to \Lambda(\Sigma)$ defined by

$$A(s,t) = d\phi_x^t(\Gamma_x(\lambda_x, \lambda'_x)(s))$$

has a boundary

$$\Gamma_{\lambda_x,T} + d\phi_x^T(\Gamma_x(\lambda_x, \lambda'_x)) - \Gamma_{\lambda'_x,T} - \Gamma_x(\lambda_x, \lambda'_x)$$

which is null homologous. Thus, if $\lambda_x$, $\lambda'_x$, $d\phi_x^T(\lambda_x)$, $d\phi_x^T(\lambda'_x)$ intersect trivially $\mathbb{E}(x)$, $\mathbb{E}(\phi^T(x))$, respectively, then we have that

$$|n_{\mathbb{E}}(\Gamma_{\lambda_x,T}) - n_{\mathbb{E}}(\Gamma_{\lambda'_x,T})| \leq |n_{\mathbb{E}}(d\phi_x^T(\Gamma_x(\lambda_x, \lambda'_x)))| + |n_{\mathbb{E}}(\Gamma_x(\lambda_x, \lambda'_x))| \leq 4n.$$

If any of $\lambda_x$, $\lambda'_x$, $d\phi_x^T(\lambda_x)$, $d\phi_x^T(\lambda'_x)$ intersect the Maslov cycle $\Lambda^{\mathbb{E}}$, the 2-chain $A$ may intersect $\Lambda^{\mathbb{E}}$ non-transversally at one of such points. In that case, the last estimate may be modified by at most $n$ at each of these four subspaces. □

Note that the bound in Lemma 2.1 can be improved with more sophisticated arguments. For instance a very elegant formulation is given in [**4**]; see also [**3**] and the proof of Theorem 4.2 below.

The *asymptotic Maslov index* of point $x$ in $\Sigma$ is the following limit when it exists:

$$\mathfrak{M}_{\mathbb{E}}(x) := \lim_{T\to+\infty} \frac{1}{T} n_{\mathbb{E}}(\Gamma_{\lambda_x,T}). \tag{3}$$

It is clear from Lemma 2.1 that this limit is independent of the choice of the Lagrangian plane $\lambda_x$ in the fiber $\Pi^{-1}(x)$.

The flow $\phi^t$ induces a flow $\Phi^t$ in the Lagrangian Grassmann bundle defined for all $x$ in $\Sigma$ and for all $\lambda_x$ in $\Pi^{-1}(x)$ by

$$\Phi^t(x, \lambda_x) := (\phi^t(x), d\phi_x^t(\lambda_x)).$$

We denote by $\mathbb{X}$ the vector field corresponding to the flow $\Phi^t$ and we say that the data $(\mathbb{E}, \mathbb{X})$ satisfies *the bounding condition* if the map

$$\begin{aligned} \Lambda(\Sigma) &\to \mathbb{R} \\ (x, \lambda_x) &\mapsto \eta_{\mathbb{E}(x,\lambda_x)}(\mathbb{X}(x, \lambda_x)) \end{aligned}$$

is uniformly bounded on $\Lambda(\Sigma)$.

From the path $\Gamma_{\lambda_x,T}$, we construct a piecewise smooth loop $\Gamma'_{\lambda_x,T}$ obtained by concatenating $\Gamma_0$, $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ in $\Lambda(\Sigma)$ defined as follows:

- $\Gamma_0 = \Gamma_{\lambda_x,T}$;
- $\Gamma_1$ is a path (as in Lemma 2.1) from $\Phi^T(x, \lambda_x)$ to $(\phi^T(x), J_{\phi^T(x)}\mathbb{E}_{\phi^T(x)})$ in $\Pi^{-1}(\phi^T(x))$, whose intersection number with the Maslov cycle $\Lambda^{\mathbb{E}}(\Sigma)$ is bounded in norm by $4n$;
- $\Gamma_2$ is the path $t \mapsto J_{\phi^t(x)}\mathbb{E}_{\phi^t(x)}$ for $t$ going from $T$ to zero;
- $\Gamma_3$ is a path (as in Lemma 2.1) from $(x, J_x\mathbb{E}_x)$ to $(x, \lambda_x)$ in $\Pi^{-1}(x)$ whose intersection number with the Maslov cycle $\Lambda^{\mathbb{E}}(\Sigma)$ is bounded in norm by $4n$.

It follows from the construction that if the limit $\mathfrak{M}_{\mathbb{E}}(x)$ exists, then it satisfies

$$\mathfrak{M}_{\mathbb{E}}(x) = \lim_{T\to+\infty} \frac{1}{T}\mathfrak{M}_{\mathbb{E}}([\Gamma'_{\lambda_x,T}]),$$

where $[\Gamma'_{\lambda_x,T}]$ stands for the homology class of the path $\Gamma'_{\lambda_x,T}$.

Using the integral version of the Maslov index, we have that

$$\mathfrak{M}_{\mathbb{E}}([\Gamma'_{\lambda_x,T}]) = \int_{\Gamma_0} \eta_{\mathbb{E}} + \int_{\Gamma_1} \eta_{\mathbb{E}} + \int_{\Gamma_2} \eta_{\mathbb{E}} + \int_{\Gamma_3} \eta_{\mathbb{E}}.$$

It is clear that $\int_{\Gamma_2} \eta_{\mathbb{E}} = 0$. If $(\mathbb{E}, \mathbb{X})$ satisfies the bounding condition, the integrals $\int_{\Gamma_1} \eta_{\mathbb{E}}$ and $\int_{\Gamma_3} \eta_{\mathbb{E}}$ are uniformly bounded in $T$. It follows that the limit $\mathfrak{M}_{\mathbb{E}}(x)$, when it exists, is also given by the average

$$\mathfrak{M}_{\mathbb{E}}(x) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T \eta_{\mathbb{E}}(\mathbb{X}(\Phi^t(x, \lambda_x)))\, dt. \tag{4}$$

PROPOSITION 2.2. *Let $\nu$ be an invariant probability measure of the flow $\phi^t$ with support in $\Sigma$ and assume that the data $(\mathbb{E}, \mathbb{X})$ satisfies the bounding condition. Then:*

(i) *for $\nu$-almost every $x$ in $\Sigma$, the limit $\mathfrak{M}_{\mathbb{E}}(x)$ exists;*

(ii) *the map $x \to \mathfrak{M}_{\mathbb{E}}(x)$ is integrable.*

We denote by $\mathfrak{M}_{\mathbb{E}}(\nu)$ the integral $\int_\Sigma \mathfrak{M}_{\mathbb{E}}(x)\, d\nu(x)$ and call it *the asymptotic Maslov index of the measure $\nu$*.

The proof of this proposition clearly relies on an ergodic theorem. The peculiarity here is that the probability measures we consider are invariant measures for the flow $\phi^t$ on $\Sigma$ and the quantities we average are computed for the flow $\Phi^t$ on $\Lambda(\Sigma)$. There are actually two ways to prove Proposition 2.2.

One way consists of forgetting a part of the dynamics of the flow $\Phi^t$ by showing that the quantity $n_{\mathbb{E}}(\Gamma_{\lambda_x,T})$ up to a bounded error is independent of the Lagrangian plane $\lambda_x$

and is quasi-additive. This allows us to define a bounded cocycle over $\Sigma$. The proof of Proposition 2.2 is then a direct application of the sub-additive ergodic theorem and can be found for instance in the work of Ruelle [**17**] and Barge and Ghys [**4**] (in a slightly different context).

The second way, which we follow in a moment, consists of lifting the invariant measures for the flow $\phi^t$ to invariant measures for the flow $\Phi^t$ and then use the standard Birkhoff ergodic theorem. The advantage of this second approach is double:

- it gives a simple proof of the continuity of the asymptotic Maslov index with respect to the invariant measure for the weak* topology;
- it gives a better understanding of the dependence of the asymptotic Maslov index with respect to the section $\mathbb{E}$.

Before proceeding to the proof and in order to better understand the way invariant measures of the flow $\phi^t$ can be lifted to invariant measures of the flow $\Phi^t$, let us recall some basic definitions and results in ergodic theory.

Let $p : X \to Y$ be a surjective continuous map between compact metric spaces and let $\Phi_t$ be a flow on $X$ and $\phi_t$ a flow on $Y$ such that $\phi_t \circ p = p \circ \Phi_t$. Let $\mathcal{M}_\Phi$ and $\mathcal{M}_\phi$ be the invariant Borel probability measures of $\Phi_t$ and $\phi_t$ respectively. It is well known that the induced map $p_* : \mathcal{M}_\Phi \to \mathcal{M}_\phi$ is surjective. If $X$ is not compact, a similar result can be obtained if we assume that $p$ is a proper map. Let $X \cup \{\infty\}$ and $Y \cup \{\infty\}$ be the one point compactification of $X$ and $Y$, respectively. The map $p$ and the flows $\Phi_t$ and $\phi_t$ extend naturally to the compatifications and we denote these extensions by $\bar{p}$, $\bar{\Phi}_t$ and $\bar{\phi}_t$, respectively. A measure $\nu \in \mathcal{M}_\phi$ induces a measure $\bar{\nu} \in \mathcal{M}_{\bar{\phi}}$ by setting $\bar{\nu}(\infty) = 0$. Since $\bar{p}_*$ is surjective, there exists $\bar{\mu}$ such that $\bar{p}_*(\bar{\mu}) = \bar{\nu}$. However, since $\bar{\mu}(\infty) = 0$, $\bar{\mu}$ in fact defines a measure $\mu$ that lifts $\nu$.

*Proof of Proposition 2.2.* The compactness of the fibers of $\Pi$ yields that the projection map $\Pi$ is proper. Hence, the discussion in the previous paragraph shows that given a $\phi$-invariant measure $\nu$, there exists a $\Phi$-invariant measure $\mu$ such that $\Pi_*\mu = \nu$.

Since the data $(\mathbb{E}, \mathbb{X})$ satisfies the bounding condition, we can apply the Birkhoff ergodic theorem to the dynamical system $(\Lambda(\Sigma), \mu, \Phi^t)$ and the observable $(x, \lambda_x) \mapsto \eta_{\mathbb{E}_x}(\mathbb{X}(x, \lambda_x))$. This yields that for $\mu$-almost every $(x, \lambda_x)$ in $\Lambda(\Sigma)$ (and thus for $\nu$-almost every $x$ in $\Sigma$) the quantity $(1/T)\int_0^T \eta_{\mathbb{E}}(\mathbb{X}(\Phi^t(x, \lambda_x)))\,dt$ converges, when $T$ goes to $+\infty$, to a limit which does not depend on $\lambda_x$ in $\Pi^{-1}(x)$ and is a $\nu$-integrable function of $x$ in $\Sigma$. □

*Remark 2.3.* The hypothesis that the data $(\mathbb{E}, \mathbb{X})$ satisfies the bounding condition is actually too strong and the proof of Proposition 2.2 works if we require only the map $(x, \lambda_x) \mapsto \eta_{\mathbb{E}_x}(\mathbb{X}(x, \lambda_x))$ to be $\mu$-integrable where $\mu$ is an invariant measure of $\Phi^t$ which is a lift of $\nu$. However, in the following we see that it is convenient to keep a hypothesis which does not depend on the invariant measure we consider.

COROLLARY 2.4. *If* $(\mathbb{E}, \mathbb{X})$ *satisfies the bounding condition and* $\nu$ *is a Borel* $\phi^t$*-invariant probability, then*

$$\mathfrak{M}_{\mathbb{E}}(\nu) = \int \eta_{\mathbb{E}}(\mathbb{X})\,d\mu,$$

*for any* $\Phi^t$*-invariant lift* $\mu$ *of* $\nu$ *to* $\Lambda(\Sigma)$.

PROPOSITION 2.5. *The asymptotic Maslov index of a probability measure depends continuously on the invariant probability measure for the weak*-topology.*

*Proof.* Suppose that $\nu_N$ is a sequence of $\phi^t$-invariant probabilities on $\Sigma$ with $\lim_{N\to\infty}\nu_N = \nu$. We have seen in the proof of Proposition 2.2 that any $\phi^t$-invariant probability measure $\nu_N$ on $\Sigma$ can be lifted to a $\Phi^t$-invariant probability measure $\mu_N$ on $\Lambda(\Sigma)$: $\Pi_*\mu_N = \nu_N$.

Consider a metric space $\mathcal{S}$, a family $\mathcal{M}$ of Borel probability measures on $\mathcal{S}$ is *tight* if for each $\varepsilon > 0$, there is a compact set $K_\varepsilon \subset \mathcal{S}$ for which $P(K_\varepsilon) > 1-\varepsilon$ for all $P \in \mathcal{M}$. Tight families are characterized by the fact that their closures are compact in the weak* topology (see, for instance, [**7**]).

Since the fibers of $\Pi$ are compact, the family $\{\mu_N\}$ is tight. Thus, given any subsequence $\langle N_k\rangle$, there exists a subsequence $\langle N_{k_\ell}\rangle$ such that the measures $\mu_{N_{k_\ell}}$ converge when $l$ goes to $+\infty$ to a $\Phi^t$-invariant measure $\mu$, which satisfies $\Pi_*\mu = \nu$. Since the data $(\mathbb{E},\mathbb{X})$ satisfies the bounding condition

$$\mathfrak{M}_{\mathbb{E}}(\nu_{N_{k_\ell}}) = \int \eta_{\mathbb{E}}(\mathbb{X})\,d\mu_{N_{k_\ell}} \xrightarrow{\ell} \int \eta_{\mathbb{E}}(\mathbb{X})\,d\mu = \mathfrak{M}_{\mathbb{E}}(\nu).$$

The subsequence $\langle N_k\rangle$ has been chosen arbitrarily and the limit $\mathfrak{M}_{\mathbb{E}}(\nu)$ does not depend on this subsequence, thus $\lim_{N\to+\infty}\mathfrak{M}_{\mathbb{E}}(\nu_N) = \mathfrak{M}_{\mathbb{E}}(\nu)$. $\square$

2.1. *Change of reference section.* Let us describe now how the asymptotic Maslov index behaves under a change of the section $\mathbb{E}$. Let $\mathbb{F}$ be another section of $\Pi : \Lambda(\Sigma) \to \Sigma$ and assume that the data $(\mathbb{F},\mathbb{X})$ also satisfies the bounding condition. From (1), we get that the map

$$\begin{aligned} \Sigma &\to \mathbb{R} \\ x &\mapsto \mathbb{F}^\star \eta_{\mathbb{E}_x}(X(x)) \end{aligned}$$

is uniformly bounded on $\Sigma$ and

$$\mathfrak{M}_{\mathbb{E}}(\nu) - \mathfrak{M}_{\mathbb{F}}(\nu) = \int (\mathbb{F}^*\eta_{\mathbb{E}})(X)\,d\nu. \tag{5}$$

Note that (5) can be made more explicit when the *Schwartzman asymptotic cycle* of a $\phi^t$-invariant measure $\nu$ [**18, 19**] can be computed. This is the case when the following two assertions† are satisfied:

- the first homology group $H_1(\Sigma,\mathbb{R})$ has a finite dimension;
- the measure $\nu$ is $X$-*tame*, i.e. the first de Rham cohomology group $H^1(\Sigma,\mathbb{R})$ is generated by a finite number of 1-forms $\omega_i$ such that for each $i$ the map

  $$\begin{aligned} \Sigma &\to \mathbb{R} \\ x &\mapsto \omega_{i\,x}(X(x)) \end{aligned}$$

  is $\nu$-integrable.

† Note that both assertions are satisfied when $\Sigma$ is compact.

In this context, the asymptotic cycle of the measure $\nu$ is the unique element $\mathcal{S}(\nu)$ in $H_1(\Sigma, \mathbb{R}) = H^1(\Sigma, \mathbb{R})^*$, which satisfies for any closed 1-form $\zeta$:

$$\langle [\zeta], \mathcal{S}(\nu) \rangle = \int \zeta(X)\, d\nu,$$

where $[\zeta]$ is the cohomology class of $\zeta$ and the brackets $\langle\ ,\ \rangle$ denote evaluation.

With these notations, (5) reads†

$$\mathfrak{M}_{\mathbb{E}}(\nu) - \mathfrak{M}_{\mathbb{F}}(\nu) = \langle [\mathbb{F}^* \eta_{\mathbb{E}}], \mathcal{S}(\nu) \rangle. \tag{6}$$

Note that the asymptotic Maslov index of a $\phi^t$-invariant measure $\nu$ does not depend on the section in the following two cases:

- if $H_1(\Sigma, \mathbb{R}) = 0$;
- or if the Schwartzman asymptotic cycle of the measure $\nu$ is zero.

If some $\Phi^t$-invariant measure $\mu$ which is a lift of $\nu$ is $\mathbb{X}$-tame, the asymptotic cycle $\mathcal{S}(\mu)$ in $H_1(\Lambda(\Sigma), \mathbb{R})$ is well defined. The asymptotic cycle $\mathcal{S}(\nu)$ in $H_1(\Sigma, \mathbb{R})$ and the Maslov index $\mathfrak{M}_{\mathbb{F}}(\nu)$ are also well defined (see Remark 2.3).

*Remark 2.6.* In the case of an odd-dimensional manifold $M$ (that we choose to be compact to make things simpler) equipped with a contact structure, there exists an analogous definition of the asymptotic Maslov index. More precisely, a contact structure is the data of a 1-form $\eta$ such that the plane field corresponding to the kernels is nowhere integrable, i.e. for every $x$ in $M$,

$$\eta_x \wedge (d\eta_x)^n \neq 0,$$

where $\dim M = 2n + 1$. It is clear that the 2-form $d\eta$ restricted to $\ker \eta$ is symplectic. To a contact structure $\eta$, one can associate a vector field $X$ whose flow $\phi^t$ is called the *Reeb flow* and which is defined by:

- $i_X \eta = 1$;
- $i_X d\eta = 0$.

This flow preserves the planes $\ker \eta$, the symplectic form $d\eta$ on $\ker \eta$ and the volume form $\eta \wedge (d\eta)^n$.

Assume that there exists a continuous section which associates to each point $x$ in $M$ a Lagrangian plane $\mathbb{E}(x)$ in $\ker \eta_x$. Then it is possible to associate to each $\phi^t$-invariant probability measure $\nu$ its asymptotic Maslov index in the same way as we did for the even-dimensional case. In particular, this can be done for the measure induced by the volume form $\eta \wedge (d\eta)^n$.

## 3. *Hamiltonian flows*

Let $\mathcal{N}$ be a $2n$-dimensional complete connected Riemannian manifold. A *Hamiltonian* on $\mathcal{N}$ is a $C^\infty$ function $H : \mathcal{N} \to \mathbb{R}$. The *Hamiltonian vector field* $X$ of $H$ is defined by

$$\omega(X, \cdot) = -dH. \tag{7}$$

† See [**11**] for a similar dependence on the trivialization in the case of non singular vector fields in a 3-dimensional manifold.

The level sets of the Hamiltonian $\Sigma = H^{-1}\{e\}$ are called the *energy levels* of $H$ and are invariant under the flow of $X$.

In the remainder of this paper we focus our attention on the dynamics of *regular energy levels* $\Sigma$, i.e. the energy levels which correspond to a regular value of $H$. Thus, $\Sigma$ is a codimension 1 submanifold of $\mathcal{N}$ and $X$ is a non-singular vector field on $\Sigma$. Furthermore, we always assume the following hypothesis.

(a) *Completeness:* the Hamiltonian vector field $X$ gives rise to a complete flow $\phi : \Sigma \times \mathbb{R} \to \Sigma$ on the energy level $\Sigma$.

If the energy level $\Sigma$ is compact, this last hypothesis is clearly satisfied.

### 3.1. *The asymptotic Maslov index for the Liouville measure.*

The Hamiltonian flow on the energy level $\Sigma$ inherits a canonical smooth invariant measure $\bar{m}$, called *the Liouville measure* which corresponds to the volume form $m = i^*\sigma$, where $\sigma$ is a form such that $\omega^n = dH \wedge \sigma$ and $i : \Sigma \hookrightarrow T^*M$ is the inclusion map [**1**].

PROPOSITION 3.1. *If the energy level $\Sigma$ is compact and the form $\omega^{n-1}$ is exact†, then the asymptotic cycle $\mathcal{S}(\bar{m})$ of the Liouville measure $\bar{m}$ is zero.*

*Proof.* Let $Y$ be a vector field $Y \in T_\Sigma\mathcal{N}$ such that $\omega(X, Y) \equiv -1$. The sequence of equalities

$$\begin{aligned} dH \wedge i_Y\,\omega^n &= -i_Y(dH \wedge \omega^n) + (i_Y\,dH) \wedge \omega^n \\ &= 0 + (i_Y i_X\,\omega)\,\omega^n \\ &= \omega^n, \end{aligned}$$

yields the explicit formulation

$$m = i^*(i_Y\,\omega^n).$$

In order to prove that the asymptotic cycle is zero, it is enough to check that for any closed 1-form $\eta$ on $\Sigma$:

$$\langle \eta, \mathcal{S}(\bar{m})\rangle = \int \eta(X)\,d\bar{m} = 0. \tag{8}$$

We have

$$\eta(X)\,m = i_X(\eta \wedge m) + \eta \wedge (i_X m) = \eta \wedge (i_X m).$$

On the other hand, since $dH = -i_X\omega = 0$ on $\Sigma$,

$$i_X m = i_X(i^* i_Y\,\omega^n) = i^*(\omega^{n-1}).$$

Since $\omega^{n-1}$ is exact, there exists a $(2n-1)$-form, $\tau$ such that $d\tau = i^*(\omega^{n-1})$. Consequently,

$$\eta(X)m = \eta \wedge d\tau = d(\eta \wedge \tau).$$

A direct application of Stokes theorem gives $\mathcal{S}(\bar{m}) = 0$. □

† We later discuss the strength of the hypothesis on the exactness of $\omega^{n-1}$.

Let $\Sigma$ and $\Sigma'$ be two regular energy levels of two Hamiltonian functions $H$ and $H'$, respectively. Let $F$ be a symplectic diffeomorphism from a neighborhood of $\Sigma$ to a neighborhood of $\Sigma'$. We say that $F$ is a symplectic conjugacy if $F$ maps $\Sigma$ onto $\Sigma'$ and conjugates the corresponding Hamiltonian flows on the energy levels.

COROLLARY 3.2. *If a regular energy level $\Sigma$ is compact and $\omega^{n-1}$ is exact, then:*

(i) *for any two continuous Lagrangian sections $\mathbb{E}$, $\mathbb{F}$,*

$$\mathfrak{M}_{\mathbb{E}}(\bar{m}) = \mathfrak{M}_{\mathbb{F}}(\bar{m});$$

(ii) *the asymptotic Maslov index of the Liouville measure is invariant under symplectic conjugacies of the Hamiltonian flow.*

*Proof.* (i) is a direct consequence of (6).

In order to prove (ii), suppose that $F$ is a symplectic conjugacy and let $\mathbb{E}$ be a continuous Lagrangian section of $\Sigma$. A direct computation yields $F_\star \bar{m} = \bar{m}'$, where $\bar{m}$ and $\bar{m}'$ stand, respectively, for the Liouville measures on $\Sigma$ and $\Sigma'$. The asymptotic Maslov index of the Liouville measure $\bar{m}$ on the energy level $\Sigma$ can be computed (thanks to (i)) using any Lagrangian section, $\mathbb{E}$ for instance.

It follows that $\mathfrak{M}_{\mathbb{E}}(\bar{m}) = \mathfrak{M}_{F_\star\mathbb{E}}(F_\star\bar{m}) = \mathfrak{M}_{F_\star\mathbb{E}}(\bar{m}')$, which is the asymptotic Maslov index of the Liouville measure $\bar{m}'$ on the energy level $\Sigma'$ computed using the Lagrangian section $F_\star\mathbb{E}$. □

## 4. *Optical Hamiltonians*

Now we restrict our attention to Hamiltonian flows whose lift to $\Lambda(\Sigma)$ cuts the Maslov cycle transversely with positive orientation. In the next two subsections, we relate the work of Duistermaat [**10**] and Bialy and Polterovich [**5**] with the asymptotic Maslov index.

### 4.1. *Positive tangent vectors.*

Consider again the Lagrangian manifold $\Lambda(n)$ introduced in §1; let us make the transverse orientation of the train of a given Lagrangian plane more explicit here.

For $t$ in $[-1, 1]$, let $t \mapsto \lambda(t) \in \Lambda(n)$ be a curve in $\Lambda(n)$ passing through the Lagrangian plane $\lambda(0) = \lambda$. There exists a curve of symplectic automorphisms of $\mathbb{R}^{2n}$, $t \mapsto Sp(t)$ such that $\lambda(t) = Sp(t)\lambda$, $Sp(0) = \mathbb{I}$.

To the pair $(\lambda, \lambda'(0))$ (where $\lambda'(0)$ stands for the tangent vector

$$\lambda'(0) = \left.\frac{d\lambda}{dt}\right|_{t=0} \in T_\lambda\Lambda(n)),$$

we can associate the bilinear form $\beta_{\lambda,\lambda'(0)}$ on $\lambda$:

$$(\xi, \eta) \longmapsto \omega(\xi, Sp'(0)\eta) \quad \text{for } \xi, \eta \in \lambda. \tag{9}$$

Note the following:

- This form is well defined: another choice of curve of symplectic transformations $t \mapsto \tilde{S}p(t)$ such that $\lambda(t) = \tilde{S}p(t)\lambda$ satisfies $\tilde{S}p(t) = Sp(t)Q(t)$, where $Q(t)$ preserves $\lambda$ and $Q(0) = \mathbb{I}$, hence $(d/dt)\omega(\xi, Q(t)\eta) = \omega(\xi, Q'(0)\eta)$, a quantity that vanishes since $\lambda$ is Lagrangian, it also depends only on $(\lambda, \lambda'(0))$ and not on the specific curve $\lambda(t)$†.

- This form is symmetric since $Sp(t)$ preserves the symplectic form $\omega$.
- The map $\lambda'(0) \mapsto \beta_{\lambda,\lambda'(0)}$ is linear since it is a quotient of the derivative of the map that takes a symplectic automorphism $S$ into the bilinear form $\omega(\xi, S\eta)$.

Recall that the train $\Lambda^{\mathbb{E}_0}(n)$ of a given Lagrangian plane $\mathbb{E}_0$ is the union for $k \geq 1$ of the collection of open submanifolds $\Lambda^k = \{\lambda \in \Lambda(n) \mid \dim(\lambda \cap \mathbb{E}_0) = k\}$. There exists a morphism from the normal bundle to $\Lambda^k$, $T_\lambda\Lambda(n)/T_\lambda\Lambda^k$, to the space $\mathcal{S}^2(\lambda \cap \mathbb{E}_0)$ of symmetric bilinear forms on $\lambda \cap \mathbb{E}_0$, which is defined as follows. To each vector $u + T_\lambda\Lambda^k$, we associate the restriction of the bilinear form $\beta_{\lambda,u}$ to the subspace $\lambda \cap \mathbb{E}_0$. In order to check that this map is well defined, it is enough to show that $\beta_{\lambda,u}|_{\lambda\cap\mathbb{E}_0} = 0$ for $u \in T_\lambda\Lambda^k$ (recall that the map $u \mapsto \beta_{\lambda,u}$ is linear), which can be proved using an arc of symplectic linear maps $t \mapsto Sp(t)$ such that $Sp(0)\lambda = \lambda$, $Sp'(0)\lambda = u$ and $Sp(t)(\lambda \cap \mathbb{E}_0) = \lambda(t) \cap \mathbb{E}_0$ for all $t$ in $[-1, 1]$. Actually, this morphism is an isomorphism: it is injective since $\beta_{\lambda,u}|_{\lambda\cap\mathbb{E}_0} \neq 0$ when $u$ is not in the symplectic orthogonal of $\lambda \cap \mathbb{E}_0$ which contains $\lambda$ and $\mathbb{E}_0$ and has dimension $2n - k$, it is onto because the source and the target have the same dimension.

We say that a vector $u \in T_\lambda\Lambda$ is *positive* if $\beta_{\lambda,u}|_{\lambda\cap\mathbb{E}_0}$ is positive definite whenever $\lambda \cap \mathbb{E}_0 \neq \{0\}$. A curve $\lambda(t) \in \Lambda(n)$ is *positive* if its tangent vectors are positive. For a positive curve $\lambda(t)$, the set of $t'$ for which $\lambda(t)$ belongs to the train of $\mathbb{E}_0$ is discrete. Indeed, if $\lambda \in \Lambda^k$, the fact that $\beta_{\lambda,u}|_{\lambda\cap\mathbb{E}_0}$ is positive definite implies that $\lambda'(t)$ is not in the tangent space over $\lambda$ of the closure of $\Lambda^p$ for all $1 \leq p \leq k$.

In particular, if $\lambda \in \Lambda^1$, i.e. $\dim(\lambda \cap \mathbb{E}_0) = 1$, $\mathcal{S}^2(\lambda \cap \mathbb{E}_0) \approx T_\lambda\Lambda/T_\lambda\Lambda^1 \approx \mathbb{R}$. A vector $u$ in $T_\lambda\Lambda(n)$ is transversal to $\Lambda^1$ if and only if $\beta_{\lambda,u}|_{\lambda\cap\mathbb{E}_0} \neq 0$. This gives a transversal orientation on the cycle $\Lambda^{\mathbb{E}_0}$. Note that for $\lambda \in \Lambda^1$, the curves $t \mapsto e^{it\mathbb{I}}\lambda$ are positive. Thus, this orientation agrees with that given in §1.

It is clear how to extend the above definitions to the case of a manifold $\mathcal{N}$ equipped with a symplectic structure, a connected submanifold $\Sigma$ of $\mathcal{N}$ and a section $\mathbb{E}$ from $\Sigma$ to the Grassmann Lagrangian bundle.

4.2. *Optical Hamiltonians.* A Hamiltonian $H : \mathcal{N} \to \mathbb{R}$ is *optical* or *positively twisted* with respect to a differentiable Lagrangian section $\mathbb{E}$ on a regular energy level $\Sigma$ if the flow lines of the lifted Hamiltonian flow $\Phi^t$ on $\Lambda(\Sigma)$ are positive curves; that is, for any $\lambda \in \Lambda^{\mathbb{E}}(\Sigma)$, the form

$$(\xi, \eta) \xrightarrow{\beta} \omega\left(\xi, \frac{d}{dt}\bigg|_{t=0} (\Phi^t\eta)\right) \quad \xi, \eta \in \lambda \cap \mathbb{E},$$

restricted to $\lambda \cap \mathbb{E}$, is positive definite.

*Examples.*

(1) A convex Hamiltonian in the cotangent bundle of a manifold equipped with its canonical symplectic structures is optical with respect to the vertical section (see Appendix A).

† This can be shown using a local parametrization of the Lagrangian Grassmann bundle near $\lambda$.

(2) A Riemannian metric with a twisted symplectic structure is optical with respect to the vertical section (see Remark B.2).
(3) A Riemannian metric with positive curvature is optical with respect to the section given by the kernel of the Riemannian connection, i.e. the horizontal bundle.
(4) $C^2$-perturbations of the above examples remain optical with respect to the perturbed sections.

From now on, we consider a regular energy level $\Sigma$ of an optical Hamiltonian system $H$, on which the flow $\phi^t$ induced by the Hamiltonian is complete and such that the data $(\mathbb{E}, \mathbb{X})$ satisfies the bounding condition. The Maslov cocycle $\mathfrak{M}_{\mathbb{E}}$ measures the oriented intersection with the Maslov cycle $\Lambda^{\mathbb{E}}$. Recall from Corollary 2.4 that if $(\mathbb{E}, \mathbb{X})$ satisfies the bounding condition (which occurs if the second derivatives of the Hamiltonian are uniformly bounded on $\Sigma$), we have that

$$\mathfrak{M}_{\mathbb{E}}(\nu) := \int \eta_{\mathbb{E}}(\mathbb{X})\, d\mu, \tag{10}$$

for any invariant probability $\nu$ on $\Sigma$ and *any* invariant lift $\mu$ of $\nu$ to $\Lambda(\Sigma)$.

To a Lagrangian plane $\lambda_x$ in the fiber $\Pi^{-1}(x)$ of $\Lambda(\Sigma)$, we associate the path $\Gamma_{\lambda_x,T} : [0, T] \to \Lambda(\Sigma)$, defined for any $t \in [0, T]$ by

$$\Gamma_{\lambda_x,T}(t) = d\phi^t{}_x(\lambda_x).$$

Since $H$ is optical, the set of intersection points of $\Gamma_{\lambda_x,T}$ with the Maslov cycle $\Lambda^{\mathbb{E}}$ is discrete and the curve $\Gamma_{\lambda_x,T}$ crosses the Maslov cycle always in the positive direction. As explained in [**3, 5, 10**], each time $t$ for which $\Gamma_{\lambda_x,T}(t)$ intersects the Maslov cycle, adds precisely $\dim(\Gamma_{\lambda_x,T}(t) \cap \mathbb{E})$ to the intersection number $n_{\mathbb{E}}(\Gamma_{\lambda_x,T})$. It follows that

$$n_{\mathbb{E}}(\Gamma_{\lambda_x,T}) = \sum_{\Gamma_{\lambda_x,T}(t)\cap\mathbb{E}\neq\{0\}} \dim(\Gamma_{\lambda_x,T}(t) \cap \mathbb{E}). \tag{11}$$

From (3) and (11), we get the following.

LEMMA 4.1. *Assume that $H$ is $\mathbb{E}$-optical and satisfies the bounding condition for $(\mathbb{E}, \mathbb{X})$. If the invariant measure $\nu$ is ergodic, then for $\nu$-almost every point $x$ in $\Sigma$ and for every $\lambda_x$ in $\Pi^{-1}(x)$, we have that*

$$\lim_{T\to+\infty} \frac{1}{T} \sum_{\Gamma_{\lambda_x,T}(t)\cap\mathbb{E}\neq\{0\}} \dim(\Gamma_{\lambda_x,T}(t) \cap \mathbb{E}) = \mathfrak{M}_{\mathbb{E}}(\nu).$$

We now recall the ergodic decomposition theorem (see for instance [**14**]). Let us consider for every point $x$ in the ambient space, the probability measure $\mu(x, T)$ which is equidistributed along the arc of orbit joining $x$ to $\phi^T(x)$. It turns out that for $\mu$-almost every $x$, the probabilities $\mu(x, T)$ converge in the weak$^\star$ topology, to an ergodic invariant probability measure $\mu_x$. Furthermore, we get a decomposition of the measure $\mu$ in the measures $\mu_x$ in the following sense: for any measurable function $f$ bounded on the ambient space,

$$\int f d\mu = \int \left( \int f d\mu_x \right) d\mu. \tag{12}$$

Let us come back now to our particular situation and consider a $\Phi^t$-invariant probability measure $\mu$ on $\Lambda(\Sigma)$ which is a lift of a $\phi^t$-invariant probability measure $\nu$ on $\Sigma$. It is clear that the pushforward by the projection $\Pi$ of any ergodic component measure $\mu_{(x,\lambda_x)}$ of $\mu$, associated to a point $(x, \lambda_x)$ is the ergodic component measure of $\nu$ associated to the point $x$. Applying (12) to the map: $(x, \lambda_x) \mapsto (\eta_{\mathbb{E}})_x(\mathbb{X}(x, \lambda_x))$ we get that for every invariant Borel probability measure $\nu$ on $\Sigma$

$$\mathfrak{M}_{\mathbb{E}}(\nu) := \int \mathfrak{M}_{\mathbb{E}}(x)\, d\nu = \int \mathfrak{M}_{\mathbb{E}}(\nu_x)\, d\nu. \tag{13}$$

4.3. *Conjugate points.* A point $x_2 \in \mathcal{N}$ is said to be $\mathbb{E}$-*conjugate* to $x_1 \in \mathcal{N}$ if $x_2 = \phi^\tau(x_1)$ and $\Phi^\tau(\mathbb{E}(x_1)) \cap \mathbb{E}(x_2) \neq \{0\}$ for some $\tau > 0$. Theorem 4.4 below provides a simple criterion for the existence of conjugate points.

We will need the following symplectic analogue of Sturm's theorem on the number of zeros of solutions of second-order differential equations. This result was proved by Arnold in [**3**] for the case of the linear symplectic space and a Hamiltonian given by kinetic energy plus a potential. However, Arnold's proof holds in our setting without any significant changes. Set $\dim \mathcal{N} = 2n$.

THEOREM 4.2. *Let $\Sigma$ be a regular energy level of an optical Hamiltonian and let $\lambda_x$ and $\lambda'_x$ be two Lagrangian planes in the fiber $\Pi^{-1}(x)$ of $\Lambda(\Sigma)$. If $t \mapsto d\phi^t_x(\lambda_x)$ has $n+1$ points of intersection with the Maslov cycle $\Lambda^{\mathbb{E}}$ (counted with multiplicity) in an interval $[t_1, t_2]$, then $t \mapsto d\phi^t_x(\lambda'_x)$ has at least one point of intersection with $\Lambda^{\mathbb{E}}$ in the same interval $[t_1, t_2]$.*

*Proof.* We will essentially reproduce Arnold's proof of the theorem on zeros in [**3**].

We begin with a useful definition. The Maslov index of the path $\Gamma$ in $\Lambda(n)$ starting at a point $\lambda_0$, which does not belong to $\Lambda^{\lambda_1}$ and ending at $\lambda_1$ is, by definition, the intersection number $n_{\lambda_1}(\Gamma')$, where $\Gamma'$ is a path close to $\Gamma$ starting at $\lambda_0$ and ending at a point $\lambda'_1$ close to $\lambda_1$ which lies on the positive domain of the Maslov cycle $\Lambda^{\lambda_1}$. The positive domain is defined as the set of all Lagrangian planes $\lambda$ in a neighborhood of $\lambda_1$ for which there is a positive curve which begins at $\lambda_1$ and ends at $\lambda$, intersecting the Maslov cycle $\Lambda^{\lambda_1}$ only at $\lambda_1$.

Let $\widetilde{\Lambda}(n)$ be the universal covering of $\Lambda(n)$. The homotopy class of a path connecting $\lambda_0$ to $\lambda_1$ in $\Lambda(n)$ can be represented by a pair of points $(\widetilde{\lambda}_0, \widetilde{\lambda}_1)$ in $\widetilde{\Lambda}(n)$. Suppose that $\lambda_0 \notin \Lambda^{\lambda_1}$ and let $m(\widetilde{\lambda}_0, \widetilde{\lambda}_1)$ be the Maslov index of such a path. Arnold showed [**3**, p. 253] that the Maslov index $m(u, v)$ of pair of points in $\widetilde{\Lambda}(n)$ has the following property:

$$m(u, v) + m(v, w) - m(u, w) = I(\pi(u), \pi(v), \pi(w)), \tag{14}$$

where $\pi : \widetilde{\Lambda}(n) \to \Lambda(n)$ is the covering projection and the index $I(\lambda_0, \lambda_1, \lambda_2)$ of a triplet of pairwise transverse Lagrangian planes is defined as the index of the following quadratic form. Write $\zeta = \xi + \eta$ where $\xi \in \lambda_0$ and $\eta \in \lambda_1$. Set

$$q_{\lambda_0,\lambda_1}(\zeta) = \omega(\xi, \eta).$$

Then $I(\lambda_0, \lambda_1, \lambda_2)$ is the index of $q_{\lambda_0,\lambda_1}$ restricted to $\lambda_2$.

Using the trivialization $I_{\mathbb{E}} : \Lambda(\Sigma) \to \Sigma \times \Lambda(n)$, the section $\mathbb{E}$ corresponds to the Lagrangian plane $q = 0$, which we call $\alpha$ (following Arnold's notation). As before, let $\tau : \Sigma \times \Lambda(n) \to \Lambda(n)$ be the projection onto the second factor.

Set $\gamma(t) = \tau \circ I_{\mathbb{E}} \circ d\phi_x^t(\lambda_x)$ and $\gamma'(t) = \tau \circ I_{\mathbb{E}} \circ d\phi_x^t(\lambda'_x)$. Consider now a family depending continuously on $t$ of paths $\delta(t)$ from $\gamma(t)$ to $\alpha$. We first assume that $\gamma(t_1)$ and $\gamma(t_2)$ are transverse to $\alpha$. Let

$$\nu := n_{\mathbb{E}}(d\phi_x^t(\lambda_x)|_{[t_1,t_2]}).$$

By (11) we have

$$\nu = \sum_{\gamma(t)\cap\alpha\neq\{0\}} \dim(\gamma(t) \cap \alpha). \tag{15}$$

Consider also a family of paths $\delta'$ connecting $\gamma'$ with $\alpha$. In the covering space we pick a point $\widetilde{\alpha}$ over $\alpha$ and we cover the paths $\delta(t)$ and $\delta'(t)$ by paths $\widetilde{\delta}$ and $\widetilde{\delta'}$ which end at $\widetilde{\alpha}$; their origins are denoted by $\widetilde{\gamma}$ and $\widetilde{\gamma'}$, respectively. Then we have

$$\nu = m(\widetilde{\alpha}, \widetilde{\gamma}(t_2)) - m(\widetilde{\alpha}, \widetilde{\gamma}(t_1)). \tag{16}$$

By (14),

$$m(\widetilde{\alpha}, \widetilde{\gamma}(t)) + m(\widetilde{\gamma}(t), \widetilde{\gamma'}(t)) - m(\widetilde{\alpha}, \widetilde{\gamma'}(t)) = I(\alpha, \gamma(t), \gamma'(t)).$$

The middle term does not depend on $t$, since the index is symplectically invariant. The right-hand side is always between zero and $n$ for all $t$. Hence, the increment of the left-hand side between $t_1$ and $t_2$ is bounded in absolute value by $n$; that is

$$|(m(\widetilde{\alpha}, \widetilde{\gamma}(t_1)) - m(\widetilde{\alpha}, \widetilde{\gamma'}(t_1))) - (m(\widetilde{\alpha}, \widetilde{\gamma}(t_2)) - m(\widetilde{\alpha}, \widetilde{\gamma'}(t_2)))| \leq n.$$

Using (16), we get $|\nu - \nu'| \leq n$ and the theorem follows from this and (15).

Now suppose that $\gamma(t_1)$ or $\gamma(t_2)$ are not transverse to $\alpha$. For $\varepsilon > 0$, define

$$\nu_\varepsilon = n_{\mathbb{E}}(d\phi_x^t(\lambda_x)|_{[t_1-\varepsilon,t_2+\varepsilon]}).$$

Since the Hamiltonian is optical, for $\varepsilon > 0$ small enough, $\gamma(t_1)$ and $\gamma(t_2)$ are transverse to $\alpha$ and $\nu_\varepsilon = \nu_0 = \nu$, counting multiplicities. Similarly, $\nu'_e = \nu'_0 = \nu$. We have proved above that $|\nu_\varepsilon - \nu'_\varepsilon| \leq n$. Hence $|\nu - \nu'| \leq n$. □

COROLLARY 4.3. *If $x_0 \in \Sigma$ and $x_1 = \phi^\tau(x_0)$ is conjugate to $x_0$, then for any Lagrangian plane $\lambda_{x_0} \in \Pi^{-1}(x_0)$ there exists $0 \leq t \leq \tau$ such that*

$$d\phi_{x_0}^\tau(\lambda_{x_0}) \cap \mathbb{E}(\phi^\tau(x_0)) \neq \{0\}.$$

*Proof.* Consider the path $\eta(t) := d\phi_{x_0}^t(\mathbb{E}(x_0))$ for $t \in [0, \tau]$. Then $\eta$ has $n$ points of intersection with $\Lambda^{\mathbb{E}}$ at $t = 0$ and at least one point of intersection with $\Lambda^{\mathbb{E}}$ at $t = \tau$. Then the curve $t \mapsto d\phi_{x_0}^t(\lambda_{x_0})$ has at least one intersection with the Maslov cycle $\Lambda^{\mathbb{E}}$ on $[0, \tau]$. □

THEOREM 4.4. *Let $\nu$ be an invariant probability measure on a regular energy level of an optical Hamiltonian $(H, \Sigma, \mathbb{E})$ which satisfies the completeness hypothesis and the bounding condition. Then $\mathfrak{M}_{\mathbb{E}}(\nu) > 0$ if and only if there are $\mathbb{E}$-conjugate points in the support of $\nu$.*

*Proof.* If the orbit of any point $x$ in $\Sigma$ has no conjugate points then for all $T > 0$, $n_{\mathbb{E}}(\Gamma_{\mathbb{E}_x,T}) = n$ and consequently, using Lemma 2.1, $0 \leq n_{\mathbb{E}}(\Gamma_{\lambda_x,T}) \leq 9n$ for any Lagrangian plane $\lambda_x$ in $\Pi^{-1}(x)$. Hence $\mathfrak{M}_{\mathbb{E}}(x) = 0$, for all $x \in \Sigma$ and thus $\mathfrak{M}(\nu) = 0$.

Conversely, suppose that there exists $x_1$ in the support of $\nu$ such that $x_2 = \phi^\tau(x_1)$ is conjugate to $x_1$. Since $H$ is optical, for any small enough flow box $U_1$ containing $x_1$ and for every $y_1$ in $U_1$, there exists a conjugate point to $y_1$ on the arc of orbit starting at $y_1$ before the next return to $U_1$. Choose now an ergodic component measure $\nu_x$ of $\nu$. Using the Birkhoff ergodic theorem, we know that for $\nu_x$-almost every $y$ in $\Sigma$,

$$\lim_{T\to+\infty} \frac{1}{T}\theta(y, T, U_1) = \nu_x(U_1),$$

where $\theta(y, T, U_1)$ is the time spent in $U_1$ by the arc of orbit with length $T$ starting at $y$. Call $\alpha$ the maximal time length of a connected component of an arc of orbit crossing $U_1$, then

$$\liminf_{T\to+\infty} \frac{1}{T} n(y, T, U_1) \geq \frac{\nu_x(U_1)}{\alpha},$$

where $n(y, T, U_1)$ is the number of times the arc of orbit starting at $y$ of length $T$ visits $U_1$. Corollary 4.3 implies that in between two conjugate points there must be a time $t$ for which $\Gamma_{\lambda_y,T}(t) \cap \mathbb{E}$ is non-trivial. Hence, the last inequality implies that for $\nu_x$-almost every $y$ in $\Sigma$ and for every $\lambda_y$ in $\Pi^{-1}(y)$ we have

$$\liminf_{T\to+\infty} \frac{1}{T} \sum_{\Gamma_{\lambda_y,T}(t)\cap\mathbb{E}\neq\{0\}} \dim(\Gamma_{\lambda_y,T}(t) \cap \mathbb{E}) \geq \frac{\nu_x(U_1)}{\alpha}.$$

Hence, using Lemma 4.1,

$$\mathfrak{M}_{\mathbb{E}}(\nu_x) \geq \frac{\nu_x(U_1)}{\alpha}.$$

Integrating against $\nu$, (13) yields

$$\mathfrak{M}_{\mathbb{E}}(\nu) \geq \frac{\nu(U_1)}{\alpha} > 0. \qquad \square$$

COROLLARY 4.5. *Let $(H, \Sigma, \mathbb{E})$ be an optical Hamiltonian with complete Hamiltonian flow and satisfying the bounding condition for $(\mathbb{E}, \mathbb{X})$. If the Liouville measure of $\Sigma$ is finite, then the asymptotic Maslov index of the Liouville measure is non-zero ($\mathfrak{M}(\bar{m}) > 0$) if and only if $\Sigma$ has conjugate points with respect to $\mathbb{E}$.*

In fact, the same result holds for any invariant probability $\nu$ with $\mathrm{supp}(\nu) = \Sigma$.

We now give the first application of the asymptotic Maslov index.

Observe that when the energy level is compact and $\omega^{n-1}$ is exact, by Corollary 3.2(i), the Maslov index of the Liouville measure does not depend on the choice of Lagrangian section.

PROPOSITION 4.6. *Let $(M, g)$ and $(N, h)$ be two closed Riemannian manifolds. Suppose that there exists a contactomorphism between the unit sphere bundle of $M$ and the unit sphere bundle of $N$ which conjugates the geodesic flows of $M$ and $N$. Then $\mathfrak{M}_M(\bar{m}) = \mathfrak{M}_N(\bar{m})$.*

*Proof.* It is well known that the geodesic flow of a Riemannian manifold is optical with respect to the vertical section which is given by the kernel of the differential of the projection map from the tangent bundle to the manifold (see §5). By Corollary 3.2 and Remark 2.6, if there exists a contactomorphism between the unit sphere bundle of $M$ and the unit sphere bundle of $N$ which conjugates the geodesic flows, we have $\mathfrak{M}_M(\bar{m}) = \mathfrak{M}_N(\bar{m})$. □

*Remark 4.7.* In particular, $M$ has conjugate points if and only if $N$ does. This was proved by Croke and Kleiner [**9**] under the much weaker assumption of the existence of a $C^0$-conjugacy between the geodesic flows. This naturally raises the question: *is it true that the asymptotic Maslov index is an invariant of $C^0$-time preserving conjugacy?*

We now give our main application of the asymptotic Maslov index.

THEOREM 4.8. *Let $(H, \Sigma, \mathbb{E})$ be an optical Hamiltonian with $\Sigma$ compact and $\omega^{n-1}$ exact. Assume that $e : \Sigma \to \Lambda(\Sigma)$ is a continuous semi-conjugacy between the Hamiltonian flow $\phi^t$ on $\Sigma$ and its lift $\Phi^t$ to $\Lambda(\Sigma)$. Then:*
(a) $\Pi(e(\Sigma))$ *has no conjugate points;*
(b) $e(z) \cap \mathbb{E}(\Pi e(z)) = \{0\}$ *for all* $z \in \Sigma$.

When $e$ is a section (i.e. $\Pi \circ e = id_\Sigma$), Mañé [**13**] gave a proof of this theorem in the case of recurrent geodesic flows and Paternain and Paternain [**16**] proved it for convex Hamiltonians.

*Remark 4.9.* Bialy and Polterovich proved in [**6**] that if $H : \mathcal{N} \to \mathbb{R}$ is proper†, bounded from below and optical then $H^{2n-2}(\mathcal{N}, \mathbb{R}) = 0$ for $n \geq 3$ and therefore any closed $2n-2$ form is exact. This implies, in particular, the exactness of the form $\omega^{n-1}$.

*Proof of Theorem 4.8.* (a) Write $f = \Pi \circ e$. Since $e$ is a semi-conjugacy, the measure $e_*\bar{m}$ is a $\Phi^t$-invariant lift of the pushforward $f_*\bar{m}$ of the Liouville measure $\bar{m}$. Using (10), we have that

$$\mathfrak{M}_{\mathbb{E}}(f_*\bar{m}) = \int \eta_{\mathbb{E}}(\mathbb{X})\, d(e_*\bar{m}) = \int \eta_{\mathbb{E}}(\mathbb{X} \circ e)\, d\bar{m} = \langle e^*\mathfrak{M}_{\mathbb{E}}, \mathcal{S}(\bar{m})\rangle = 0,$$

where $e^*\mathfrak{M}_{\mathbb{E}}$ is the cohomology class obtained from the map induced by $\Sigma \xrightarrow{e} \Lambda(\Sigma)$ and the last equality follows from Proposition 3.1. (To see why the third equality holds when $e$ is only continuous, just view $e^*\mathfrak{M}_{\mathbb{E}}$ as the class of a function $\Sigma \to S^1$ differentiable along the flow and given by $\varphi \circ e$, where $\varphi : \Lambda(\Sigma) \to S^1$ represents the Maslov class.) Since $\mathrm{supp}(f_*\bar{m}) = \Pi(e(\Sigma))$, Theorem 4.4 completes the proof of (a).

(b) Write $f = \Pi \circ e$. Suppose that $e(z) \cap \mathbb{E}(f(z)) \neq \{0\}$ for some $z \in \Sigma$. Since $H$ is optical, there exists a small flow box $U$ for $\phi_t$ containing $z$ such that for every $w \in U$, the path $t \mapsto \Phi_t(e(w)) = e(\phi_t(w))$ crosses the Maslov cycle at least once. Since almost every point for the Liouville measure is recurrent we can choose a point $w \in U$ such that its orbit under $\phi_t$ returns infinitely many times to $U$. Therefore, for $T$ large enough, the path $[0, T] \ni t \mapsto \Phi_t(e(w))$ intersects the Maslov cycle at least $n + 1$ times. It follows from Theorem 4.2 that there must be conjugate points along the orbit

† In particular with compact energy levels.

of $f(w)$. This contradicts (a). (Alternatively, instead of Theorem 4.2, we could have used Lemma 2.1 by taking $T$ large enough so that the path $[0, T] \ni t \mapsto \Phi_t(e(w))$ intersects the Maslov cycle at least $8n + 1$ times.) $\square$

When the energy level is non-compact, the conclusion of Theorem 4.8 does not hold. An example is given by the geodesic flow of the paraboloid of revolution (cf. Paternain and Paternain [**16**]). In this case, $e = \langle X \rangle \oplus \langle Y \rangle$ is an invariant Lagrangian section, where $X = (\partial/\partial t)\psi_t$ is the vector field of the geodesic flow and $Y = (\partial/\partial t)\, dR_t$, where $R_t$ is the flow on the paraboloid given by rotation isometries. However, this example has infinite Liouville measure. In Appendix B, we give a more elaborate example that shows that the compactness hypothesis in Theorem 4.8 cannot be replaced just by finite Liouville measure.

4.4. *Anosov flows.* A complete flow $\phi^t$ on a manifold $V$ is *Anosov* if there exists a continuous splitting of the tangent space over each point $x$ in $V$:

$$TV_x = E^s_x \oplus E^u_x \oplus Y_x,$$

(where $Y$ stands for the direction of the vector field induced by $\phi^t$) such that, for some Riemannian metric on $V$, $d\phi^t|_{E^s}$ (respectively $d\phi^t|_{E^u}$) is uniformly exponentially contracting as $t$ goes to $+\infty$ (respectively $-\infty$).

Anosov flows possess a very rich and well-understood dynamics and, consequently, it is important to know whether the flow of a Hamiltonian which is complete on a regular energy level can be Anosov.

LEMMA 4.10. *Let $\Sigma$ be a regular energy level with complete Hamiltonian flow $\phi^t$ and finite Liouville measure such that the flow $(\Sigma, \phi^t)$ is Anosov. Then, for all $x$ in $\Sigma$, the spaces $E^{cs}_x := E^s_x \oplus Y_x$ and $E^{cu} := E^u_x \oplus Y_x$ are Lagrangian planes.*

The proof is already standard in slightly different contexts. We give it here for the sake of completeness and to make it fit with our hypotheses.

*Proof.* Since the sum of the dimensions of $E^{cs}$ and $E^{cu}$ is $2n$, it is enough to prove that both spaces are isotropic. Actually, from the definition of the Hamiltonian vector field (7), $\omega(Y_x, \cdot) \equiv 0$ on $\Sigma = H^{-1}\{e\}$, so we only have to prove that $E^s_x$ and $E^u_x$ are isotropic.

Choose a neighborhood $U$ of the point $x$ in $\Sigma$ whose closure $\bar{U}$ is compact. Recall that since the Liouville measure is finite, then $\bar{m}$-almost every point is recurrent. Thus, it is possible to find a recurrent point $z$ in $U$. We consider a sequence of times $(t_n)$ such that $t_n$ goes to $+\infty$ with $n$ and $\phi^{t_n}(z)$ is in $U$. Now choose two vectors $v_1$ and $v_2$ in $E^s_z$. The quantity $\omega_{\phi^{t_n}(z)}(d\phi^{t_n}{}_z(v_1), d\phi^{t_n}{}_z(v_2))$ decreases in norm at least exponentially at $n$ goes to $+\infty$ (since $\omega$ is bounded on $\bar{U}$). On the other hand, the same quantity is independent of $n$ since $\phi^t$ preserves $\omega$. Thus $\omega_z(v_1, v_2) = 0$. A similar argument works for $E^u_z$. Since recurrent points are dense, we get that $E^s_x$ and $E^u_x$ are isotropic for all $x$ in $\Sigma$. $\square$

Lemma 4.10 shows that a prerequisite to find a flow $\phi^t$ of a Hamiltonian, complete and Anosov on a regular energy level $\Sigma$, is to find a continuous $d\phi^t$-invariant Lagrangian section $e$. Theorem 4.8 shows some clear obstructions to the existence of such sections when $H$ is optical.

COROLLARY 4.11. *Let* $(\Sigma, H, \mathbb{E})$ *be an optical Hamiltonian with* $\Sigma$ *compact and* $\omega^{n-1}$ *exact and such that* $(\Sigma, \phi^t)$ *is Anosov. Then* $E^s$ *and* $E^u$ *are transversal to* $\mathbb{E}$ *and* $\Sigma$ *has no conjugate points.*

Corollary 4.11 was proved by Klingenberg [**12**] for geodesic flows on compact manifolds, by Mañé [**13**] for geodesic flows whose non-wandering set is the whole unit sphere bundle and by Paternain and Paternain [**16**] for convex Hamiltonians.

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## A. *Appendix. Convex Hamiltonians*

An important special class of optical Hamiltonians is given by convex Hamiltonians on the cotangent bundle $T^*M$.

Let $M$ be a $n$-dimensional, connected manifold without boundary, $\mathcal{N} = T^\star M$ its cotangent bundle and $\pi : \mathcal{N} \to M$ the standard projection. The cotangent bundle is equipped with a canonical 1-form, called the Liouville 1-form, defined by:

$$\Theta_p(\zeta) := p(d\pi(\zeta)),$$

for all $p \in T^*M$ and $\zeta \in T_pT^*M$. The 2-form $\omega = d\Theta$ is a symplectic form† on $\mathcal{N}$. The Lagrangian Grassmann bundle $\Lambda(\mathcal{N})$ is also equipped with a canonical smooth section $\mathbb{V} = \ker d\pi$, called the *vertical* section.

A Hamiltonian $H : T^*M \to \mathbb{R}$ is *convex* if for all $q \in M$, $p \in T_q^*M$, the Hessian matrix $(\partial^2 H/\partial p_i \partial p_j)(q, p)$ (calculated with respect to linear coordinates on $T_q^*M$) is positive definite. Convex Hamiltonian systems play a central role in physics and have been extensively studied.

LEMMA A.1. [**5**]. *A convex Hamiltonian is optical with respect to the vertical section* $\mathbb{V}$.

*Proof.* Locally, $(T^*M, \omega, \mathbb{V})$ can be identified with $(\mathbb{R}^{2n}, dp \wedge dq, q = 0)$. We consider a Lagrangian plane in the train of the plane $q = 0$ and in order to fix the notation, we assume that $\dim(\lambda \cap \mathbb{V}) = k$ (note that it is not restrictive to also assume that $\lambda \cap [p = 0] = \{0\}$). As we have already seen in §1, $\lambda$ written in coordinates is a graph: $\lambda = \{ q = Ap \mid p \in \mathbb{R}^n \}$, where $A$ is a (symmetric) linear map, $A : \mathbb{R}^n \to \mathbb{R}^n$. Then $\lambda \cap \mathbb{V} = \{ (q, p) \mid q = 0,\ p \in \ker A \}$. Consider now a curve $t \mapsto \lambda(t)$ passing through $\lambda$ at $t = 0$ and write $\lambda(t) = S(t)\lambda$ with

$$S(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}.$$

† In this situation, $\omega^{n-1}$ is exact.

The symmetric 2-form $\beta_{\lambda,\lambda'(0)}|_{\lambda\cap\mathbb{V}}$ reads

$$\beta_{\lambda,\lambda'(0)}|_{\lambda\cap\mathbb{V}}(\xi,\xi) = p^{\mathsf{T}}\dot{b}(0)p,$$

where $p \in \ker A$ is the unique vector in $\mathbb{R}^n$ such that $\xi = (0, p)$.

The Hamiltonian vector field (7) is given by $X = (H_p, -H_q) = (\dot{q}, \dot{p})$. From the equation $(d/dt)\, d\psi_t = DX\, d\psi_t$, we get that the orbit of the Lagrangian plane $\lambda$ under the lifted Hamiltonian flow is a curve $t \mapsto S(t)\lambda$ passing through $\lambda$ at $t = 0$ such that

$$S'(t) = DX = \begin{bmatrix} H_{pq} & H_{pp} \\ -H_{qq} & -H_{qp} \end{bmatrix}.$$

In particular, $\dot{b}(0) = H_{pp}$ is positive definite. □

*Remark A.2.* Note that our definition of $\mathbb{V}$-conjugate points coincides with the usual definition of conjugate points for convex Hamiltonians and also with the more standard definition that uses Jacobi fields.

In the case of convex Hamiltonians, Theorem 4.8 can be improved as follows.

COROLLARY A.3. *Assume that $\phi^t$ is a convex Hamiltonian flow on a regular energy level $\Sigma \subset T^*M$ with $\Sigma$ compact. If there is a continuous $\Phi^t$-invariant Lagrangian section $e : \Sigma \to \Lambda(\Sigma)$, then:*

(a) *$\Sigma$ has no conjugate points;*

(b) *$e(z) \cap \mathbb{V}(z) = \{0\}$ for all $z \in \Sigma$;*

(c) *$\pi(\Sigma) = M$.*

The fact that (c) is a necessary condition for the existence of an Anosov flow on energy levels was first observed in [**16**].

*Proof.* We only have to prove (c). We have to show that the map $\pi : \Sigma \to M$ is a submersion since, in such a case, $\pi(\Sigma)$ is open and closed on $M$ (which is connected) and then equal to $M$. Note that the existence of a continuous $d\phi^t$-invariant Lagrangian section $e$ implies the existence of a continuous $d\phi^t$-invariant Lagrangian section $\mathbb{F}$ such that for all $x$ in $\Sigma$, $\mathbb{F}_x \subset T\Sigma_x$. Indeed, this section $\mathbb{F}$ is defined by $\mathbb{F}_x = e(x) \cap T\Sigma_x \oplus Y(x)$ where $Y(x)$ is the direction in $T\Sigma_x$ tangent to the flow $\phi^t$. Let $x$ be a point in $\Sigma$ where $\pi$ is not a submersion. Then $\mathbb{V}_{x_0} \subset T\Sigma_{x_0}$ and, consequently,

$$\dim \mathbb{V}_{x_0} \cap \mathbb{F}_{x_0} \geq 1,$$

a contradiction to (b). □

In the previous corollary, the proof of (a) is even simpler than that in Theorem 4.8: observe that $e_*\bar{m}$ is an invariant lift of $\bar{m}$ whose support is disjoint from the section $Je$. By the invariance of the asymptotic Maslov index with respect to the section (cf. Corollary 3.2.(ii)), $\mathfrak{M}_{\mathbb{V}}(\bar{m}) = \mathfrak{M}_{Je}(\bar{m}) = 0$. Now use Theorem 4.4.

## B. *Appendix*

The aim of this appendix is to prove the following result.

THEOREM B.1. *There exists a convex Hamiltonian on a surface with a regular energy level that satisfies the following properties:*
(a) *the flow is complete on this energy level;*
(b) *the Liouville measure of the energy level is finite;*
(c) *there exists a smooth invariant Lagrangian section;*
(d) *every orbit in the energy level has conjugate points.*

We know from Theorem 4.8 that if a compact energy level of a convex Hamiltonian admits a continuous invariant Lagrangian section, then there are no conjugate points in the level. On the other hand, it is also known (see [**16**]) that this result extends to a special class of Hamiltonians (namely the *symmetric* Hamiltonians) when the energy level is not compact but has a finite Liouville measure. Theorem B.1 shows that the symmetry in the Hamiltonian cannot be removed.

B.1. *Magnetic flows in general.* Let $M^n$ be a closed $n$-dimensional manifold endowed with a $C^\infty$ Riemannian metric $g$, and let $\pi : TM \to M$ be the canonical projection. The symplectic form on $TM$ obtained by pulling back the canonical symplectic form of $T^*M$ via the Riemannian metric is denoted by $\omega_0$. Consider a closed 2-form $\Omega$ of $M$ and the new symplectic form $\omega_1$ defined by

$$\omega_1 \stackrel{\text{def}}{=} \omega_0 + \pi^*\Omega.$$

The 2-form $\omega_1$ is a symplectic form. We say that it defines a *twisted symplectic structure*.

Let $E : TM \to \mathbb{R}$ be given by

$$E(x, v) = \tfrac{1}{2} g_x(v, v).$$

The Hamiltonian flow of $E$ with respect to $\omega_1$ models the motion of a particle of unit mass and charge under the effect of a magnetic field, whose Lorentz force $Y : TM \to TM$ is the bundle map determined by

$$\Omega_x(u, v) = g_x(Y_x(u), v),$$

for all $x \in M$ and all $u$ and $v$ in $T_xM$. In other words, the curve

$$t \mapsto (\gamma(t), \dot\gamma(t)) \in TM$$

is an orbit of the Hamiltonian flow iff

$$\frac{D\dot\gamma}{dt} = Y_\gamma(\dot\gamma), \tag{17}$$

where $D$ stands for the covariant derivative of $g$. The Hamiltonian flow of $E$ with respect to $\omega_1$ leaves the unit sphere bundle $SM = E^{-1}(\frac{1}{2})$ invariant and, therefore, it defines a flow

$$\phi^t : SM \to SM,$$

that we call the *magnetic flow* of the pair $(g, \Omega)$. The magnetic flow of the pair $(g, 0)$ is the geodesic flow of the Riemannian metric $g$. A curve $\gamma$ that satisfies (17) will be called a *magnetic geodesic*.

*Remark B.2.* The magnetic flow is locally conjugate to the Lagrangian flow of the Lagrangian $L(x, v) = \frac{1}{2}\langle v, v\rangle_x + \eta_x(v)$, where $\eta_x(v)$ is a local 1-form such that $d\eta = \Omega$. The conjugacy preserves the vertical section. This Lagrangian flow, in turn, is conjugate to the Hamiltonian flow of the convex Hamiltonian $H(x, p) = \frac{1}{2}\|p - \eta_x\|_x^2$, in $T^*M$ with the canonical symplectic structure. In particular, the twisted geodesic flow is optical with respect to the vertical section. A computation also shows that the Liouville measure on a regular energy level of the twisted geodesic flow coincides with the Liouville measure on a regular energy level of the geodesic flow itself.

B.2. *Magnetic flows for constant fields on rotationally symmetric surfaces.* Let $M$ be an oriented surface. Given $(x, v) \in SM$, let $iv$ be the unique vector in $T_xM$ such that $\{v, iv\}$ is a positively oriented orthonormal basis of $T_xM$. The area form $\Omega$ is given by

$$\Omega_x(u, v) = g_x(iu, v),$$

hence the Lorentz force $Y$ is given by

$$Y_x(v) = iv.$$

Define the $\lambda$-*magnetic flow* as the magnetic flow of $(g, \lambda\Omega)$. It follows from (17) that $t \mapsto \gamma(t)$ is a $\lambda$-magnetic geodesic iff

$$\frac{D\dot{\gamma}}{dt} = \lambda\, i\dot{\gamma}.$$

In other words, $\gamma$ is a $\lambda$-magnetic geodesic iff $\gamma$ has constant geodesic curvature $\lambda$.

Suppose now that $M = \mathbb{R} \times S^1$ is a rotationally symmetric surface, i.e. if $(s, \varphi)$ are the obvious coordinates in $M$, then the Riemannian metric of $M$ in these coordinates has the expression

$$g = ds^2 + r(s)^2\, d\varphi^2,$$

where $r : \mathbb{R} \to (0, \infty)$ is a smooth function.

We orient $S^1$ counterclockwise and we give $M$ the product orientation. This means that

$$\left\{\frac{\partial}{\partial s}, \frac{\partial}{\partial \varphi}\right\}$$

is a positively oriented basis of $M$ and, hence, the area form $\Omega$ is given by

$$\Omega = r\, ds \wedge d\varphi.$$

Define

$$R(s) := \int_0^s r(u)\, du.$$

LEMMA B.3. *A curve $t \mapsto (s(t), \varphi(t))$ is a $\lambda$-magnetic geodesic iff*

$$\ddot{s} = r\dot{\varphi}(r'\dot{\varphi} - \lambda)$$

$$\frac{d}{dt}(r^2\dot{\varphi} - \lambda R) = 0,$$

*where a dot indicates a derivative with respect to $t$ and a prime indicates a derivative with respect to the $s$-parameter. In particular the quantity $r(s)^2\dot{\varphi} - \lambda\, R(s)$ is a first integral of the flow called the Clairaut integral.*

*Proof.* Let us consider the Lagrangian

$$L(s,\varphi,\dot{s},\dot{\varphi}) = \tfrac{1}{2}(\dot{s}^2 + r(s)^2\dot{\varphi}^2) - \lambda\, R(s)\dot{\varphi}.$$

Note that

$$d(R\,d\varphi) = r\,ds \wedge d\varphi = \Omega.$$

Hence, the extremals of $L$ are the λ-magnetic geodesics. The lemma follows from a simple computation derived from the Euler–Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0. \qquad \square$$

LEMMA B.4. *The parallel $t \mapsto (s_0, at)$ $(a \neq 0)$ is a unit speed λ-magnetic geodesic iff*

$$|a^{-1}| = r(s_0)$$
$$\frac{r'(s_0)}{r(s_0)} = \pm\lambda,$$

*where the positive sign holds if $a > 0$ and the negative signs holds if $a < 0$.*

*Proof.* The parallel has unit speed iff $r^2(s_0)\,a^2 = 1$. Using Lemma B.3 we see that the parallel is a λ-magnetic geodesic iff $r'(s_0)a = \lambda$. $\square$

Let $\theta \in [0,\pi]$ be the angle that a unit speed λ-magnetic geodesic makes with the parallel $t \mapsto (s, t/r(s))$ at the point $(s,\varphi)$. We have

$$r\cos\theta = r^2\dot{\varphi},$$

and, therefore, the Clairaut integral from Lemma B.3 reads

$$r\cos\theta - \lambda R = c, \tag{18}$$

where $c$ is a constant that depends only on the λ-magnetic geodesic.

Define

$$f_{+,\lambda}(s) := r(s) - \lambda R(s),$$
$$f_{-,\lambda}(s) := -r(s) - \lambda R(s).$$

LEMMA B.5. *Suppose that there exists a λ-magnetic geodesic for which $\theta = 0$ at the values $r(s_1)$ and $r(s_2)$ with $s_1 < s_2$. Then there exists $s_0 \in (s_1, s_2)$ on which $f_{+,\lambda}$ has a local maximum and such that the parallel $t \mapsto (s_0, t/r(s_0))$ is a λ-magnetic geodesic.*

*Similarly, suppose that there exists a λ-magnetic geodesic for which $\theta = \pi$ at the values $r(s_1)$ and $r(s_2)$ with $s_1 < s_2$. Then there exists $s_0 \in (s_1, s_2)$ on which $f_{-,\lambda}$ has a local minimum and such that the parallel $t \mapsto (s_0, -t/r(s_0))$ is a λ-magnetic geodesic.*

*Proof.* We only give a proof of the first statement here (the proof of the second one is completely analogous).

If, for all $s \in (s_1, s_2)$, the λ-magnetic geodesic is always tangent to the parallel at $s$, the parallel is an integral curve and the lemma is proved. If not, there exists $s_3 \in (s_1, s_2)$ such that the angle $\theta_3$ at which the $\lambda$-magnetic geodesic crosses the parallel at $s_3$ is not zero and then $\cos\theta_3 < 1$.

Since $r(s_3) > 0$, using the Clairaut integral (Lemma B.3), we get

$$f_{+,\lambda}(s_3) > f_{+,\lambda}(s_1) = f_{+,\lambda}(s_2).$$

Consequently, there exists $s_0 \in (s_1, s_2)$ such that $f_{+,\lambda}$ presents a local maximum at $s_0$. However,

$$f'_{+,\lambda}(s) = r'(s) - \lambda\, r(s) = r(s)\left(\frac{r'(s)}{r(s)} - \lambda\right).$$

Hence $f'_{+,\lambda}(s_0) = 0$ iff $r'(s_0)/r(s_0) = \lambda$. By Lemma B.4 this implies that the parallel $t \mapsto (s_0, t/r(s_0))$ is a $\lambda$-magnetic geodesic. □

*Proof of Theorem B.1.* The example that achieves the properties described in Theorem B.1 is chosen among $\lambda$-magnetic flows associated to rotationally symmetric surfaces.

Let us first construct the rotationally symmetric surface. Consider a smooth function $u : \mathbb{R} \to \mathbb{R}$ with the following properties:

(1) $u$ is odd, i.e. $u(s) = -u(-s)$ for all $s \in \mathbb{R}$;
(2) for all $s \in \mathbb{R}$, $-1 < u(s) < 1$;
(3) for all $s > 3$, $u(s) = -2/s$.

Now let $r(s) : \mathbb{R} \to (0, \infty)$ be defined by

$$\frac{r'(s)}{r(s)} = u(s),$$
$$r(0) = r_0 > 0.$$

In other words,

$$r(s) = r_0 \exp\left(\int_0^s u(t)\, dt\right).$$

Observe that $r(s)$ is an even function of $s$ and that for $s > 3$, we have

$$r(s) = r(3)9/s^2. \tag{19}$$

Let $M = \mathbb{R} \times S^1$ be the rotationally symmetric surface determined by such a function $s \mapsto r(s)$. The estimate (19) implies that the area of $M$ is finite:

$$\int_M \Omega = 2\pi \int_{-\infty}^{+\infty} r(s)\, ds = 4\pi\, R(+\infty).$$

Consequently (using Remark B.2), the Liouville measure of the $\lambda$-magnetic flow on the unit tangent bundle $SM$ is also finite. This ensures Theorem B.1(b).

We now consider the $\lambda$-magnetic flow on $SM$ which corresponds to $\lambda = 1$ and prove (a), (c) and (d) for this particular flow.

Since the vector field generated by the circle action is tangent to the energy levels, above each point in $SM$ the vector space generated by the magnetic vector field and the vector field generated by the circle action is isotropic. Since $u(s) \in (-1, 1)$, Lemma B.4 ensures that no parallel is a magnetic geodesic. In other words, no magnetic geodesic is an orbit of the circle action and, consequently, we have constructed an invariant Lagrangian section $E$

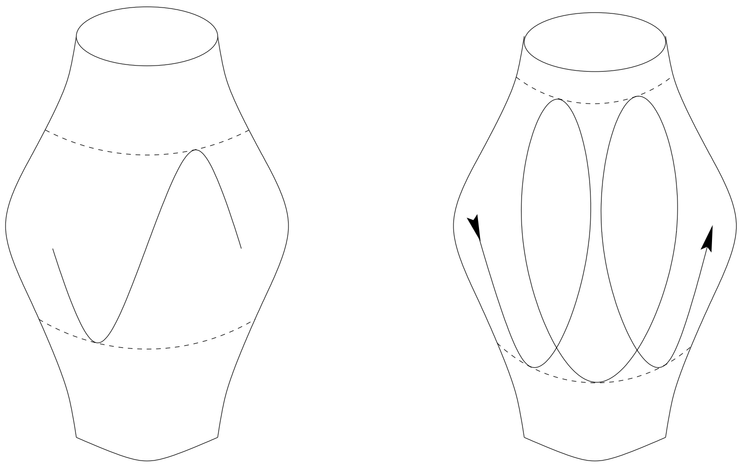

FIGURE B.1. The figure on the left-hand side shows a typical geodesic while the figure on the right-hand side shows a typical magnetic geodesic that 'curls'.

spanned by the magnetic vector field and the lift of the circle action that generates the symmetry. This ensures item (c) of Theorem B.1(c).

Observe that the Clairaut integral $C : SM \to \mathbb{R}$ and the Lagrangian section $E$ satisfy

$$E(x, v) = \ker d_{(x,v)}C$$

for all points $(x, v)$ in $SM$. In particular, $C$ does not have critical points†.

Note that

$$f_{-,1}(s) \leq C(s, \varphi, \dot{s}, \dot{\varphi}) \leq f_{+,1}(s),$$

for all $s \in \mathbb{R}$ and that $f_{-,1}$ and $f_{+,1}$ are strictly decreasing functions which have the same finite limit when $s \to +\infty$ (respectively when $s \to -\infty$). Hence the projection to $M$ of any level set of $C$ has to be contained in a compact set. Therefore, this level set is compact and, hence, its connected components are finitely many tori. In conclusion, the energy level $SM$ is foliated by tori on which the Clairaut integral is constant. This implies in particular that the magnetic vector field is complete: Theorem B.1(a).

Actually, it is possible to give a better visualization of the dynamical behavior of the magnetic flow. The connected components of the level sets of the Clairaut integral $C$ are 2-tori that project on $M$ onto strips bounded by two parallels. The magnetic geodesics on these tori oscillate between these two parallels. The difference with the case of the geodesic flow of a surface of revolution comes from the fact that in our example a parallel cannot be a magnetic geodesic. It follows from Lemma B.5 that a magnetic geodesic makes an angle $\theta = 0$ with the bottom parallel and an angle $\theta = \pi$ with the top one. In other words it 'curls' as in Figure B.1.

Let us now prove Theorem B.1(d). More precisely let us show that *every* magnetic geodesic has conjugate points.

Consider a magnetic geodesic. The invariant section $E$ (spanned by the magnetic vector field and the vector field induced by the circle action) intersects non-trivially the

† The image by the derivative $dC$ of a tangent vector to a meridian is non-zero.

vertical fibers above the points of the magnetic geodesic at which the latter is tangent to the parallels. Consequently, by Lemma 2.1 the magnetic geodesic has conjugate points. □

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