# HOMOGENIZATION ON MANIFOLDS

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ABSTRACT. We present a theorem by Contreras, Iturriaga and Siconolfi [8] in which we give a setting to generalize the homogenization of the Hamilton-Jacobi equation from tori to other manifolds.

A *homogenization problem* consists of a Partial Differential Equation (PDE) with a fast (oscillating) variable $\varepsilon$ and a slow variable. The homogenization result is that when the oscillating period $\varepsilon$ tends to zero, there is a limit of the solutions $u^\varepsilon$ of the PDE to a solution of an homogenized or "averaged" PDE.

An example of the homogenization result that we present here is the convergence of the average distance in the universal cover of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ to the distance in the stable norm in $H_1(\mathbb{T}^2, \mathbb{R}) = \mathbb{R}^2$, when the diameter of the fundamental domain $\varepsilon$ tends to zero (see fig. 1).
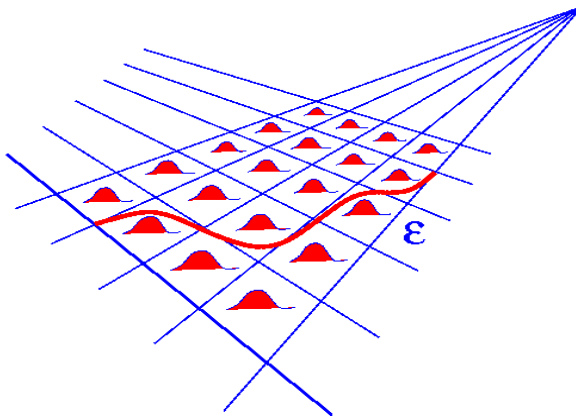


FIGURE 1. Convergence to the stable norm.

In higher dimensions the minimal geodesics may not converge. This is related to the flats of the stable norm as in Hedlund's example [14] in figure 2.
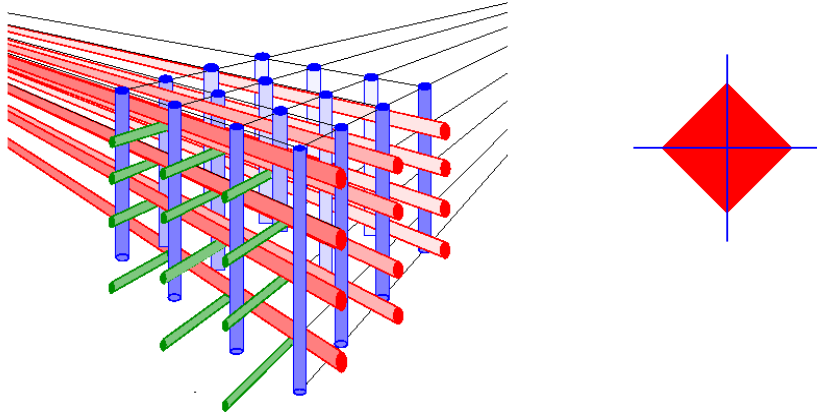
FIGURE 2. Hedlund's example and its stable norm.

Hedlund's example is a 3-torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, in which the Riemannian metric is deformed in three disjoint tubes of different homological directions in which the central closed geodesics are very short. In the example, minimal geodesics follow the tubes with at most two jumps and the stable norm is

$$\|(x, y, z)\| = |x| + |y| + |z|.$$

In Hedlund's example the minimal geodesics do not converge as $\varepsilon \to 0$. There is a convergence as "holonomic measures" to an invariant measure supported on three periodic orbits on the tubes. The fact that there is no ergodic minimizing measure in a given homology class implies that the stable norm is flat on that class.

An important observation in this geodesic example of homogenization is that the average minimal distance can be computed from the geodesics of the stable norm, which are straight lines. One expects that the homogenized or averaged problem is much simpler and computable than the original problem. Another application of homogenization theory is to obtain macroscopic laws from microscopic data.

Homogenization theory has mostly been done in a periodic setting (i.e. on the torus $\mathbb{T}^n$) or in quasi-periodic tilings or random media on $\mathbb{R}^n$. In the case of the Hamilton-Jacobi equation, the limiting objects are well known and naturally defined on arbitrary manifolds: the effective Lagrangian is Mather's minimal action function $\beta$ and the effective (or homogenized) Hamiltonian is its dual $\beta^*$, also known as Mañé's critical value. Nevertheless this homogenizations have only been made in $\mathbb{T}^n$.

We will show how to extend the homogenization result for the Hamilton-Jacobi equation from the torus $\mathbb{T}^n$ to an arbitrary compact manifold. We hope that the setting presented here can be applied to many other homogenization results.

## 1. Homogenization of the Hamilton-Jacobi equation.

Let $M$ be a compact manifold without boundary. A *Tonelli Lagrangian* is a $C^2$ function $L : TM \to \mathbb{R}$ satisfying:

(i) *Convexity:* $\quad \frac{\partial L}{\partial v \, \partial v}(x, v)$ is positive definite $\forall (x, v) \in TM$.

(ii) *Superlinearity:* $\quad \lim_{|v| \to +\infty} \frac{L(x,v)}{|v|} = +\infty$ uniformly on $x \in M$.

Examples of Tonelli Lagrangians are

(1) *The kinetic energy:* $\quad L(x, v) = \frac{1}{2} \|v\|_x$, which gives the geodesic flow and whose homogenization is equivalent to the examples given above.

(2) The Mechanical Lagrangian: $L(x, v) = \frac{1}{2} \|v\|_x - U(x) = $ kinetic energy - potential energy. This Lagrangian gives rise to Newton's law with force $F = -\nabla U(x)$.

The *action* of a smooth curve $\gamma : [0, T] \to M$ is

$$A_L(\gamma) = \int_0^T L\big(\gamma(t), \dot{\gamma}(t)\big) \, dt.$$

Critical points of $A_L$ satisfy the *Euler-Lagrange* equation

(1)
$$\frac{d}{dt} \frac{\partial L}{\partial v} = \frac{\partial L}{\partial x}.$$

The Euler-Lagrange equation is a second order equation whose solutions give rise to the *Lagrangian Flow:* $\varphi_t : TM \to TM$,

$$\varphi_t(x, v) = (\gamma(t), \dot{\gamma}(t)),$$

where $\gamma$ is the solution of (1) with initial conditions $(\gamma(0), \dot{\gamma}(0)) = (x, v)$.

The convex dual of the Lagrangian is the Hamiltonian $H : T^*M \to \mathbb{R}$

$$H(x, p) = \sup_{v \in T_x M} \big\{ p(v) - L(x, v) \big\}.$$

The *Legendre Transform* $L_v : TM \to T^*M$, $L_v(x, v) = \frac{\partial L}{\partial v}(x, v)$, converts the Euler-Lagrange equation (1) into the Hamiltonian equations:

$$\frac{d}{dt} L_v = L_x \quad \xrightarrow{\; L_v \;} \quad \begin{cases} \dot{x} &= H_p \\ \dot{p} &= -H_x \end{cases}$$

and conjugates the Lagrangian and Hamiltonian flows.

The *Hamilton-Jacobi* equation

(2)
$$\partial_t u + H(x, \partial_x u) = 0$$

encodes the minimal (Lagrangian) action cost. A solution $u : M \times \mathbb{R}_+ \to \mathbb{R}$, to the Hamilton-Jacobi equation with initial condition

$$u(x,0) = f(x)$$

is given by the Lax formula

$$u(x,t) = \inf \left\{ f(\gamma(0)) + \int_0^t L(\gamma, \dot{\gamma}) \;\Big|\; \gamma \in C^1([0,t], M), \; \gamma(t) = x \right\}.$$

The characteristics of the Hamilton-Jacobi equation are *Tonelli minimizers* i.e. minimizers of the action with fixed endpoints and fixed time interval. The value of the solution is the initial value + the action along these minimizers. Tangent vectors to the characteristics are related to $\partial_x u$ through the Legendre Transform $L_v$:

(3) $$\partial_x u = L_v(\gamma, \dot{\gamma}).$$

Usually there are no global classical solutions of the Hamilton-Jacobi equation due to crossing of characteristics as in figure 3. Indeed, from (3) at a crossing point there are various candidates for $\partial_x u$, and hence $\partial_x u$ does not exist.
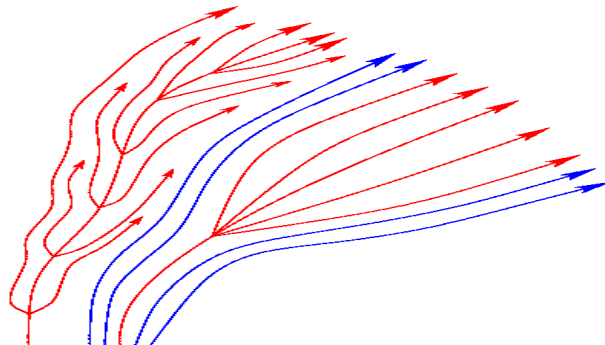


FIGURE 3. Crossing of characteristics.

There are two popular types of weak solutions in PDEs:

- Weak solutions with *weakly differentiable* functions and Sobolev Spaces are inspired on the formula of integration by parts.
- The *viscosity solution* is inspired on the maximum principle for PDEs.

The first definition of viscosity solutions was made by L.C. Evans in 1980 [12]. Subsequently the definition and properties of the viscosity solutions of Hamilton-Jacobi equations were refined by Crandall, Evans and Lions in [9]. The existence and uniqueness of the *viscosity solution* of the initial value problem for the Hamilton-Jacobi equation was proved by Crandall and Lions in [10].

A continuous function is a *viscosity solution* of

$$\partial_t u + H(x, \partial_x u) = 0$$

if for every open set $U \subset M$ and any $\phi \in C^1(U \times \mathbb{R}_+, \mathbb{R})$:

- if $u - \phi$ attains a local maximum at $(y_0, t_0) \in U \times \mathbb{R}_+$, then $\partial_t \phi(y_0, t_0) + H(y_0, \partial_x(y_0, t_0)) \leq 0$.
- if $u - \phi$ attains a local minimum at $(y_0, t_0) \in U \times \mathbb{R}_+$, then $\partial_t \phi(y_0, t_0) + H(y_0, \partial_x(y_0, t_0)) \geq 0$.

**1.1. Theorem** (Lions, Papanicolaou, Varadhan [15], Evans [13])**.**

*Let $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a $\mathbb{Z}^n$-periodic Tonelli Hamiltonian. For $\varepsilon$ small let $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. Consider the Cauchy problem for the Hamilton-Jacobi equation*

(4)
$$\partial_t u^\varepsilon + H\left(\tfrac{x}{\varepsilon}, \partial_x u^\varepsilon\right) = 0,$$
$$u^\varepsilon(x, 0) = f_\varepsilon(x).$$

*If $\lim_\varepsilon f_e = f$ uniformly then $\lim u^\varepsilon = u$ uniformly, where $u$ is the solution to*

$$\partial_t u + \overline{H}(\partial_x u) = 0,$$
$$u(x, 0) = f(x).$$

*The function $\overline{H} : \mathbb{R}^n \to \mathbb{R}$, called the effective Hamiltonian is convex, superlinear and is independent of the variable $x$.*

The solutions to the homogenized problem can be easily written because the characteristics are straight lines and $p = \partial_x u$ is constant along them

$$\begin{cases} \dot{p} &= -\overline{H}_x = 0, \\ \dot{x} &= \overline{H}_p = \text{constant}. \end{cases}$$

Thus

$$u(y, t) = \min_{x \in \mathbb{R}^n} \left\{ f(x) + t\,\overline{L}\left(\tfrac{y-x}{t}\right) \right\},$$

where

$$\overline{L}(x, v) = \max_{p \in \mathbb{R}^n} \left\{ p(v) - \overline{H}(p) \right\}$$

is the *Effective Lagrangian*.

It turns out that the Effective Lagrangian $\overline{L} = \beta$ is Mather's minimal action function $\beta : H_1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$. The Effective Hamiltonian is related to Mañé's critical value by

$$\overline{H}(P) = \alpha(P) = c(L - P), \qquad P \in \mathbb{H}^1(\mathbb{T}^n, \mathbb{R}),$$

here $(L - P)(x, v) := L(x, v) - \omega_x(v)$, where $\omega$ is a closed 1-form in the cohomology class $P$. As such, it has several interpretations (see [7]):

(i) $\alpha$ is the convex dual of $\beta$.

(ii) $\alpha(P) = \inf \left\{ k \in \mathbb{R} \mid \oint_\gamma (L - P + k) \geq 0 \quad \forall \text{ closed curve } \gamma \text{ in } \mathbb{T}^n \right\}$.

(iii) $\alpha(P) = \inf \left\{ k \in \mathbb{R} \mid \Phi_k > -\infty \right\}$, where $\Phi_k : M \times M \to \mathbb{R}$ is

$$\Phi_k(x, y) := \inf \left\{ \oint_\gamma (L - P + k) \mid \gamma \text{ curve in } \mathbb{T}^n \text{ from } x \text{ to } y \right\},$$

i.e. the minimal action with free time interval.[1]

(iv) $\alpha(P) = -\inf \left\{ \int (L - P) \, d\mu \mid \mu \text{ is an invariant measure for } L \right\}$.

(v) $\alpha(P)$ is the energy level containing the support of the invariant measures $\mu$ which minimize $\int (L - P) \, d\mu$.

(vi) $\alpha(P) = \displaystyle\min_{u \in C^1(\mathbb{T}^n, \mathbb{R})} \max_{x \in \mathbb{T}^n} H(x, P + d_x u)$.

(vii) $\alpha(P)$ is the minimum of the energy levels which contain a Lagrangian graph in $T^*\mathbb{T}^n$ with cohomology class $P$.

(viii) From Fathi's weak KAM theory, $\alpha(P)$ is the unique constant for which there are global viscosity solutions of the Hamilton-Jacobi equation

$$H(x, P + d_x v) = \alpha(P), \qquad x \in \mathbb{T}^n.$$

We explain briefly why Theorem 1.1 and (viii) imply that the Effective Hamiltonian $\overline{H}$ is Mather's alpha function $\alpha$. Consider the case of affine initial conditions. The problem

(5)
$$\left\{ \begin{array}{l} f(x) = u(x, 0) = a + P \cdot x \\ \partial_t u + \alpha(\partial_x u) = 0 \end{array} \right\}$$

has solution

$$u(x, t) = a + P \cdot x - \alpha(P) t.$$

Let $v : \mathbb{T}^n \times \mathbb{R}_+ \to \mathbb{R}$ be a $\mathbb{Z}^n$-periodic solution to the *"cell problem"*:

$$H(x, P + d_x v) = \alpha(P), \qquad v : \mathbb{T}^n \times \mathbb{R}_+ \to \mathbb{R}.$$

Let

$$u^\varepsilon(x, t) := u(x, t) + \varepsilon \, v\left(\tfrac{x}{\varepsilon}\right),$$

$$F_\varepsilon(x) := u^\varepsilon(x, 0) = f(x) + \varepsilon \, v\left(\tfrac{x}{\varepsilon}\right).$$

Then $u^\varepsilon$ solves

$$\left\{ \begin{array}{l} \partial_t u^\varepsilon + H\left(\tfrac{x}{\varepsilon}, \partial_x u^\varepsilon\right) = -\alpha(P) + H\left(\tfrac{x}{\varepsilon}, P + \partial_y v\left(\tfrac{x}{\varepsilon}\right)\right) = 0, \\ u^\varepsilon(x, 0) = f_\varepsilon(x). \end{array} \right.$$

Also we have that $f_\varepsilon \to f$ and $u^\varepsilon \to u$ uniformly and by (5) $u$ satisfies a Hamilton-Jacobi equation with Hamiltonian $\alpha$. Therefore Theorem 1.1 implies that $\overline{H}(P) = \alpha(P)$.

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[1]The function $\Phi_k$ is called Mañé's action potential.

1.1. **The Problems.**

The generalization of Theorem 1.1 to other manifolds has three problems:

1. It is not clear how to choose the generalization of $\frac{x}{\varepsilon}$.

2a. Equation (4) is the Hamilton-Jacobi equation for the Hamiltonian $H_\varepsilon(x,p) := H(\frac{x}{\varepsilon},p)$, where $p$ *"remains the same"*. It is not clear how to do it in non-parallelizable manifolds where the parallel transport depends on the path.

2b. The effective Hamiltonian $\overline{H}(P)$ "does not depend on $x$". This is another version of the same problem 2a.

3. The candidate for effective Hamiltonian is Mather's $\alpha$ function $\alpha : H^1(M,\mathbb{R}) \to \mathbb{R}$. But in general $\dim H^1(M,\mathbb{R}) \neq \dim M$, i.e. the limit PDE would be in a space with different dimension, the differential structure would be destroyed.

In fact, the Hamilton-Jacobi equation is an encoding of a variational principle (the minimal cost function) that will be stable under the change of space.

The torus $M = \mathbb{T}^n$ has many coincidences that allow to formulate Theorem [15]:

(1) Its universal cover satisfies

$$\widetilde{\mathbb{T}^n} = \mathbb{R}^n = H_1(\mathbb{T}^n,\mathbb{R}) = H^1(\mathbb{T}^n,\mathbb{R}).$$

The effective Hamiltonian $\overline{H} = \alpha : H^1(\mathbb{T}^n,\mathbb{R}) = \mathbb{R}^n \to \mathbb{R}$ and the effective Lagrangian $\overline{L} = \beta : H_1(M,\mathbb{R}) \to \mathbb{R}$ are defined in the same space as the original periodic Hamiltonian. Thus the original PDE and the limit equation are in the same space.

(2) The cotangent bundle is trivial: $T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$ and the parallel transport does not depend on the path. Thus we can talk of a Hamiltonian that does not depend on $x$ and the Hamilton-Jacobi equation for the effective Hamiltonian $\partial_t u + \overline{H}(\partial_x u) = 0$ makes sense.

1.2. **The solution.**

**Problem 3.** We start with the solution to problem 3: the space for the family of PDEs. Let $M$ be a compact manifold without boundary and $H : T^*M \to \mathbb{R}$ a Tonelli Hamiltonian. Consider the Hurewicz homomorphism $\mathbf{h} : \pi_1(M) \to H_1(M,\mathbb{R})$ which sends the homotopy class of a curve to its homology class with real coefficients. The maximal free abelian cover $\widetilde{M}$ is the covering map $\widetilde{M} \to M$ with group of Deck transformations

$$\mathrm{Deck}(\widetilde{M}) = \mathbb{Z}^k = \mathrm{Im}(\mathbf{h}) \subset H_1(M,\mathbb{R}),$$

where $k = \dim H_1(M,\mathbb{R})$ and $\pi_1(\widetilde{M}) = \ker \mathbf{h}$.

**Problem 1.**$[x \mapsto \frac{x}{\varepsilon}]$ Let $d$ be the metric induced on $\widetilde{M}$ by the lift of the Riemannian metric on $M$. For problem 1 we use the metric spaces $M_\varepsilon := (\widetilde{M}, \varepsilon d)$.

The maximal free abelian cover $\widetilde{M}$ has the structure of $\mathbb{Z}^k$, i.e. it is (perhaps a complicated) fundamental domain which is repeated as the points in $\mathbb{Z}^k$, as in figure 4. The
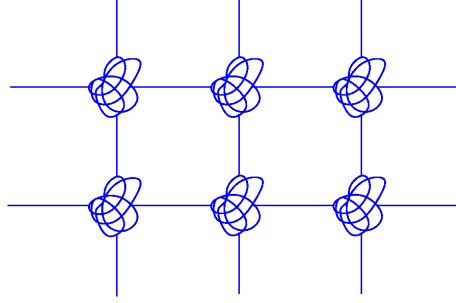


FIGURE 4. The structure of $\widetilde{M}$.

space $\widetilde{M}_\varepsilon$ has a "large scale structure" as $\varepsilon \mathbb{Z}^k \hookrightarrow \mathbb{R}^k = H_1(M, \mathbb{R})$.

We think of $M_\varepsilon \xrightarrow{\varepsilon} H_1(M, \mathbb{R})$ as of $\varepsilon \mathbb{Z}^k \longrightarrow \mathbb{R}^k$. For example: "linear maps on $\widetilde{M}$" shall correspond to "integrals of closed 1-forms". Our solutions of the $\varepsilon$-oscillation Hamilton-Jacobi equation will be uniformly Lipschitz on $\widetilde{M}_\varepsilon$, i.e. $\varepsilon K$-Lipschitz on $M_\varepsilon$. So that a solution $U^\varepsilon$ on $M_\varepsilon$ will define a function $rv^\varepsilon$ on $\varepsilon \mathbb{Z}^k$ which is $K$-Lipschitz. By an Arzelá-Ascoli argument we will obtain a convergence $v^\varepsilon \to v$ on $\mathbb{R}^k = H_1(M, \mathbb{R})$.
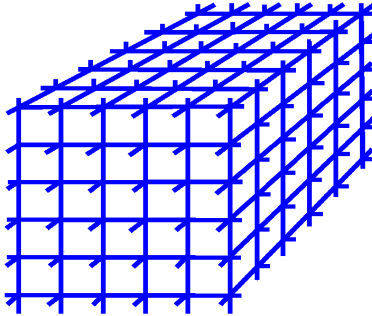


FIGURE 5. Example of a free abelian cover of a surface $M = \mathbb{T}^2 \# \mathbb{T}^2$ with group of Deck transformations $\mathbb{Z}^3$. It is not the maximal free abelian cover of $M$, because $\dim H_1(M, \mathbb{R}) = 4$. The limit space $\lim_\varepsilon M_\varepsilon = \mathbb{R}^3$ has higher dimension than $M$.

**Problem 2.** [$\overline{H}$ **independent of** $x$] The solution to problem 2 consists on transforming the equation to an equivalent PDE. In the case of $\mathbb{R}^n$ as in problem (4) define $v^\varepsilon : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ by

$$u^\varepsilon(x,t) =: v^\varepsilon(\tfrac{x}{e}, t).$$

From (4) we obtain that $v^\varepsilon$ is a solution to the problem

(6) $$\partial_t v^\varepsilon + H\left(y, \tfrac{1}{\varepsilon}\partial_y v^\varepsilon\right) = 0,$$

(7) $$v^\varepsilon(y, 0) = f_\varepsilon(\varepsilon y).$$

Now equation (6) makes sense on any manifold. Equation (7) will make sense with the following definition of convergence of spaces.

## 1.3. Convergence of spaces.

This is inspired in Gromov's Hausdorff convergence but it is made ad hoc for our homogenization problem. We will only need quasi-isometries because since we are doing analysis, just the equivalence class of the norms matter.

Let $(M, d)$, $(M_n, d_n)$ be metric spaces and $F_n : (M_n, d_n) \to (M, d)$ a continuous function. We say that $\lim_n (M_n, d_n, F_n) = (M, d)$ if

(a) There are $B, A_n > 0$, with $\lim_n A_n = 0$ such that

$$\forall x, y \in M_n : \qquad B^{-1} d_n(x, y) - A_n \le d\left(F_n(x), F_n(y)\right) \le B d_n(x, y).$$

(b) For all $y \in M$ and $n$ there are $x_n \in M_n$ with $\lim_n x_n = y$.

Observe that (b) is a kind of surjectivity condition. And (a) implies that

$$\forall y \in M : \qquad \operatorname{diam} F_n^{-1}\{y\} \le B A_n \xrightarrow{n} 0,$$

a kind of injectivity condition.

If $\lim_n(M_n, d_n, F_n) = (M, d)$, and $f_n(M_n, d_n) \to \mathbb{R}$, $F(M, d) \to \mathbb{R}$ are continuous, we say that $\lim_n f_n = f$ *uniformly on compact sets* if for every compact set $K \subset M$

$$\lim_n \sup_{x \in F_n^{-1}(K)} |f_n(x) - f(F_n(x))| = 0.$$

And we say that the family $\{f_n\}$ is equicontinuous if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\forall n : \qquad x, y \in M_n, \quad d_n(x, y) < \delta \quad \implies \quad |f_n(x) - f_n(y)| < \varepsilon.$$

Fix a basis $c_1, \ldots c_k$ for $H^1(M, \mathbb{R})$. Fix closed 1-forms $\omega_i$ on $M$ such that $c_i = [\omega_i]$. Define $G : \widetilde{M} \to H_1(M, \mathbb{R}) = H^1(M, \mathbb{R})^*$ by

$$G(x) \cdot c_i = \oint_{x_0}^{x} \tilde{\omega}_i \,,$$

where $\tilde{\omega}_i$ is the pullback of $\omega_i$ on $\widetilde{M}$. Let $F_\varepsilon : (M_\varepsilon, d_\varepsilon) \to H_1(M, \mathbb{R})$ be $F(x) := \varepsilon \, G(x)$.

**1.2. Proposition.** $\lim_{\varepsilon \to 0}(\widetilde{M}, \varepsilon d, F_\varepsilon) = H_1(M, \mathbb{R})$

In the homogenized or averaged problem we will have that the (limit) positions are in the configuration space $H_1(M, \mathbb{R})$ and the momenta $p$ and differentials $\partial_x u$ are in the dual of the configuration space $H_1^* = H^1(M, \mathbb{R})$.

This explains why the effective Lagrangian $\overline{L} = \beta : H_1(M, \mathbb{R}) \to \mathbb{R}$ is defined in the *homology* group $H_1(M, \mathbb{R})$ but the effective Hamiltonian is defined in the *cohomology* group $H^1(M, \mathbb{R})$.

**1.3. Theorem** (Contreras, Iturriaga, Siconolfi [8])**.**

*Let $M$ be a closed Riemannian manifold. Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian and $f_\varepsilon : (M_\varepsilon, d_\varepsilon) \to \mathbb{R}$ continuous functions such that $\lim_\varepsilon f_\varepsilon = f$ uniformly, with $f : H_1(M, \mathbb{R}) \to \mathbb{R}$ Lipschitz.*

*Let $\widetilde{H}$ be the lift of $H$ to $\widetilde{M}$ and let $v^\varepsilon$ be the solution to the problem*

$$\partial_t v^\varepsilon + \widetilde{H}\big(y, \tfrac{1}{\varepsilon}\partial_y v^\varepsilon\big) = 0,$$

$$v^\varepsilon(y, 0) = f_\varepsilon(y).$$

*Then the family $v^\varepsilon : \widetilde{M}_\varepsilon \times ]0, +\infty[ \to \mathbb{R}$ is equicontinuous and*

$$\lim_{\varepsilon \to 0} v^\varepsilon = u : H_1(M, \mathbb{R}) \to \mathbb{R}$$

*uniformly on compact sets of $H_1(M, \mathbb{R}) \times ]T_0, +\infty[$, for any $T_0 > 0$, where $u$ is the solution to*

$$\partial_t u + \overline{H}(\partial_x u) = 0,$$

$$u(x, 0) = f(x);$$

*and $\overline{H} : H^1(M, \mathbb{R}) \to \mathbb{R}$ is $\overline{H} = \alpha$ Mather's alpha function.*

**1.4. Subcovers.**

On abelian covers $\widehat{M}$ with Deck transformation group $\mathbb{D}$ of the form $\mathbb{D} = \mathbb{Z}^k \oplus \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_p}$ the limit $\lim_\varepsilon(\widehat{M}, \varepsilon d)$ will kill the torsion $\mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_p}$, as in figure 6. Thus we may restrict to free abelian covers with group of Deck transformation without torsion $\mathbb{D} = \mathbb{Z}^k$. These are sub covers of $\widetilde{M}$.
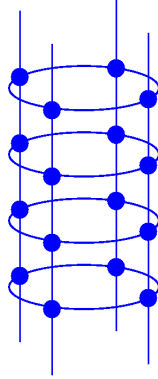
FIGURE 6. The limit process kills the torsion: $\varepsilon\left(\mathbb{Z}_4 \oplus \mathbb{Z}\right) \to \mathbb{R}$.

Using equivariance properties of the Hamilton-Jacobi equation, we obtain as a corollary of Theorem 1.3 a similar result for other free abelian covers.
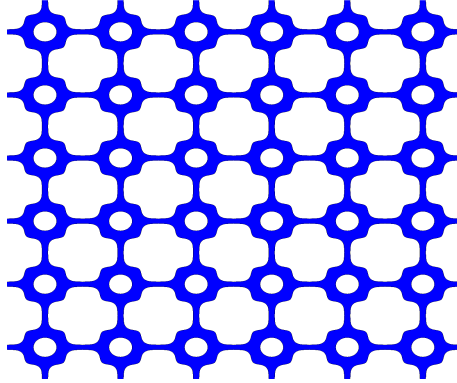


FIGURE 7. $\mathbb{Z}^2$-cover of the surface $M_3$ of genus 3. In this case $H_1(M_3, \mathbb{R}) = \mathbb{R}^3$.

## 1.5. **Speculations.**

There are generalizations of Aubry-Mather theory which can be interpreted as a homogenization besides $\mathbb{T}^n$ or $\mathbb{Z}^n$ and should give results in the setting presented above. On a generalization originated by Moser [17], Caffarelli, de la Llave and Valdinocci extend Aubry-Mather theory to higher dimensions on very general manifolds, see [11, remark 2.6], [5], [4]. There is also an extension by Candel and de la Llave [6] of the Aubry-Mather theory in statistical mechanics to configuration sets more general than $\mathbb{Z}^n$. Viterbo's symplectic homogenization [19] has also been extended to general manifolds by Monzner, Vichery and Zapolsky [16].

Most of the homogenization theory is made only for the torus $\mathbb{T}^n$. Some PDE's techniques go through this setting despite the destruction of the differential structure in the limit. For example in the homogenization of the Hamilton-Jacobi equation, Evans perturbed test function method goes through to give a proof of the same result.

The translation of homogenization results to manifolds can give interesting geometric objects. We have the following examples:

- The homogenization of the geodesic flow gives the stable norm.

    The stable norm was used by Burago and Ivanov in their proof of the Hopf conjecture [2]. Bangert [1, Th. 6.1] proves that a metric on $\mathbb{T}^2$ whose stable norm is euclidean is the flat metric on $\mathbb{T}^2$. Osuna [18] proves that if $\mathbb{T}^n$ has the 1-dimensional and $(n-1)$-dimensional stable norms Euclidean then the metric is flat.

- The homogenization of the Hamilton-Jacobi equation gives Mather's alpha function or Mañé's critical value as the effective Hamiltonian.

    In this case the limiting object $\overline{H}(P)$ was known independently of homogenization and had many interesting characterizations besides homogenization: variational, ergodic, geometric, symplectic as in (i)–(viii).

Another example of a possible result is the homogenization of the Riemannian Laplacian. Let $M$ be a closed manifold and $\Omega \subset H_1(M, \mathbb{R})$ a domain. Let $f : \partial\Omega \to \mathbb{R}$ and $F : \Omega \to \mathbb{R}$ be continuous functions. Choose a basis $[\omega_i]$ for $H^1(M, \mathbb{R})$ and let $G_\varepsilon : \widetilde{M}_\varepsilon \to H_1(M, \mathbb{R})$ be

$$G_\varepsilon(x) \cdot [\omega_i] = \varepsilon \oint_{x_0}^x \omega_i.$$

Let $v_\varepsilon$ be the solution to the problem

$$\Delta v_\varepsilon = F \circ G_\varepsilon \qquad \text{on} \quad G_\varepsilon^{-1}(\Omega),$$

$$v_\varepsilon = f \circ G_\varepsilon \qquad \text{on} \quad \partial G_\varepsilon^{-1}(\Omega).$$

Prove that $v_\varepsilon \to u$ where

$$\sum_{ij} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = F \qquad \text{on} \quad \Omega,$$

$$u = f \qquad \text{on} \quad \partial\Omega.$$

In this homogenized Laplacian we should have that

$$A_{ij} = \int_M \langle \eta_i(x), \eta_j(x) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the induced inner product in $T^*M$ and $\eta_i$ is the harmonic 1-form in the class $[\omega_i]$.
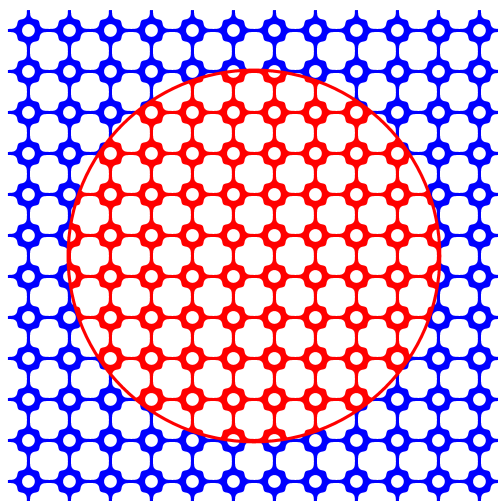
FIGURE 8. Homogenization of the Riemannian Laplacian.

Other questions can be:

- Homogenization of the eigenvalue problem for the Riemannian Laplacian.
- Probabilistic proofs of the homogenization of the Laplacian.
- Homogenization of the discretization of the Laplacian on graphs.
- Does it always give the same effective Laplacian?
  Also for the wave and heat equations?
- What about quasi-periodic arrays of manifolds?
- What about non-abelian covers?

For non-abelian covers we have some work in progress with Alfonso Sorrentino. The Gromov-Hausdorff tangent cone of the covering [3] should give the effective space.

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