

A GENERIC PROPERTY OF FAMILIES OF LAGRANGIAN SYSTEMS

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- M compact manifold, $\partial M = \emptyset$.
- $\mathbb{T} = (\mathbb{R}/\mathbb{Z}, +)$ or $(\{0\}, +)$.

A **Tonelli lagrangian** is a C^2 function $L : \mathbb{T} \times TM \rightarrow \mathbb{R}$ such that

- (1) *Convex* in fibers of $\mathbb{T} \times TM \rightarrow \mathbb{T} \times M$.
- (2) fiberwise *superlinear* $\lim_{|\theta| \rightarrow \infty} \frac{L(t, \theta)}{|\theta|} = +\infty, \quad (t, \theta) \in \mathbb{T} \times TM$.
- (3) The E-L. eq. gives a *complete* flow:

$$\frac{d}{dt} L_v = L_x, \quad \varphi : \mathbb{R} \times (\mathbb{T} \times TM) \rightarrow \mathbb{T} \times TM.$$

$L : \mathbb{T} \times TM \rightarrow \mathbb{R}$ is a **strong** Tonelli Lagrangian if

$\forall u \in C^\infty(\mathbb{T} \times M, \mathbb{R}) : L+u$ is a Tonelli Lagrangian (i.e. complete).

For simplicity: autonomous lagrangian $\mathbb{T} = (\{0\}, +)$.

Minimizing measures

- $\mathcal{P}(L)$ = Borel prob. meas. on TM invariant under E-L flow.
- Action: $A_L : \mathcal{P}(L) \rightarrow \mathbb{R} \cup \{+\infty\}$.

$$A_L(\mu) = \langle L, \mu \rangle = \int_{TM} L \, d\mu.$$

$$\mathcal{M}(L) = \arg \min_{\mu \in \mathcal{P}(L)} A_L(\mu) = \text{minimizing measures.}$$

- The ergodic components of a minimizing measure are also minimizing and they are all **mutually singular**

$\implies \mathcal{M}(L) = \text{simplex}$ whose extremal points are the ergodic minimizing measures.

i.e. *Every ergodic minimizing measure gives a linearly independent direction in $\mathcal{M}(L)$.*

Example:

- If $\mu_0, \mu_1, \mu_2, \mu_3$ are different ergodic minimizing measures.
- Take A_{ij} such that $\mu_i(A_{ij}) = 1, \mu_j(A_{ij}) = 0$.
- If $\nu = a_0 \mu_0 + \dots + a_3 \mu_3$, then

$$\nu(A_{01} \cap A_{02} \cap A_{03}) = a_0.$$

$$\nu(A_{10} \cap A_{12} \cap A_{13}) = a_1.$$

- i.e. the measure ν determines its coefficients $a_0, \dots, a_3$.

□

In general we may have $\dim \mathcal{M}(L) = +\infty$.

But not in the "generic" case:

Theorem. $\mathbb{A} =$ *finite dimensional convex family of (strong) Tonelli Lagrangians.*

$\implies$

$\exists$ *residual* $\mathcal{O} \subset C^\infty(\mathbb{T} \times M, \mathbb{R})$ s.t.

$$u \in \mathcal{O}, \quad L \in \mathbb{A} \implies \dim \mathcal{M}(L - u) \leq \dim \mathbb{A},$$

i.e. $L - u$ has at most $1 + \dim \mathbb{A}$ ergodic minimizing measures.

APPLICATIONS

Definition. A property $\mathbb{P}$ is **generic** for Lagrangians if

$\forall$ strong Tonelli Lagrangian $L : \mathbb{T} \times TM \rightarrow \mathbb{R}$

$\exists$ residual set $\mathcal{O}_L \subset C^\infty(\mathbb{T} \times M, \mathbb{R})$ s.t.

$\mathbb{P}$ holds for $L - u, \quad \forall u \in \mathcal{O}_L$.

Corollary 1 (Mañé). $\mathbb{A} = \{pt\}$.

A generic Lagrangian has a unique minimizing measure.

λ closed 1-form on $M \implies$

$\mathcal{M}(L - \lambda)$ depends only on the cohom. class $c = [\lambda]$.

$$\mathcal{M}(L - c) := \mathcal{M}(L - \lambda).$$

Corollary 2. *A generic lagrangian satisfies:*

$\forall c \in H^1(M, \mathbb{R})$ the Lagrangian $L - c$ has at most

$1 + \dim H^1(M, \mathbb{R})$ ergodic minimizng measures.

Quotient Aubry set.

- “Quotient Aubry set $\overline{\mathcal{A}} = \{ \text{static classes} \}$ ”.
- static classes are disjoint subsets of TM .
- Each static class supports at least one ergodic minimizing measure”.

Mather results:

$$(1) \begin{matrix} No \\ sing. \end{matrix} + \left\{ \begin{array}{ll} \mathbb{T} = \mathbb{R}/\mathbb{Z} & \& \dim M \leq 2 \\ & or \\ \mathbb{T} = \{0\} & \& \dim M \leq 3 \end{array} \right\} \implies \begin{matrix} quotient \\ set \end{matrix} \text{ Aubry is } \begin{matrix} Totally \\ disconnected. \end{matrix}$$

(2) $\exists$ examples whose quotient Aubry set $\approx$ interval.

Corollary 3. *For a generic Lagrangian:*

$$\forall c \in H^1(M, \mathbb{R})$$

the quotient Aubry set $\mathcal{A}_c$ of $L - c$ satisfies

$$\#\mathcal{A}_c \leq 1 + \dim H^1(M, \mathbb{R}).$$

Mather Questions:

- RIO 1999. For a generic lagrangian:
is $HD(\mathcal{A}_c) = 0$ for all $c \in H^1(M, \mathbb{R})$?
- PALO ALTO 2003. Is $\mathcal{A}_c$ totally disconnected?

Closed measures (with compact support).

$\mathcal{P}(TM)$ = Borel probabilities on TM .

OBS: If $f : M \rightarrow \mathbb{R}$ then $df : TM \rightarrow \mathbb{R}$.

Closed probabilities $\mathcal{C}$:

$$\mathcal{C} := \left\{ \nu \in \mathcal{P}(TM) : \int_{TM} df \, d\nu = 0, \quad \forall f \in C^1(M, \mathbb{R}) \right\}$$

$B > 0$:

$$\mathcal{C}_B := \left\{ \nu \in \mathcal{C} : \nu(|v| > B) = 0 \right\}.$$

Theorem (Mañé & Fathi-Siconolfi).

If μ minimizes the action A_L on $\bigcup_{B>0} \mathcal{C}_B$,

then $\mu \in \mathcal{M}(L)$.

$\mathcal{P}(M)$ = Borel probabilities on M with weak* topology,
(is a compact metric sapce).

$$\pi : TM \rightarrow M,$$

$$\pi_* : \mathcal{C}_B \rightarrow \mathcal{P}(M), \quad \mathcal{M}_{\mathcal{C}_B}(L) := \arg \min_{\mathcal{C}_B} L,$$

$$\mathcal{M}_{\mathcal{P}(M)}(L, B) := \pi_*(\mathcal{M}_{\mathcal{C}_B}(L)).$$

Proposition.

- $\mathbb{A}$ = *finite dimensional convex family of lagrangians on M .*
- $B > 0$.

$\implies$

$$\exists \text{ residual } \mathcal{O}(\mathbb{A}, B) \subset C^\infty(M, \mathbb{R}) \text{ s.t.}$$

$$L \in \mathbb{A}, \quad u \in \mathcal{O}(\mathbb{A}, B) \implies \dim_{\mathcal{P}(M)}(L, B) \leq \dim \mathbb{A}.$$

Prop $\implies$ Theo:

$$\mathcal{O}(\mathbb{A}) := \cap_{B \in \mathbb{N}} \mathcal{O}(\mathbb{A}, B) \text{ is residual in } C^\infty(M, \mathbb{R}),$$

$$\exists B_0 > 0 : \quad \mathcal{M}(L) = \arg \min_{\mathcal{C}_B} L = \mathcal{M}_{\mathcal{C}_B}(L), \quad \forall B > B_0.$$

$$\text{Mather's Graph Property} \left[\pi|_{\bigcup_{\mu \in \mathcal{M}(L)} \text{supp}(\mu)} \text{ is 1-1} \right]$$

$\implies$

$$\dim \mathcal{M}(L) = \dim \pi_*(\mathcal{M}(L)) = \dim \mathcal{M}_{\mathcal{P}(K)}(L, B) \quad \forall B > B_0.$$

□

Proof of the Prop:

- If $W \subset \mathcal{P}(M)$ define ε -nbhd

$$W_\varepsilon := \bigcup_{\mu \in W} B(\mu, \varepsilon) \quad \leftarrow \text{open } \varepsilon\text{-ball}$$

- $D \subset \mathbb{A}, \quad k \in \mathbb{N}, \quad \varepsilon > 0$

$$\mathcal{O}(D, \varepsilon, k) := \{ u \in C^\infty(M, \mathbb{R}) :$$

$$\forall L \in D \quad \text{convex } \mathcal{M}_{\mathcal{P}(M)}(L, B) \subset \left. \begin{array}{l} \varepsilon\text{-nbhd of some} \\ k\text{-dim convex} \\ \text{subset of } \mathcal{P}(M) \end{array} \right\}$$

- **Claim:** Prop. holds with

$$\mathcal{O}(\mathbb{A}, B) = \bigcap_{\varepsilon > 0} \mathcal{O}(\mathbb{A}, \varepsilon, \dim \mathbb{A}).$$

PROOF:

$$\text{If } u \in \mathcal{O}(\mathbb{A}, B) \implies \dim \mathcal{M}_{\mathcal{P}}(L, B) \leq \dim \mathbb{A}.$$

$$\text{Because if not} \quad \left[\dim \mathcal{M}_{\mathcal{P}}(L, B) \leq \dim \mathbb{A} \right]$$

$$\implies \exists L \in \mathbb{A} \text{ s.t. } \mathcal{M}_{\mathcal{P}}(L, B) \text{ contains a ball of dim } 1 + \dim \mathbb{A}.$$

$$\implies \mathcal{M}_{\mathcal{P}}(L, B) \not\subset \varepsilon\text{-nbhd of any } k\text{-dim convex.} \\ \text{if } \varepsilon \text{ is suff. small}$$

□

∴ Enough to prove that $\mathcal{O}(\mathbb{A}, B)$ is residual.

Enough $\forall D \subset \mathbb{A}$ cpct : $\mathcal{O}(D, \varepsilon, \dim \mathbb{A})$ open & dense.

OPEN:

$k \in \mathbb{N}^+, \varepsilon > 0, D \subset \mathbb{A}$ *cpct.*

$$\implies \mathcal{O}(D, \varepsilon, k) \text{ open in } C^\infty(M, \mathbb{R})$$

is a consequence of the semicontinuity of

$$\mathbb{A} \times C^\infty(M, \mathbb{R}) \ni (L, u) \longmapsto \mathcal{M}_{\mathcal{C}_B}(L - u) \subset \mathcal{C}_B.$$

DENSE:

$$\forall \varepsilon > 0, \quad \mathcal{O}(\mathbb{A}, \varepsilon, \dim \mathbb{A}) \text{ is dense in } C^\infty(M, \mathbb{R}).$$

Finite dimensional approximation:

Consider functions $w \in C^\infty(M, \mathbb{R})$ as

$$\begin{aligned} w : \mathcal{P}(M) &\longrightarrow \mathbb{R} \\ \mu &\longmapsto w(\mu) = \int_M w \, d\mu \end{aligned}$$

Lemma. $\exists m \in \mathbb{N}^+ \quad \exists T_m = (w_1, \dots, w_m) : \mathcal{P}(M) \rightarrow \mathbb{R}^m$
with $w_i \in C^\infty(M, \mathbb{R})$ s.t.

$$\forall x \in \mathbb{R}^m \quad \text{diam } T_m^{-1}\{x\} < \varepsilon.$$

PROOF:

Choose $\langle w_n \rangle_{n \in \mathbb{N}} \subset C^\infty(M, \mathbb{R})$ but dense in $C^0(M, \mathbb{R})$.

If the Lemma is false $\exists \mu_n, \nu_n, \quad T_n(\mu_n) = T_n(\nu_n), \quad d(\mu_n, \nu_n) \geq \varepsilon,$

$$\mu_n \rightarrow \mu, \quad \nu_n \rightarrow \nu, \quad \text{but} \quad \int w_n \, d\mu = \int w_n \, d\nu, \quad \forall n.$$

$$\implies d(\mu, \nu) \geq \varepsilon \quad \& \quad \int w \, d\mu = \int w \, d\nu, \quad \forall w \in C^\infty(M, \mathbb{R}).$$

$$(\implies \Leftarrow)$$

□

Fix $w \in C^\infty(M, \mathbb{R})$

want to prove $w \in \overline{\mathcal{O}(\mathbb{A}, \varepsilon, \dim \mathbb{A})}$

$$F_m : \mathbb{A} \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$$

$$\begin{aligned} F_m(L, x) &= \min_{\substack{\mu \in \mathcal{C}_B \\ T_m \circ \pi(\mu) = x}} (L - w)(\mu) & \text{if } x \in T_m(\mathcal{C}_B), \\ &= +\infty & \text{if } x \notin T_m(\mathcal{C}_B). \end{aligned}$$

For $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ let

$$M_m(L, y) := \arg \min_{x \in \mathbb{R}^m} [F(L, x) - y \cdot x] \subset \mathbb{R}^m$$

Obs.

- (1) $\mathcal{M}_{\mathcal{P}}(L - w - y_1 w_1 - \dots - y_m w_m, B) \subset T_m^{-1}(M_m(L, y)).$
- (2) $M_m(L, y) = \text{subdifferentials of the Legendre transform of } F_m.$

$$\mathcal{O}_m(w) := \{y \in \mathbb{R}^m : \forall L \in \mathbb{A} \quad \dim M_m(L, y) \leq \dim \mathbb{A}\}$$

Lemma $\implies$

$$y \in \mathcal{O}_m(w) \implies w + y_1 w_1 + \dots + y_m w_m \in \mathcal{O}(\mathbb{A}, \varepsilon, \dim \mathbb{A})$$

$\therefore$ For $w \in \overline{\mathcal{O}(\mathbb{A}, \varepsilon, \dim \mathbb{A})}$

enough to prove $0 \in \overline{\mathcal{O}_m(w)}$

Proposition. $\mathcal{O}_m(\varepsilon)$ is dense in $\mathbb{R}^m$.

PROOF: $G_m =$ Legendre transform of F_m .

$$\begin{aligned} G_m(L, y) &= \max_{x \in \mathbb{R}^m} y \cdot x - F_m(L, x), \\ &= \max_{\mu \in \mathcal{C}_B} \int (w + y_1 w_1 + \cdots + y_m w_m - L) d\mu. \end{aligned}$$

$$\therefore G_m \text{ convex, finite } \quad (\Rightarrow C^0 \text{ on } \mathbb{A} \times \mathbb{R})$$

in both L, y

$\partial G_m :=$ subdifferential of G_m

$$\tilde{\Sigma} := \{ (L, y) \in \mathbb{A} \times \mathbb{R}^m : \dim \partial G_m(L, y) \geq 1 + \dim \mathbb{A} \}.$$

Ambrosio + Alberti $\Rightarrow$

$$HD(\tilde{\Sigma}) \leq (m + \dim \mathbb{A}) - (1 + \dim \mathbb{A}) = m - 1.$$

$$\Sigma := \text{proj}_{\mathbb{R}^m}(\tilde{\Sigma})$$

$$\Rightarrow HD(\Sigma) \leq m - 1 \quad \Rightarrow \quad \mathbb{R}^m \setminus \Sigma \text{ dense in } \mathbb{R}^m.$$

$$\text{ENOUGH: } y \notin \Sigma \quad \Rightarrow \quad y \in \mathcal{O}_m(w).$$

i.e.

$$\text{ENOUGH: } y \notin \Sigma \quad \Rightarrow \quad \forall L \in \mathbb{A} : \dim M_m(L, y) \leq \dim \mathbb{A}.$$

$$\text{def } \Sigma \quad \Rightarrow \quad \left[y \notin \Sigma, L \in \mathbb{A} \Rightarrow \dim \partial G_m(L, y) \leq \dim \mathbb{A} \right]$$

$$\therefore \text{ enough to prove } \dim M_m(L, y) \leq \dim \partial G_m(L, y).$$

$$\text{True because } M_m(L, y) \subseteq \partial_y G_m(L, y).$$

∂_y partial differential

□

Singular sets of convex functions.

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ *convex*

subdifferential at $x \in \mathbb{R}^n$ is

$$\partial f(x) := \{ \ell : \mathbb{R}^n \rightarrow \mathbb{R} : f(y) \geq f(x) + \ell(y-x), \forall y \in \mathbb{R}^n \}.$$

$\partial f(x) \subset \mathbb{R}^n$ *is convex*

$$k \in \mathbb{N}, \quad \Sigma_k(f) := \{ x \in \mathbb{R}^n : \dim \partial f(x) \geq k \}.$$

Proposition.

$$\left. \begin{array}{l} f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ convex} \\ 0 \leq k \leq n \end{array} \right\} \implies HD(\Sigma_k(f)) \leq n - k.$$

Obs.

*Adding $|x|^2$ if necessary (which does not change Σ_k)
can assume f superlinear and*

$$f(y) \geq f(x) + \ell(y-x) + \frac{1}{2} |y-x|^2, \quad \forall x, y \in \mathbb{R}^n, \quad \forall \ell \in \partial f(x).$$

Lemma.

$$\ell \in \partial f(x), \quad \ell' \in \partial f(x') \implies |x - x'| \leq \|\ell - \ell'\|.$$

Obs. f *superlinear* $\implies \partial f$ *surjective*.

Corollary. $\exists$ Lipschitz function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ s.t.

$$\ell \in \partial f(x) \implies x = F(\ell).$$

Proof of the Proposition.

Let $A_k \subset \mathbb{R}^n$ be a set s.t.

- $HD(A_k) = n - k$.
- A_k intersects any convex subset of dimension k .

┌ For example:

$$A_k = \{ x \in \mathbb{R}^n : x \text{ has at least } k \text{ rational coordinates} \}.$$

└

$$x \in \Sigma_k \implies \partial f(x) \text{ intersects } A_k \implies x \in F(A_k).$$

$$\therefore \Sigma_k \subset F(A_k)$$

$$F \text{ Lipschitz} \implies HD(\Sigma_k) \leq HD(A_k) = n - k.$$

□

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