A generic property of **families** of lagrangian systems

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Setting.

- M compact manifold, $\partial M = \emptyset$.
- $\mathbb{T} = (\mathbb{R}/_{\mathbb{Z}}, +)$ or $(\{0\}, +)$.

A **Tonelli lagrangian** is a C^2 function $L : \mathbb{T} \times TM \to \mathbb{R}$ such that

1. *Convex* in fibers of $\mathbb{T} \times TM \to \mathbb{T} \times M$.
2. fiberwise *superlinear*
   $\lim_{|\theta|\to\infty} \frac{L(t,\theta)}{|\theta|} = +\infty, \qquad (t, \theta) \in \mathbb{T} \times TM.$
3. The E-L. eq. gives a *complete* flow:

$$\frac{d}{dt} L_v = L_x, \qquad \varphi : \mathbb{R} \times (\mathbb{T} \times TM) \to \mathbb{T} \times TM.$$

$L : \mathbb{T} \times TM \to \mathbb{R}$ is a **strong** Tonelli Lagrangian if

$\forall u \in C^\infty(\mathbb{T}\times M, \mathbb{R}) :\; L+u$ is a Tonelli Lagrangian (i.e. complete).

For simplicity: autonomous lagrangian $\mathbb{T} = (\{0\}, +)$.

Minimizing measures

- $\mathcal{P}(L) =$ Borel prob. meas. on *TM* invariant under E-L flow.
- Action: $A_L : \mathcal{P}(L) \to \mathbb{R} \cup \{+\infty\}$.

$$A_L(\mu) = \langle L, \mu \rangle = \int_{TM} L \, d\mu.$$

$$\mathcal{M}(L) = \underset{\mu \in \mathcal{P}(L)}{\arg\min} \; A_L(\mu) \quad = \text{ minimizing measures.}$$

- The ergodic components of a minimizing measure are also minimizing and they are all **mutually sigular**

$\Longrightarrow \quad \mathcal{M}(L) =$ **simplex** whose extremal points are the ergodic minimizing measures.

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Every ergodic minimizing measure gives a linearly independent direction in $\mathcal{M}(L)$.

Example:

- If μ_0, μ_1, μ_2, μ_3 are different ergodic minimizing measures.
- Take A_{ij} such that $\mu_i(A_{ij}) = 1$, $\mu_j(A_{ij}) = 0$.
- If $\nu = a_0\,\mu_0 + \cdots + a_3\,\mu_3$, then

$$\nu(A_{01} \cap A_{02} \cap A_{03}) = a_0.$$

$$\nu(A_{10} \cap A_{12} \cap A_{13}) = a_1.$$

- i.e. the measure ν determines its coefficients $a_0, \ldots, a_3$.

In general we may have $\dim \mathcal{M}(L) = +\infty$.

But not in the "generic" case:

Theorem

$\mathbb{A}$ = *finite dimensional* **convex** *family of (strong) Tonelli Lagrangians.*

$\Longrightarrow$

$\exists$ *residual* $\mathcal{O} \subset C^\infty(\mathbb{T} \times M, \mathbb{R})$ *s.t.*

$$u \in \mathcal{O}, \quad L \in \mathbb{A} \quad \Longrightarrow \quad \dim \mathcal{M}(L-u) \leq \dim \mathbb{A},$$

i.e. $L - u$ *has at most* $1 + \dim \mathbb{A}$ *ergodic minimizing measures.*

Applications

Definition

A property $\mathbb{P}$ is **generic** for Lagrangians if

$$\begin{gathered}\forall \text{ strong Tonelli Lagrangian } L : \mathbb{T} \times TM \to \mathbb{R} \\ \exists \text{ residual set } \mathcal{O}_L \subset C^\infty(\mathbb{T} \times M, \mathbb{R}) \text{ s.t.} \\ \mathbb{P} \text{ holds for } L - u, \quad \forall u \in \mathcal{O}_L.\end{gathered}$$

Corollary (Mañé)

$\mathbb{A} = \{pt\}$.

A generic Lagrangian has a unique minimizing measure.

λ closed 1-form on $M \implies$
$\mathcal{M}(L-\lambda)$ depends only on the cohom. class $c = [\lambda]$.

$$\mathcal{M}(L-c) := \mathcal{M}(L-\lambda).$$

Corollary

A generic lagrangian satisfies:

$\forall c \in H^1(M,\mathbb{R})$ *the Lagrangian* $L - c$ *has at most* $1 + \dim H^1(M,\mathbb{R})$ *ergodic minimizng measures.*

Applications: Quotient Aubry set

- "Quotient Aubry set $\overline{\mathcal{A}} = \{$ static classes $\}$".
- static classes are disjoint subsets of *TM*.
- Each static class supports at least one ergodic minimizing measure".

Mather

1. *No sing.* $+ \left\{ \begin{matrix} \mathbb{T} = \mathbb{R}/\mathbb{Z} & \& & \dim M \leq 2 \\ & \textit{or} & \\ \mathbb{T} = \{0\} & \& & \dim M \leq 3 \end{matrix} \right\} \Longrightarrow$

   quotient Aubry set *is* *Totally disconnected.*

2. $\exists$ examples whose quotient Aubry set $\approx$ interval.

Corollary

For a generic Lagrangian:

$$\forall c \in H^1(M, \mathbb{R})$$

the quotient Aubry set $\mathcal{A}_c$ of $L - c$ satisfies

$$\#\mathcal{A}_c \leq 1 + \dim H^1(M, \mathbb{R}).$$

Mather Questions:

- Rio 1999. For a generic lagrangian:
  is $HD(\mathcal{A}_c) = 0$ for all $c \in H^1(M, \mathbb{R})$?
- Palo Alto 2003. Is $\mathcal{A}_c$ totally disconnected?

Closed measures (with compact support).

$\mathcal{P}(TM)$ = Borel probablilities on TM.

OBS: If $f : M \to \mathbb{R}$ then $df : TM \to \mathbb{R}$.

Cosed probablilities $\mathcal{C}$:

$$\mathcal{C} := \{ \nu \in \mathcal{P}(TM) \ : \ \int_{TM} df \, d\nu = 0, \ \ \forall f \in C^1(M, \mathbb{R}) \}$$

$B > 0$:

$$\mathcal{C}_B := \{ \nu \in \mathcal{C} \ : \ \nu([|v| > B]) = 0 \}.$$

Theorem (Mañé & Fathi-Siconolfi)

If μ minimizes the action A_L on $\bigcup_{B>0} \mathcal{C}_B$,

then $\mu \in \mathcal{M}(L)$.

$\mathcal{P}(M)$= Borel probabilities on M with weak* topology,
(is a compact metric space).

$\pi : TM \to M,$

$\pi_* : \mathcal{C}_B \to \mathcal{P}(M),$

$$\mathcal{M}_{\mathcal{C}_B}(L) := \arg\min_{\mathcal{C}_B} L,$$

$$\mathcal{M}_{\mathcal{P}(M)}(L, B) := \pi_*\big(\mathcal{M}_{\mathcal{C}_B}(L)\big).$$

Proposition

- $\mathbb{A}$= *finite dimensional convex family of lagrangians on M.*
- $B > 0.$

$\Longrightarrow$

$$\exists \;\; \textit{residual} \;\; \mathcal{O}(\mathcal{A}, B) \subset C^\infty(M, \mathbb{R}) \;\; s.t.$$

$$L \in \mathbb{A}, \quad u \in \mathcal{O}(\mathbb{A}, B) \implies \dim_{\mathcal{P}(M)}(L, B) \leq \dim \mathbb{A}.$$

Prop $\Longrightarrow$ Teo:

$$\mathcal{O}(\mathbb{A}) := \cap_{B\in\mathbb{N}}\mathcal{O}(\mathbb{A},B) \text{ is residual in } C^\infty(M,\mathbb{R}),$$

$$\exists B_0 > 0 \;: \qquad \mathcal{M}(L) = \operatorname*{arg\,min}_{\mathcal{C}_B} L = \mathcal{M}_{\mathcal{C}_B}(L), \quad \forall B > B_0.$$

Mather's Graph Property $\left[\pi\Big|_{\bigcup_{\mu\in\mathcal{M}(L)}\text{supp}(\mu)} \text{ is 1–1}\right]$
$\Longrightarrow$

$$\dim \mathcal{M}(L) = \dim \pi_*\big(\mathcal{M}(L)\big) = \dim \mathcal{M}_{\mathcal{P}(K)}(L,B) \quad \forall B > B_0.$$

$\square$

Proof of the Prop:

- If $W \subset \mathcal{P}(M)$ define ε-nbhd

$$W_\varepsilon := \bigcup_{\mu \in W} B(\mu, \varepsilon) \quad \leftarrow \text{open } \varepsilon\text{-ball}$$

- $D \subset \mathbb{A}, \quad k \in \mathbb{N}, \quad \varepsilon > 0$

$$\mathcal{O}(D,\varepsilon,k) := \Big\{ u \in C^\infty(M,\mathbb{R}) \ : \quad \forall L \in D \quad \text{convex } \mathcal{M}_{\mathcal{P}(M)}(L,B) \subset \begin{array}{l} \varepsilon\text{-nbhd of some} \\ k\text{-dim convex} \\ \text{subset of } \mathcal{P}(M) \end{array} \Big\}$$

- **Claim:** *Prop. holds with*

$$\mathcal{O}(\mathbb{A}, B) = \bigcap_{\varepsilon > 0} \mathcal{O}(\mathbb{A}, \varepsilon, \dim \mathbb{A}).$$

Proof.

If $u \in \mathcal{O}(\mathbb{A}, B) \implies \dim \mathcal{M}_{\mathcal{P}}(L, B) \leq \dim \mathbb{A}$.

Because if not $\left[\dim \mathcal{M}_{\mathcal{P}}(L, B) \leq \dim \mathbb{A}\right]$

$\implies \exists L \in \mathbb{A}$ s.t. $\mathcal{M}_{\mathcal{P}}(L, B)$ contains a ball of dim $1 + \dim \mathbb{A}$.

$\implies \mathcal{M}_{\mathcal{P}}(L, B) \not\subset$ ε-nbhd of any k-dim convex.
if ε is suff. small

□

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□

$\therefore$ Enough to prove that $\mathcal{O}(\mathbb{A}, B)$ is residual.

Enough $\forall D \subset \mathbb{A}$ cpct : $\mathcal{O}(D, \varepsilon, \dim \mathbb{A})$ open & dense.

OPEN:

$k \in \mathbb{N}^+$, $\varepsilon > 0$, $D \subset \mathbb{A}$ cpct.

$$\Longrightarrow \quad \mathcal{O}(D, \varepsilon, k) \text{ open in } C^\infty(M, \mathbb{R})$$

is a consequence of the semicontinuity of

$$\mathbb{A} \times C^\infty(M, \mathbb{R}) \ni \quad (L, u) \longmapsto \mathcal{M}_{\mathcal{C}_B}(L - u) \quad \subset \mathcal{C}_B.$$

DENSE:

$$\forall \varepsilon > 0, \quad \mathcal{O}(\mathbb{A}, \varepsilon, \dim \mathbb{A}) \textit{ is dense in } C^\infty(M, \mathbb{R}).$$

Finite dimensonal approximation:

Consider functions $w \in C^\infty(M, \mathbb{R})$ as

$$\begin{array}{rccc} w: & \mathcal{P}(M) & \longrightarrow & \mathbb{R} \\ & \mu & \longmapsto & w(\mu) = \int_M w \, d\mu \end{array}$$

Lemma

$\exists \; m \in \mathbb{N}^+ \;\; \exists \; T_m = (w_1, \ldots, w_m) : \mathcal{P}(M) \to \mathbb{R}^m$

$$\textit{with } w_i \in C^\infty(M, \mathbb{R}) \textit{ s.t.}$$

$$\forall x \in \mathbb{R}^m \quad \operatorname{diam} T_m^{-1}\{x\} < \varepsilon.$$

Proof.

Choose $\langle w_n \rangle_{n\in\mathbb{N}} \subset C^\infty(M,\mathbb{R})$ but dense in $C^0(M,\mathbb{R})$.

If the Lemma is false $\quad \exists \mu_n,\, \nu_n, \quad T_n(\mu_n) = T_n(\nu_n)$,

$d(\mu_n,\nu_n) \geq \varepsilon$,

$$\mu_n \to \mu, \quad \nu_n \to \nu, \quad \text{but} \quad \int w_n\, d\mu = \int w_n\, d\nu, \ \ \forall n.$$

$$\Longrightarrow \quad d(\mu,\nu) \geq \varepsilon \quad \& \quad \int w\, d\mu = \int w\, d\nu, \quad \forall w \in C^\infty(M,\mathbb{R}).$$

$$(\Rightarrow\Leftarrow)$$

□

DENSE:

$$\forall \varepsilon > 0, \quad \mathcal{O}(\mathbb{A}, \varepsilon, \dim \mathbb{A}) \textit{ is dense in } C^\infty(M, \mathbb{R}).$$

Fix $w \in C^\infty(M, \mathbb{R})$

want to prove $\quad w \in \overline{\mathcal{O}(\mathbb{A}, \varepsilon, \dim \mathbb{A})}$.

$F_m : \mathbb{A} \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$

$$\begin{aligned} F_m(L, x) &= \min_{\substack{\mu \in \mathcal{C}_B \\ T_m \circ \pi(\mu) = x}} (L - w)(\mu) && \text{if} \quad x \in T_m(\mathcal{C}_B), \\ &= +\infty && \text{if} \quad x \notin T_m(\mathcal{C}_B). \end{aligned}$$

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For $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ let

$$M_m(L, y) := \underset{x \in \mathbb{R}^m}{\arg\min} \left[F(L, x) - y \cdot x \right] \quad \subset \mathbb{R}^m$$

Obs.

1. $\mathcal{M}_{\mathcal{P}}(L - w - y_1 w_1 - \cdots - y_m w_m, B) \subset T_m^{-1}(M_m(L, y))$.
2. $M_m(L, y) =$ subdiferentials of the Legendre transform of F_m.

$$\mathcal{O}_m(w) := \{ y \in \mathbb{R}^m \ : \ \forall L \in \mathbb{A} \quad \dim M_m(L, y) \leq \dim \mathbb{A} \}$$

Lemma $\Longrightarrow$

$$y \in \mathcal{O}_m(w) \implies w + y_1 w_1 + \cdots + y_m w_m \in \mathcal{O}(\mathbb{A}, \varepsilon, \dim \mathbb{A})$$

$\therefore$ For $\quad w \in \overline{\mathcal{O}(\mathbb{A}, \varepsilon, \dim \mathbb{A})}$ enough to prove $\quad 0 \in \overline{\mathcal{O}_m(w)}$.

Proposition

$\mathcal{O}_m(\varepsilon)$ is dense in $\mathbb{R}^m$.

Proof: G_m = Legendre transform of F_m.

$$G_m(L, y) = \max_{x \in \mathbb{R}^m} \; y \cdot x - F_m(L, x),$$
$$= \max_{\mu \in \mathcal{C}_B} \int \big(w + y_1 w_1 + \cdots + y_m w_m - L \big) \, d\mu.$$

$\therefore \quad \underset{\text{in both } L,\, y}{G_m \text{ convex, finite}} \qquad (\Rightarrow C^0 \text{ on } \mathbb{A} \times \mathbb{R})$

∂G_m:= subdifferential of G_m

$$\widetilde{\Sigma} := \left\{ (L, y) \in \mathbb{A} \times \mathbb{R}^m \; : \; \dim \partial G_m(L, y) \geq 1 + \dim \mathbb{A} \right\}.$$

Ambrosio + Alberti $\Longrightarrow$

$$HD(\tilde{\Sigma}) \leq (m + \dim \mathbb{A}) - (1 + \dim \mathbb{A}) = m - 1.$$

$\Sigma := \mathrm{proj}_{\mathbb{R}^m}(\widetilde{\Sigma})$

$$\implies \quad HD(\Sigma) \leq m-1 \quad \implies \quad \mathbb{R}^m \setminus \Sigma \text{ dense in } \mathbb{R}^m.$$

ENOUGH: $\quad y \notin \Sigma \quad \implies \quad y \in \mathcal{O}_m(w).$
i.e.
ENOUGH: $\quad y \notin \Sigma \quad \implies \quad \forall L \in \mathbb{A} : \quad \dim M_m(L,y) \leq \dim \mathbb{A}.$

$$\text{def } \Sigma \quad \implies \quad \left[\; y \notin \Sigma, \; L \in \mathbb{A} \implies \dim \partial G_m(L,y) \leq \dim \mathbb{A} \; \right]$$

$\therefore$ enough to prove $\quad \dim M_m(L,y) \leq \dim \partial G_m(L,y).$

True because $\quad M_m(L,y) \subseteq \partial_y G_m(L,y).$

☞partial differential

□

Singular sets of convex functions.

subdifferential at $x \in \mathbb{R}^n$ is

$$\partial f(x) := \{\, \ell : \mathbb{R}^n \to \mathbb{R} \;:\; f(y) \geq f(x) + \ell(y - x),\ \forall y \in \mathbb{R}^n \,\}.$$

$\partial f(x) \subset \mathbb{R}^n$ is convex

$$k \in \mathbb{N}, \quad \Sigma_k(f) := \{\, x \in \mathbb{R}^n \;:\; \dim \partial f(x) \geq k \,\}.$$

Proposition

$$\left.\begin{matrix} f : \mathbb{R}^n \to \mathbb{R} \textit{ convex} \\ 0 \leq k \leq n \end{matrix}\right\} \quad \Longrightarrow \quad HD\big(\Sigma_k(f)\big) \leq n - k.$$

Obs.
Adding $|x|^2$ if necessary (which does not change Σ_k) can assume f superlinear and

$$f(y) \geq f(x) + \ell(y-x) + \frac{1}{2}|y-x|^2, \qquad \forall x, y \in \mathbb{R}^n, \ \forall \ell \in \partial f(x).$$

Lemma

$$\ell \in \partial f(x), \quad \ell' \in \partial f(x') \quad \Longrightarrow \quad |x - x'| \leq \left\| \ell - \ell' \right\|.$$

Obs. $\qquad f$ superlinear $\quad \Longrightarrow \quad \partial f$ surjective.

Corollary
$\exists$ *Lipschitz function* $F : \mathbb{R}^n \to \mathbb{R}^n$ *s.t.*

$$\ell \in \partial f(x) \quad \Longrightarrow \quad x = F(\ell).$$

Proof of the Poposition.

Let $A_k \subset \mathbb{R}^n$ be a set s.t.

- $HD(A_k) = n - k$.
- A_k intersects any convex subset of dimension k.

For example:
$A_k = \{ x \in \mathbb{R}^n \ : \ x \text{ has at least } k \text{ rational coordinates} \}$.

$$x \in \Sigma_k \implies \partial f(x) \text{ intersects } A_k \implies x \in F(A_k).$$

$\therefore \Sigma_k \subset F(A_k)$

$$F \text{ Lipschitz} \implies HD(\Sigma_k) \leq HD(A_k) = n - k.$$

□