MAXIMIZING MEASURES FOR EXPANDING TRANSFORMATIONS.

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Abstract. Let $\sigma : \Sigma^+ \hookleftarrow$ be a one-sided subshift of finite type. We show that for a generic α-Hölder continuous function $A : \Sigma^+ \to \mathbb{R}$, the supremum

$$m(A) = \sup\{\textstyle\int A\, d\nu \,|\, \nu \text{ is a } \sigma\text{-invariant Borel probability }\}$$

is achieved by a unique invariant probability. In the set $\cup_{\beta>\alpha} C^\beta(\Sigma^+, \mathbb{R})$, with the C^α-topology, generically the maximizing measure is supported on a periodic orbit. This proves a version of a conjecture of R. Mañé. We also show that these maximizing measures can be obtained as weak limits of equilibrium states.

We apply these theorems to the class $\mathcal{F}_\lambda(\alpha)$ of $C^{1+\alpha}$ endomorphisms of the circle $f : S^1 \to S^1$ which are coverings of degree 2, expanding $f'(x) > \lambda > 1$, $\forall x \in S^1$ and orientation preserving. We prove that generically on $f \in \mathcal{F}_\lambda(\alpha)$, the invariant probability which maximizes the Lyapunov exponent $\int \log f'\, d\nu$ is unique, and that on $\cup_{\beta>\alpha}\mathcal{F}_\lambda(\beta)$ (with the $C^{1+\alpha}$-topology) this (unique) maximizing measure is suported on a periodic orbit.

Introduction.

Let $\sigma : \Sigma^+ \hookleftarrow$ be a one-sided topologically mixing subshift of finite type and $A : \Sigma^+ \to \mathbb{R}$ a Hölder continuous function. In this paper we are interested on σ-invariant probability measures μ which maximize the integral $\int A\, d\nu$ among all Borel σ-invariant probability measures.

Fix $0 < \lambda < 1$ and endow Σ^+ with the metric $d(\mathbf{x}, \mathbf{y}) = \frac{1}{\lambda^n}$, where $\mathbf{x}$, $\mathbf{y} \in \Sigma^+$, $\mathbf{x} = (x_0, x_1, \dots) \in \Sigma^+$, $\mathbf{y} = (y_0, y_1, \dots)$, and $n = \min\{k \geq 0 \,|\, x_k \neq y_k\}$. Let $0 < \alpha \leq 1$, given an α-Hölder function $A : \Sigma^+ \to \mathbb{R}$, write

$$\mathrm{Hold}_\alpha(A) = \sup_{0<d(x,y)\leq 1} \frac{|A(x) - A(y)|}{d(x,y)^\alpha}, \qquad \|A\|_0 = \sup_{x\in\Sigma^+} |A(x)|$$

and define the α-Hölder norm of A by

$$\|A\|_\alpha = \mathrm{Hold}_\alpha(A) + \|A\|_0 \,.$$

Denote by $C^\alpha(\Sigma^+, \mathbb{R})$ the set of α-Hölder continuous functions $A : \Sigma^+ \to \mathbb{R}$ endowed with the α-Hölder norm $\|\ \|_\alpha$. In view of applications, we shall restrict ourselves to

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Hölder functions with zero topological pressure. The results below hold also without this restriction. Denote by $C_0^\alpha(\Sigma^+,\mathbb{R})$ the subset of functions $A\in C^\alpha(\Sigma^+,\mathbb{R})$ which have zero topological pressure. We shall prove

Theorem A.
There is a generic set $\mathcal{G}_1\subseteq C_0^\alpha(\Sigma^+,\mathbb{R})$ such that if $A\in\mathcal{G}_1$ then A has a unique maximizing measure whose support is uniquely ergodic.

The problem we consider here is in some sense analogous to the problems considered in the Aubry-Mather theory (see [7], [12]) for Lagrangian flows. In particular our result is analogous to a recent result of Mañé [11] on Lagrangian flows, where he shows that generically (on the Lagrangian and on the homological position) there is a unique action minimizing measure. The main difference among these theories is that in the lagrangian setting the dynamics is defined by variational properties and hence minimizing properties imply invariance under the Euler-Lagrange flow. In our setting we have to impose somehow the invariance under the shift. The analogous to fix the homology in the Aubry-Mather theory in our setting is to consider side conditions, like $\int\psi_i\,d\nu=c_i$, $i=1,2,\ldots,k$, where $\psi_i\in C^\alpha(\Sigma^+,\mathbb{R})$ and $c_i\in\mathbb{R}$ are constants, in the maximization problem for A. We obtain the same results in this case because by means of the Legendre transform this problem is equivalent to maximizing $A+\sum_{i=1}^k x_i\,\psi_i$ for certain fixed values of $x_i\in\mathbb{R}$ (which depend on the c_i's).

In [12] and [13], R. Mañé conjectured that generically the unique minimizing measure is supported on a periodic orbit. In our case we can prove this conjecture in a subset of $C_0^\alpha(\Sigma^+,\mathbb{R})$ of functions which have slightly more regularity. Let $C_0^{\alpha+}(\Sigma^+,\mathbb{R})$ be the closure in the α-Hölder topology of $\cup_{\gamma>\alpha}C_0^\gamma(\Sigma^+,\mathbb{R})$. Given a periodic point $p\in\operatorname{Fix}\sigma^n\subset\Sigma^+$, let ν_p be the σ-invariant probability supported on the positive orbit of p.

Theorem B.
Let $\mathcal{G}_2\subset C_0^{\alpha+}(\Sigma^+,\mathbb{R})$ be the set of $A\in C_0^{\alpha+}(\Sigma^+,\mathbb{R})$ such that there is a unique minimizing measure which is supported on a periodic orbit and it is locally constant. Then $\mathcal{G}_2$ is open and dense in $C_0^{\alpha+}(\Sigma^+,\mathbb{R})$.

Here locally constant means that there is a periodic point $p\in\Sigma^+$ and a neighbourhood $\mathcal{U}\ni A$ such that for all $B\in\mathcal{U}$, the unique maximizing measure for B is ν_p, where ν_p is the σ-invariant probability supported on the positive orbit of p.

The techniques used to prove this theorem involve the analogous to the *action potential* defined by Mañé for Lagrangians in [13]. Here we define this potential by

$$S(x,y):=\lim_{\varepsilon\to0}\,\sup\Big\{\sum_{k=0}^{n}\big[A(\sigma^k z)-m_0\big]\,\Big|\,n>0,\ \sigma^n z=y,\ d(z,x)<\varepsilon\Big\},$$

for $x,\ y \in \Sigma^+$, where $m_0 = \inf\{\, \int A\, d\nu \,|\, \nu$ a σ-invariant Borel probability$\,\}$. In general, the function $S(x,y)$ is highly discontinuous (cf. proposition 3.5), but if e.g. x is in the support of a maximizing measure, then the function $y \mapsto S(x,y)$ is α-Hölder continuous. Writing $V(y) = S(x,y)$ in this case, it is staright forward from the definition of S that

$$V(\sigma y) \geq V(y) + A(y) - m_0 \tag{1}$$

for all $y \in \Sigma^+$. In the Lagrangian case this V corresponds to the existence of a subsolution of the Hamilton-Jacobi equation (cf. Fathi [5], [6], also [4]). Writing $B(y) = A(y) + V(y) - V(\sigma y)$, then $B(y)$ is α-Hölder and $\int B\, d\nu = \int A\, d\nu$ for any σ-invariant probability. Hence we can replace A by B in the maximization problem, with the advantage that $B \leq m_0 = \max_\nu \int B\, d\nu$. This implies that the inequality (1) in in fact an equality on the support of any minimizing measure. This, in turn, implies the

Coboundary Property.

The function A is cohomologous to a constant on the support of any maximizing measure by (the same) a Hölder continuous coboundary function, i.e. $A = V - V \circ \sigma + m_0$ *on* $\operatorname{supp}(\mu)$ *for any maximizing measure* μ*. In particular any any measure supported on a support of a maximizing measure is maximizing.*

In fact, the coboundary property can be extended to the set $\mathfrak{S} = \{\, x \in \Sigma^+ \,|\, W(x,x) = 0 \,\}$, which contains the support of all maximizing measures (cf. proposition 3.1 item [4]).

It is possible to construct examples in which there is a unique maximizing measure with positive entropy. In particular not supported on a periodic orbit.

There is an example in [8] of an invariant measure μ on the full 2-shift $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$ whose support is uniquely ergodic. If $A : \Sigma_2^+ \to \mathbb{R}$ is an α-Hölder function which attains its maximum exactly at $\operatorname{supp}(\mu)$, then μ is the unique maximizing measure for A. By adding a constant we can make $P(A) = 0$.

Another important ingredient in the proof of theorem B is the continuously varying support property, that we state now. Let $A \in C_0^\alpha(\Sigma^+, \mathbb{R})$ and μ a maximizing measure for A. We say that a sequence of probability measures ν_n *strongly converges* to a probability μ if $\nu_n \to \mu$ weakly* and $\operatorname{supp}(\nu_n) \to \operatorname{supp}(\mu)$ in the Hausdorff metric. We say that the pair (A,μ) has the *continuously varying support property* if given a sequence $A_n \to A$ in $C_0^\alpha(\Sigma^+, \mathbb{R})$, an maximizing measures μ_n for A_n, then μ_n strongly converges to μ.

Theorem C.

There is a dense subset $\mathcal{D} \subset C_0^{\alpha+}(\Sigma^+, \mathbb{R})$ *such that any* $A \in \mathcal{D}$ *has a unique maximizing measure* μ *and the pair* (A,μ) *has the continuously varying support property.*

Finally, we relate our maximization problem with the thermodynamic formalism. Let $A \in C^\alpha(\Sigma^+, \mathbb{R})$ and for each $t \in \mathbb{R}$ let $\widehat{\mu}_t$ be the equilibrium state for $t\,A$. The following proposition appeared in a slightly different form in [10]:

Proposition D.
Suppose that the maximizing measure μ_A for $A \in C^\alpha(\Sigma^+, \mathbb{R})$ is unique and $A > 0$. Then $\mu_A = \lim_{t \to +\infty} \widehat{\mu}_t$ in the weak topology.*

Recall the variational principle for the topological pressure $P(tA) = \max_\nu h(\nu) + t \int A\, d\nu$, where the maximum is along the σ-invariant probabilities, $P(tA)$ is the topological pressure and $h(\nu)$ is the metric entropy of ν. The result above shows that when $t \to +\infty$ in the variantional principle, one is putting less strength in the entropy of the measure and more stress in the integral of the measure. However, the integral-maximizing measures do not seem to inherit properties from the approximating equilibrium states.

Expanding maps of the circle.

We can apply the results above to concrete situations using symbolic dinamics. An example that motivated us is the case of invariant probabilities maximizing the Lyapunov exponent on a degree 2 expanding maps of the circle.

Consider the class $\mathcal{F} = \mathcal{F}_\lambda(\alpha)$ of $C^{1+\alpha}$ endomorphisms of the circle $f : S^1 \to S^1$ which are coverings of degree 2, expanding $f'(x) > \lambda > 1$, $\forall x \in S^1$ and orientation preserving. For a $C^{1+\alpha}$ endomorphism $f \in \mathcal{F}$, denote its $C^{1+\alpha}$ norm by

$$\|f\|_{1+\alpha} = \|f\|_{C^1} + \mathrm{Hold}_\alpha(f').$$

We say that an f-invariant Borel probability is a *Lyapunov maximizing measure* or simply a *maximizing measure* if it maximizes the integral

$$\int \log f'\; d\nu \tag{2}$$

among all f-invariant probabilities. The Lyapunov exponent of an invarinat measure ν is given by the integral (2). It expresses the mean value of the rate of expansiveness of points in the support of ν. We are looking for measure with maximal sensitivity dependence on initial conditions.

Theorem A1
Generically on the $C^{1+\alpha}$-topology for maps $f \in \mathcal{F}$, there exists a unique f-invariant Lyapunov maximizing measure μ_f. Moreover, the support of μ_f is uniquely ergodic.

Let $\mathcal{F}(\alpha+)$ be the closure of $\cup_{\beta>\alpha}\mathcal{F}_\lambda(\beta)$ in $\mathcal{F}$ (with the $C^{1+\alpha}$-topology).

Theorem B1

There is a generic set $\mathcal{G}_2 \subset \mathcal{F}(\alpha+)$ such that for $f \in \mathcal{G}_2$ there is a unique f-invariant Lyapunov maximizing measure and it is supported on a periodic orbit.

In section § 1 we prove theorem A. On section § 2 we show some preliminary shadowing lemmas. On section § 3 we define and prove the properties of the action potential and state the coboundary property. On section § 4 we prove theorem C. On section § 5 we prove theorem A. On section § 6 we prove theorem D. On section 7 we prove theorems A1 and B1 and give an equivalence between $C^{1+\alpha}$ expanding dynamics on the circle and α-Hölder maps on the shift.

1. Generic uniqueness of the maximizing measure.

In this section we prove theorem A. We start with nome notation. Denote by $K(\sigma)$ the set of σ-invariant Borel probabilities on Σ^+, endowed with the weak* topology. Given $A \in C_0^\alpha(\Sigma^+, \mathbb{R})$, set

$$m(A) = \max\big\{ \textstyle\int A\, d\nu \,\big|\, \nu \in K(\sigma) \big\}$$

$$\mathcal{M}(A) = \big\{ \mu \in K(A) \,\big|\, \textstyle\int A\, d\mu = m(A) \big\}$$

A measure μ in $\mathcal{M}(A)$ is called a *maximizing measure*.

The arguments in this section rely on Banach space techniques. Since $C_0^\alpha(\Sigma^+, \mathbb{R})$ is a Banach manifold, we need to do some conversions:

Proof of theorem A:

The topological pressure $P : C^\alpha(\Sigma^+, \mathbb{R}) \to \mathbb{R}$ is real analytic (cf. [14]). It is also a submersion because $P(A+r) = P(A)+r$ for any $r \in \mathbb{R}$. Hence the set $C_0^\alpha(\Sigma^+, \mathbb{R}) = P^{-1}\{0\}$ is a Banach manifold.

Given $A_0 \in C_0^\alpha(\Sigma^+, \mathbb{R})$, the derivative of the pressure at A_0 is given by $DP(A_0) \cdot B = \int B\, d\widehat{\mu}$, where $\widehat{\mu}$ is the equilibrium state for A_0 and $B \in C^\alpha(\Sigma^+, \mathbb{R})$ (see [11, corollary 1.4 and 1.7] or [14]). Hence the tangent space at A_0 to $C_0^\alpha(\Sigma^+, \mathbb{R})$ is the set of functions $A_0 + B$ where $B \in C^\alpha(\Sigma^+, \mathbb{R})$ and $\int B\, d\widehat{\mu} = 0$. This space does not intersect 0 because $\int A_0\, d\widehat{\mu} = -$entropy of $\widehat{\mu}$.

Locally, near A_0, there is a homeomorphism $\mathcal{Y}$ associating $A \in C_0^\alpha(\Sigma^+, \mathbb{R})$ to $A_0 + B$ $B \in T_{A_0} C_0^\alpha(\Sigma^+, \mathbb{R})$. This homeomorphism is of the form $\mathcal{Y}(A) = c_A\, A = A_0 + B$, where c_A is a positive constant. Therefore the maximizing measures for A or $A_0 + B = c_A\, A$ are the same. Thus to show the generic properties of maximizing measures for A or $A_0 + B = c_A\, A$ is the same problem.

So, we have to show that generically on functions B close to zero and such that $\int B\, d\widehat{\mu} = 0$, the function $A_0 + B$ has a unique maximizing measure. This is proven on theorem 1.1 below.

□

Fix a Borel probability measure $\widehat{\mu}$ on Σ^+ and $A_0 \in C_0^\alpha(\Sigma^+,\mathbb{R})$. Define

$$\mathcal{H} = \{\, A_0 + B \mid \int B\, d\widehat{\mu} = 0 \,\}$$

1.1. **Theorem.** *There exists a residual set $\mathcal{G}_1 \subset \mathcal{H}$ such that for all $A \in \mathcal{G}_1$, the set $\mathcal{M}(A) = \{\mu\}$ has a unique element. Moreover the support of μ is uniquely ergodic.*

Proof:

The proof will require two lemmas. The idea is to show that for any $\varepsilon > 0$, the open set

$$\mathcal{O}_\varepsilon = \{\, A \in \mathcal{H} \mid \operatorname{diam}\mathcal{M}(A) \le \varepsilon \,\}$$

is dense. Considering ε of the form $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$, we obtain from Baire's Theorem that there is a residual set whith a unique maximizing measure. Item [4] of proposition 3.1 implies that any invariant measure on the support of a maximizing measure is maximizing. Hence if there is a unique maximizing measure, its support must be uniquely ergodic.

Consider K_0 a subset of $K(\sigma)$ (the set of invariant measures for σ) and define

$$m_0(A) := \sup\{\, \int A\, d\nu \mid \nu \in K_0 \,\},$$

$$\mathcal{M}_0(A) := \big\{\, \mu \in K_0 \mid \int A\, d\mu = m_0(A) \,\big\}.$$

We say that u is an extremal point of the convex set C, if u is not a mid point of a segment where the endpoints are in C. A point p in the convex set C is said a strictly extremal point for C, if there exists a linear map h on the set E such that the supremum of h restricted to C is attained at p and only at p.

A classical result in convex analysis (see Strazewicz's Theorem in [16]) states that any extremal u can be approximated by a strictly extremal p.

1.2. **Lemma.**

If μ_0 is an extremal point of a compact set K_0, the for all $\varepsilon > 0$, there exists $w \in \mathcal{H}$ such that $\operatorname{diam}\mathcal{M}_0(w) < 2\,\varepsilon$ *and* $d(\mu_0, \mathcal{M}_0(w)) < \varepsilon$.

Proof:

Consider a sequence w_n, $n \in \mathbb{N}$, of functions in $\mathcal{H}$ that define a metric $\widetilde{d}$ on K_0 by

$$\widetilde{d}(\nu,\mu) = \sum_{j=1}^{\infty} \frac{1}{2^j} \left| \int w_j\, d\mu - \int w_j\, d\nu \right|,$$

compatible with the weak convergence on the compact space of probabilities on Σ^+. For each $n \in \mathbb{N}$, define $P_n : K_0 \to \mathbb{R}^n$ by

$$P(\mu) = \big(\int w_1\, d\mu, \dots, \int w_n\, d\mu \big).$$

From the definition of $\widetilde{d}$ and the compactness of K_0, it is easy to see that for all $\varepsilon > 0$, there exist $\delta > 0$ and $n > 0$ such that, if $S \subset \mathbb{R}^n$ and $\operatorname{diam} S < \delta$, then

$$\operatorname{diam}(P_n^{-1})(S) < \varepsilon. \tag{3}$$

Note that $u = P_n(\mu_0)$ is an extremal point of $P_n(K_0) = C$. From Strazewicz's Theorem applied to C, let p be a strictly extremal point such that $d(p, P_n(\mu_0)) < \delta$. Then by (3), we have that

$$\operatorname{diam}(P^{-1}(p), \mu_0) < \varepsilon. \tag{4}$$

Consider $w = \sum_{i=1}^n \lambda_i\, w_i$, where the λ_i are such that $h(x_1, x_2, \dots, x_n) = \sum_{i=1}^n \lambda_i\, x_i$. It is easy to see that $\mathcal{M}_0(w) = P^{-1}(p)$. Therefore, by (4), $\widetilde{d}(\mu_0, \mathcal{M}_0(w)) < \varepsilon$ and $\operatorname{diam}(\mathcal{M}_0(w)) < 2\,\varepsilon$. This shows the lemma.

$\square$

1.3. **Lemma.**

Suppose that $A \in \mathcal{H}$ and consider and extremal point μ_0 of $\mathcal{M}(A)$. Then for any neighbourhood U of μ_0 in $K(\sigma)$, and every $\varepsilon > 0$, there exists $A_1 \in U$ such that $d(\mu_0, \mathcal{M}(A_1)) < \varepsilon$ and $\operatorname{diam}\mathcal{M}(A_1) < 2\,\varepsilon$.

Proof:

Applying lemma 1.2 to $K_0 = \mathcal{M}(A)$, for any $\varepsilon > 0$ there exists w such that $\operatorname{diam}\mathcal{M}_0(w) < \varepsilon$ and $d(\mu_0, \mathcal{M}_0(w) < \varepsilon$. Let

$$m = m_0(w) = \sup\big\{ \textstyle\int w\, d\mu \,\big|\, \mu \in \mathcal{M}(A) \big\},$$
$$m_0 = m(A) = \sup\big\{ \textstyle\int A\, d\mu \,\big|\, \mu \in K(\sigma) \big\}.$$

Denote by f_0 and f_1 the functions defined on $\mu \in K(\sigma)$ by

$$f_0(\mu) = \int A\, d\mu - m_0,$$
$$f_1(\mu) = \int w\, d\mu - m.$$

Then

$$f_1(\mu) = 0, \qquad \text{for all } \mu \in \mathcal{M}_0(\mu), \tag{5}$$
$$f_0(\mu) = 0, \qquad \text{for all } \mu \in \mathcal{M}_0(\mu), \tag{6}$$

(because $\mathcal{M}_0(w) \subset K_0 = \mathcal{M}(A)$).

Reciprocally, observe that if $\mu \in \mathcal{M}(A) = K_0$ and if $f_1(\mu) = 0$, then

$$\mu \in \mathcal{M}_0(w). \tag{7}$$

For $\mu \in K(\sigma)$, if $f_0(\mu) = 0$, then

$$\mu \in K_0 = \mathcal{M}(A). \tag{8}$$

Observe that by the definition of m and m_0,

$$f_1(\mu) \leq 0 \quad \text{for all } \mu \in K_0 \quad \text{and} \quad f_0(\mu) \leq 0 \quad \text{for all } \mu \in K(\sigma). \tag{9}$$

Now define$f_\lambda = f_0 + \lambda\, f_1$ for all $\lambda > 0$. Let

$$m(\lambda) = \max_{\nu \in K(\sigma)} f_\lambda(\nu),$$

$$\mathcal{M}_\lambda = \{\, \mu \in K(\sigma) \,|\, f_\lambda(\mu) = m(\lambda)\, \}.$$

Observe that

$$m(\lambda) \geq 0, \tag{10}$$

because by (5) and (6), if $\mu \in \mathcal{M}_0(\mu)$, then $f_\lambda(\mu) = 0$.

Claim: $\lim_{\lambda\to 0}\ \operatorname{diam}(\mu_0, \mathcal{M}(\lambda)) < \varepsilon$.

If this claim is true, taking $A_1 = f_\lambda$ for λ small the lemma is proved.

Suppose that the claim is false. The there exist a sequence $\lambda_n \to 0$ and $\mu_n \in \mathcal{M}(\lambda_n)$ such that

$$\inf_{n\in\mathbb{N}} \widetilde{d}(\mu_n, \mu_0) \geq \varepsilon. \tag{11}$$

Consider a limit $\overline{\mu}$ of a subsequence of μ_n. Then by (11)

$$\widetilde{d}(\overline{\mu}, \mu_0) \geq \varepsilon. \tag{12}$$

If we prove that $\overline{\mu} \in \mathcal{M}_0(w)$, then from (12), we obtain a contradiction. Note that

$$f_1(\mu_n) \geq 0, \tag{13}$$

because from (10) and (9)

$$0 \leq m(\lambda_n) = f_0(\mu_n) + \lambda\, f_1(\mu_n) \leq \lambda_n\, f_1(\mu_n).$$

Note also that

$$\lim_{n\to\infty} m(\lambda_n) = \lim_{n\to\infty} \max_{\mu\in K(\sigma)} f_{\lambda_n}(\mu) = \max_{\mu\in K(\sigma)} f_0(\mu) = 0.$$

Since $f_{\lambda_n} = f_0(\mu_n) + \lambda_n\, f_1(\mu_n) = m(\lambda_n)$, by continuity,

$$f_0(\overline{\mu}) = f_0\big(\lim_{n\to\infty} \mu_n\big) = \lim_{n\to\infty} \big(m(\lambda_n) - \lambda_n\, f_1(\mu_n)\big) = 0.$$

Therefore by (8), $\overline{\mu} \in \mathcal{M}(A)$. Now, by (13), $f_1(\mu_n) \geq 0$ and then $f_1(\overline{\mu}) = \lim_{n\to\infty} \geq 0$. Since $\overline{\mu} \in K_0 = \mathcal{M}(A)$, then by (9), $f_1(\overline{\mu}) \leq 0$. Therefore $f_1(\overline{\mu}) = 0$. Finally, from (7) $\overline{\mu} \in \mathcal{M}_0(A)$. This contradicts $\widetilde{d}(\overline{\mu}, \mu_0) \leq \varepsilon$.

□

2. Shadowing Lemmas.

Let $\sigma : \Sigma^+ \hookleftarrow$ be a positive subshift of finite type. For $\mathbf{x}, \mathbf{y} \in \Sigma^+$, $\mathbf{x} = (x_0, x_1, \dots) \in \Sigma^+$, $\mathbf{y} = (y_1, y_2, \dots)$, write $d(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{1}}{\lambda^{\mathbf{n}}}$, where $n = \min\{k \geq 0 \,|\, x_k \neq y_k\}$. Let $\varepsilon_0 = \frac{1}{\lambda} > 0$, so that

[1] If $y = (y_0, y_1, \dots) \in \Sigma^+$ and $(x_0, y_0, y_1, \dots) \in \Sigma^+$ then the local inverse $\psi_{x_0}(z) = (x_0, z_0, z_1, \dots)$ is defined on all $\{\, z \,|\, d(z, y) < \varepsilon_0 \,\}$.

[2] If $x,\ y \in \Sigma^+$ and $d(x, y) < \varepsilon$, then $d(\sigma x, \sigma y) = \lambda\, d(x, y)$.

For $x \in \Sigma^+$ and $r > 0$ write $B(x, r) = \{\, y \in \Sigma^+ \,|\, d(x, y) < r \,\}$.

We say that a sequence $\{x_0, \dots, x_n\}$ is a *δ-pseudo-orbit with M jumps*, if $d(\sigma x_i, \sigma x_{i+1}) \leq \delta$ for all $0 \leq i \leq N-1$ and $\#\{\, 0 \leq i \leq N-1 \,|\, x_{i+1} \neq \sigma(x_i) \,\} = M$. We say that a δ-pseudo-orbit $\{\, x_0, \dots, x_N \,\}$ is *ε-shadowed by $p \in \Sigma^+$*, if $d(\sigma^k p, x_k) < \varepsilon$ for all $0 \leq k \leq N$.

2.1. **Lemma.** *Let $\varepsilon_1 := (1 - \lambda^{-1})\, \varepsilon_0$. For all $A : \Sigma^+ \to \mathbb{R}$ α-Hölder continuous, there exists $K_1 = K(A, \lambda) > 0$ such that if $0 < \delta < \varepsilon_1$ and $\{\, x_0, \dots, x_N \,\}$ is a δ-pseudo orbit with M jumps, then there exists $p \in \Sigma^+$ that $\big(\frac{\delta}{1 - \lambda^{-1}}\big)$-shadows $\{x_i\}_{i=1}^N$ and for all $0 \leq i \leq j \leq N$*

$$\left| \sum_{k=i}^{j} A(\sigma^k p) - \sum_{k=i}^{j} A(x_k) \right| \leq M\, K_1\, \delta^\alpha.$$

Moreover,

[1] *The point p can be taken such that $\sigma^N(p) = x_N$.*

[2] *If the pseudo-orbit is periodic (i.e. $x_N = x_0$), then the point p can be taken N-periodic: $\sigma^N(p) = p$.*

Proof:

For $1 \leq n \leq N$ let $\varphi_n : B(x_n, \varepsilon_0) \to \Sigma^+$ be the branch of the inverse of σ such that $\varphi_n(\sigma x_{n-1}) = x_{n-1}$. Then $\psi_N := \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_N$ is a contraction with Lipschitz constant $\lambda^{-N} < 1$. Moreover, $\varphi_n(B(x_n, r)) \subseteq \varphi_n(B(\sigma x_{n-1}, r + \delta)) \subseteq B(x_{n-1}, r)$ for $r = \frac{\delta}{\lambda - 1}$, $r + \delta < \varepsilon_0$. [This gives $\delta < (1 - \lambda^{-1})\, \varepsilon_0 =: \varepsilon_1$.] In particular $\psi_N(B(x_N, r)) \subseteq B(x_0, r)$.

[1] Let $p = \psi_N(x_N) \in B(x_0, r)$.

[2] Let $p \in B(x_0, r) = B(x_n, r)$ be the fixed point of ψ_N.

Then $d(\sigma^k p, x_k) \leq r = \frac{\delta}{\lambda - 1}$.

Let $0 < a_1 < a_2 < a_3 < \dots < a_M \leq N$ be the indices such that $\sigma(x_{a_i}) \neq x_{a_i + 1}$. Let $a_0 = 0$, $a_{M+1} = N$ and $b_i = a_{i+1} - a_i$, $0 \leq i \leq M$. Then, for $0 \leq j < b_i$, we have that

$$\begin{aligned} d(\sigma^{a_i + j} p, x_{a_i + j}) &\leq \lambda^{j - b_i}\, d(\sigma^{a_{i+1}} p, \sigma^{b_i} x_{a_i}) \\ &\leq \lambda^{j - b_i} \left[d(\sigma^{a_{i+1}} p, x_{a_{i+1}}) + d(\sigma(x_{a_{i+1} - 1}), x_{a_{i+1}}) \right] \\ &\leq \lambda^{j - b_i} \left[\tfrac{\delta}{\lambda - 1} + \delta \right] = \lambda^{j - b_i}\, \tfrac{1}{1 - \lambda^{-1}}\, \delta. \end{aligned}$$

$$\begin{aligned}\Big|\sum_{k=i}^{j} A(\sigma^k p) - \sum_{k=i}^{j} A(x_k)\Big| &\le \sum_{k=0}^{N-1} \big|A(\sigma^k p) - A(x_k)\big| \\ &\le \sum_{k=0}^{M}\sum_{j=0}^{b_k-1} \mathrm{Hold}_\alpha(A)\, \lambda^{-j\alpha}\, \tfrac{1}{(1-\lambda^{-1})^\alpha}\, \delta^a \\ &\le (M+1)\, \mathrm{Hold}_\alpha(A)\, \tfrac{1}{1-\lambda^{-\alpha}}\, \tfrac{1}{(1-\lambda^{-1})^\alpha}\, \delta^\alpha.\end{aligned}$$

□

3. The Action Potential.

Given x,$y \in \Sigma^+$ and $\delta > 0$, define

$$S_\delta(x,y) := \sup\left\{ \sum_{k=0}^{n-1} \big[A(\sigma^k(z) - m_0\big] \;\Big|\; \sigma^n z = y,\ d(z,x) < \delta \right\},$$

where $m_0 = m(A)$. Since Σ^+ is topologically transitive, the backward orbit of any point $y \in \Sigma^+$ is dense in σ. Hence the set in the definition above is non-empty and thus $S_\delta(x,y) > -\infty$ for any $\delta > 0$. We will show below that $\sup_{x,y\in\Sigma^+,\delta>0} S_\delta(x,y) < +\infty$. Since the function $\delta \mapsto S_\delta(x,y)$ is increasing, we can define

$$S(x,y) = \lim_{\delta\to 0^+} S_\delta(x,y).$$

We get that $-\infty \le S(x,y) \le Q$. In fact the value $S(x,y) = -\infty$ is possible and in general the function $S(x,y)$ is highly discontinuous. We quote the properties of $S(x,y)$ in the following proposition:

3.1. **Proposition.**

[1] *There is $Q > 0$ such that $S_\delta(x,y) < Q$ for all x $y \in \Sigma^+$ and all $\delta > 0$.*

[2] *For all $x \in \Sigma^+$, $S(x,x) \le 0$.*

[3] *For all $x,\ y,\ z \in \Sigma^+$, $S(x,y) + S(y,z) \le S(x,z)$.*

[4] *Let*

$$\mathfrak{S} := \{\, x \in \Sigma^+ \mid S(x,x) = 0 \,\}.$$

Then $\mathfrak{S}$ is closed and foward invariant. A measure μ is maximizing if and only if $\mathrm{supp}(\mu) \subseteq \mathfrak{S}$. *In particular $\mathfrak{S} \ne \emptyset$.*

[5] *If $x \in \mathfrak{S}$ then the function $W : \Sigma^+ \to \mathbb{R}$, $W(y) = S(x,y)$ is finite and α-Hölder continuous with* $\mathrm{Hold}_\alpha(W) \le C(\lambda)\,\mathrm{Hold}_\alpha(A)$. *Moreover, $W(y) - W(x) \ge S(x,y)$ for all $x,\ y \in \Sigma^+$.*

3.2. **Corollary.**

[1] *The α-Hölder continuous function $B(x) := A(x) - m_0 + W(x) - W(\sigma x)$ satisfies $B \geq 0$, $\int B\, d\nu = \int A\, d\nu$ for any invariant measure and $\int B\, d\mu = 0$ for any maximizing measure.*

[2] *If μ is a maximizing measure, then any invariant measure ν with* $\operatorname{supp}(\nu) \subseteq \operatorname{supp}(\mu)$ $(\subset \mathfrak{S})$ *is maximizing. In particular if A has a unique maximizing measure, then the set $\mathfrak{S}$ (and hence also* $\operatorname{supp}(\mu)$*) is uniquely ergodic.*

Proof: Item [1] follows from 3.1 [5] because

$$W(\sigma x) - W(x) \geq S(\sigma x, x) \geq A(x).$$

□

A subset $K \subseteq \Sigma^+$ is said *ε-separated* if $d(x, y) > \varepsilon$ for all $x,\ y \in K$ with $x \neq y$. Given a periodic point $p \in \operatorname{Fix}(\sigma^n)$, let ν_p be the probability measure defined by

$$\int f\ d\nu_p = \frac{1}{n} \sum_{k=0}^{n-1} f(\sigma^k x),$$

for any continuous function $f : \Sigma^+ \to \mathbb{R}$.

Proof of proposition 3.1:

[1] Let $\varepsilon = \varepsilon_2$ from lemma 2.1. Let

$$M(\varepsilon) := \max\{\, B \subseteq \Sigma^+ \mid B \text{ is } \varepsilon\text{-separated}\,\}.$$

Let $N > M(\varepsilon)$ and $x \in \Sigma^+$. Let

$$k_0 = \max\{\, 0 \leq k \leq N \mid \{x, \sigma x, \dots, \sigma^k x\} \text{ is } \varepsilon\text{-separated}\,\},$$

$$k_1 = \max\{\, 0 \leq k \leq N \mid \{\sigma^k x, \sigma^{k+1} x, \dots, \sigma^N x\} \text{ is } \varepsilon\text{-separated}\,\}.$$

Then $k_0 \leq M(\varepsilon)$ and $N - k_1 \leq M(\varepsilon)$. The set $\{\sigma^j x \mid 0 \leq j \leq k_0,\ k_1 \leq j \leq N\,\}$ is not ε-separated. Hence there are $0 \leq i \leq k_0$, $k_1 \leq j \leq N$ such that $d(\sigma^i x, \sigma^j x) < \varepsilon$. By lemma 2.1,

$$\sum_{k=i}^{j} \big[A(\sigma^k x) - m_0\big] \leq \sup_{p \in \operatorname{Fix}\sigma^n} n \int \big[A - m_0\big]\ d\nu_p + K\,\varepsilon^\alpha \leq K\,\varepsilon^a.$$

and

$$\sum_{k=1}^{N-1} \big[A(\sigma^k x) - m_0\big] \leq K\,\varepsilon^\alpha + 2\, M(\varepsilon)\ \|A - m_0\|_0\,,$$

for all $x \in \Sigma^+$ and *all* $N > 0$. Thus

$$S_\delta(x, y) \leq K\,\varepsilon^\alpha + 2\, M(\varepsilon)\ \|A - m_0\|_0$$

for all $\delta > 0$, $x,\ y \in \Sigma^+$. This implies item[1].

[2] If $0 < \delta < \varepsilon_2$, $d(x, y) < \delta$ and $\sigma^n y = x$, then by lemma 2.1,

$$\left|\sum_{k=0}^{n-1} \big[A(\sigma^k y) - m_0\big] - \sum_{k=0}^{n-1} \big[A(\sigma^k p) - m_0\big]\right| \le K\,(2\delta)^\alpha$$

for some periodic point $p \in \operatorname{Fix} \sigma^n$. Since $\int [A - m_0]\, d\nu_p \le 0$, then

$$\sum_{k=0}^{n-1} \big[A(\sigma^k y) - m_0\big] \le n \int \big[A - m_0\big]\, d\nu_p + K\,(2\delta)^\alpha \le K\,(2\delta)^a.$$

Hence $S_\varepsilon(x, y) \le K\,(2\delta)^a = K\,(2(1-\lambda)\varepsilon)^a$ for $e = \frac{\delta}{1-\lambda}$. Letting $\varepsilon \to 0$, we obtain that $S(x, x) \le 0$.

[3] Given $\delta > 0$ let $a,\ b \in \Sigma^+$ be such that $d(x, a) < \delta$, $\sigma^n a = y$; $d(y, b) < \delta$, $\sigma^m b = z$ for some $n,\ m > 0$ and

$$\sum_{k=0}^{n-1} \big[A(\sigma^k a) - m_0\big] \ge S_\delta(x, y) - \delta \tag{14}$$

$$\sum_{k=0}^{m-1} \big[A(\sigma^k b) - m_0\big] \ge S_\delta(y, z) - \delta \tag{15}$$

Then $\{a, \sigma(a), \dots, \sigma^{n-1}a, b, \dots, \sigma^m b = z\}$ is a 2δ-pseudo-orbit with 1 jump. By lemma 2.1, there is $p \in \Sigma^+$ which $\big[\frac{2\delta}{1-\lambda}\big]$-shadows the pseudo-orbit, $\sigma^{n+m}p = \sigma^m b = z$ and

$$\sum_{k=0}^{n+m-1} \big[A(\sigma^k p) - m_0\big] - \left[\sum_{k=0}^{n-1} \big[A(\sigma^k a) - m_0\big] + \sum_{k=0}^{m-1} \big[A(\sigma^k b) - m_0\big]\right] \ge -K\,(2\delta)^\alpha.$$

Since $d(x, p) \le d(x, a) + d(a, p) \le \delta\left[\frac{2}{1-\lambda} + 1\right] =: \varepsilon(\delta)$, and $\sigma^{n+m}p = z$, then, using (14) and (15), we have that

$$S_{\varepsilon(\delta)}(x, z) \ge [S_\delta(x, y) - \delta] + [S_\delta(y, z) - \delta] - 2^\alpha\, K\, \delta^\alpha.$$

Letting $\delta \to 0$, then $\varepsilon(\delta) \to 0$ and

$$S(x, z) \ge S(x, y) + S(y, z).$$

In order to prove item [5] we need the following

3.3. **Lemma.** *If $S(x, x) = 0$, then for all $\varepsilon > 0$ and $M > 0$ there exists $w \in \Sigma^+$ and $n > M$ such that $d(w, x) < \varepsilon$, $\sigma^n w = y$ and*

$$\sum_{k=0}^{n-1} \big[A(\sigma^k w) - m_0\big] \ge S(x, y) - \varepsilon. \tag{16}$$

Proof:

Let $\delta > 0$ be such that

$$M\,\delta + \delta + M\,K\,\delta^{\alpha} < \varepsilon,$$

$$\tfrac{\delta}{1-\lambda} + \delta < \varepsilon.$$

Let $a \in \Sigma^{+}$ and $n > 0$ be such that $d(a,x) < \delta$, $\sigma^{n} a = y$ and

$$\sum_{k=1}^{n-1} \big[A(\sigma^{k} a) - m_0\big] \geq S_{\delta}(x,y) - \delta \geq S(x,y) - \delta.$$

Since $S(x,x) = 0$ then there is $b \in \Sigma^{+}$ and $n > 0$ such that $d(b,x) < \delta$, $\sigma^{m} b = x$ and

$$\sum_{k=1}^{m-1} \big[A(\sigma^{k} b) - m_0\big] \geq S_{\delta}(x,x) - \delta \geq S(x,x) - \delta \geq -\delta.$$

The ordered set $\{b,\ldots,\sigma^{m-1}b\},\overset{M\text{ times}}{\ldots\ldots\ldots}\{b,\ldots,\sigma^{m-1}b\},\{a,\ldots,\sigma^{n}a\}$ is a 2δ-pseudo-orbit with M jumps. By lemma 2.1 there is $w \in \Sigma^{+}$ such that $d(b,w) < \frac{\delta}{1-\lambda}$, $\sigma^{mM+n} w = \sigma^{n} a = y$ and

$$\Bigg| \sum_{k=0}^{mM+n-1} \big[A(\sigma^{k} w) - m_0\big] - \Bigg\{ M \sum_{k=0}^{m-1} \big[A(\sigma^{k} b) - m_0\big] + \sum_{k=0}^{n-1} \big[A(\sigma^{k} a) - m_0\big] \Bigg\} \Bigg| \leq M\,K\,\delta^{\alpha}.$$

Then

$$\begin{aligned} \sum_{k=0}^{mM+n-1} \big[A(s^{k} w) - m_0\big] &\geq -M\,\delta + S(x,y) - \delta - M\,K\,\delta^{\alpha} \\ &\geq S(x,y) - \varepsilon. \end{aligned}$$

Moreover $d(w,x) \leq d(w,b) + d(b,x) \leq \frac{\delta}{1-\lambda} + \delta < \varepsilon$, $s^{mM+n} w = y$ and $mM + n > M$.

□

[5] Now we prove item [5]. Let $z,\ y \in \Sigma^{+}$ and $d(y,z) = d$ small. Given $\varepsilon > 0$ let $M = M(\varepsilon) > 0$ be such that $\lambda^{M}(\varepsilon + d) < \varepsilon$. Let $w \in \Sigma^{++}$ and $n > M(\varepsilon)$ be as in lemma 3.3. Since $d(\sigma^{n} w, z) \leq d(\sigma^{n} w, y) + d(y,z) \leq \varepsilon + d$, then the ordered set $\{w, \sigma w, \ldots, \sigma^{n-1} w, z\}$ is an $(\varepsilon + d)$-pseudo-orbit with 1 jump. By lemma 2.1 there exists $p \in \Sigma^{+}$ such that $s^{n} p = z$, $d(w,p) < \lambda^{n}(d + \varepsilon)$ and

$$\Bigg| \sum_{k=0}^{n-1} \big[A(\sigma^{k} p) - m_0\big] - \sum_{k=0}^{n-1} \big[A(\sigma^{k} w) - m_0\big] \Bigg| \leq K\,(d + \varepsilon)^{\alpha}. \tag{17}$$

Since $n > M(\varepsilon)$ we have that $d(p,x) \le d(p,w) + d(w,x) \le \varepsilon + \lambda^n(d+\varepsilon) < 2\varepsilon$. Then, using (17) and (16), we have that

$$S_{2\varepsilon}(x,z) \ge \sum_{k=0}^{n-1} \big[A(\sigma^k p) - m_0\big] \ge S(x,y) - \varepsilon - K\,(d+\varepsilon)^\alpha,$$

where $K = C(\lambda)\,\mathrm{Hold}_\alpha(A)$. Letting $\varepsilon \to 0$ we get that

$$S(x,z) \ge S(x,y) - K\,d^\alpha.$$

Interchanging the roles of y and z we obtain that

$$|W(y) - W(z)| = |S(x,y) - S(x,z)| \le K\,d^\alpha.$$

Now, by the triangle inequality, we have that

$$W(z) - W(y) = S(x,y) - S(x,y) \ge S(y,z).$$

[4] We now prove item [4]. We first prove that if μ is an invariant measure with $\mathrm{supp}(\mu) \subseteq \mathfrak{S}$ then it is maximizing. Fix $x \in \mathfrak{S}$ and define $W(y) = S(x,y)$ and $B(y) = A(y) - m_0 + W(y) - W(\sigma y)$. By item [4] we have that $W(\sigma y) - W(y) \ge S(y,\sigma y) \ge A(y)$. Hence $B(y) \le 0$ for all $y \in \Sigma^+$ and $\int B\,d\mu = \int (A - m_0)\,d\mu$.

To see that μ is maximizing, it is enough to show that $B \equiv 0$ on $\mathfrak{S}$. Let $y \in \mathfrak{S}$. Then $S(y,y) = 0$ and for any $\delta > 0$ there exists $z = z(\delta) \in \mathfrak{S}$ and $n > 0$ such that $d(z,y) < \delta$, $\sigma^n z = y$ and

$$\sum_{k=0}^{n-1} \big[A(\sigma^k z) - m_0\big] > S_\delta(y,y) - \delta.$$

Then

$$S_{\lambda\delta}(\sigma y, y) \ge \sum_{k=1}^{n-1} \big[A(\sigma^k z) - m_0\big] > S_\delta(y,y) - \delta A(z) + m_0.$$

Letting $\delta \to 0$, we get that

$$S(y,y) \le S(\sigma y, y) + A(y) - m_0 \le S(\sigma y, y) + S(y, \sigma y) \le S(y,y).$$

Thus $S(\sigma y, y) = -A(y) + m_0$ and $S(y,\sigma y) = A(y) - m_0$. Now

$$\begin{aligned} S(x,y) \ge S(x,\sigma y) + S(\sigma y, y) &= S(x,\sigma y) - A(y) + m_0 \\ &\ge S(x,y) + S(y,\sigma y) - A(y) - m_0 \ge S(x,y). \end{aligned}$$

Hence $S(x,y) - S(x,\sigma y) = -A(y) + m_0$, and then $B(y) = A(y) - m_0 + S(x,y) - S(x,\sigma y) = 0$.

Now we prove that if μ is a maximizing measure then $\mathrm{supp}(\mu) \subseteq \mathfrak{S}$. A proof of the following lemma is supplied below:

3.4. **Lemma.** *(Mañé [12])*

Let $(X, \mathcal{B}, \mu, f)$ be an ergodic measure preserving dynamical system and $F : X \to \mathbb{R}$ an integrable function. Given $A \in \mathcal{B}$ with $\mu(A) > 0$, denote by $\widehat{A}$ the set of points $x \in A$ such that for all $\varepsilon > 0$ there exists an integer $N > 0$ such that $f^N(x) \in A$ and

$$\Big| \sum_{k=0}^{N-1} F(f^k(x)) - N \int F \, d\mu \Big| < \varepsilon.$$

Then $\mu(A) = \mu(\widehat{A})$.

Let μ be a maximizing measure and $y \in \text{supp}(\mu)$. Let $\delta > 0$, $z \in \Sigma^+$ and $n > 0$ such that $d(y, z) < \delta$, $d(\sigma^n z, y) < \delta$ and

$$\sum_{k=0}^{n-1} \big[A(\sigma^k z) - m_0 \big] > -\delta.$$

The set $\{y, \sigma z, \sigma^2 z, \dots, \sigma^{n-1} z, y\}$ is a δ-pseudo-orbit with 2 jumps. By lemma 2.1, there is $w \in \Sigma^+$ with $d(w, y) < \frac{\delta}{1-\lambda}$, $\sigma^n w = y$ and

$$\Big| \sum_{k=0}^{n-1} \big[A(\sigma^k w) - m_0 \big] - \sum_{k=0}^{n-1} \big[A(\sigma^k z) - M_0 \big] \Big| \le 2\, K\, \delta^\alpha.$$

Hence

$$S_{\frac{\delta}{1-\lambda}}(y, y) \ge -\delta - 2\, K\, \delta^\alpha.$$

Letting $\delta \to 0$ we get that $S(y, y) = 0$.

□

Proof of Lemma 3.4:

We may assume that $\int F \, d\mu = 0$. For $\varepsilon > 0$ let

$$A(\varepsilon) := \big\{\, p \in A \;\big|\; \exists N > 0,\ f^N(p) \in A,\ \big| \textstyle\sum_{k=0}^{N-1} F(f^k p) \big| < \varepsilon \,\big\}.$$

Let $x \in A$ be a point such that Birkhoff's Theorem holds for F and the characteristic functions 1_A and $1_{A(\varepsilon)}$. It is enough to prove that $\mu(A(\varepsilon)) = \mu(A)$ because $\widehat{A} = \cap_{n>0} A(1/n)$.

Let $N_1 < N_2 < \cdots$ be the integers for which $F^{N_i}(x) \in A$. Define $\delta(k) \ge 0$ by $N_k \cdot \delta(k) = |\sum_{i=0}^{N_k - 1} F(f^i x)|$. Then $\lim_{k \to +\infty} \delta(k) = 0$.

Let $c_j := \sum_{i=0}^{N_j - 1} F(f^i x)$ and

$$S(k) := \big\{\, 1 \le j \le k-1 \,|\, \nexists \ell > j \text{ with } |c_\ell - c_j| < \varepsilon \,\big\}.$$

Then $\varepsilon \, \# S(k) \le 2\, \delta(k)\, N_k$.

If $j \notin S(k)$ then $|c_\ell - c_j| = |\sum_{N_j}^{N_\ell - 1} F(f^i x)| < \varepsilon$ for some $\ell > j$, hence $f^{N_j}(x) \in A(\varepsilon)$.

We have that

$$\frac{1}{N_k}\,\#\{\,0\le j<N_k\,\big|\,f^j(x)\in A\setminus A(\varepsilon)\,\}\le\frac{1}{N_k}\,\#S(k)$$
$$\le\frac{1}{N_k}\cdot\frac{2\delta(k)}{\varepsilon}\,N_k=\frac{2\,\delta(k)}{\varepsilon}\xrightarrow{\;k\;}0.$$

The choice of x implies that $\mu(A\setminus A(\varepsilon))=0$.

□

To give an idea of how discontinuous the functions $S(x,y)$ and $y\mapsto S(x,y)$ $(x\notin\mathfrak{S})$ may be, we show the following proposition:

3.5. **Proposition.**
Given $x\in\Sigma^+$ and $0<N\le\min\{k>0\,|\,\sigma^k(x)=x\,\}\le+\infty$, then

$$S(x,\sigma^N x)=\sum_{k=0}^{N-1}\big[A(\sigma^k x)-m_0\big]$$

and $S(x,x)=S(x,\sigma^N x)+S(\sigma^N x,x)$.

Proof:
Fix $x\in\Sigma^+$ and $N>0$ as in the statement of proposition 3.5. Let $\varepsilon>0$ be small and $0<\delta<\varepsilon$ such that if $d(z,x)<\delta$ then

$$d(\sigma^k z,\sigma^k x)<\varepsilon\qquad\text{for all } 0\le k\le N. \tag{18}$$

Let $w\in\Sigma^+$ and $M>0$ be such that $d(w,x)<\delta$, $\sigma^M w=x$ and

$$\sum_{k=0}^{M-1}\big[A(\sigma^k w)-m_0\big]\ge S_\delta(x,x)-\delta.$$

If $0<2\varepsilon<\min\{\,d(\sigma^i x,\sigma^j x)\,|\,0\le i<j\le N\,\}=:D$, then $M>N$ because for $0<k\le N$ we have that

$$d(\sigma^k w,\ x)\ge d(\sigma^k x,x)-d(\sigma^k x,\sigma^k w)>D-\varepsilon>\delta.$$

From (18), we have that

$$\sum_{k=0}^{N-1}\big[A(\sigma^k x)-m_0\big]\ge\sum_{k=0}^{N-1}\big[A(\sigma^k w)-m_0\big]-N\,K\,\varepsilon^\alpha,$$

where K is an α-Hölder constant for A. Then

$$\begin{aligned}
S(x,\sigma^N x) + S_\varepsilon(\sigma^N x, x) &\geq \sum_{k=0}^{N-1}\big[A(\sigma^k x) - m_0\big] + S_\varepsilon(\sigma^N x, x) \\
&\geq \sum_{k=0}^{N-1}\big[A(\sigma^k w) - m_0\big] - N\,K\,\varepsilon^\alpha + \sum_{k=N}^{M-1}\big[A(\sigma^k w) - m_0\big] \\
&\geq S_\delta(x,x) - \delta - N\,K\,\varepsilon^\alpha.
\end{aligned}$$

Letting $\varepsilon \to 0$, we have that

$$\begin{aligned}
S(x,x) &\geq S(x,\sigma^N x) + S(\sigma^N x, x) \\
&\geq \sum_{k=0}^{N-1}\big[A(\sigma^k x) - m_0\big] + S(\sigma^N x, x) \\
&\geq S(x,x).
\end{aligned}$$

□

4. The continuously varying support property.

Definition: We say that a pair $(A,\mu) \in C_0^\alpha(\Sigma^+,\mathbb{R}) \times \mathcal{M}(\sigma)$ has the *semi-continuously varying support property* if for any neighbourhood $U \subseteq \Sigma^+$ of $\operatorname{supp}(\mu)$ there exists a neighbourhood $\mathcal{V} \ni A$ of A in the C^0-topology, such that if $\phi \in \mathcal{V}$, and ν is a maximizing measure for $A+\phi$, then $\operatorname{supp}(\nu) \subseteq U$.

4.1. **Lemma.**
If a function $A \in C_0^\alpha(\Sigma^+,\mathbb{R})$ has a unique minimizing measure μ and the semi-continuously varying support property, then $\operatorname{supp}(\mu)$ *is uniquely ergodic and μ has the continuously varying support property.*

Proof:
The unique ergodicity follows from item [4] of proposition 3.1. To obtain the continuously varying support property we have to show that the map $C_0^\alpha(\Sigma^+,\mathbb{R}) \ni A \to \mathcal{M}(\Sigma^+)$ is continuous in the strong topology. By the hypothesis of semi-continuity, it is enough to prove that if $\psi_n \in C^\alpha(\Sigma^+,\mathbb{R})$ and $\|\psi\|_0 \to 0$, then $\nu_n \to \mu$ weakly*, where ν_n is a maximizing measure for $A+\psi_n$.

Choose a limit $\widetilde{\nu}$ of a subsequence of ν_n. Then $\int(L+\psi_n)\,d\mu \leq \int(L+\psi_n)\,d\nu_n$ and hence $\int L\,d\mu \leq \int L\,d\widetilde{\nu}$. Thus $\widetilde{\nu}$ is maximizing for A and hence $\widetilde{\nu} = \mu$.

□

Theorem A combined with the following proposition give a proof of theorem C.

4.2. **Proposition.**

Let $A^* \in C_0^\alpha(\Sigma^+, \mathbb{R})$ admiting a unique maximizing measure μ^*. Let $\Psi : \Sigma^+ \to \mathbb{R}$ be a continuous function such that $\Psi(x) = 0$ for $x \in \operatorname{supp}(\mu^*)$ and $\Psi(x) < 0$ for $x \notin \operatorname{supp}(\mu^*)$. Then $(A^* + \Psi, \mu^*)$ has the semi-continuously varying support property.

Proof:

Write $A := A^* + \Psi$. By lemma 4.1, it is enoungh to prove the semi-continuosly varying support property. Suppose that it does not hold. Then there is a neighbourhood U of $\operatorname{supp}(\mu^*)$ and a sequence $\langle A_n \rangle_{n\geq 0} \subset C_0^\alpha(\Sigma^+, \mathbb{R})$ of Hölder functions converging to A and maximizing measures μ_n for A_n such that $K_n = \operatorname{supp}(\mu_n) \not\subseteq U$. We may assume that μ_n converges weakly to μ_∞ and $\langle K_n \rangle_{n\geq 0}$ converges in the Hausdorff metric to a compact set K_∞.

Step one: Let $\lambda_n = \int A_n \, d\mu_n$ and $\lambda^* = \int A \, d\mu^*$. We prove that $\lambda_n \to \lambda^*$ and that $\langle \mu_n \rangle_{n\geq 0}$ converges weakly* to μ^*.

We have that $\lambda^n \geq \int A_n \, d\mu^*$, hence

$$\liminf_n \lambda_n \geq \int A \, d\mu^* = \lambda^*.$$

Moreover,

$$\begin{aligned}\lambda^* = \int A \, d\mu^* &\geq \int A \, d\mu_n \\ &\geq \int A_n \, d\mu_n - \|A - A_n\|_0 = \lambda_n - \|A - A_n\|_0 .\end{aligned}$$

Letting $n \to \infty$, we get that $\lim_n \sup \lambda_n \leq \lambda^*$.

Step two: We how that we can extend the coboundary equation for A to K_∞.

Fix $\overline{x} \in \Sigma^+$. Let $V_n \in C^\alpha(\Sigma^+, \mathbb{R})$ be a function given by proposition 3.1[5] for A_n. By adding a constant we may assume that $V_n(\overline{x}) = 0$. By proposition 3.1[5], $\operatorname{Hold}_\alpha(V_n)$ is uniformly bounded on n. By Arzela-Ascoli theorem, there is a convergent subsequence $V_n \xrightarrow{\| \|_0} W$ to an α-Hölder function W. Since

$$A_n = \lambda_n + V_n - V_n \circ \sigma \qquad \text{on } K_n,$$

then

$$A = \lambda^* + W - W \circ \sigma \qquad \text{on } K_\infty. \tag{19}$$

Similarly,

$$A \leq \lambda^* + W - W \circ \sigma \qquad \text{on all } \Sigma^+. \tag{20}$$

Since $\int A \, d\mu^* = \int A^* \, d\mu^* = \lambda^*$, then

$$A = \lambda^* + W - W \circ \sigma \qquad \text{on } \operatorname{supp}(\mu^*). \tag{21}$$

Step three:

Since $K_n = \text{supp}\mu_n$ then for all $x \in K_n$ there is a complete foward orbit in K_n containing x, i.e. there is $\langle x_k \rangle_{k\in\mathbb{Z}}$ such that $x_0 = x$ and $\sigma(x_k) = x_{k+1}$ for all $k \in \mathbb{Z}$. Then K_∞ has also this property.

Let $y \in K_\infty \setminus U$ and $\langle y_k \rangle_{k\in\mathbb{Z}} \subseteq K_\infty$ such that $y_0 = y$ and $\sigma(y_k) = y_{k+1}$, $\forall k \in \mathbb{Z}$. By the cohomology property (19), any invariant measure supported on K_∞ is maximizing and thus it is μ^*. Hence there are sequences $M,\ N \to +\infty$ such that

$$\tfrac{1}{N}\sum_{k=0}^{N-1} \delta_{y_k} \xrightarrow{w^*} \mu^* \qquad \text{and} \qquad \tfrac{1}{M}\sum_{k=-M}^{-1} \delta_{y_k} \xrightarrow{w^*} \mu^*,$$

where δ_y is the Dirac probability supported on $\{y\}$ and the convergences are in the weak* topology. In particular, we may assume that $d(y_N, \text{supp}\,\mu^*) \to 0$ and $d(y_{-M}, \text{supp}\,\mu^*) \to 0$. Since μ^* is uniquely minimizng, then in is ergodic. By the ergodicity of μ^*, there is $z = z(N, M) \in \text{supp}(\mu^*)$ and $K = K(N, M) > 0$ such that $d(z, y_N) \to 0$ and $d(\sigma^K z, y_{-M}) \to 0$. The sequence $y_{-M}, \ldots, y_0, \ldots, y_{N-1}, z, \ldots, \sigma^{K-1}z$ is a closed ε-pseudo orbit with 2 jumps and with $e = \varepsilon_{N,M} \to 0$.

Let $B = A - \lambda^* + W - W \circ \sigma \le 0$. By (19) and (21), $B = 0$ on $K_\infty \cup \text{supp}(\mu^*)$. By lemma 2.1[2], there is a periodic point $p \in \Sigma^+$ such that $d(p, y_0) < \frac{\varepsilon}{1-\lambda^a}$ and

$$\begin{aligned} -\sum_{k=0}^{M+N+K-1} B(\sigma^k p) &= \sum_{k=-M}^{N-1} B(y_k) + \sum_{k=0}^{K-1} B(\sigma^k z) - \sum_{k=0}^{M+N+K-1} B(\sigma^k p) \\ &< 2\,K_1\,\varepsilon^\alpha. \end{aligned}$$

Now,

$$\begin{aligned} \sum_{k=0}^{M+N+K-1} \big[A^*(\sigma^k p) - \lambda^*\big] &= \sum_{k=0}^{M+N+K-1} B(\sigma^k p) - \sum_{k=0}^{M+N+K-1} \Psi(\sigma^k p) \\ &\ge -\Psi(p) + 2\,K_1\,\varepsilon^\alpha. \end{aligned}$$

Since $p \to y_0$ and $\Psi(y_0) < 0$ then, for $\varepsilon > 0$ small, we have that

$$\sum_{k=0}^{M+N+K-1} \big[A^*(\sigma^k p) - \lambda^*\big] > 0. \tag{22}$$

If ν_p is the invariant measure supported on the positive orbit of p, then (22) implies that $\int A^*\, d\nu_p > \lambda^*$. This contradicts the choice of μ^*.

□

4.3. **Remark.** If in proposition 4.2 we need B and $B + \Psi$ to have pressure zero, we can replace $B + \Psi$ by $t\,(B + \Psi)$ such that $P\big(t\,(B + \Psi)\big) = 0$. Since the function $f(t, \Psi) = P\big(t\,(B + \Psi)\big)$ is analytic on $\mathbb{R} \times C^a(\Sigma^+, \mathbb{R})$, then Ψ can be chosen C^α-arbitrarily close to 0 and t arbitrarily close to 1. In particular, $t\,(B + \Psi)$ can be made C^α arbitrarily close to B for any $0 < \alpha \le 1$.

5. Maximizing measures for generic potentials.

Let $C_0^\alpha(\Sigma^+, \mathbb{R})$ be the set of α-Hölder continuous functions $A : \Sigma^+ \to \mathbb{R}$ which have topological entropy $P(A) = 0$, endowed with the α-Hölder norm $\|A\|_\alpha := \|A\|_0 + \|A\|_\alpha$. Let $C_0^{\alpha+}(\Sigma^+, \mathbb{R})$ be the closure in the α-Hölder topology of $\cup_{\gamma>\alpha} C_0^\gamma(\Sigma^+, \mathbb{R})$.

If $p \in \operatorname{Fix}\sigma^N$, let ν_p be the probabiliy measure defined by

$$\int f \, d\nu_p = \frac{1}{N} \sum_{k=0}^{N-1} f(\sigma^k p).$$

For convenience of the reader we rephrase theorem B.

Theorem B.
Let $\mathcal{G}_2 \subset C_0^{\alpha+}(\Sigma^+, \mathbb{R})$ be the set of $A \in C_0^{\alpha+}(\Sigma^+, \mathbb{R})$ such that there is a neighbourhood $\mathcal{U} \ni A$ such that for all $B \in \mathcal{U}$, the unique maximizing measure for B is ν_p. Then $\mathcal{G}_2$ is open and dense in $C_0^{\alpha+}(\Sigma^+, \mathbb{R})$.

Proof:
Let $\mathcal{H} \subset C_0^{\alpha+}(\Sigma^+, \mathbb{R})$ be the set of $A \in C_0^{\alpha+}(\Sigma^+, \mathbb{R})$ such that there is a unique maximizing measure for A which is supported on a periodic orbit. By proposition 5.1, the set $\mathcal{H}$ is dense on $C_0^{\alpha+}(\Sigma^+, \mathbb{R})$. By proposition 4.2 and remark 4.3 there is a dense subset $\mathcal{A} \subseteq \mathcal{H}$ such that any $A \in \mathcal{A}$ has the semi-continuously varying support property. Then $\mathcal{A}$ is dense in $C_0^{\alpha+}(\Sigma^+, \mathbb{R})$. We show now that $\mathcal{A} = \mathcal{G}_2$ and, in particular, that it is open on $C_0^{\alpha+}(\Sigma^+, \mathbb{R})$. Let $A \in \mathcal{A}$ and let $p \in \Sigma^+$ be a periodic point such that the maximizing measure for A is ν_p. There exists a neighbourhood U of $\mathcal{O}(p)$ such that the unique invariant measure supported on U is ν_p. Since A has the continuously varying support property, then there is a neighbourhood $\mathcal{U}(A) \subset C_0^{\alpha+}(\Sigma^+, \mathbb{R})$ such that the (unique) maximizing measure for any $B \in \mathcal{U}(A)$ is ν_p.

$\square$

5.1. **Proposition.**
The set $\mathcal{H}$ of functions $A \in C_0^{\alpha+}(\Sigma^+, \mathbb{R})$ such that A has a unique minimizing measure and this measure is supported on a periodic orbit is dense on $C_0^{\alpha+}(\Sigma^+, \mathbb{R})$.

Proof:
Let $F \in C_0^{\alpha+}(\Sigma^+, \mathbb{R})$, then for any $\rho > 0$ there is $\alpha < \gamma < 1$ and $A \in C_0^\gamma(\Sigma^+, \mathbb{R})$ such that $\|A - F\|_\alpha < \rho$. Let $\alpha < \beta < \gamma$, we will find $G = A + \Psi \in C_0^\beta(\Sigma^+, \mathbb{R})$ such that $\|\Psi\|_\beta < \rho$. Then $G \in C_0^{\alpha+}(\Sigma^+, \mathbb{R})$ and $\|\Psi\|_\alpha \le \|\Psi\|_\beta < \rho$.

Let μ be a maximizing measure for A. Suppose that there are no periodic points on $\operatorname{supp}(\mu)$. Then for all $n>0$, $\min_{z\in\operatorname{supp}(\mu)} d(z,\sigma^n z)>0$. Because otherwise $\operatorname{Fix}\sigma^n\cap\operatorname{supp}(\mu)\neq\emptyset$. We will first find a periodic point sufficiently close to $\operatorname{supp}(\mu)$.

Let $\eta:=\frac{1}{2}(1-\lambda)$ and let $K>0$ be such that

$$\frac{1}{1-\lambda^K}<\tfrac{3}{2}, \tag{23}$$

$$1-\frac{\lambda+\lambda^K}{1-\lambda^K}>\eta. \tag{24}$$

and let $D>0$ be such that

$$\min\{\, d(z,\sigma^j z)\,|\, z\in\operatorname{supp}(\mu),\ 0<j\leq K\,\}>3D.$$

Since $\cup_{n\leq K}\operatorname{Fix}\sigma^n$ is finite, there is $0<\varepsilon_1<D$ such that

$$\inf\{\, d(z,\sigma^j z)\,|\, d(z,\operatorname{supp}(\mu))<2\varepsilon_1,\ 0<j\leq K\,\}>2D. \tag{25}$$

Given $0<\varepsilon<\varepsilon_1$, let $z\in\operatorname{supp}(\mu)$ and $n>0$ be such that

$$d:=d(z,\sigma^n z)=\min\{\, d(\sigma^i z,\sigma^j z)\,|\, 0\leq i<j\leq n\,\}<\varepsilon.$$

By (25), we have that $n>K$. Using lemma 2.1, we get that there exists $p\in\operatorname{Fix}\sigma^n$ such that $d(p,\sigma^n z)\leq\frac{d}{1-\lambda^n}$ and for $0\leq j\leq n$,

$$d(\sigma^j p,\sigma^j z)\leq\frac{d\,\lambda^{n-j}}{1-\lambda^n}\leq\tfrac{3}{2}\,d<\tfrac{3}{2}\,\varepsilon<2\,\varepsilon_1. \tag{26}$$

Given $0\leq i<j\leq n-1$ by (26) and (25), we have that

$$d(\sigma^i p,\sigma^j p)>2D>\eta\, d \qquad \text{if } j\leq i+K, \tag{27}$$

and using (26) and (24),

$$\begin{aligned} d(\sigma^i p,\sigma^j p) &\geq d(\sigma^i z,\sigma^j z)-d(\sigma^i z,\sigma^i p)-d(\sigma^j z,\sigma^j p)\\ &> d-\frac{\lambda^{n-i}\,d}{1-\lambda^n}-\frac{\lambda^{n-j}}{1-\lambda^n}\\ &> \left[1-\frac{\lambda+\lambda^K}{1-\lambda^n}\right]d>\eta\, d \qquad \text{if } i+K\leq j. \end{aligned} \tag{28}$$

Fix $q\in\operatorname{supp}(\mu)\subseteq\mathfrak{S}$ and $W:\Sigma^+\to\mathbb{R}$, $W(y)=S(q,y)$. Then W is γ-Hölder continuous and

$$W(\sigma x)-W(x)\geq S(x,\sigma x)\geq A-m_0 \qquad \text{for all } x\in\Sigma^+.$$

Hence

$$W\circ\sigma-W=A-m_0 \qquad \text{on } \operatorname{supp}(\mu).$$

Let $B(x) := A(x) - m_0 + W(x) - W(\sigma x) \leq 0$. Let $K_1 > 0$ be an γ-Hölder constant for B. Let

$$\delta = \tfrac{1}{4}\,\eta\, d \qquad \text{and} \qquad Q = K_1 \left[\tfrac{4}{\eta}\right]^{\gamma} > K_1.$$

If $d(x,y) < \delta$ and $0 < \beta < a$, then

$$|B(x) - B(y)| \leq K_1\, d(x,y)^{\gamma} < K_1\, \delta^{\gamma-\beta}\, d(x,y)^{\beta}.$$

For $x \in \Sigma^+$, define $|x| := \min\{d(x, \sigma^k p) \,|\, k = 1, \ldots, n\}$ and $p_x = \sigma^k p$ such that $d(x, p_x) = |x|$. Let

$$\Phi(x) = \max\Big\{\, 0\,,\ \Big[3\,Q\,\delta^{\gamma-\beta} - \tfrac{B(p_x)}{\delta^{\beta}}\Big]\,(\delta^{\beta} - |x|^{\beta})\Big\}. \tag{29}$$

We show that $\max_x B(x) + \Phi(x) = Q\delta^{\gamma} = B(\sigma^k x) + \Phi(\sigma^k x)$ for all $k = 1, \ldots, n$. Indeed, for $|x| < \delta$

$$|B(x) - B(p_x)| \leq K_1\ |x|^{\gamma} \leq Q\ |x|^{\gamma} \leq (Q\,\delta^{\gamma-\beta})\ |x|^{\beta}\,.$$

If $p_x = \sigma^i p$, then

$$\begin{aligned} |B(p_x)| &\leq \big|B(\sigma^i z)\big| + K_1\, d(\sigma^i z, p_x)^{\gamma} \\ &\leq 0 + K_1\ \Big[\tfrac{\lambda}{1-\lambda^n}\Big]^{\gamma}\ d^{\gamma} \leq K_1\, d^{\gamma} \leq Q\,\delta^{\gamma}. \end{aligned}$$

Hence

$$\begin{aligned} B(x) + \Phi(x) &\leq B(p_x) + Q\,\delta^{\gamma-\beta}\ |x|^{\beta} + 3\,Q\delta^{\gamma} - B(p_x) \\ &\qquad\qquad - 3\,Q\,\delta^{\gamma-\beta}\ |x|^{\beta} + \frac{Q\,\delta^{\gamma}}{\delta^{\beta}}\ |x|^{\beta} \\ &\leq 3\,Q\,\delta^{\gamma} - Q\delta^{\gamma-\beta}\ |x|^{\beta} \leq 3\,Q\,\delta^{\gamma}. \end{aligned}$$

Also $B(p_x) + \Phi(p_x) = 3\,Q\,\delta^{a}$ and $B(x) + \Phi(x) = B(x) \leq 0 < 3\,Q\,\delta^{a}$ for $|x| > \delta$.

If $\nu \neq \nu_p$ is a σ-invariant probability, we have that

$$\int A\ d\nu = \int \big[A + W - W\circ\sigma\big]\ d\nu \leq \int B(x)\ d\nu_p + m_0 < \int B\ d\nu_p + m_0.$$

We now prove that the β-Hölder norm of Φ can be made arbitrarily small. We have that

$$\|\Phi\|_0 := \sup_{x\in\Sigma^+} |\Phi(x)| \leq 3\,Q\,\delta^{\gamma} + \max_{0\leq i\leq n-1} \big|B(\sigma^i p)\big| \leq 4\,Q\,\delta^{\gamma}.$$

Observe that if $d(x,y) \leq \delta$ and $|y| \leq \delta$ then by (27) and (28) we have that $p_x = p_y$. If $|y| \leq |x| \leq 2\delta$ and $0 < \beta < 1$, then

$$|x|^{\beta} - |y|^{\beta} \leq \big(|x| - |y|\big)^{\beta} \leq d(x,y)^{\beta}\,.$$

And if $|y| \le |x| \le \delta$, then

$$|\Phi(x) - \Phi(y)| \le (3\,Q\,\delta^{\gamma-\beta} + Q\,\delta^{\gamma-\beta})\big(|x|^\beta - |y|^\beta\big)$$
$$\le 4\,Q\,\delta^{\gamma-\beta}\,d(x,y)^\beta.$$

If $|y| \le \delta < |x|$ and $d(x,y) \le \delta$, then

$$|\Phi(x) - \Phi(y)| \le 4\,Q\,\delta^{\gamma-\beta}\left(\delta^\beta - |y|^\beta\right) \le 4\,Q\,\delta^{\gamma-\beta}\left(|x|^\beta - |y|^\beta\right)$$
$$\le 4\,Q\,\delta^{\gamma-\beta}\,d(x,y)^\beta.$$

If $d(x,y) \ge \delta$ then

$$|\Phi(x) - \Phi(y)| \le |\Phi(x)| \le 4\,Q\,\delta \le 4\,Q\,\delta^{\gamma-\beta}\,d(x,y)^\beta.$$

Hence

$$\mathrm{Hold}_\gamma(\Phi) := \sup_{0<d(x,y)\le 1} \frac{|\Phi(x) - \Phi(y)|}{d(x,y)^\beta} \le 4\,Q\,\delta^{\gamma-\beta}.$$

If we let $\varepsilon \to 0$ then $\delta \to 0$, $\|\Phi\|_0 \to 0$ and $\mathrm{Hold}_\beta(\Phi) \to 0$ for any $0 < \beta < \min\{1, \gamma\}$.

In the case when there is a periodic point $p \in \mathrm{Fix}\,\sigma^n \cap \mathrm{supp}(\mu) \neq \emptyset$, choose $D > 0$ such that $d(\sigma^i p, \sigma^j p) > 2D$ for $0 \le i < j \le n-1$ and define $\Phi(x)$ by the same formula as (29). In this case $B(p_x) \equiv 0$. The rest of the proof is the same.

Finally, we need to pertub A among the β-Hölder functions with pressure zero. Let $t = t(\Phi) \in \mathbb{R}$ be such that $P(A + \Phi + t(\Phi)) = 0$. Since $|P(A+\Phi) - P(A)| \le \|\Phi\|_0$, $P(A + \Phi + t) = P(A + \Phi) + t$ and $P(A) = 0$, then $|t| \le \|\Phi\|_0$. The perturbing function $\Psi = \Phi + t(\Phi)$ has $\|\Psi\|_0 \le 2\,\|\Phi\|_0 \le 4\,Q\,\delta^\gamma$ and $\mathrm{Hold}_\beta(\Psi) = \mathrm{Hold}_\beta(\Phi) \le 4\,Q\,\delta^{\gamma-\beta}$.

□

6. Relations with the Thermodynamic Formalism.

Proof of Proposition D.

Let $\widehat{\mu}_t$ be the equilibrium state for tA. Suppose that $\widehat{\mu}_t$ does not converges weakly* to μ_A, then for $\varepsilon > 0$ small and a subsequence t_n,

$$0 < \int A\,d\widehat{\mu}_{t_n} < \int A\,d\mu_A - \varepsilon.$$

Take t_n large enough such that $t_n\varepsilon - h_{top}(\sigma) > 0$. Then

$$h(\mu_A) + t_n \int A\,d\mu_A \ge h(\mu_A) - t_n\Big[\int A\,d\widehat{\mu}_{t_n} + \varepsilon\Big]$$
$$\ge h(\widehat{\mu}_{t_n}) - h_{top}(\sigma) + t_n\,\varepsilon + t_n \int A\,d\widehat{\mu}_{t_n}$$
$$> h(\widehat{\mu}_{t_n}) + t_n \int A\,d\widehat{\mu}_{t_n}.$$

□

7. Expanding maps of the circle.

In this section we prove theorems A1 and B1. The idea is to show in proposition 7.1 below, a homeomorphism among the $C^{1+\alpha}$ expanding dynamics on S^1 and C^α functions on the correspondig shift Σ^+, and then to apply theroems A and B.

Consider a point $y_0 \in S^1$. In order to prove theorems A1 and B1, it is enough to prove their claims for the class of maps $f \in \mathcal{F}(\alpha)$ (resp. $\mathcal{F}(\alpha+)$) that fix the point y_0. We will also denote by $\mathcal{F}(\alpha)$ (resp. $\mathcal{F}(\alpha+)$) this new class of maps.

We need to consider an abstract model that will be played by the transformation $T : S^1 \to S^1$, given by $T(x) = 2x \pmod 1$. This map is equivalent to the full one-sided shift in two symbols with identifications. We will use the diadic notation for points in the circle without stressing the equivalence of both systems.

We call x_0 the fixed point of T. Given f, we will define a bi-Hölder map θ_f which conjugates f and T, that is, $f \circ \theta_f = \theta_f \circ T$. In particular $\theta_f(y_0) = x_0$.

Construction of θ_f:

Given a map f, let z be the unique pre-image of y_0 different from y_0. Each point $t \in S^1$, $t \neq y_0$, has two different preimages in $S^1 \setminus \{y_0\}$. These preimages t_0 and t_1 aqre ordered by the order of the interval $S^1 \setminus \{y_0\}$, that is, $t_0 < t_1$.

We will order and code all pre-images $z_{\alpha_1,\alpha_2,\dots,\alpha_n}(f)$ (where $\alpha_i \in \{0,1\}$ and $n \in \mathbb{N}$) of z in the following way: if $z_{\alpha_1,\alpha_2,\dots,\alpha_n}(f)$ is defined, then $z_{\alpha_1,\alpha_2,\dots,\alpha_n,0}(f)$ and $z_{\alpha_1,\alpha_2,\dots,\alpha_n,1}(f)$ are ordered by the previous procedure.

We do the same for T (substituting y_0 by x_0) and obtain a set of coded points $z_{\alpha_1,\alpha_2,\dots,\alpha_n}(T)$ where $\alpha_i \in \{0,1\}$ and $n \in \mathbb{N}$. Denote by $Z(f)$ and $Z(T)$ the set of preimages defined above respectively for f and T.

Define first θ_f in these points, by associating the corresponding points $Z(f)$ and $Z(T)$ with the same code. Then θ_f extends continuously to S^1 in a unique way, because both sets of preimages are dense on S^1. The map θ_f is a homeomorphism. By usual bounded distortion arguments, we obtain that θ_f is bi-Hölder.

Consider the set $\mathcal{H}_\lambda(\alpha)$ of α-Hölder continuous functions $A : S^1 \to \mathbb{R}$ wich are smaller than $-\log\lambda$. Observe that the topological pressure of $-\log f' \circ \theta_f$ is zero (see [14] for a definition of topological pressure). Denote by $\mathcal{H}^0_\lambda(\alpha)$ the set of functions in $\mathcal{H}_\lambda(a)$ with topological pressure zero and let $\mathcal{H}^0_\lambda(\alpha+)$ be the clousure in the C^α-topology of $\cup_{\beta>\alpha}\mathcal{H}^0_\lambda(\alpha)$.

Define the transformation $\mathcal{G} : \mathcal{F}(\alpha) \to \mathcal{H}^0_\lambda(\alpha)$ by $\mathcal{G}(f) = -\log f' \circ \theta_f$, where $f \in \mathcal{F}_\lambda$. Similarly define $\mathcal{G} : \mathcal{F}(\alpha+) \to \mathcal{H}^0_\lambda(a+)$. Observe that θ_f depends on f in the definition of $\mathcal{G}$.

Theorems A1 and B1 follow from theorems A, B and the next proposition:

7.1. **Proposition.** *The transformations $\mathcal{G}$ are homeomorphisms from $\mathcal{F}(\alpha)$ [resp. $\mathcal{F}(\alpha+)$] (with the $C^{1+\alpha}$ distance) to $\mathcal{H}^0_\lambda$ [resp. $\mathcal{H}^0_\lambda(\alpha+)$] (with the C^α distance).*

Proof:

We shall prove that $\mathcal{G} : \mathcal{F}(\alpha) \to \mathcal{H}^0_\lambda(\alpha)$ is a homeomorphism for any $0 < \alpha < 1$. This implies that $\mathcal{G} : \mathcal{F}(a+) \to \mathcal{H}^0_\lambda(\alpha+)$ is a homeoporhism for any $0 < \alpha < 1$.

We show first that $\mathcal{G}$ is surjective. We have to find f and θ_f as above for each given $A \in \mathcal{H}^0_\lambda$.

Denote by $K(T)$ the set of invariant measures for T. For a given Hölder potential A with pressure zero, denote by $\widehat{\mu}_A$ the eigenmeasure of the dual of the Ruelle-Perron-Frobenius operator of the potential A, that is $\mathcal{L}^*_A \widehat{\mu}_A = \widehat{\mu}_A$ (see [14] for references on Thermodinamic Formalism). Note that the maximal eigenvalue of $\mathcal{L}^*_A$ is 1, because the pressure of A is zero and that $\widehat{\mu}_A$ is not necessarily an invariant measure in $K(T)$.

Now we define a Hölder homeomorphism $\theta_A : S^1 \to S^1$. By definition $\theta_A(x_0) := y_0$. For $x \neq x_0$ define $\theta_A(x) = y$ in such way that length$(y_0, y) = \widehat{\mu}_A(x_0, x)$. The map θ_A is well defined because $\widehat{\mu}_A$ is a probability with no atoms which is positive on open sets and the circle is oriented and has lenght one.

Let $f = \theta_A \circ T \circ {\theta_A}^{-1}$. Since θ_A preserves orientation, then the two sets of preimages $Z(f)$ and $Z(T)$ are ordered in the same way. This proves that $\theta_A = \theta_f$.

The Jacobian of T with respect to the measure $\widehat{\mu}_A$ is e^{-A}. By definition, the pushed measure of $\widehat{\mu}_A$ by θ_A is the Lebesgue measure. Since f was defines by the change of coordinates θ_A, then f', the Jacobian of f satisfies $f' = \mathrm{e}^{-A}\,\theta_A^{-1}$. Therefore f' exists and it is Hölder. This shows that $\mathcal{G}$ is surjective.

Now we show that $\mathcal{G}$ is injective. Suppose that two maps f and g satisfy $\mathcal{G} = A_f = A_g = \mathcal{G}(g)$. Consider the respective changes of coordinates θ_f and θ_g.

Note that $h = \theta_f^{-1} \circ \theta_g$ conjugates f and g, because θ_f conjugates f and T and θ_g conjugates g and T. Since $\theta_f = \theta_g$, because $A_f = A_g$, then g is the identity and hence $f = g$. This implies that $\mathcal{G}$ is injective.

From the definition of $\mathcal{G}$ and the reasoning above, it is easy to see that the map $\mathcal{G}$ is an homeomorphism.

□

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