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Calculus of Variations

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The Palais-Smale condition on contact type energy levels for convex Lagrangian systems

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Abstract We prove that for a uniformly convex Lagrangian system L on a compact manifold M, almost all energy levels contain a periodic orbit. We also prove that below Mañé's critical value of the lift of the Lagrangian to the universal cover, $c_u(L)$, almost all energy levels have conjugate points.

We in addition prove that if an energy level is of contact type, projects onto M and $M \neq \mathbb{T}^2$, then the free time action functional of $L + k$ satisfies the Palais-Smale condition.

1 Introduction

In this paper we continue the study of the Morse theory of the free time action functional for convex lagrangian systems that we begun in [7]. This time we try to include the case of low energy levels, where very little is known. The main problem with the free time action functional is that it may fail to satisfy the Palais-Smale condition, usually required for variational methods. Here we prove that if an energy level is of contact type, projects onto the whole configuration space M and $M \neq \mathbb{T}^2$ is not the 2-torus, then it satisfies the Palais-Smale condition. We also prove that when an energy level projects onto M and is below Mañe's critical value of the universal cover, the set of closed loops has a mountain pass geometry. An adaptation of an argument by Struwe to the mountain pass geometry shows the existence of convergent Palais-Smale sequences for almost all energy levels. This implies that for almost all energy levels which project onto M the Euler-Lagrange flow has a periodic orbit, has closed orbit loops starting at any $x \in M$, and has conjugate points if the energy is below Mañé's critical value of the universal cover.

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The same holds for an energy level which satisfies the Palais-Smale condition, and hence in particular for contact type energy levels.

In [7] we proved that high energy levels have a periodic orbit. Very low energy levels which do not project onto M are displaceable, and then, by results of Frauenfelder and Schlenk [10, 28], they have finite Hofer-Zehnder capacity. Combining these results we get that almost all energy levels have a periodic orbit. Our class of Lagrangian systems includes exact magnetic flows on compact manifolds.

1.1 Critical energy values

Let M be a closed Riemannian manifold with $\dim M \geq 2$. Let $\pi : TM \to M$ be the projection. A Lagrangian on M is a C^∞ function $L : TM \to \mathbb{R}$. We shall assume that L is (uniformly) *convex:* there is $a > 0$ such that

$$w^* \cdot \frac{\partial^2 L}{\partial v\, \partial v}\bigg|_{(x,v)} \cdot w > a\ |w|_x^2 \qquad \text{for all } x \in M,\ v, w \in T_x M.$$

This uniform convexity and the compactness of M imply (see e.g. Lemma 3.1 below) that L is superlinear:

$$\lim_{|v|_x \to +\infty} \frac{L(x,v)}{|v|_x} = +\infty \qquad \text{uniformly on } TM.$$

Since M is compact and L is autonomous, the *Euler-Lagrange equation*

$$\frac{d}{dt}\frac{\partial L}{\partial v}(x,\dot{x}) = \frac{\partial L}{\partial x}(x,\dot{x}) \tag{E-L}$$

defines a complete flow φ_t on TM called the *Euler-Lagrange flow* of L. The *energy function* $E : TM \to \mathbb{R}$,

$$E(x,v) := \frac{\partial L}{\partial v}(x,v) \cdot v - L(x,v),$$

is invariant under the Euler-Lagrange flow.

The *action* of an absolutely continuous curve $\gamma \in C^{ac}([a,b], M)$ is defined by

$$A_L(\gamma) = \int_a^b L(\gamma(s), \dot{\gamma}(s))\, ds.$$

As noticed by Dias Carneiro [1] and Mañé [18], critical points for the action of $L + k$ among curves with free time interval are solutions of the Euler-Lagrange equation which have energy $E \equiv k$. The most direct way to obtain critical points is to look for minima. It turns out that if k is low enough there are no minima because then the action of $L + k$ is not bounded from below. The exact threshold is given by *Mañé's critical value*:

$$c(L) = \min\{k \in \mathbb{R} \mid A_{L+k}(\gamma) \geq 0 \text{ for all closed curves } \gamma \text{ on } M\}.$$

The action functional A_{L+k} is bounded from below on the space of curves with fixed endpoints and on the space of closed curves if and only if $k \geq c(L)$. It is also known [3, 18] that

$$c(L) \geq e_0(L) := \min\{k \in \mathbb{R} \mid \pi(E^{-1}\{k\}) = M\}.$$

If $p : N \to M$ is a covering map and $L_1 = L \circ dp$ is the lift of the Lagrangian, it is easy to check that $c(L_1) \leq c(L)$. Thus we have that

$$e_0(L) \leq c_u(L) \leq c_0(L) \leq c(L),$$

where c_u and $c_a = c_0$ are the critical values of the lifts of L to the universal cover and the abelian cover. The number $c_0(L)$ is also called the *strict critical value* and has the following characterization [27]:

$$\begin{aligned} c_0(L) &= -\min\left\{\int L\,d\mu \,\middle|\, \mu \text{ is a } \varphi_t\text{-invariant probability with homology } \rho(\mu)=0\right\} \\ &= \min\left\{c(L-\omega) \;\middle|\; [\omega] \in H^1(M,\mathbb{R})\right\}, \qquad (1) \end{aligned}$$

where the homology $\rho(\mu) \in H_1(M,\mathbb{R}) \approx H^1(M,\mathbb{R})^*$ of an invariant measure with compact support μ is defined by

$$\langle [\omega], \rho(\mu)\rangle = \int_{TM} \omega_x(v)\, d\mu(x,v)$$

for any closed 1-form ω on M. Here $[\omega] \in H^1(M,\mathbb{R})$ is the cohomology class of ω.

Given a covering map $p : N \to M$ let $L_1 = L \circ dp$ be the lift of the Lagrangian L to TN and $c_1 = c(L_1)$ its critical value. The *Peierls barrier* $h_{c_1} : N \times N \to \mathbb{R}$ is defined by

$$\begin{aligned} h_{c_1}(q_0, q_1) &:= \liminf_{T\to+\infty} \Phi_{c_1}(q_0, q_1; T), \\ \Phi_{c_1}(q_0, q_1; T) &:= \inf\{A_{L_1+c_1}(\gamma) \mid \gamma \in C^{ac}([0,T], N), \gamma(0) = q_0, \gamma(T) = q_1\}. \end{aligned}$$

1.2 The Palais-Smale condition

We describe now our setting for the Morse theory of the free time action functional. Let $\mathcal{H}^1(M)$ be the set of absolutely continuous curves $x : [0,1] \to M$ such that

$$\int_0^1 |\dot{x}(s)|^2_{x(s)}\, ds < \infty.$$

Then $\mathcal{H}^1(M)$ is a Hilbert manifold and its tangent space at x consists of weakly differentiable vector fields along x whose covariant derivative is bounded in $\mathcal{L}^2$. Set $\mathbb{R}^+ := \{t \in \mathbb{R} \mid t > 0\}$. We shall use the Hilbert manifold $\mathcal{H}^1(M) \times \mathbb{R}^+$ with the Riemannian metric

$$\begin{aligned} \langle(\xi,\alpha),(\eta,\beta)\rangle_{(x,T)} &= \alpha\,\beta + f(T)\,\langle \xi(0), \eta(0)\rangle_{x(s)} \\ &\quad + g(T) \int_0^1 \left\langle \frac{D}{ds}\xi(s), \frac{D}{ds}\eta(s)\right\rangle_{x(s)} ds, \qquad (2) \end{aligned}$$

where $\frac{D}{ds}$ is the covariant derivative along $x(s)$ and $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ are smooth positive functions such that $\max\{f, g\} \le 2$,

$$f(T) = \begin{cases} T^2 & \text{if } T \le 1, \\ 1 & \text{if } T \ge 10. \end{cases} \quad \text{and} \quad g(T) = \begin{cases} T^2 & \text{if } T \le 1, \\ \dfrac{1}{T}\, e^{-4T^2} & \text{if } T \ge 10. \end{cases}$$

We shall discuss this choice of metric in more detail later on. Observe that this metric is locally equivalent to the metric obtained when $f \equiv g \equiv 1$. In particular, the set of differentiable functions on $\mathcal{H}^1(M) \times \mathbb{R}^+$ is the same for this metric and for the one with $f \equiv g \equiv 1$.

Given $k \in \mathbb{R}$ define the *free time action functional* $\mathcal{A}_k : \mathcal{H}^1(M) \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\mathcal{A}_k(x, T) = \int_0^1 \left[L\left(x(s), \frac{\dot{x}(s)}{T} \right) + k \right] T \; ds.$$

Observe that if $y(t) := x(t/T)$ then

$$\mathcal{A}_k(x, T) = A_{L+k}(y).$$

We say that L is *Riemannian at infinity* if there exists $R > 0$ such that $L(x, v) = \frac{1}{2}\,|v|_x^2$ if $|v|_x > R$. In [7, prop. 18] it is proven that given a uniformly convex lagrangian L and $k \in \mathbb{R}$, there exists a convex lagrangian L_0 such that $L = L_0$ on $[E \le k+1]$ and L_0 is Riemannian at infinity. In [7, Lemma 19] it is proven that if $L = L_0$ on $[E \le c(L)+1]$ then $c(L) = c(L_0)$. Thus if our objective is to find solutions of the Euler-Lagrange equation with prescribed energy, we can assume that L is Riemannian at infinity.

Given $q_0,\ q_1 \in M$ let $\Omega_M(q_0, q_1)$ be the set of curves $(x, T) \in \mathcal{H}^1(M) \times \mathbb{R}^+$ with endpoints $x(0) = q_0$ and $x(1) = q_1$. Also, let Λ_M be the set of closed curves in $\mathcal{H}^1(M) \times \mathbb{R}^+$. The sets $\Omega_M(q_0, q_1)$ and Λ_M are Hilbert submanifolds of $\mathcal{H}^1(M) \times \mathbb{R}^+$. A connected component of $\Omega_M(q_0, q_1)$ (resp. Λ_M) consists of closed curves in the same homotopy class with fixed endpoints (resp. in the same free homotopy class).

A theorem of Smale [29] implies that $\mathcal{A}_k$ is C^2 on $\mathcal{H}^1(M) \times \mathbb{R}^+$ for the metric with $f \equiv g \equiv 1$, and hence also for the metric (2). We show in Lemma 2.1 that a critical point of $\mathcal{A}_k$ restricted to $\Omega_M(q_0, q_1)$ or to Λ_M is a solution of the Euler-Lagrange equation with energy $E \equiv k$.

We say that $\mathcal{A}_k$ satisfies the *Palais-Smale condition on* $\Omega_M(q_0, q_1)$ [resp. on Λ_M] or that the *energy level k satisfies the Palais-Smale condition on* $\Omega_M(q_0, q_1)$ [resp. on Λ_M] if every sequence (x_n, T_n) in *the same connected component* of $\Omega_M(q_0, q_1)$ [resp. on Λ_M] such that $|\mathcal{A}_k(x_n, T_n)|$ is bounded and $\lim_n \|d_{(x_n,T_n)}\mathcal{A}_k\|_{(x_n,T_n)} = 0$ has a convergent subsequence.

We shall prove

Theorem A *If L is Riemannian at infinity and $\mathcal{A}_k$ does not satisfy the Palais-Smale condition on $\Omega_M(q_0, q_1)$, or on Λ_M, then there exists a Borel probability measure μ, invariant under the Euler-Lagrange flow, supported in a connected* *component of the energy level $E \equiv k$, which has homology $\rho(\mu) = 0$ and whose $(L+k)$-action is zero:*

$$A_{L+k}(\mu) = \int [L+k]\, d\mu = 0.$$

In Appendix A we give an example in which the measure obtained in Theorem A can not be ergodic. In [7, th. C] we found counterexamples to the Palais-Smale condition at $k = c(L)$, but in [7] we didn't require the Palais-Smale sequences to be in the same connected component of the space of curves. Combining the arguments in [7] with those of Theorem A we get the following Corollary B. The novelty is that it allows curves with trivial homotopy class.

Corollary B *If L is Riemannian at infinity, then $\mathcal{A}_k$ satisfies the Palais-Smale condition for all $k > c_u(L)$. On $\Omega_M(q_0, q_1)$, $\mathcal{A}_{c_u}$ satisfies the Palais-Smale condition if and only if the Peierls barrier on the universal cover is $h_{c_u} \equiv +\infty$.*

Another example is the lagrangian $\mathbb{L} : T\mathbb{D} \to \mathbb{R}$ on the hyperbolic disc $\mathbb{D} \subset \mathbb{C}$, where $\mathbb{L}(x, v) = \frac{1}{2}|v|_x^2 + \eta_x(v)$, $|\cdot|_x$ is the hyperbolic metric and η is a 1-form on $\mathbb{D}$ whose differential $d\eta$ is the hyperbolic area form. In this case the Peierls barrier at $k = c(\mathbb{L}) = c_u(\mathbb{L})$ is finite (cf. [2, ex. 6.2]) and $\mathcal{A}_{c_u}$ does not satisfy the Palais-Smale condition. If M is a compact surface with constant curvature $K \equiv -1$, the Euler-Lagrange flow of $\mathbb{L}$ projects to a (non-exact) magnetic flow on TM. At the energy level $k = c_u(\mathbb{L})$ the projection of the Euler-Lagrange flow of $\mathbb{L}$ is the horocycle flow[1] for M which has no closed orbits.

The idea of the proof of Theorem A is the following. Let (x_n, T_n) be a Palais-Smale sequence in the same connected component of Λ_M or $\Omega_M(q_0, q_1)$. We first prove in Proposition 3.12, similar to [7], that if the times T_n are bounded away from 0 and $+\infty$ then there is a convergent subsequence. In Corollary 3.6 we prove that if $q_0 \neq q_1$ and $(x_n, T_n) \in \Omega_M(q_0, q_1)$ then T_n is bounded away from zero. In Proposition 3.8 we prove that if $\liminf_n T_n = 0$ and $(x_n, T_n) \in \Lambda_M$ or $(x_n, T_n) \in \Omega_M(q_0, q_0)$ then x_n converges to a singularity $(q_0, 0) \in TM$ of the Euler-Lagrange flow with zero action $L(q_0, 0) + k = 0$ and energy k. In this case the measure μ is the Dirac probability supported at the point $(q_0, 0)$.

The most delicate case is when $\lim_n T_n = +\infty$. Since the gradient of $-\mathcal{A}_k$ at (x_n, T_n) converges to zero one expects that the curves $y_n(sT_n) := x_n(s)$ are approximate solutions of the Euler-Lagrange equation with average energy k. If μ_n is the probability measure on TM defined by

$$\int_{TM} f\, d\mu_n = \int_0^1 f\left(x_n, \frac{\dot{x}_n}{T_n}\right) ds = \frac{1}{T_n} \int_0^{T_n} f(y_n, \dot{y}_n)\, dt, \quad [y_n(sT_n) := x_n(s)],$$

we prove that μ_n converges to an invariant probability for the Euler-Lagrange flow with support in the energy level k. Since the $L+k$ action of the curves y_n is bounded and $\lim_n T_n = +\infty$ their average action converges to zero. Since their homotopy class is fixed, and $\lim_n T_n = +\infty$, their average homology class tends to zero.

[1] When $k > c_u(L)$ the flow is Anosov on $d\pi(\mathbb{E}^{-1}\{k\})$ and for $k < c_u$ the energy level $d\pi(\mathbb{E}^{-1}\{k\})$ is foliated by contractible periodic orbits.

We use the functions f and g in the definition of the metric (2) to deal with the cases $\lim_n T_n = 0$ and $\lim_n T_n = +\infty$. In order to motivate their choice observe that by suitably expanding the metric near the endpoints any bounded function on the open interval $]-1, 1[$ can be made not to satisfy the Palais-Smale condition. For example let $\psi(x) = x^2$ on $|x| < 1$. Let $h : \mathbb{R} \to]-1, 1[$ be a diffeomorphism. Then $\psi \circ h$ does not satisfy the Palais-Smale condition because $\lim_{x\to\pm\infty} d_x(\psi \circ h) = 0$. So, if one is going to obtain any conclusion from the fact that the Palais-Smale condition does not hold, one needs to use an appropriate metric. Since our metric is locally equivalent to the usual one with $f = g \equiv 1$, the critical points are still solutions of the Euler-Lagrange equation and also the change of metric does not prevent finding Palais-Smale sequences by, say, a minimax argument.

1.3 The mountain pass geometry

We show that for low energy levels $e_0(L) < k < c_u(L)$, the action functional $\mathcal{A}_k$ exhibits a mountain pass geometry on the space of loops $\Omega_M(q_0, q_0)$ and closed curves Λ_M. This result is suggested by Taĭmanov in [33, p. 362] for a different action functional for magnetic flows saying that "one-point curves form the manifold of local minima of the functional ℓ". S. Bolotin (cf. [33, p. 362]) observed that the results of the papers [20–24, 32] may not be valid because the Palais-Smale condition could fail. The approach in this paper recovers the (a.e.)–validity of some of those results.

Let $k < c_u(L)$. By the definition of $c_u(L)$, there are a closed curve $(x_1, T_1) \in \Lambda_M$ and for any $q_0 \in \pi(E^{-1}\{k\})$ a loop $(x_2, T_2) \in \Omega_M(q_0, q_0)$, both with trivial homotopy class and negative $(L + k)$-action.

Proposition C

1. *Let $q_0 \in M$ and $k > E(q_0, 0)$. Then there exists $c > 0$ such that if $\Gamma : [0, 1] \to \Omega_M(q_0, q_0)$ is a continuous path joining a constant loop $\Gamma(0) = q_0 : [0, T] \to \{q_0\} \subset M$ (with any $T > 0$) to any closed loop $\Gamma(1) \in \Omega_M(q_0, q_0)$ with negative $(L + k)$-action, $A_{L+k}(\Gamma(1)) < 0$, then*
$$\sup_{s\in[0,1]} A_{L+k}(\Gamma(s)) > c > 0.$$
2. *Let $k > e_0(L)$. Then there exists $c > 0$ such that if $\Gamma : [0, 1] \to \Lambda_M$ is a continuous path joining any constant curve $\Gamma(0) = q_0 : [0, T] \to \{q_0\} \subset M$ to any closed curve $\Gamma(1)$ with negative $(L + k)$-action, $A_{L+k}(\Gamma(1)) < 0$, then*
$$\sup_{s\in[0,1]} A_{L+k}(\Gamma(s)) > c > 0.$$

Standard critical point theory gives contractible periodic orbits on any energy level $e_0(L) < k < c_u(L)$ where the Palais-Smale condition holds. Since the failure of the Palais-Smale condition can only be due to one direction of noncompactness, namely the time parameter T on $\mathcal{H}^1(M) \times \mathbb{R}^+$, an argument originally due to Struwe in [30] (see also Struwe [31], Jeanjean [14] and Jeanjean, Toland [15]) can be applied to the mountain pass geometry of Proposition C to overcome the Palais-Smale condition for almost every k.

Previous results on higher energy levels (cf. [3, 7, 18]) give that $E^{-1}\{k\}$ has a periodic orbit for every $k > c_u(L)$. When the energy level does not project onto the whole configuration space M (i.e. $k < e_0(L)$) we show that the displacement energy of $[E \leq k]$ is finite. Then by results of U. Frauenfelder and F. Schlenk [10, 28], the π_1-sensitive Hofer-Zehnder capacity of $[E \leq k]$ is finite and so standard arguments (cf. [13]) show that almost any energy level $E^{-1}\{k\}$, $k < e_0(L)$ has a contractible periodic orbit. We summarize this in the following:

Theorem D

a. *There is a total Lebesgue measure set $A \subset \mathbb{R}$ such that for all $k \in A$ either the energy level $E^{-1}\{k\}$ is empty or it contains a periodic orbit.*
   Moreover,
   - *The set A contains $]c_u(L), +\infty[$.*
   - *If $k < c_u(L)$ and $k \in A$, then this periodic orbit is contractible.*
   - *If $e_0(L) < k < c_u(L)$ and $k \in A$, this periodic orbit has positive $(L+k)$-action and it is not a strict local minimizer of $\mathcal{A}_k$ on Λ_M.*

b. *For any $q_0 \in M$, there is a total Lebesgue measure subset $]c_u(L), +\infty[\subset B \subset]E(q_0, 0), +\infty[$ such that for all $k \in B$ there is a solution of the Euler-Lagrange equation in $\Omega_M(q_0, q_0)$ with energy k. If $E(q_0, 0) < k < c_u(L)$ and $k \in B$, this solution is not a strict local minimizer of $\mathcal{A}_k$ on $\Omega_M(q_0, q_0)$.*

c. *The above items hold for a specific $k \in]e_0(L), c_u(L)[$ (resp. $k \in]E(q_0, 0), c_u(L))$ if the energy level k satisfies the Palais-Smale condition.*

As an example in Appendix C we prove that a lagrangian with no magnetic term has a closed orbit on every energy level.

Two points θ_0, $\theta_1 \in TM$, are said to be *conjugate* if there is $\tau \in \mathbb{R}$ such that $\theta_1 = \varphi_\tau(\theta_0)$ and $\mathbb{V}(\theta_1) \cap d_{\theta_0}\varphi_\tau(\mathbb{V}(\theta_0)) \neq \{0\}$, where $\mathbb{V} \subset T(TM)$ is the vertical sub-bundle $\mathbb{V}(\theta) = \ker d_\theta \pi$. R. Mañé asked whether if $k < c_0(L)$ there is always an orbit with energy k and conjugate points. G. Paternain and M. Paternain in [27] showed examples of magnetic flows with Anosov energy levels without conjugate points with energy $k \in]c_u(L), c_0(L)[$. At $k = c_u(L)$ these examples do not have conjugate points. The question remains open for $k < c_u(L)$.

In [7, p. 663] we gave an example of an orbit segment without conjugate points which is not a local minimizer of the free time action functional. In Proposition 9.1 we prove that in an energy level without conjugate points every orbit segment is a strict local minimizer of the action functional. Since a mountain pass critical point can not be a strict local minimizer we get

Theorem E *Let $e_m(L) = \inf_{(x,v)\in TM} E(x, v)$.*

There is an open subset with total Lebesgue measure $A \subset [e_m(L), c_u(L)[$ such that if $k \in A$ then there is an orbit with energy k and conjugate points.

If $e_m(L) < k < c_u(L)$ and $\mathcal{A}_k$ satisfies the Palais-Smale condition, then the energy level k has conjugate points.

In [6, Prop. 8] and in [5, Cor. 1.13] we proved that if k is a regular value of the energy function E and $k < e_0(L)$ then $E^{-1}\{k\}$ has conjugate points.

We don't know if the following holds:

Question: Is it true that for the universal cover $\tilde{M}$,

$$c_u(L) = \inf\{k \in \mathbb{R} \mid \forall x, y \in \tilde{M}\ \exists \text{ orbit } \gamma \in \Omega_M(x, y),\ E(\gamma, \dot{\gamma}) = k\}?$$

An *exact magnetic flow* is the lagrangian flow of

$$L(x, v) = \frac{1}{2}\,|v|_x^2 - \eta_x(v),$$

where $|\cdot|_x$ is the Riemannian metric of M and η_x is a non-closed 1-form on M. Thus for exact magnetic flows we get periodic orbits for almost all energy levels and in particular for contact type energy levels, as seen below.

1.4 Contact type energy levels

We now concentrate on a property that ensures the Palais-Smale condition. Let $H : T^*M \to \mathbb{R}$ be the hamiltonian associated to L:

$$H(x, p) = \max_{v \in T_x M} [p(v) - L(x, v)], \tag{3}$$

and let $\omega = dp \wedge dx$ be the canonical symplectic form on T^*M. The *hamiltonian vector field* X on T^*M is defined by $i_X\omega = -dH$. The induced *hamiltonian flow* is conjugate to the lagrangian flow of L by the Legendre transform $\mathcal{L} : TM \to T^*M$, $\mathcal{L}(x, v) = L_v(x, v)$. The energy function satisfies $E = H \circ \mathcal{L}$, so that energy levels for L are sent to level sets of H.

An energy level $\Sigma = H^{-1}\{k\}$ is said to be of *contact type* if there exists a 1-form λ on Σ such that $d\lambda = \omega|_{T\Sigma}$ and $\lambda(X) \neq 0$. We call such a form λ a *contact-type* form for Σ.

Proposition F *If $\Sigma = H^{-1}\{k\}$ is of contact type and $\pi_* : H_1(\Sigma, \mathbb{R}) \to H_1(M, \mathbb{R})$ is injective, then $\mathcal{A}_k$ satisfies the Palais-Smale condition.*

Corollary G *If $[H = k]$ is of contact type,* $\dim M \geq 2$, $M \neq \mathbb{T}^2$ *and $k > e_0$, then $\mathcal{A}_k$ satisfies the Palais-Smale condition.*

In Sect. 2 we introduce the space of curves with free time interval and the action functional and compare various metrics on the space of curves. In Sect. 3 we prove Theorem A. In Sect. 4 we prove Corollary B. In Sect. 5 we prove Proposition C on the mountain pass geometry. In Sect. 6 we prove some results in Morse theory that we need and the relative completeness of the gradient flow of the action functional. In Sect. 7 we give the argument to overcome the Palais-Smale condition in a mountain pass geometry for the action functional. In Sect. 9 we prove Theorems D and E and in Sect. 10 we prove Proposition F and Corollary G. In Appendix A we give an example in which the measure of Theorem A can not be ergodic. In Appendix B we show energy levels of non-contact type. In Appendix C we prove that non-magnetic lagrangians have periodic orbits on every energy level.

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2 The action functional and the space of curves

Given a Riemannian metric on M, by Nash's Theorem there exists an isometric embedding of M into some $\mathbb{R}^N$. Let

$$\begin{aligned}\mathcal{H}^1 &:= \mathcal{H}^1(\mathbb{R}^N)\\ &:= \left\{\xi : [0,1] \to \mathbb{R}^N \text{ absolutely continuous } \;\middle|\; \int_0^1 |\dot{\xi}(s)|^2\, ds < +\infty\right\}\end{aligned}$$

be endowed with the metric

$$\langle \xi, \eta\rangle_{\mathcal{H}^1} := \langle \xi(0), \eta(0)\rangle + \int_0^1 \langle \dot{\xi}(s), \dot{\eta}(s)\rangle\, ds.$$

The corresponding norm is given by

$$\|\xi\|^2_{\mathcal{H}^1} := |\xi(0)|^2 + \int_0^1 |\dot{\xi}(s)|^2\, ds.$$

On $\mathcal{H}^1 \times \mathbb{R}^+$ we shall use the Riemannian metric

$$\langle\, (\xi, \alpha), (\eta, \beta)\, \rangle_{(x,T)} = \alpha\,\beta + f(T)\,\langle \xi(0), \eta(0)\rangle + g(T) \int_0^1 \langle \dot{\xi}(s), \dot{\eta}(s)\rangle\, ds, \quad (4)$$

where f and g are smooth positive functions such that $\max\{f,\ g\} \le 2$,

$$f(T) = \begin{cases} T^2 & \text{if } T \le 1,\\ 1 & \text{if } T \ge 10.\end{cases} \quad \text{and} \quad g(T) = \begin{cases} T^2 & \text{if } T \le 1,\\ \dfrac{1}{T}\, e^{-4T^2} & \text{if } T \ge 10.\end{cases}$$

Let $\zeta(t) := \xi(t/T)$, $0 \le t \le T$. Then $\dot{\xi}(t/T) = T \cdot \dot{\zeta}(t)$ and

$$\int_0^1 |\dot{\xi}(s)|^2\, ds = T \cdot \int_0^T |\dot{\zeta}(t)|^2\, dt.$$

In the variables (ζ, α) the Riemannian metric above is written as

$$\begin{aligned}\|(\xi,\alpha)\|^2_{(x,T)} &= \alpha^2 + T^2\,|\zeta(0)|^2 + T^3 \int_0^T |\dot{\zeta}|^2\, dt \qquad \text{if } T \le 1,\\ \|(\xi,\alpha)\|^2_{(x,T)} &= \alpha^2 + |\zeta(0)|^2 + e^{-4T^2} \int_0^T |\dot{\zeta}|^2\, dt \qquad \text{if } T \ge 10.\end{aligned} \quad (5)$$

This metric is locally equivalent to the metric of the product Hilbert space $\mathcal{H}^1 \times \mathbb{R}$.

Given $q_0, q_1 \in M$, let

$$\begin{aligned}\Omega(q_0, q_1) &:= \{(x, T) \in \mathcal{H}^1 \times \mathbb{R}^+ | x(0) = q_0,\ \ x(1) = q_1\},\\ \Lambda &:= \{(x, T) \in \mathcal{H}^1 \times \mathbb{R}^+ | x(0) = x(1)\}.\end{aligned}$$

Their tangent spaces at (x, T) are given by

$$T_{(x,T)}\Omega(q_0, q_1) = \{(\xi, \alpha) \in \mathcal{H}^1 \times \mathbb{R} |\ \xi(0) = \xi(1) = 0\},$$
$$T_{(x,T)}\Lambda = \{(\xi, \alpha) \in \mathcal{H}^1 \times \mathbb{R} |\ \xi(0) = \xi(1)\}.$$

Endow $\Omega(q_0, q_1)$ and Λ with the Riemannian metric (4).

Let

$$\mathcal{H}^1(M) := \{x \in \mathcal{H}^1(\mathbb{R}^N) \mid x([0, 1]) \subset M\},$$
$$\Omega_M(q_0, q_1) := \Omega(q_0, q_1) \cap \mathcal{H}^1(M) \times \mathbb{R}^+,$$
$$\Lambda_M := \Lambda \cap \mathcal{H}^1(M) \times \mathbb{R}^+.$$

Then $\mathcal{H}^1(M) \times \mathbb{R}^+$, $\Omega_M(q_0, q_1)$ and Λ_M are Hilbert submanifolds of $\mathcal{H}^1(\mathbb{R}^N) \times \mathbb{R}^+$, $\Omega(q_0, q_1)$ and Λ respectively. A connected component of Λ_M is given by closed curves in the same free homotopy class. A connected component of $\Omega_M(q_0, q_1)$ is given by the curves (x, T) in $\Omega_M(q_0, q_1)$ belonging to a given homotopy class with fixed endpoints.

On $\mathcal{H}^1(M) \times \mathbb{R}^+$ we shall use the intrinsic Riemannian metric defined by

$$\begin{aligned} &\langle(\xi, \alpha), (\eta, \beta)\rangle_{(x,T)} \\ &:= \alpha\, \beta + f(T)\, \langle \xi(0), \eta(0)\rangle_{x(0)} + g(T) \int_0^1 \left\langle \frac{D}{ds}\xi(s), \frac{D}{ds}\eta(s)\right\rangle_{x(s)} ds \end{aligned} \tag{6}$$

where $\langle\cdot, \cdot\rangle_x$ is the Riemannian metric on M and $\frac{D}{ds}$ are covariant derivatives. Since M is isometrically embedded into $\mathbb{R}^N$, the covariant derivative $\frac{D}{ds}\xi(s) = \mathbb{P}\big(\dot{\xi}(s)\big)$ is the orthogonal projection $\mathbb{P} : T_x\mathbb{R}^N \to T_xM$ of the derivative $\dot{\xi}(s)$ taken in $\mathbb{R}^N$. Thus the norm in $\mathcal{H}^1(M)\times\mathbb{R}^+$ is smaller than the induced norm from $\mathcal{H}^1(\mathbb{R}^N) \times \mathbb{R}^+$. Formulas analogous to (5) hold for the norm on $\mathcal{H}^1(M) \times \mathbb{R}^+$. We also want to compare the metric on $\mathcal{H}^1(M) \times \mathbb{R}^+$ with the metric induced by $\mathcal{H}^1(\mathbb{R}^m) \times \mathbb{R}^+$ on a local chart $\mathbb{R}^m \supset U \hookrightarrow M$. In Lemma 2.2 below we shall prove that the three norms are locally equivalent.

Given $k \in \mathbb{R}$, define the action functionals $\mathcal{A}_k : \Omega_M(q_0, q_1) \to \mathbb{R}$ and $\mathcal{A}_k : \Lambda_M \to \mathbb{R}$ of $L + k$ by

$$\mathcal{A}_k(x, T) = \int_0^1 T \left[L\left(x(s), \frac{\dot{x}(s)}{T}\right) + k \right] ds.$$

Writing $y(t) := x(\frac{t}{T})$, $0 \le t \le T$, we have that

$$\mathcal{A}_k(x, T) = \int_0^T [L(y, \dot{y}) + k]\ dt =: A_{L+k}(y).$$

We say that a lagrangian L is *quadratic at infinity* if there is $R > 0$, a 1-form θ_x on M and $a, \psi \in C^\infty(M, \mathbb{R})$, $a > 0$, such that $L(x, v) = \frac{1}{2}\, a(x)\, |v|_x^2 + \theta_x(v) + \psi(x)$ for all $|v|_x \ge R$, where $|v|_x$ is the Riemannian norm of v in *TM*.

We say that L is *Riemannian at infinity* if there exists $R > 0$ such that $L(x, v) = \frac{1}{2}|v|_x^2$ for all $|v|_x > R$. Since we are assuming that M is isometrically embedded in $\mathbb{R}^N$, this is equivalent to $L(x, v) = \frac{1}{2}\, |v|^2$ for $|v| > R$, where

$|v|$ is the euclidean norm of v and $(x, v) \in TM \subset \mathbb{R}^N \times \mathbb{R}^N$. In a coordinate chart such a lagrangian is given as $L(x, v) = \frac{1}{2}\, v^* G(x) v$, when $|v|$ is large enough, where $G(x)$ is the matrix of the Riemannian metric in the chart. Then in coordinate charts L is quadratic at infinity.

It follows from a result of Smale [29] that if L is Riemannian at infinity then the action functional $\mathcal{A}_k$ is C^2 on $\mathcal{H}^1(M) \times \mathbb{R}^+$ with the metrics with $f \equiv g \equiv 1$. Since the Riemannian metrics (4), (6) are locally equivalent to the metrics with $f \equiv g \equiv 1$, then $\mathcal{A}_k$ is C^2 on $\Omega_M(q_0, q_1)$ and on Λ_M, with respect to all three Riemannian metrics.

The derivative of $\mathcal{A}_k$ is given by

$$\begin{aligned} d_{(x,T)}\mathcal{A}_k(\xi, \alpha) &= \int_0^1 T\left[L_x\left(x, \frac{\dot{x}}{T}\right)\xi + L_v\left(x, \frac{\dot{x}}{T}\right)\frac{\dot{\xi}}{T}\right] ds + \alpha \int_0^1 \left[k - E\left(x, \frac{\dot{x}}{T}\right)\right] ds \\ &= \int_0^T [L_x(y, \dot{y})\,\zeta + L_v(y, \dot{y})\,\dot{\zeta}]\, dt + \frac{\alpha}{T}\int_0^T [k - E(y, \dot{y})]\, dt, \end{aligned} \tag{7}$$

where $y(t) = x(\frac{t}{T})$, $\zeta(t) = \xi(\frac{t}{T})$, for $0 \le t \le T$ and $E : TM \to \mathbb{R}$,

$$E(x, v) = v\, L_v(x, v) - L(x, v)$$

is the energy function. The formulas (7) can be interpreted either in local charts with usual derivatives or in covariant derivatives. In the latter case,

$$L_x\, \xi = \langle \nabla_x L, \xi \rangle_{x(s)} \quad \text{and} \quad L_v\, \dot{\xi} = \left\langle \nabla_v L, \frac{D}{ds}\xi \right\rangle_{x(s)},$$

where $\nabla_x L$ and $\nabla_v L$ are the projections of the gradient of L in the splitting $T_{(x,\dot{x})}TM = H \oplus V$ and $\frac{D}{ds}\xi$ is the covariant derivative of ξ. The splitting $T_\theta TM = H(\theta) \oplus V(\theta)$ is described after Proposition 3.13 below.

Fix $C_1 > 0$. We say that $f : U \subset \mathbb{R}^m \to M$ is a *bounded chart* if f is an embedding such that the pull-back f^*g of the Riemannian metric has matrix $G(x)$ such that G and G^{-1} have C^1 norm bounded by C_1. Fix a finite atlas of bounded charts $\mathcal{U} = \{U_i\}_{i \in I}$ such that each $U_i \subset \mathbb{R}^m$ is a convex set.

We fix some constants used repeatedly later on. Observe that the property of being quadratic at infinity is invariant under transformations by bounded charts. The following constants are taken to hold in any bounded chart of our finite atlas, i.e. in Eqs. (8)–(11) below the same constants are assumed to hold when the norm $|\cdot|_x$ is interpreted as either the riemannian metric on M, the euclidean norm on $\mathbb{R}^N \supset M$ or the euclidean norm on any bounded chart $U_i \subset \mathbb{R}^m$ of our atlas. The norms $|L_x|$, $|L_{xv}|$ are interpreted as the euclidean norm in any bounded chart $U_i \in \mathcal{U}$.

Since L is uniformly convex,

$$a_0 := \inf_{(x,v) \in TM} \frac{v \cdot L_{vv}(x, v) \cdot v}{|v|_x^2} > 0. \tag{8}$$

Since L is quadratic at infinity there are $a_1,\ a_2,\ A_0,\ b_1 > 0$ such that

$$L(x, v) \ge a_1\, |v|_x^2 - a_2, \qquad \text{for all } (x, v) \in TM, \tag{9}$$

$$A_0 := \sup_{(x,v)} \|L_{vv}(x,v)\|_x < +\infty, \tag{10}$$

$$b_1 := \sup_{x\in M} \|d_x\psi\|_x\,, \qquad \text{where } \psi(x) := L(x,0). \tag{11}$$

Since L is quadratic at infinity,

$$b_2 := \sup_{(x,v)\in TM} \frac{|L_x(x,v)|}{1+|v|_x^2} < +\infty, \tag{12}$$

$$b_3 := \sup_{(x,v)\in TM} \frac{|L_{xv}(x,v)|}{1+|v|_x} < +\infty. \tag{13}$$

Lemma 2.1 *If* $(x,T) \in \Omega_M(q_0,q_1)$ [resp. $(x,T)\in\Lambda_M$] *is a critical point of the action functional* $\mathcal{A}_k : \Omega_M(q_0,q_1) \to \mathbb{R}$ [resp. $\mathcal{A}_k : \Lambda_M \to \mathbb{R}$] *then the curve* $y:[0,T]\to M$, $y(t) := x(t/T)$ *is a differentiable solution of the Euler-Lagrange equation* $\frac{d}{dt}L_v(y,\dot y) = L_x(y,\dot y)$ *with* $y(0)=q_0$, $y(T)=q_1$, [resp. $(y,\dot y)$ is a closed orbit of the Euler-Lagrange flow] *with energy* $E(y,\dot y)\equiv k$.

Proof Cover the image $y([0,T])$ by images $\widetilde{U}_i \subset M$ of charts $U_i \in \mathcal{U}$, $U_i \subset \mathbb{R}^m$. It is enough to prove that y is a solution of the Euler-Lagrange equation on each intersection $y([0,T])\cap\widetilde{U}_i$. Assume for a while that $y([0,T])\subset U_i\subset\mathbb{R}^m$. Using the same notation as in (7), we have that

$$\begin{aligned} d_{(x,T)}\mathcal{A}_k(\xi,0) &= \int_0^T [L_x(y,\dot y)\,\zeta + L_v(y,\dot y)\,\dot\zeta]\,dt \\ &= \mathbb{L}_x\cdot\zeta\big|_0^T + \int_0^T [L_v(y,\dot y) - \mathbb{L}_x(t)]\cdot\dot\zeta\,dt \\ &= 0, \end{aligned} \tag{14}$$

where $\mathbb{L}_x(t) := \int_0^t L_x\big(y(s),\dot y(s)\big)\,ds$. Since $L_x(y,\dot y) \le b_2(1+|\dot y|_x^2)$ and $x\in\mathcal{H}^1(M)$, we have that $L_x(y,\dot y)\in\mathcal{L}^1([0,T],\mathbb{R}^m)$ and that $\mathbb{L}_x$ is continuous in view of Lebesgue's theorem.

Since for both Λ_M and $\Omega_M(q_0,q_1)$ we can choose $\zeta(0)=\zeta(T)=0$,

$$\int_0^T [L_v(y,\dot y) - \mathbb{L}_x(t)]\cdot\dot\zeta\,dt = 0$$

for all $\dot\zeta\in\mathcal{L}^2([0,T],\mathbb{R}^m)$ with $\int_0^T \dot\zeta\,dt = 0$.

This implies that $L_v(y,\dot y) - \mathbb{L}_x$ is constant a.e. in $[0,T]$. Since $\mathbb{L}_x(t)$ is continuous, it is bounded on $[0,T]$. Since L is superlinear and $\mathbb{L}_x$ is bounded, $\dot y$ is bounded by a constant almost everywhere. Since L is convex, $v\mapsto L_v(y,v)$ is a continuous bijection. Hence we can uniquely extend $\dot y$ to $[0,T]$ so that $L_v(y,\dot y)-\mathbb{L}_x$ is constant on all $t\in[0,T]$. Since $\mathbb{L}_x(t)$ is continuous, $L_v(y,\dot y)$ is also continuous and hence $\dot y(t)$ is continuous.

We have that

$$L_v(y(t),\dot y(t)) = A + \int_0^t L_x(y,\dot y)\,dt \tag{15}$$

for some constant $A \in \mathbb{R}^m$. Since $\dot{y}(t)$ is continuous, the right hand side of (15) is differentiable and

$$\frac{d}{dt} L_v(y, \dot{y}) = L_x(y, \dot{y}).$$

Hence $y(t)$ is a differentiable solution of the Euler-Lagrange equation. The theory of ordinary differential equations implies that y is C^r if L is C^{r+2}.

Since $y(t)$ is a solution of the Euler-Lagrange equation, its energy $E(y(t), \dot{y}(t))$ is constant. Since

$$\left.\frac{\partial \mathcal{A}_k}{\partial T}\right|_{(x,T)} = \frac{1}{T} \int_0^T [\, k - E(y, \dot{y})]\, dt = 0,$$

$E(y, \dot{y}) \equiv k$. This completes the case of $\Omega_M(q_0, q_1)$.

For the case of Λ_M it remains to prove that $\dot{y}(0) = \dot{y}(T)$. We choose a chart $U_i \subset \mathcal{U}$ whose image contains $y(0) = y(T)$ and restrict ourselves to vector fields ζ over y with support in the connected component of $y([0, T]) \cap \widetilde{U}_i$ containing $y(0)$. Since we already know that $(y, \dot{y})$ is a differentiable solution of the Euler-Lagrange equation, integrating by parts in (14) we have that

$$\begin{aligned} d_{(x,T)}\mathcal{A}_k(\xi, 0) &= L_v\, \zeta\big|_0^T + \int_0^T \left(L_x - \frac{d}{dt} L_v\right) \zeta\, dt \\ &= [L_v(y(T), \dot{y}(T)) - L_v(y(0), \dot{y}(0))] \cdot \zeta(0) + 0 \qquad (16) \end{aligned}$$

whenever $\zeta(0) = \zeta(T) \in \mathbb{R}^m$. Then $L_v\big(y(T), \dot{y}(T)\big) = L_v\big(y(0), \dot{y}(0)\big)$. Since $y(T) = y(0)$ and $v \mapsto L_v(y(0), v)$ is injective, $\dot{y}(T) = \dot{y}(0)$. □

The following lemma shows that the intrinsic Riemannian metric $\|\cdot\|_{(x,T)}^{\mathcal{H}^1(M)\times\mathbb{R}^+}$ given by (6) on $\mathcal{H}^1(M) \times \mathbb{R}^+$ and the induced metric from $\mathcal{H}^1(\mathbb{R}^N) \times \mathbb{R}^+$ are locally equivalent. Also the metric on $\mathcal{H}^1(M) \times \mathbb{R}^+$ and the metric on $\mathcal{H}^1(\mathbb{R}^m) \times \mathbb{R}^+$, $m = \dim M$, on a bounded coordinate chart are locally equivalent.

Lemma 2.2

1. *Given* $A_1, T_1 > 10$, *there exists* $B = B(A_1, T_1, k, \{f, g\}) > 0$ *such that if* $(x, T) \in \mathcal{H}^1(M) \times \mathbb{R}^+$, $|\mathcal{A}_k(x, T)| < A_1$ *and* $T < T_1$, *then for all* $(\xi, \alpha) \in T_{(x,T)}\Omega_M(q_0, q_1) \cup T_{(x,T)}\Lambda_M$,

$$\frac{1}{B}\ \|(\xi,\alpha)\|_{(x,T)}^{\mathcal{H}^1(\mathbb{R}^N)\times\mathbb{R}^+} \le \|(\xi,\alpha)\|_{(x,T)}^{\mathcal{H}^1(M)\times\mathbb{R}^+} \le \|(\xi,\alpha)\|_{(x,T)}^{\mathcal{H}^1(\mathbb{R}^N)\times\mathbb{R}^+}. \qquad (17)$$

2. *Let* $\psi : U \subset \mathbb{R}^m \to M$ *be an immersion such that the pull-back* $\psi^* g_M(v, w) = v^* G(x)\, w$ *of the Riemannian metric on M has matrix* $G(x)$ *which is bounded in the* C^1*-norm:*

$$\max\left\{ \|G\|_{C^1(U,\mathbb{R}^{m\times m})}, \|G^{-1}\|_{C^1(U,\mathbb{R}^{m\times m})}\right\} < C_1.$$

For all $A_1, T_1 > 10$ *there exists* $B = B(A_1, T_1, C_1, k, \{f, g\}) > 0$ *such that if* $(x, T) \in \mathcal{H}^1(\mathbb{R}^m) \times \mathbb{R}^+$, $|\mathcal{A}_k(\psi \circ x, T)| < A_1$ *and* $T < T_1$, *then for all* $(d_x\psi \circ \xi, \alpha) \in T_{(x,T)}\Omega_M(q_0, q_1) \cup T_{(x,T)}\Lambda_M$,

$$\frac{1}{B}\ \|(\xi,\alpha)\|_{(x,T)}^{\mathcal{H}^1(\mathbb{R}^m)\times\mathbb{R}^+} \le \|(d_x\psi\circ\xi,\alpha)\|_{(x,T)}^{\mathcal{H}^1(M)\times\mathbb{R}^+} \le B\ \|(\xi,\alpha)\|_{(x,T)}^{\mathcal{H}^1(\mathbb{R}^m)\times\mathbb{R}^+}.$$

Proof (1). Let $y(sT) := x(s)$. Then

$$A_1 > A_{L+k}(y) \geq \int_0^T \left(a_1\,|\dot y|_y^2 - a_2 + k\right)\,dt = a_1 \int_0^T |\dot y|_y^2\,dt - (a_2 - k)\,T,$$

$$\int_0^T |\dot y(t)|_y^2\,dt \leq \frac{A_1 + (a_2 - k)\,T}{a_1}, \tag{18}$$

$$\int_0^1 |\dot x(s)|_x^2\,ds = T \int_0^T |\dot y(t)|_y^2\,dt \leq T\left[\frac{A_1 + (a_2 - k)\,T}{a_1}\right]. \tag{19}$$

Let $\mathbb{T}(s,r) : T_{x(r)}M \to T_{x(s)}M$ be the parallel transport along $x(s)$. Then

$$\xi(s) = \mathbb{T}(s,0)\cdot\xi(0) + \int_0^s \mathbb{T}(s,r)\cdot\frac{D}{dr}\xi(r)\,dr.$$

Thus

$$\begin{aligned}
|\xi(s)|_{x(s)} &\leq |\xi(0)|_{x(0)} + \left\|\frac{D}{ds}\xi\right\|_{\mathcal{L}^1([0,1])} \leq |\xi(0)|_{x(0)} + \left\|\frac{D}{ds}\xi\right\|_{\mathcal{L}^2([0,1])}\\
&\leq \max\left\{\frac{1}{\sqrt{f(T)}},\,\frac{1}{\sqrt{g(T)}}\right\}\ \sqrt{2}\ \|(\xi,0)\|_{(x,T)}^{\mathcal{H}^1(M)\times\mathbb{R}^+}\\
&\leq \sqrt{2}\ \max\left\{T^{-1},\,\alpha^{-\frac{1}{2}},\,T_1^{\frac{1}{2}}e^{2T_1^2}\right\}\ \|(\xi,0)\|_{(x,T)}^{\mathcal{H}^1(M)\times\mathbb{R}^+}
\end{aligned}\tag{20}$$

where

$$\alpha := \min_{1\leq t\leq 10}\{f(t),\ g(t)\}.$$

By their definition $\max\{f,\ g\} \leq 2$, hence

$$g(T) \leq \min\{2\,T^2,\ 2\}. \tag{21}$$

Observe that for $\xi, \eta \in T_xM$ the first two terms in (4), (6) are equal, so we only have to bound the $\mathcal{L}^2$ norm of the derivatives $\dot\xi$ and $\frac{D}{ds}\xi$.

Let U be a small tubular neighbourhood of M in $\mathbb{R}^N$ and let $F : U \to M$ be the orthogonal projection onto M. Since $\xi(s) \in T_{x(s)}M$, we have that

$$d_{x(s)}F\cdot\xi(s) = \xi(s).$$

Differentiating this equation with respect to s we get that

$$d^2_{x(s)}F(\dot x(s),\xi(s)) + d_{x(s)}F\cdot\dot\xi(s) = \dot\xi(s).$$

The second term is the projection of $\dot\xi$ to T_xM:

$$d_{x(s)}F\cdot\dot\xi(s) = \mathbb{P}\cdot\dot\xi(s) = \frac{D}{ds}\xi(s);$$

and the first term is the projection of $\dot\xi$ to the orthogonal complement $T_xM^\perp$ of T_xM:

$$\dot\xi^\perp(s) := d^2_{x(s)}F(\dot x(s),\xi(s)).$$

Let $c_1 := \sup_{x\in M} \|d_x^2 F\|^2$, then using (19), (20) and (21) we have that

$$
\begin{aligned}
g(T)\int_0^1 |\dot\xi^{\perp}(s)|^2\, ds &\le g(T)\int_0^1 c_1\, |\dot x(s)|^2\, |\xi(s)|^2\, ds\\
&\le c_1\, g(T)\, \|\xi\|_\infty^2 \int_0^1 |\dot x(s)|^2\, ds\\
&\le c_1\, \min\{2\,T^2, 2\}\; 2\, \max\left\{T^{-2}, \alpha^{-1}, T_1\, e^{4T_1^2}\right\}\\
&\quad\times \left[\|(\xi,0)\|_{(x,T)}^{\mathcal{H}^1(M)\times\mathbb{R}^+}\right]^2 T \left[\frac{A_1+(a_2-k)\,T}{a_1}\right]\\
&\le c_1\, B_1\, \left[\|(\xi,0)\|_{(x,T)}^{\mathcal{H}^1(M)\times\mathbb{R}^+}\right]^2, \qquad (22)
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &= B_1(A_1, a_1, a_2, T_1, k, \{f, g\})\\
&:= 4\,T_1 \left[\frac{A_1+(a_2+|k|)\,T_1}{a_1}\right] \max\left\{\alpha^{-1}, T_1\, e^{4T_1^2}\right\}.
\end{aligned}
$$

Observe that the bound $c_1\, B_1$ above holds for all $0 < T < T_1$.

Since $\dot\xi = \mathbb{P}\,\dot\xi + \dot\xi^{\perp}$, we have that

$$
\begin{aligned}
g(T)\, \|\dot\xi\|_{\mathcal{L}^2}^2 &\le g(T)\left[\|\mathbb{P}\,\dot\xi\|_{\mathcal{L}^2}^2 + \|\dot\xi^{\perp}\|_{\mathcal{L}^2}^2\right]\\
&\le g(T)\left\|\frac{D}{ds}\xi\right\|_{\mathcal{L}^2}^2 + c_1\, B_1\, \left[\, \|(\xi,0)\|_{(x,T)}^{\mathcal{H}^1(M)\times\mathbb{R}^+}\,\right]^2.
\end{aligned}
$$

Then, for all $0 < T < T_1$,

$$
\left[\, \|(\xi,\alpha)\|_{(x,T)}^{\mathcal{H}^1(\mathbb{R}^N)\times\mathbb{R}^+}\,\right]^2 \le [1 + c_1\, B_1]\left[\, \|(\xi,\alpha)\|_{(x,T)}^{\mathcal{H}^1(M)\times\mathbb{R}^+}\,\right]^2.
$$

Since $|\frac{D}{ds}\xi| = |\mathbb{P}\,\dot\xi| \le |\dot\xi|$, then $\|\frac{D}{ds}\xi\|_{\mathcal{L}^2} \le \|\dot\xi\|_{\mathcal{L}^2}$. This implies the second inequality in (17).

(2). We have that

$$
\frac{D}{ds}\xi = \dot\xi(s) + \sum_{ijk} \Gamma_{ij}^k(x(s))\, \dot x_i(s)\, \xi_j(s)\, e_k,
$$

where the $\Gamma_{ij}^k(x)$ are the Christoffel symbols for the Riemannian metric of M in the coordinate chart ψ^{-1} and e_k is the k-th vector of the canonical basis of $\mathbb{R}^m$. Our hypothesis on ψ implies that $c_2 = c_2(C_1) := m^6 \sup_{ijk, x\in U} |\Gamma_{ij}^k(x)|^2$ is finite. Then

$$
\left|\frac{D}{ds}\xi\right| \le |\dot\xi(s)| + \sqrt{c_2}\, \|\xi\|_\infty\, |\dot x(s)|\,.
$$

Similar calculations as in (20) and (22) using $\dot\xi$ in (20) instead of the covariant derivative show that if $0 < T < T_1$, then

$$
g(T)\left\|\frac{D}{ds}\xi\right\|_{\mathcal{L}^2}^2 \le 2\, g(T)\, \|\dot\xi\|_{\mathcal{L}^2}^2 + 2\, c_2\, B_2\left(\, \|(\xi,0)\|_{(x,T)}^{\mathcal{H}^1(\mathbb{R}^m)\times\mathbb{R}^+}\,\right)^2,
$$

where $B_2 := B_1(A_1, \overline{a}_1, \overline{a}_2, T_1, k, \{f, g\})$ and $\overline{a}_1, \overline{a}_2$ are constants such that the inequality (9) holds in our coordinate system ψ for the euclidean metric in $U \subset \mathbb{R}^m$ instead of the riemannian metric $|\cdot|_x$ on M. Then, if $0 < T < T_1$, we have that

$$\|(\xi,\alpha)\|_{(x,T)}^{\mathcal{H}^1(M)\times\mathbb{R}^+} \leq \sqrt{2}\,[1 + c_2\,B_2]^{\frac{1}{2}}\ \ \|(\xi,\alpha)\|_{(x,T)}^{\mathcal{H}^1(\mathbb{R}^m)\times\mathbb{R}^+}.$$

Now write

$$\dot{\xi}(s) = \frac{D}{ds}\xi - \sum_{ijk} \Gamma_{ij}^k(x(s))\ \dot{x}_i(s)\ \xi_j(s)\ e_k,$$

$$\left|\dot{\xi}(s)\right| \leq \left|\frac{D}{ds}\xi\right| + \sqrt{c_2}\,\sqrt{C_1}\ |\xi(s)|_{x(s)}\ \ |\dot{x}(s)|\,.$$

Writing $c_3 := c_2\,C_1$, the same calculations as in (20) and (22) give

$$g(T)\ \|\dot{\xi}\|_{\mathcal{L}^2}^2 \leq 2\,g(T)\ \left\|\frac{D}{ds}\xi\right\|_{\mathcal{L}^2}^2 + 2\,c_3\,B_2\left(\,\|(\xi,0)\|_{(x,T)}^{\mathcal{H}^1(M)\times\mathbb{R}^+}\,\right)^2.$$

And then

$$\|(\xi,\alpha)\|_{(x,T)}^{\mathcal{H}^1(\mathbb{R}^m)\times\mathbb{R}^+} \leq \sqrt{2}\,[1 + c_3\,B_2]^{\frac{1}{2}}\ \ \|(\xi,\alpha)\|_{(x,T)}^{\mathcal{H}^1(M)\times\mathbb{R}^+}. \qquad \square$$

In the next lemma, we write d for the distance $d_{\mathcal{H}^1(M)\times\mathbb{R}^+}$ on $\Omega_M(q_0, q_1)$ or on Λ_M.

Lemma 2.3 *Given $T_0>0$ there exists $C = C(T_0)>0$ and $\varepsilon = \varepsilon(T_0)>0$ such that if $T \in [\frac{1}{T_0}, T_0]$, (x,T), $(y,S) \in \Omega_M(q_0,q_1) \cup \Lambda_M$ and $d((x,T),(y,S)) < \varepsilon$ then for the Hausdorff distance d_H induced by the Riemannian metric, we have that*

$$d_H\big(x([0,1]),\, y([0,1])\big) < C\ d\big((x,T),(y,S)\big).$$

Proof Let

$$A_3 := 4\ \max\left\{\frac{1}{f(t)},\ \frac{1}{g(t)}\ \Big|\ t \in \left[\frac{1}{2T_0}, 2T_0\right]\right\},$$

where $f(t)$ and $g(t)$ are as given in the definition of the Riemannian metric on $\mathcal{H}^1(M)$. Let $0 < \varepsilon_0 < 1$ be such that

$$\frac{1}{T_0} - 2\varepsilon_0 > \frac{1}{2T_0} \quad \text{ and } \quad T_0 + 2\varepsilon_0 < 2T_0.$$

Let $0 < \varepsilon = \varepsilon(T_0) < \varepsilon_0$ and write $\delta := d\big((x,T),(y,S)\big) < \varepsilon$. There is a curve $\Gamma(\lambda) = (z_\lambda, T_\lambda)$, $\lambda \in [0,1]$, from (x,T) to (y,S) in $\Omega_M(q_0,q_1)$ or in Λ_M such that

$$\text{length}(\Gamma) = \int_0^1 \left\|\frac{d}{d\lambda}\Gamma(\lambda)\right\|\ d\lambda < 2\delta.$$

We can reparametrize Γ so that the norm of its tangent vector is constant:

$$\left\|\frac{d}{d\lambda}\Gamma(\lambda)\right\|^2 = \left|\frac{d\,T_\lambda}{d\lambda}\right|^2 + f(T_\lambda)\ \left|\frac{\partial z_\lambda(0)}{\partial\lambda}\right|_{z_\lambda(0)}^2 + g(T_\lambda)\int_0^1 \left|\frac{D}{d\lambda}\dot{z}_\lambda\right|_{z_\lambda(s)}^2\ ds < 4\,\delta^2.$$

Since $|T_\lambda - T| \le d\big(\Gamma(\lambda),(x,T)\big) \le 2\delta < 2\varepsilon_0$, then $S = T_{\lambda=1},\ T_\lambda,\ T \in [\frac{1}{2T_0}, 2T_0]$. Hence

$$
\begin{aligned}
\left|\frac{\partial z_\lambda(0)}{\partial\lambda}\right|^2_{z_\lambda(0)} &< A_3\,\delta^2 \qquad \text{for } \lambda\in[0,1],\\
\int_0^1 \left|\frac{D}{d\lambda}\dot z_\lambda\right|^2_{z_\lambda(s)} ds &< A_3\,\delta^2 \qquad \text{for } \lambda\in[0,1].
\end{aligned}
\tag{23}
$$

Let

$$F(s) := \frac{1}{2}\int_0^1 \left|\frac{\partial z_\lambda(s)}{\partial\lambda}\right|^2_{z_\lambda(s)} d\lambda.$$

From (23), $|F(0)| < A_3\,\delta^2$. We have that

$$
\begin{aligned}
F(s)-F(0) &= \int_0^s \frac{d}{ds}F(s)\,ds\\
&= \int_0^s\int_0^1 \left\langle \frac{D}{ds}\frac{\partial}{\partial\lambda}z_\lambda(s),\ \frac{\partial}{\partial\lambda}z_\lambda(s)\right\rangle_{z_\lambda(s)} d\lambda\,ds\\
&= \int_0^s\int_0^1 \left\langle \frac{D}{d\lambda}\dot z_\lambda(s),\ \frac{\partial}{\partial\lambda}z_\lambda(s)\right\rangle_{z_\lambda(s)} d\lambda\ \ ds\ ,\\
|F(s)-F(0)| &\le \left[\int_0^1\int_0^1 \left|\frac{D}{d\lambda}\dot z_\lambda(s)\right|^2 ds\,d\lambda\right]^{\frac12}\left[\int_0^s\int_0^1 \left|\frac{\partial z_\lambda(s)}{\partial\lambda}\right|^2_{z_\lambda(s)} d\lambda\,ds\right]^{\frac12}\\
&\le \sqrt{A_3\,\delta^2}\,\left[2\int_0^s F(s)\,ds\right]^{\frac12}.\\
F(s) &\le A_3\,\delta^2 + \sqrt{2A_3}\,\delta\left[\int_0^s F(s)\,ds\right]^{\frac12}.
\end{aligned}
$$

Write

$$u(s) := \left[\int_0^s F(t)\,dt\right]^{\frac12}.$$

Then

$$\frac{d}{ds}\,u(s)^2 \le A_3\,\delta^2 + \delta\sqrt{2A_3}\ u(s).$$

Let

$$J := \left\{ t\in[0,1]\ \middle|\ u(s)\le\sqrt{2A_3}\,\delta\left(s+\frac12\right),\ \forall s\in[0,t]\right\}.$$

Then J is a closed interval. We show that J is also open in [0, 1]. If $t \in J$, then

$$\begin{aligned} \frac{d}{ds} u(s)^2 &\leq A_3\,\delta^2 + 2\,A_3\,\delta^2 \left(s + \frac{1}{2}\right), \\ &\leq 2\,A_3\,\delta^2\,(s+1), \qquad \forall s \in [0,t]. \\ u(t)^2 &\leq 2\,A_3\,\delta^2 \left(\frac{1}{2}\,t^2 + t\right), \qquad \forall s \in [0,t], \\ u(s) &\leq \sqrt{2\,A_3}\,\delta \sqrt{\frac{1}{2}s^2 + s}, \qquad \forall s \in [0,t]. \end{aligned}$$

Since $\sqrt{\frac{1}{2}s^2 + s} \;< s + \frac{1}{2}$ for all $s > 0$, we have that

$$u(t) < \sqrt{2\,A_3}\,\delta\,\big(t + \frac{1}{2}\big).$$

Then J contains an open neighbourhood of t in [0, 1]. Hence $J = [0, 1]$.

Therefore, for all $s \in [0, 1]$,

$$\begin{aligned} F(s) &\leq A_3\,\delta^2 + \sqrt{2A_3}\,\delta\,u(s) \\ &\leq A_3\,\delta^2 + 2\,A_3\,\delta^2 \sqrt{\frac{3}{2}} =: \frac{1}{2}\,C^2\,\delta^2. \end{aligned}$$

We have that

$$d_M(y(s), x(s)) = d_M(z_1(s), z_0(s)) \leq \int_0^1 \left|\frac{\partial z_\lambda(s)}{\partial \lambda}\right| \,d\lambda \leq \sqrt{2\,F(s)} \leq C\;\delta$$

for all $s \in [0, 1]$. This implies the lemma. □

3 The Palais-Smale condition

In this section we are interested in the validity of the Palais-Smale condition for the action functional $\mathcal{A}_k$ on a connected component Ω_1 (resp. Λ_1) of $\Omega_M(q_0, q_1)$ (resp. Λ_M).

Theorem A *If L is Riemannian at infinity and $\mathcal{A}_k$ does not satisfy the Palais-Smale condition on $\Omega_M(q_0, q_1)$, or on Λ_M, then there exists a Borel probability measure μ, invariant under the Euler-Lagrange flow, supported in a connected component of the energy level $E \equiv k$, which has homology $\rho(\mu) = 0$ and whose $(L + k)$-action is zero:*

$$A_{L+k}(\mu) = \int [L + k]\,d\mu = 0.$$

Proof of Theorem A Let (x_n, T_n) be a sequence in a connected component Ω_1 of $\Omega_M(q_0, q_1)$ (resp. Λ_1 of Λ_M) such that

$$|\mathcal{A}_k(x_n, T_n)| < A_1 \quad \text{and} \quad \left\| d_{(x_n,T_n)}\mathcal{A}_k \right\| < \frac{1}{n}.$$

Assume that (x_n, T_n) does not have an accumulation point in $\Omega_M(q_0, q_1)$ (resp. Λ_M). Then Proposition 3.12 implies that either $\liminf_n T_n = 0$ or $\limsup_n T_n = +\infty$.

If $\limsup_n T_n = +\infty$, Proposition 3.13 implies the thesis of the theorem. So assume that $\liminf_n T_n = 0$.

If $\langle (x_n, T_n) \rangle \subset \Omega_M(q_0, q_1)$ with $q_1 \neq q_0$ then Corollary 3.6 shows that $\liminf_n T_n > 0$. This contradicts our assumption. Hence $q_0 = q_1$. If either $\langle (x_n, T_n) \rangle \subset \Omega_M(q_0, q_1)$ with $q_0 = q_1$ or $\langle (x_n, T_n) \rangle \subset \Lambda_M$, then Proposition 3.8 and Remark 3.9 imply the thesis of the theorem. □

3.1 Preliminary lemmas

Lemma 3.1 *If L is convex and quadratic at infinity, then*

$$\frac{1}{2}\, a_0\, |v|_x^2 + \theta_x(v) + \psi(x) \le L(x, v) \le \frac{1}{2}\, A_0\, |v|_x^2 + \theta_x(v) + \psi(x),$$
$$-\psi(x) + \frac{1}{2}\, a_0\, |v|_x^2 \le E(x, v) \le -\psi(x) + \frac{1}{2}\, A_0\, |v|_x^2,$$

where $\theta_x(v) := L_v(x, 0) \cdot v$ and $\psi(x) := L(x, 0)$.

Proof Let $L_0(x, v) := L(x, v) - \theta_x(v) - \psi(x)$. Let $f(t) := L_0(x, tv)$. Then $f(0) = 0$, $f'(0) = 0$ and $f''(t) = v \cdot \frac{\partial L}{\partial v^2}(x, tv) \cdot v$, so that $a_0\, |v|_x^2 \le f''(t) \le A_0\, |v|_x^2$. Hence

$$L_0(x, v) = \int_0^1 \int_0^t f''(s)\, ds\, dt \ge \int_0^1 \int_0^t a_0\, |v|_x^2\, ds\, dt \ge \frac{1}{2}\, a_0\, |v|_x^2$$
$$\le \frac{1}{2}\, A_0\, |v|_x^2.$$

Now let $g(t) := E(x, tv) = tv \cdot L_v(x, tv) - L(x, tv)$. Then $g(0) = -\psi(x)$ and $g'(t) = t\ v \cdot L_{vv}(x, tv) \cdot v$, so that $t\, a_0 |v|_x^2 \le g'(t) \le t\, A_0 |v|_x^2$. Therefore,

$$E(x, v) = g(1) = g(0) + \int_0^1 g'(t)\, dt \ge -\psi(x) + \frac{1}{2}\, a_0\, |v|_x^2$$
$$\le -\psi(x) + \frac{1}{2}\, A_0\, |v|_x^2\, . \qquad \square$$

Let $\lambda > 0$ be a Lebesgue number for our finite atlas $\mathcal{U} = \{U_i\}$ of bounded charts.

Lemma 3.2 *Suppose that L is quadratic at infinity.*

If $U_i \in \mathcal{U}$, $x, y \in U_i$ and $d_M(x, y) < \lambda$ then in the chart U_i we have:

(i) $[L_v(x,v)-L_v(y,w)]\cdot\zeta+b_3(|v|+|w|+1)\,|\zeta|\,|x-y|+A_0\,|\zeta|\,|v-w|\geq 0.$

(ii) $a_0\,|v-w|^2\leq[L_v(x,v)-L_v(y,w)]\cdot(v-w)+b_3\,(|v|+|w|+1)\,|v-w|\,|x-y|.$

Proof Recall that the domains $U_i\subset\mathbb{R}^m$ are convex. We work in local coordinates as if L were defined in $TU_i\subset\mathbb{R}^{2n}$.

$$\begin{aligned}\int_0^1 &\zeta\cdot L_{vv}(t\,(x,v)+(1-t)\,(y,w))\cdot(v-w)\,dt\\ &=(L_v(x,v)-L_v(y,w))\cdot\zeta\\ &\quad-\int_0^1\zeta\cdot L_{xv}(t\,(x,v)+(1-t)\,(y,w))\cdot(x-y)\,dt.\end{aligned}$$

This implies (i). Using $\zeta=v-w$ one gets (ii). □

Lemma 3.3 *Let $\mathcal{C}:=C^\infty([0,1],M)\times\mathbb{R}^+$.*
The subsets $\mathcal{C}\cap\Omega_M(q_0,q_1)$ and $\mathcal{C}\cap\Lambda_M$ are dense in $\Omega_M(q_0,q_1)$ and Λ_M respectively.

Proof We prove the lemma for $\Omega_M(q_0,q_1)$. The proof for Λ_M is similar. Let $\lambda>0$ be a Lebesgue number for our finite atlas $\mathcal{U}$.

Suppose first that length$(x)<\lambda$. Then the image of x lies inside of a domain of a chart $U_i\in\mathcal{U}$ and by Lemma 2.2 we can assume that $M=U_i\subset\mathbb{R}^m$. Extend $x:[0,1]\to M$ to $\mathbb{R}$ by setting $x(t)=x(0)$ for $t<0$ and $x(t)=x(1)$ for $t>1$. Then the extension is also in the Sobolev space $W^{1,2}_{\text{loc}}(\mathbb{R},\mathbb{R}^m)$, see [9, Sect. 4.1]. Let $\eta\in C^\infty(\mathbb{R},\mathbb{R})$ be

$$\eta(t):=\begin{cases}C\,\exp\left(\dfrac{1}{t^2-1}\right) & \text{if } |t|\leq 1,\\ 0 & \text{if } |t|\geq 1,\end{cases}$$

where the constant C is chosen such that $\int_{\mathbb{R}}\eta\,dt=1$. For $\varepsilon>0$ let $\eta_\varepsilon(t):=\frac{1}{\varepsilon}\,\eta(\frac{t}{\varepsilon})$. Define

$$x^\varepsilon(t):=\int_{\mathbb{R}}\eta_\varepsilon(s-t)\,x(s)\,ds.$$

Then $x^\varepsilon\in C^\infty(\mathbb{R},\mathbb{R}^m)$, $x^\varepsilon\to x$ uniformly on compact subsets and $\dot{x}^\varepsilon\to\dot{x}$ in $\mathcal{L}^2([0,1],\mathbb{R}^m)$, see [9, Sect. 4.2.1]. Let

$$y^\varepsilon(s):=x^\varepsilon(s)+(1-s)(x(0)-x^\varepsilon(0))+s(x(1)-x^\varepsilon(1)).$$

Then $(y^\varepsilon,T)\in\Omega_M(q_0,q_1)\cap\mathcal{C}$. Since $\lim_{\varepsilon\to 0}x^\varepsilon(0)=x(0)$, $\lim_{\varepsilon\to 0}x^\varepsilon(1)=x(1)$ and $\dot{x}^\varepsilon\to\dot{x}$ in $\mathcal{L}^2([0,1],\mathbb{R}^m)$ we have that $\lim_{\varepsilon\to 0}(y^\varepsilon,T)=(x,T)$ in $\Omega_M(q_0,q_1)$.

Now assume that length$(x)>\lambda$. Let $0=s_0<s_1<\cdots<s_N=1$ be such that length$(x|_{[s_{i-1},s_i]})<\frac{\lambda}{8}$. Let $x_i(t)=x\big(s_i+t\,(s_{i+1}-s_i)\big)$, $t\in[0,1]$. For each i do the construction above and obtain a C^∞ curve y_i with the same endpoints as x_i and which is near x_i in $\mathcal{H}^1(M)$. The curve $y=y_1*\cdots*y_N$, appropriately parametrized on $[0,1]$, is piecewise C^∞ and is near x in $\mathcal{H}^1(M)$. Let $0=s_0<t_1<s_1<t_2<\cdots<t_N<s_N=1$ be such that length$(y|_{[t_j,t_{j+1}]})<\frac{\lambda}{4}$. Then

$y|_{[t_j,t_{j+1}]}$ is in the domain of a chart $U_j \subset \mathbb{R}^m$ and it is C^∞ in neighbourhoods of t_j and t_{j+1}. Let $z_j(s) = y\big(t_j + s\,(t_{j+1} - t_j)\big)$, $s \in [0, 1]$. Let $c_j : [0, 1] \to U_j$ be a C^∞ curve such that $c_j = z_j$ in neighbourhoods of 0 and 1. Extend $(z_j - c_j)$ to $\mathbb{R}$ by setting $(z_j - c_j)(s) = 0$ if $s \in \mathbb{R} \setminus [0, 1]$. Then $(z_j - c_j)$ is C^∞. Let η_ε be as above and let

$$w_j^\varepsilon(t) := \int_{\mathbb{R}} \eta_\varepsilon(s - t) \cdot (z_j - c_j)(s)\, ds.$$

Then $w^\varepsilon \in C^\infty(\mathbb{R}, \mathbb{R}^m)$ and w^ε is near $(z_j - c_j)$ in $\mathcal{H}^1(\mathbb{R}^m)$. Since $(z_j - c_j) \equiv 0$ in neighbourhoods of 0 and 1 and $\text{supp}(\eta_\varepsilon) \subset [-\varepsilon, \varepsilon]$, if ε is small enough then $w^\varepsilon = 0$ in neighbourhoods of 0 and 1. Let

$$z_j^\varepsilon(t) := c_j(t) + w^\varepsilon(t), \qquad t \in [0, 1].$$

Then z_j^ε is C^∞, it is near z_j in $\mathcal{H}^1(\mathbb{R}^m)$ and coincides with z_j in neighbourhoods of 0 and 1. Let $z^\varepsilon := y|_{[0,t_1]} * z_1 * \cdots * z_{N-1} * y|_{[t_N,1]}$ appropriately parametrized on $[0, 1]$. Then z^ε is C^∞, and (z^ε, T) is near (x, T) in $\Omega_M(q_0, q_1)$. □

Lemma 3.4 *Suppose that the injectivity radius of M is larger than 2. There is $K > 0$ such that if $\gamma : [0, 1] \to M$ is a geodesic with $|\dot\gamma| \le 1$ and J is a Jacobi field along γ, then*

$$\max_{s\in[0,1]} \{|J(s)|, |J'(s)|, |J''(s)|\} \le K\,[|J(0)| + |J(1)|]$$

Proof We first prove that there is $K_0 > 0$ such that if J is a Jacobi field along γ and $J(0) = 0$ then

$$|J(s)| \le K_0\,|J(1)|.$$

Suppose that K_0 does not exist. Then for all $N \in \mathbb{N}$ there is a geodesic $\gamma_N : [0, 1] \to M$ with $|\dot\gamma_N| \le 1$, a Jacobi field J_N along γ_N with $J_N(0) = 0$, and $s_N \in [0, 1]$ such that $|J_N(s_N)| > N\,|J_N(1)|$. Since M is compact, taking a subsequence of $\langle s_N\rangle$ we can assume that the limits $s_0 = \lim_N s_N \in [0, 1]$, $(\overline{x}, \overline{v}) = \lim_N\,(\gamma_N(0), \dot\gamma_N(0)) \in TM$ and $\lim_N \frac{J_N(s_N)}{|J_N(s_N)|} \in TM$ exist. Since Jacobi fields are the projection of the derivative of the geodesic flow, which is C^1, the map $u \mapsto \frac{J_N(u)}{|J_N(s_N)|}$ converges to a Jacobi field $I(u)$ along the geodesic $\delta(u)$ with $(\delta(0), \delta'(0)) = (\overline{x}, \overline{v})$ such that $I(0) = 0$ and

$$|I(1)| = \lim_N \frac{|J_N(1)|}{|J_N(s_N)|} \le \lim_N \frac{|J_N(1)|}{N\,|J_N(1)|} = 0.$$

Also,

$$|I(s_0)| = \lim_N \frac{|J_N(s_N)|}{|J_N(s_N)|} = 1.$$

Then I is a non-trivial Jacobi field along a geodesic δ of length $|\overline{v}| \le 1$, which is zero at the endpoints. Therefore the geodesic δ has conjugate points. This contradicts[2] the hypothesis that the injectivity radius of M is larger than 2.

[2] When $\overline{v} = 0$ and δ is a constant geodesic, the Jacobi equation along δ is $J'' = 0$, which has no conjugate points.

Now let J be any Jacobi field along γ. Let $A(s)$, $B(s)$ be the Jacobi fields along γ satisfying $A(0) = 0$, $A(1) = J(1)$ and $B(0) = J(0)$, $B(1) = 0$. Since length(γ) ≤ 1 and the injectivity radius of M is larger than 2, the geodesic γ has no conjugate points. This implies that such Jacobi fields A and B exist.

By the estimate above $|A(s)| \leq K_0 \; |J(1)|$. Considering the Jacobi field $\widetilde{B}(s) := B(-s)$ along the geodesic $\tilde{\gamma}(s) := \gamma(-s)$ we get that $|B(s)| \leq K_0 \; |J(0)|$. Since $J(s) = A(s) + B(s)$, we get that

$$|J(s)| \leq K_0 \, [|J(0)| + |J(1)|] \quad \text{ for all } s \in [0, 1].$$

Since $|\dot{\gamma}| \leq 1$, from the Jacobi equation $J'' + R(\dot{\gamma}, J)\,\dot{\gamma} = 0$, we get that

$$|J''(s)| \leq b\,|J(s)| \leq b\,K_0 \;(|J(0)| + |J(1)|)\,,$$

for some $b = b(M) > 0$.

We have that

$$J(1) = T_0 \cdot J(0) + T_0 \cdot J'(0) + \int_0^1 \int_0^t T_s \cdot J''(s)\; ds\; dt,$$

where $T_s : T_{\gamma(s)}M \to T_{\gamma(1)}M$ is the parallel transport along γ. Then

$$\begin{aligned} |J'(0)| &\leq |J(0)| + |J(1)| + \int_0^1 \int_0^t |J''(s)|\; ds\; dt \\ &\leq (1 + bK_0)\,(|J(0)| + |J(1)|). \end{aligned}$$

Also,

$$\begin{aligned} |J'(t)| &\leq |J'(0)| + \int_0^t |J''(s)|\; ds \\ &\leq (1 + bK_0 + bK_0)\,(|J(0)| + |J(1)|). \end{aligned}$$

Now take $K = \max\{K_0,\ 1 + 2\,bK_0\}$. □

3.2 Palais-Smale sequences

During the rest of this section (x_n, T_n) will be a Palais-Smale sequence. This is, (x_n, T_n) will be a sequence in a fixed connected component of Λ_M or $\Omega_M(q_0, q_1)$ such that

$$|\mathcal{A}_k(x_n, T_n)| < A_1 \quad \text{ and } \quad \left\| d_{(x_n,T_n)}\mathcal{A}_k \right\|^{\mathcal{H}^1(M)\times\mathbb{R}^+}_{(x_n,T_n)} < \frac{1}{n}.$$

Also, L will be a convex lagrangian on a compact manifold M, Riemannian at infinity.

Write $y_n(t) := x_n(t/T_n)$, $0 \le t \le T_n$. As in (7) we compute

$$
\begin{aligned}
&d_{(x_n,T_n)}\mathcal{A}_k(\xi,\alpha)\\
&\quad = \int_0^1 \left[L_x\left(x_n, \frac{\dot{x}_n}{T_n}\right)\xi + L_v\left(x_n, \frac{\dot{x}_n}{T_n}\right)\frac{\dot{\xi}}{T_n}\right] T_n\, ds + \alpha \int_0^1 \left[k - E\left(x_n, \frac{\dot{x}_n}{T_n}\right)\right] ds\\
&\quad = \int_0^{T_n} [L_x(y_n,\dot{y}_n)\,\zeta + L_v(y_n,\dot{y}_n)\,\dot{\zeta}]\, dt + \frac{\alpha}{T_n}\int_0^{T_n} [k - E(y_n,\dot{y}_n)]\, dt,
\end{aligned}
\tag{24}
$$

where $\zeta(t) := \xi(t/T_n)$.

Lemma 3.5 *There exists $B = B(k, A_1, A_2) > 0$ such that if $x_n \in \mathcal{H}^1(M)$, $\mathcal{A}_k(x_n, T_n) \le A_1$ and $T_n \le A_2$, then*

$$\frac{\ell_n^2}{T_n} \le \int_0^{T_n} |\dot{y}_n|^2\, dt = \frac{1}{T_n}\int_0^1 |\dot{x}_n|^2\, ds < B,$$

where $\ell_n :=$ length(x_n) and $y_n(t) = x_n(t/T_n)$. In particular if $T_n \to 0$, then $\lim_n \ell_n = 0$.

Proof Using (9),

$$A_1 \ge \mathcal{A}_k(x_n, T_n) = A_{L+k}(y_n) \ge a_1 \int_0^{T_n} |\dot{y}_n|^2\, ds - (a_2 + |k|)\,T_n.$$

Let $\ell_n :=$ length(y_n). By the Cauchy-Schwartz inequality,

$$\ell_n^2 = \left(\int_0^{T_n} |\dot{y}_n|\, dt\right)^2 \le T_n \cdot \int_0^{T_n} |\dot{y}_n|^2\, dt.$$

The inequalities above imply the lemma with

$$B = 1 + \frac{1}{a_1}[A_1 + (a_2 + |k|)A_2] \qquad \square$$

.

Corollary 3.6 *If $(x_n, T_n) \in \Omega_M(q_0, q_1)$, $q_0 \ne q_1$ and $\mathcal{A}_k(x_n, T_n) < A_1$, then T_n is bounded away from zero.*

Corollary 3.7 *If Λ_1 is a connected component of Λ_M with a non-trivial free homotopy class, then for all $A_1 > 0$,*

$$\inf\{T > 0 \,|\, (x, T) \in \Lambda_1,\ \mathcal{A}_k(x, T) < A_1\} > 0.$$

Proof Since M is compact, $\inf\{\text{length}(x) \,|\, (x, T) \in \Lambda_1\}$ is positive. Now use Lemma 3.5. $\square$

Recall that $e_0(L) = \inf\{k \in \mathbb{R} \mid \pi(E^{-1}\{k\}) = M\}$. Observe that the radial derivative of the energy function is positive:

$$\frac{d}{ds} E(x, sv)\Big|_{s=1} = v \cdot L_{vv}(x, v) \cdot v > 0.$$

Therefore

$$\min_{v \in T_x M} E(x, v) = E(x, 0).$$

This implies that

$$e_0(L) = \max_{x \in M} E(x, 0).$$

Proposition 3.8 *If a sequence $(x_n, T_n) \in \Lambda_M$ satisfies $\mathcal{A}_k(x_n, T_n) < A_1$, $\|d_{(x_n,T_n)}\mathcal{A}_k\| < \frac{1}{n}$ and $T_n \to 0$, then there is $q_0 \in M$ and a subsequence x_{n_i} such that $q_0 = \lim_i x_{n_i}(s)$ for all $s \in [0, 1]$ and*

(i) $\lim_i \mathcal{A}_k(x_{n_i}, T_{n_i}) = 0$.
(ii) $(q_0, 0)$ *is a singularity of the Euler-Lagrange flow.*
(iii) $E(q_0, 0) = k$. *In particular $k \le e_0(L)$.*
(iv) $\lim_i \frac{1}{T_{n_i}^2} \int_0^1 |\dot{x}_{n_i}(s)|^2\, ds = 0$.

In particular, the Dirac probability measure supported on $(q_0, 0)$ is an invariant measure, supported on the energy level $E^{-1}\{k\}$, whose $(L + k)$-action is zero and has trivial homology.

Also, the (singular) energy level $E = k$ does not satisfy the Palais-Smale condition.

Remark 3.9 Proposition 3.8 will be applied to sequences in Λ_M and also to sequences in $\Omega_M(q_0, q_1)$ with $q_1 = q_0$. This means that $\|d\mathcal{A}_k\|$ is to be understood as the norm of the derivative $d_{(x_n,T_n)}\mathcal{A}_k$ restricted to the subspace $T_{(x_n,T_n)}\Omega(x_n(0), x_n(1)) \subset T_{(x_n,T_n)}\Lambda_M$ given by variational vector fields which are zero at the endpoints.

Proof Assume that $1 \ge T_n \to 0$. Let $\ell_n := \text{length}(y_n)$. By Lemma 3.5, we have that $\lim_n \ell_n = 0$. Since M is compact, taking a subsequence, we can assume that $\lim_n y_n(0) = q_0 \in M$.

Since $\lim_n \ell_n = 0$ and $\lim_n y_n(0) = q_0$, we can assume that all the curves y_n are in the domain $U_i \subset \mathbb{R}^m$, $m = \dim M$, of a bounded chart $U_i \in \mathcal{U}$. By Lemma 2.2 we can assume that on the chart U_i we have $\|d_{(x_n,T_n)}\mathcal{A}_k\|_{\mathcal{H}^1(\mathbb{R}^m)\times\mathbb{R}^+} < \frac{1}{n}$ for all n. From now on we work on the chart U_i as if $M = \mathbb{R}^m$.

Let $\xi(s) := x_n(s) - x_n(0)$. Then $\xi(0) = \xi(1) = 0$. Observe that[3] $(\xi, 0) \in T_{(x_n,T_n)}\Lambda_M$. Let $\zeta(t) := \xi(t/T_n)$, $t \in [0, T_n]$. Then $\zeta(0) = \zeta(T_n) = 0$, $\dot{\zeta}(t) = \dot{y}_n(t)$. Using $T_n \le 1$ and (5) we find

$$\left|\, d_{(x_n,T_n)}\mathcal{A}_k \cdot (\xi, 0)\,\right| \le \frac{1}{n}\left[T_n^3 \int_0^{T_n} |\dot{\zeta}|^2\, dt\right]^{\frac{1}{2}} \le \frac{T_n}{n}\left[\int_0^{T_n} |\dot{y}_n|^2\, dt\right]^{\frac{1}{2}}.$$

[3] Since $\xi(0) = \xi(1) = 0$ this tangent vector is also in $T_{(x_n,T_n)}\Omega_M(q, q)$, where $q = x_n(0) = x_n(1)$.

Using Lemma 3.2.(ii) with $w = 0$, we get that

$$L_v(x, v) \cdot v \geq L_v(y, 0) \cdot v - b_3\, |v|\, |x - y| - b_3\, |v|^2\, |x - y| + a_0\, |v|^2, \tag{25}$$

for all $(x, v) \in TU_i$. Using inequality (25) with $(x, v) = (y_n, \dot{y}_n)$ and $y = y_n(0)$, we get

$$d_{(x_n, T_n)}\mathcal{A}_k \cdot (\xi, 0) = \int_0^{T_n} [L_x(y_n, \dot{y}_n) \cdot \zeta + L_v(y_n, \dot{y}_n) \cdot \dot{\zeta}]\, dt\,,$$

$$\begin{aligned}
&\frac{T_n}{n}\left[\int_0^{T_n} |\dot{y}_n|^2\, dt\right]^{\frac{1}{2}} \\
&\quad \geq -b_2 \int_0^{T_n} [1 + |\dot{y}_n|^2]\, |y_n - y_n(0)|\, dt + \theta_{y_n(0)} \cdot \left(\int_0^{T_n} \dot{y}_n\, dt\right) \\
&\qquad -b_3 \int_0^{T_n} |\dot{y}_n|\, |y_n - y_n(0)|\, dt - b_3 \int_0^{T_n} |\dot{y}_n|^2\, |y_n - y_n(0)|\, dt \\
&\qquad + a_0 \int_0^{T_n} |\dot{y}_n|^2\, dt\,, \\
&\quad \geq -b_2\, \ell_n\, T_n + 0 - b_3\, \ell_n^2 - (b_3 + b_2)\, \ell_n \int_0^{T_n} |\dot{y}_n|^2\, dt + a_0 \int_0^{T_n} |\dot{y}_n|^2\, dt\,,
\end{aligned}$$

where $\theta_x := L_v(x, 0)$ and b_2 is from (12).

Dividing the last inequality by T_n we have that

$$-b_2\, \ell_n - b_3\, \frac{\ell_n^2}{T_n} + [a_0 - (b_3 + b_2)\, \ell_n]\left[\frac{1}{T_n} \int_0^{T_n} |\dot{y}_n|^2\, dt\right] \leq \frac{1}{n}\left[\int_0^{T_n} |\dot{y}_n|^2\, dt\right]^{\frac{1}{2}}. \tag{26}$$

Since $\mathcal{A}_k(x_n, T_n) < A_1$, from Lemma 3.5 in the chart U_i we get

$$\limsup_n \frac{\ell_n^2}{T_n} \leq \limsup_n \int_0^{T_n} |\dot{y}_n|^2\, dt < +\infty.$$

From (26) and Lemma 3.5 we get that

$$\limsup_n \frac{\ell_n^2}{T_n^2} \leq \limsup_n \frac{1}{T_n} \int_0^{T_n} |\dot{y}_n|^2\, dt < +\infty. \tag{27}$$

Since $\lim_n T_n = 0$, we get that

$$\lim_n \ell_n = 0, \quad \lim_n \frac{\ell_n^2}{T_n} = 0, \quad \lim_n \int_0^{T_n} |\dot{y}_n|^2\, dt = 0 \text{ and } \frac{1}{T_n} \int_0^{T_n} |\dot{y}_n|^2\, dt \text{ is bounded.}$$

Hence, from inequalities (27) and (26), we get that

$$\limsup_n \left[\frac{1}{T_n} \int_0^{T_n} |\dot{y}_n|\, dt\right]^2 = \limsup_n \frac{\ell_n^2}{T_n^2} \leq \lim_n \frac{1}{T_n} \int_0^{T_n} |\dot{y}_n|^2\, dt = 0. \tag{28}$$

Changing variables in the integral, this proves item (iv).

(i). By Lemma 3.1, for all $(x, v) \in TM$

$$|L(x, v) + k| \le \frac{1}{2} A_0 \, |v|_x^2 + |\theta_x(v)| + |\psi(x)| + |k|.$$

Therefore,

$$|\mathcal{A}_k(x_n, T_n)| \le \frac{1}{2} A_0 \int_0^{T_n} |\dot{y}_n|_{y_n}^2 \; dt + \ell_n \sup_{x \in M} \|\theta_x\| + T_n \big[|k| + \sup_{x \in M} |\psi(x)| \big].$$

Hence $\lim_n \mathcal{A}_k(x_n, T_n) = 0$.

(ii). Let $h : [0, 1] \to [0, 2]$ be a smooth function such that $h(0) = h(1) = 0$ and $\int_0^1 h(s) \; ds = 1$. Let $\xi(s) := h(s) \, d\psi(q_0) \in \mathbb{R}^m$, $s \in [0, 1]$, and $\zeta(t) := \xi(t/T_n)$. We have that

$$d_{(x_n, T_n)} \mathcal{A}_k \cdot (\xi, 0) = \int_0^{T_n} [L_x(y_n, \dot{y}_n) \cdot \zeta + L_v(y_n, \dot{y}_n) \cdot \dot{\zeta}] \, dt \le \frac{1}{n} \, \|(\xi, 0)\|_{(x_n, T_n)} \, . \tag{29}$$

Using (13),

$$\begin{aligned} L_x(x, v) &= L_x(x, 0) + \int_0^1 \frac{d}{ds} L_x(x, sv) \; ds \\ L_x(x, v) \cdot \zeta &\ge d\psi(x) \cdot \zeta - b_3 \, (1 + |v|_x) \, |v|_x \, |\zeta|. \end{aligned} \tag{30}$$

Write $\theta_{q_0} = L_v(q_0, 0)$. Using (30) and Lemma 3.2.(i) with $(x, v) = (y_n, \dot{y}_n)$, $(y, w) = (q_0, 0)$ in the inequality in (29), we get that

$$\begin{aligned} & T_n \int_0^1 d\psi(x_n(s)) \cdot h(s) \cdot d\psi(q_0) \; ds - b_3 \; \|\zeta\|_\infty \int_0^{T_n} (|\dot{y}_n| + |\dot{y}_n|^2) \; dt \\ & \quad + \theta_{q_0} \left(\int_0^{T_n} \dot{\zeta} \; dt \right) - b_3 \, \|\dot{\zeta}\|_\infty \; \|y_n - q_0\|_\infty \int_0^{T_n} (1 + |\dot{y}_n|) \; dt \\ & \quad - A_0 \, \|\dot{\zeta}\|_\infty \int_0^{T_n} |\dot{y}_n| \; dt \\ & \le \frac{T_n}{n} \, |d\psi(q_0)| \; \|\dot{h}\|_{\mathcal{L}^2}. \end{aligned}$$

In the inequality above the third term is zero. Also $\lim_n \|y_n - q_0\|_\infty = 0$. Dividing by T_n, letting $n \to +\infty$ and using (28), we get

$$0 \le |d\psi(q_0)|^2 = \lim_{n \to +\infty} \int_0^1 d\psi(x_n(s)) \cdot h(s) \cdot d\psi(q_0) \; ds \le 0.$$

Hence $(q_0, 0)$ is a singularity of the Euler-Lagrange flow.

(iii). We now show that $E(q_0, 0) = k$. From (11),

$$|\psi(y_n(t)) - \psi(y_n(0))| \le b_1 \, \ell_n \quad \text{for all } \; t \in [0, T_n].$$

The hypothesis $\|d_{(x_n,T_n)}\mathcal{A}_k\| < \frac{1}{n}$ implies that

$$\frac{1}{n} \geq \left| \frac{\partial \mathcal{A}_k}{\partial T}\Big|_{(x_n,T_n)} \right| = \frac{1}{T_n} \left| \int_0^{T_n} [E(y_n, \dot{y}_n) - k]\, dt \right|. \tag{31}$$

Using Lemma 3.1 we get that

$$- [\psi(y_n(0)) + k] - b_1\, \ell_n + \frac{a_0}{2\,T_n} \int_0^{T_n} |\dot{y}_n|^2\, dt \leq \frac{1}{n},$$
$$-\frac{1}{n} \leq - [\psi(y_n(0)) + k] + b_1\, \ell_n + \frac{A_0}{2\,T_n} \int_0^{T_n} |\dot{y}_n|^2\, dt.$$

Then from inequality (28), we get

$$E(q_0, 0) = -\psi(q_0) = \lim_n -\psi\big(y_n(0)\big) = k.$$

We have found that $E(q_0, 0) = k = -\psi(q_0)$ and $d\psi(q_0) = 0$, hence the point $(q_0, 0) \in TM$ is a singularity of the Euler-Lagrange flow in the energy level $E = k$. The Dirac measure supported on $(q_0, 0)$ is an invariant measure whose $(L + k)$-action is zero and has trivial homology.

This (singular) energy level $E = k$ does not satisfy the Palais-Smale condition because the curves (x_n, T_n), where $x_n(t) \equiv q_0$, $T_n = n$, are in the same connected component in Λ_M of closed curves with trivial homotopy class, they satisfy[4] $\mathcal{A}_k(x_n, T_n) \equiv 0$ and $d\mathcal{A}_k(x_n, T_n) \equiv 0$ but they do not have an accumulation point in the topology of Λ_M.

In the case $(x_n, T_n) \in \Omega_M(q_0, q_0)$, the same choice $x_n(s) \equiv q_0$, $T_n = n$ is an unbounded Palais-Smale sequence in $\Omega_M(q_0, q_0)$. □

Corollary 3.10 *If* $(q_0, 0) \in T_{q_0}M$ *is not a singularity of the Euler-Lagrange flow and a sequence* $(x_n, T_n) \in \Omega_M(q_0, q_0)$ *satisfies* $\mathcal{A}_k(x_n, T_n) < A_1$ *and* $\|d_{(x_n,T_n)}\mathcal{A}_k\| < \frac{1}{n}$*, then* $\liminf_n T_n > 0$.

Remark 3.11 Observe that the Hilbert manifolds Λ_M and $\Omega_M(q_0, q_1)$ are not complete with our riemannian metric (4) because they do not contain the points $(x, 0) \in \mathcal{H}^1(M) \times \{0\}$ that would be at finite distance from $(x, 1)$. The discussion above shows that in order to prevent a Palais-Smale sequence (x_n, T_n) from leaving the space at $\mathcal{H}^1(M) \times \{0\}$ we can either

- work on a connected component Λ_1 of Λ_M or $\Omega_M(q_0, q_1)$ with a non-trivial homotopy class.
- work on $\Omega_M(q_0, q_1)$ with $q_0 \neq q_1$.
- work on $\Omega_M(q_0, q_0)$ where $(q_0, 0)$ is not a fixed point of the Euler-Lagrange flow.
- ask that $E^{-1}\{k\}$ is not a singular energy level.
- ask that $\lim_n \mathcal{A}_k(x_n, T_n) \neq 0$.

[4] Since $E(q_0, 0) = -L(q_0, 0) = k$, we have that $\mathcal{A}_k(x_n, T_n) = 0$. They satisfy $d\mathcal{A}_k(x_n, T_n) \equiv 0$ because in formula (7), $E(q_0, 0) = k$, $L_x(q_0, 0) = d\psi(q_0) = 0$ and $L_v(q_0, 0)$ is constant. Alternatively, because the curves $y_n(t) = x_n(t/T_n)$ are solutions of the Euler-Lagrange equation with energy k, see formula (16).

On a given connected component Λ_1 of Λ_M or $\Omega_M(q_0, q_1)$ a singular energy level may not satisfy the Palais-Smale condition as one sees by looking at sequences of curves (x_n, T_n) with $\lim_n T_n = +\infty$ which spend much time near the singularity, as it happens, for instance if q_0, q_1 are, respectively, in the projections of the unstable and stable manifolds of a hyperbolic singularity. In such an example Theorem A says that the measure μ_n defined at the end of Sect. 1.2 converges to the Dirac measure at the singularity.

Proposition 3.12 *If a sequence* $\{(x_n, T_n)\}_{n\in\mathbb{N}} \subset \Omega_M(q_0, q_1)$ *or* $\{(x_n, T_n)\}_{n\in\mathbb{N}} \subset \Lambda_M$ *satisfies*

$$\mathcal{A}_k(x_n, T_n) < A_1, \quad \left\| d_{(x_n,T_n)}\mathcal{A}_k \right\| < \frac{1}{n} \quad \textit{and} \quad 0 < \liminf_n T_n < +\infty;$$

then there exists a convergent subsequence.

Proof Taking a subsequence we can assume that $T = \lim_n T_n \in \mathbb{R}^+$ exists. Since M is compact, after taking a sub-subsequence, we can assume that $q_0 := \lim_n x_n(0)$, $q_1 := \lim_n x_n(1)$ and $T = \lim_n T_n \in \mathbb{R}^+$ exist. In the case of Λ_M we then have $q_0 = q_1$.

We will extract a Cauchy sequence from $\{(x_n, T_n)\}_{n\in\mathbb{N}}$. Since $T > 0$, such a Cauchy sequence has a limit in Λ_M (resp. in $\Omega_M(q_0, q_1)$). By Lemma 3.3, there are smooth curves $\tilde{x}_n$ such that $d[(x_n, T_n), (\tilde{x}_n, T_n)] < \frac{1}{n}$. Hence we can assume that the curves x_n are C^∞, for if $\{(\tilde{x}_n, T_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, so is $\{(x_n, T_n)\}_{n\in\mathbb{N}}$ and since $\mathcal{A}_k$ is C^1, also $\lim_n \|d\mathcal{A}_k(\tilde{x}_n, T_n)\| = 0$. Similarly, since $d[(x_n, T_n), (x_n, T)] \le |T_n - T| \xrightarrow{n} 0$, we can assume that $T_n = T$ for all n. Also, since we are assuming that $T_n = T$ is fixed, it is equivalent to use the metric (6) with $f(T) = g(T) = 1$.

Let $y_n(t) := x_n(t/T)$ and let $\alpha_n : [0, 1] \to M$ and $\beta_n : [T+1, T+2] \to M$ be minimal geodesics joining $\alpha_n(0) = q_0$, $\alpha_n(1) = y_n(0)$; $\beta_n(T+1) = y_n(T)$, $\beta_n(T+2) = q_1$. Taking a subsequence we can assume that $d(x_n(0), q_0) < 1$ and $d(x_n(1), q_1) < 1$. Then $|\dot{\alpha}_n| \le 1$ and $|\dot{\beta}_n| \le 1$. Define

$$w_n(t) = \begin{cases} \alpha_n(t) & \text{if} \quad 0 \le t \le 1, \\ y_n(t-1) & \text{if} \quad 1 \le t \le T+1, \\ \beta_n(t) & \text{if } T+1 \le t \le T+2. \end{cases}$$

Then all the curves $w_n : [0, T+2] \to M$ join q_0 to q_1. Their action is uniformly bounded because

$$\begin{aligned} A_{L+k}(w_n) &= \mathcal{A}_k(x_n, T) + A_{L+k}(\alpha_n) + A_{L+k}(\beta_n) \\ &\le A_1 + 2 \cdot \sup_{|v|\le 1} [L(x, v) + k] =: A_2. \end{aligned}$$

By the Cauchy-Schwartz inequality and Lemma 3.5,

$$\begin{aligned} d(w_n(t_1), w_n(t_2)) &\le \int_{t_1}^{t_2} |\dot{w}_n(s)| \; ds \le \sqrt{|t_2 - t_1|} \left[\int_0^{T+2} |\dot{w}_n|^2 \; ds \right]^{\frac{1}{2}} \\ &\le B(k, A_2, T+2)^{\frac{1}{2}} \; |t_2 - t_1|^{\frac{1}{2}}. \end{aligned}$$

So the family $\{w_n\}$ is equicontinuous. By the Arzelá-Ascoli Theorem there is a convergent subsequence of $\{w_n\}$ in the C^0 topology. This implies that also $\{x_n\}$ has a convergent subsequence in the C^0 topology. In the sequel we work with a convergent subsequence of $\{x_n\}$.

We can assume that the injectivity radius of M is larger than 2. For n, m large enough $d(x_n(s), x_m(s)) < 1$ for all $s \in [0, 1]$. Let $\gamma_s : [0, 1] \to M$ be the minimizing geodesic joining $\gamma_s(0) = x_n(s)$ to $\gamma_s(1) = x_m(s)$. Let $\Gamma : [0, 1] \times [0, 1] \to M$ be defined by $\Gamma(s, r) := \gamma_s(r)$. Then

$$\left|\frac{\partial \Gamma}{\partial r}(s, r)\right| = |\dot{\gamma}_s(r)| = d_M\big(x_n(s), x_m(s)\big) \le d_{n,m}, \tag{32}$$

where $d_{n,m} = \sup_{s\in[0,1]} d(x_n(s), x_m(s))$. Observe that $J(r) := \frac{\partial \Gamma}{\partial s}(s, r)$ is a Jacobi field along γ_s with $J(0) = \dot{x}_n(s)$ and $J(1) = \dot{x}_m(s)$. Since $|\dot{\gamma}_s| \le 1$, by Lemma 3.4,

$$\left|\frac{\partial \Gamma}{\partial s}(s, r)\right| = |J(r)| \le K\,[|\dot{x}_n(s)| + |\dot{x}_m(s)|], \tag{33}$$

$$\left|\frac{D}{ds}\frac{\partial \Gamma}{\partial r}(s, r)\right| = \left|\frac{D}{dr}\frac{\partial \Gamma}{\partial s}(s, r)\right| = |J'(r)| \le K\,[|\dot{x}_n(s)| + |\dot{x}_m(s)|], \tag{34}$$

By Lemma 3.5,

$$\begin{aligned} \|\dot{x}_n\|_{\mathcal{L}^1} \le \|\dot{x}_n\|_{\mathcal{L}^2} &\le T\,B(k, A_1, T) =: B_1, \\ \left\|\frac{\dot{x}_n}{T}\right\|_{\mathcal{L}^2} = \|\dot{y}_n\|_{\mathcal{L}^2} &\le B(k, A_1, T) =: B_2. \end{aligned} \tag{35}$$

Let $\eta_{n,m}(s) := \frac{\partial \Gamma}{\partial r}(s, 1) \in T_{x_m(s)}M$ and $\xi_{n,m}(s) := \frac{\partial \Gamma}{\partial r}(s, 0) \in T_{x_n(s)}M$. We have that

$$\begin{aligned} \|\eta_{n,m}\|^2_{\mathcal{H}^1(M)} &= |\eta_{n,m}(0)|^2 + \int_0^1 |\dot{\eta}(s)|^2\,ds \\ &= \left|\frac{\partial \Gamma}{\partial r}(0, 1)\right|^2 + \int_0^1 \left|\frac{D}{ds}\frac{\partial \Gamma}{\partial r}(s, 1)\right|^2\,ds. \end{aligned}$$

From (32), (34) and (3.2),

$$\begin{aligned} \|\eta_{n,m}\|^2_{\mathcal{H}^1(M)} &\le d(x_n(0), x_m(0)) + K\left(2\,\|\dot{x}_n\|^2_{\mathcal{L}^2} + 2\,\|\dot{x}_m\|^2_{\mathcal{L}^2}\right) \\ &\le 1 + 4\,K\,B_1 =: K_1. \end{aligned}$$

Similarly,

$$\|\xi_{n,m}\|^2_{\mathcal{H}^1(M)} \le K_1.$$

Also

$$\|\eta_{n,m}\|_\infty \le d_{n,m} \quad \text{and} \quad \|\xi_{n,m}\|_\infty \le d_{n,m}.$$

Since $\lim_n \|d_{(x_n,T)}\mathcal{A}_k\|_{(x_n,T)} = 0$, for the product norm $\|\cdot\|_{\mathcal{H}^1(M)\times\mathbb{R}}$, with $f(T) \equiv g(T) \equiv 1$, we also have that

$$\lim_n \left\|\, d_{(x_n,T)}\mathcal{A}_k \right\|_{\mathcal{H}^1(M)\times\mathbb{R}} = 0, \tag{36}$$

where the derivative is restricted to the tangent space $T_{(x_n,T)}\Omega_M(q_0,q_1)$ [resp. $T_{(x_n,T)}\Lambda_M$]. Therefore given $\varepsilon>0$ there is $N>0$ such that

$$\left\| \partial_{x_n}\mathcal{A}_k\big|_{(x_n,T)}\cdot\eta\right\|_{\mathcal{H}^1(M)} < \frac{1}{2}\,\varepsilon$$

for every $n\geq N$ and $\|\eta\|_{\mathcal{H}^1(M)}\leq K_1$ with $\eta(0)=\eta(1)=0$ when $(x_n,T)\in\Omega_M(q_0,q_1)$ and $\eta(0)=\eta(1)$ when $(x_n,T)\in\Lambda_M$. We can take $\eta=\eta_{n,m}$ and $\eta=\xi_{n,m}$ defined above over (x_m,T) and (x_n,T) respectively. Therefore

$$\left\| \partial_{x_m}\mathcal{A}_k\big|_{(x_m,T)}\cdot\eta_{n,m}-\partial_{x_n}\mathcal{A}_k\big|_{(x_n,T)}\cdot\xi_{n,m}\right\|_{\mathcal{H}^1(M)}<\varepsilon$$

From formula (7) for $\partial_x\mathcal{A}_k$, we have that

$$\begin{aligned}\Bigg|\int_0^1 T\left[\left\langle\nabla_x L\left(x_m,\frac{\dot x_m}{T}\right),\,\eta_{n,m}\right\rangle-\left\langle\nabla_x L\left(x_n,\frac{\dot x_n}{T}\right),\,\xi_{n,m}\right\rangle\right]ds\,+\\ +\int_0^1\left[\left\langle\nabla_v L\left(x_m,\frac{\dot x_m}{T}\right),\,\dot\eta_{n,m}\right\rangle-\left\langle\nabla_v L\left(x_n,\frac{\dot x_n}{T}\right),\,\dot\xi_{n,m}\right\rangle\right]ds\Bigg|<\varepsilon\end{aligned} \tag{37}$$

for $m,n>N$. Since L is quadratic at infinity,

$$b_2:=\sup_{(x,v)\in TM}\frac{|\nabla_x L(x,v)|_x}{1+|v|_x^2}<+\infty.$$

The first term in (37) is bounded by $2\,T\,b_2\,(1+B_2^2)\,d_{n,m}$. Consequently the second integral in (37) is small for big m,n.

Observe that the integrand in the second term of (37) is

$$\begin{aligned}&\left\langle\nabla_v L\left(x_m,\frac{\dot x_m}{T}\right),\,\dot\eta_{n,m}\right\rangle-\left\langle\nabla_v L\left(x_n,\frac{\dot x_n}{T}\right),\,\dot\xi_{n,m}\right\rangle\\ &\quad=\left\langle\nabla_v L\left(\Gamma(s,r),\frac{1}{T}\frac{\partial\Gamma}{\partial s}(s,r)\right),\;\frac{D}{ds}\frac{\partial\Gamma}{\partial r}(s,r)\right\rangle\Bigg|_{r=0}^{r=1}\\ &\quad=\int_0^1\frac{D}{dr}\left\langle\nabla_v L\left(\Gamma(s,r),\frac{1}{T}\frac{\partial\Gamma}{\partial s}(s,r)\right),\;\frac{D}{ds}\frac{\partial\Gamma}{\partial r}(s,r)\right\rangle dr\\ &\quad=\int_0^1\left\langle\partial_x\nabla_v L\left(\Gamma,\frac{1}{T}\frac{\partial\Gamma}{\partial s}\right)\cdot\frac{\partial\Gamma}{\partial r}+\partial_v\nabla_v L\left(\Gamma,\frac{1}{T}\frac{\partial\Gamma}{\partial s}\right)\cdot\frac{1}{T}\frac{D}{dr}\frac{\partial\Gamma}{\partial s}\,,\;\frac{D}{ds}\frac{\partial\Gamma}{\partial r}\right\rangle dr\\ &\quad=\int_0^1\left\langle\partial_x\nabla_v L\left(\Gamma,\frac{1}{T}\frac{\partial\Gamma}{\partial s}\right)\cdot\frac{\partial\Gamma}{\partial r}\,,\;\frac{D}{ds}\frac{\partial\Gamma}{\partial r}\right\rangle dr\\ &\qquad+\int_0^1\left\langle\partial_v\nabla_v L\left(\Gamma,\frac{1}{T}\frac{\partial\Gamma}{\partial s}\right)\cdot\frac{1}{T}\frac{D}{dr}\frac{\partial\Gamma}{\partial s}\,,\;\frac{D}{ds}\frac{\partial\Gamma}{\partial r}\right\rangle dr,\end{aligned} \tag{38}$$

here $\partial_x\nabla_v L$ and $\partial_v\nabla_v L$ are the partial derivatives of the second component of $TM\ni(x,v)\mapsto\big(x,\nabla_v L(x,v)\big)\in TM$ with respect to the splitting $T_{(x,v)}TM=H\oplus V$ described below. The partial derivative $\partial_v\nabla_v L(x,v)$ coincides with the second derivative of $v\mapsto L(x,v)\in T_xM$ in the vector space T_xM.

Since L is quadratic at infinity,

$$b_3 := \sup_{(x,v)\in TM} \frac{\|\partial_x \nabla_v L(x,v)\|}{1+|v|_x} < +\infty.$$

Then, by (33), (32) and (34),

$$\begin{aligned}|\text{first term in } (38)| &\le \int_0^1 b_3 \left[1+\left|\frac{1}{T}\frac{\partial\Gamma}{\partial s}\right|\right] \left|\frac{\partial\Gamma}{\partial r}\right| \left|\frac{D}{ds}\frac{\partial\Gamma}{\partial r}\right| dr \\ &\le b_3\left[1+\frac{1}{T}K\big(|\dot{x}_n(s)|+|\dot{x}_m(s)|\big)\right] d_{n,m} K\big(|\dot{x}_n(s)|+|\dot{x}_m(s)|\big).\end{aligned}$$

By (3.2) and Cauchy-Schwartz inequality,

$$\int_0^1 |\text{first term in } (38)|\, ds \le b_3\left(2\,K B_1 + \frac{1}{T}\,4\,K^2 B_1^2\right) d_{n,m} \xrightarrow{n,m} 0.$$

Since $\frac{D}{dr}\frac{\partial\Gamma}{\partial s} = \frac{D}{ds}\frac{\partial\Gamma}{\partial r}$, from (8) we have that

$$\int_0^1 [\text{second term of } (38)]\, ds \ge \int_0^1\int_0^1 a_0\,\frac{1}{T}\left|\frac{D}{ds}\frac{\partial\Gamma}{\partial r}\right|^2 dr\, ds.$$

The integral of (38) corresponds to the second term in the left of (37). Since the first term in (37) is small, we get that

$$\lim_{n,m\to+\infty}\int_0^1\int_0^1 \left|\frac{D}{ds}\frac{\partial\Gamma}{\partial r}\right|^2 dr\, ds = 0. \tag{39}$$

Using (32), in $\mathcal{H}^1(M)\times\mathbb{R}^+$ we have that

$$\begin{aligned} d[(x_n,T),\ (x_m,T)]^2 &\le \int_0^1 \left\|\frac{\partial\Gamma}{\partial r}\right\|^2_{(\Gamma(\cdot,r),T)} dr \\ &= \int_0^1 \left|\frac{\partial\Gamma}{\partial r}(0,r)\right|^2 dr + \int_0^1\int_0^1 \left|\frac{D}{ds}\frac{\partial\Gamma}{\partial r}\right|^2 ds\, dr \\ &\le d_{n,m}^2 + \int_0^1\int_0^1 \left|\frac{D}{ds}\frac{\partial\Gamma}{\partial r}\right|^2 ds\, dr.\end{aligned}$$

By (39), $\{(x_n,T)\}$ is a Cauchy sequence. □

The remainder of this section is devoted to the proof of

Proposition 3.13 *Suppose that L is Riemannian at infinity. Let Λ_1 be a connected component of $\Omega_M(q_0,q_1)$ or Λ_M. If a sequence $\{(x_n,T_n)\}_{n\in\mathbb{N}}\subset\Lambda_1$ satisfies*

$$|\mathcal{A}_k(x_n,T_n)| < A_1, \quad \left\|d_{(x_n,T_n)}\mathcal{A}_k\right\| < \frac{1}{n} \quad \textit{and} \quad \limsup_n T_n = +\infty,$$

then there exists a Borel probability measure μ, invariant under the Euler-Lagrange flow, supported on a connected component of the energy level $E\equiv k$, which has homology $\rho(\mu)=0$ and whose $(L+k)$-action is zero.

In the proof of this proposition we cannot use Lemma 2.2.(2) on the equivalence of the metric of $\mathcal{H}^1(M) \times \mathbb{R}^+$ to the metric of $\mathcal{H}^1(\mathbb{R}^m) \times \mathbb{R}^+$ on local charts because the times T_n are not bounded. Here we shall use strongly that the Lagrangian is Riemannian at infinity and not only quadratic at infinity.

We first fix the notation used in the proof of Proposition 3.13. There is a canonical splitting of the tangent space

$$T_\theta TM = H(\theta) \oplus V(\theta)$$

where the *vertical subspace* $V(\theta)$ is the kernel of the derivative $d_\theta\pi$ of the projection $\pi : TM \to M$ and the *horizontal subspace* $H(\theta)$ is the kernel of the connection map $K : T_\theta TM \to T_{\pi(\theta)}M$. Both subspaces are naturally identified with $T_{\pi(\theta)}M \approx H(\theta) \approx V(\theta)$ in the following way: a tangent vector $\zeta = (h, v) \in H \oplus V$ has horizontal and vertical components given by $h = d_\theta\pi(\zeta) \in T_{\pi(\theta)}M \approx H(\theta)$ and $v = K(\zeta) \in T_{\pi(\theta)}M \approx V(\theta)$.

The Sasaki metric on *TM* is given by

$$\begin{aligned} \langle \zeta_1, \zeta_2 \rangle_\theta &:= \langle d_\theta\pi(\zeta_1), d_\theta\pi(\zeta_1) \rangle_{\pi(\theta)} + \langle K(\zeta_1), K(\zeta_2) \rangle_{\pi(\theta)} \\ &= \langle h_1, h_2 \rangle_{\pi(\theta)} + \langle v_1, v_2 \rangle_{\pi(\theta)}, \end{aligned}$$

where $\zeta_i = (h_i, v_i) \in H(\theta) \oplus V(\theta)$, $i = 1, 2$.

We endow T^*M with the metric for which the identification $TM \longleftrightarrow T^*M$ induced by the Riemannian metric $(x, v) \leftrightarrow \langle v, \cdot \rangle_x = p$ is an isometry. Under this identification, the canonical symplectic form $\omega = dp \wedge dx$ on T^*M is sent to the form

$$\begin{aligned} \omega_\theta(\zeta_1, \zeta_2) &= \langle K(\zeta_1), d\pi(\zeta_2) \rangle_x - \langle K(\zeta_2), d\pi(\zeta_1) \rangle_x \\ &= \langle v_1, h_2 \rangle_{\pi(\theta)} - \langle v_2, h_1 \rangle_{\pi(\theta)}. \end{aligned}$$

We shall ambiguously use this identification in the rest of this section.

Let $H : TM \to \mathbb{R}$ be the hamiltonian associated to L:

$$H(x, p) := \max_{v \in T_x M} [\langle p, v \rangle_x - L(x, v)], \qquad p \in T_x M.$$

The hamiltonian vector field X on *TM* is given by $i_X\omega = -dH$. Its flow lines solve the hamiltonian equations

$$\dot{x} = \nabla_p H(x, p)\,, \qquad \frac{D}{dt}\, p = -\nabla_x H(x, p).$$

When seen in *TM*, $\nabla_x H$ and $\nabla_p H$ are the projections to the horizontal and vertical subspaces of the gradient of the hamiltonian H with respect to the Sasaki metric. The hamiltonian flow of H is conjugate to the Euler-Lagrange flow of L by the *Legendre transform* $\mathcal{L}(x, v) = (x, \nabla_v L(x, v)) = (x, p)$. This is, $\langle p, \cdot \rangle_x = \frac{\partial}{\partial v} L(x, v)$, in the vector space $T_x M$.

Observe that

$$\nabla_x L(x, v) = -\nabla_x H(x, p), \qquad \text{if } p = \nabla_v L(x, v).$$

If $(x,T)\in\mathcal{H}^1(M)\times\mathbb{R}^+$, $x\in C^\infty([0,1],M)$ and $\xi\in T_{(x,T)}\Omega_M(x(0),x(1))$, the partial derivative of the action functional is given by

$$d_{(x,T)}\mathcal{A}_k(\xi,0)=\int_0^T\left\langle p,\frac{D}{dt}\zeta\right\rangle_y-\langle\nabla_x H(y,p),\zeta\rangle_y\ \ dt, \tag{40}$$

where $y(sT):=x(s)$, $\zeta(sT):=\xi(s)$ and $p(t)=\nabla_v L(y(t),\dot y(t))$. Let $\mathcal{T}(t,s):T_{y(s)}M\to T_{y(t)}M$ be the parallel transport along $y(t)$. For $t\in[0,T]$ let

$$\mathbb{H}_x(t):=\mathcal{T}(t,0)\cdot a+\int_0^t\mathcal{T}(t,s)\cdot\nabla_x H(y(s),p(s))\,ds,$$

where the constant $a\in T_{y(0)}M$ is chosen such that

$$\int_0^T\mathcal{T}(T,t)\cdot[\,p(t)+\mathbb{H}_x(t)]\,dt=0. \tag{41}$$

Let

$$\rho(t):=p(t)+\mathbb{H}_x(t). \tag{42}$$

Integrating by parts in (40) we have that

$$d_{(x,T)}\mathcal{A}_k(\xi,0)=-\langle\mathbb{H}_x,\zeta\rangle_y\Big|_0^T+\int_0^T\left\langle\rho(t),\frac{D}{dt}\zeta(t)\right\rangle_{y(t)}dt. \tag{43}$$

Define $\zeta_1(t)$ by

$$\zeta_1(t):=\int_0^t\mathcal{T}(t,s)\cdot[\,p(s)+\mathbb{H}_x(s)]\,ds.$$

By (41), $\zeta_1(0)=\zeta_1(T)=0$. Then if $\xi_1(s):=\zeta_1(sT)$ we have that $(\xi_1,0)\in T_{(x,T)}\Omega_M(x(0),x(1))$ and also $(\xi_1,0)\in T_{(x,T)}\Lambda_M$ if $x(0)=x(1)$. Observe that

$$\frac{D}{dt}\zeta_1(t)=\rho(t)=p(t)+\mathbb{H}_x(t).$$

Applying (43), we get

$$\begin{aligned}
d_{(x,T)}\mathcal{A}_k(\xi_1,0)&=\int_0^T|\rho(t)|^2_{y(t)}\ dt\\
&\le\left\|d_{(x,T)}\mathcal{A}_k\right\|\cdot\|\xi_1\|_{(x,T)}\\
&\le\left\|d_{(x,T)}\mathcal{A}_k\right\|\cdot e^{-2T^2}\left[\int_0^T\left|\frac{D}{dt}\zeta_1\right|^2_{y(t)}dt\right]^{\frac12}\qquad\text{if } T\ge 10,\\
&\le\left\|d_{(x,T)}\mathcal{A}_k\right\|\cdot e^{-2T^2}\left[\int_0^T|\rho(t)|^2_{y(t)}\,dt\right]^{\frac12}\qquad\text{if } T\ge 10.
\end{aligned}$$

Therefore[5]

$$\|\rho\|_{\mathcal{L}^2([0,T])} \le e^{-2T^2} \left\| d_{(x,T)}\mathcal{A}_k \right\| , \qquad \text{if } T \ge 10. \tag{44}$$

Proof of Proposition 3.13 We can assume that $T_n \to +\infty$. Moreover, we can assume that

$$T_n \ge n \ge 10.$$

Since $\mathcal{A}_k$ and $d\mathcal{A}_k$ are continuous on $\Omega_M(q_0, q_1)$ and Λ_M, by Lemma 3.3 we can assume that $x_n : [0,1] \to M$ is C^∞ for all n. Observe that if $q_0 = q_1$, $T_{(x,T)}\Omega_M(q_0,q_1) \subset T_{(x,T)}\Lambda_M$. In the sequel we shall only use tangent vectors in $T_{(x,T)}\Omega_M(q_0,q_1)$, so that the arguments apply for both $\Omega_M(q_0,q_1)$ and Λ_M.

Let $y_n(t) := x_n(t/T_n)$, $t \in [0, T_n]$, and

$$p_n(t) := \nabla_v L(y_n(t), \dot y_n(t)) \in T_{y_n} M.$$

Let $\rho_n(t)$ be defined as in (42) for the path $(y_n(t), p_n(t))$:

$$\rho_n(t) := p_n(t) + \mathbb{H}_x(t).$$

In particular

$$\frac{D}{dt}\,\rho_n = \frac{D}{dt}\,p_n + \nabla_x H(y_n, p_n). \tag{45}$$

Since $\|d_{(x_n,T_n)}\mathcal{A}_k\| < \frac{1}{n}$, from (44) we have that

$$\|\,\rho_n\|_{\mathcal{L}^2([0,T_n])} = \left[\int_0^{T_n} |\rho_n(t)|^2_{y_n(t)}\ dt\right]^{\frac{1}{2}} < \frac{e^{-2T_n^2}}{n}. \tag{46}$$

Let

$$A_n := \left\{ t \in [0, T_n] \,\middle|\, |\rho_n(t)|_{y_n(t)} < e^{-\frac{3}{2}T_n^2} \right\}.$$

If m is the Lebesgue measure on $[0, T_n]$, we have that

$$e^{-3T_n^2}\, m\big(A_n^c\big) \le \int_0^{T_n} |\rho_n(t)|^2\ dt < \frac{1}{n^2}\, e^{-4T_n^2}.$$

Thus $m(A_n^c) < \frac{1}{n^2}\, e^{-T_n^2}$ and hence

$$m(A_n) > T_n - \frac{1}{n^2}\, e^{-T_n^2}. \tag{47}$$

We assume that the Riemannian metric on M has injectivity radius larger than 2. Since the Lagrangian is Riemannian at infinity then the Hamiltonian is also Riemannian at infinity: $H(x,p) = \frac{1}{2}\ |p|_x^2$ when $|p|_x \ge R$. Define $R_1 \ge R$ by

$$\frac{1}{2}\,R_1^2 = \sup\left\{ H(x,p)\,,\ \frac{1}{2}\,|\nabla_p H(x,p)|_x^2 \ \middle|\ |p|_x \le R \right\} \ge \frac{1}{2}\,R^2.$$

[5] Inequality (44) is our fundamental estimate for the rest of the section. Observe that since, by definition, $p(t) = \nabla_v L(y, \dot y)$, the first hamiltonian equation $\dot y = \nabla_p H(y,p)$ follows from the Legendre transform of H. Thus the quantity $\rho(t) = p(t) + \mathbb{H}_x(t)$ measures the deviation of $(y(t), p(t))$ from being a solution of the second hamiltonian equation $\dot p = -\nabla_x H(x,p)$. Our problem now is to obtain a true invariant measure from this $\mathcal{L}^2$ estimate.

Let $d_1 > |k|$ be such that

$$H(x,p) > \frac{1}{2}\,|p|_x^2 - d_1, \quad \text{for all } (x,p) \in TM. \tag{48}$$

Choose $R_0 > 0$ such that

$$R_0 > \max\{R_1\,,\ 10(|k| + d_1 + 1)\}. \tag{49}$$

In particular

$$H(x,p) = \frac{1}{2}\,|p|_x^2 \quad \text{if } |p|_x > R_0.$$

Also

$$\nabla_p H(x,p) = p \quad \text{and} \quad \nabla_x H(x,p) = 0 \quad \text{if } |p|_x > R_0.$$

Lemma 3.14

$$|p|_x < R_0 \iff H(x,p) < \frac{1}{2}\,R_0^2 \iff |v|_x < R_0,\ \textit{where } v = \nabla_p H(x,p).$$

Proof If $H(x,p) \geq \frac{1}{2}\,R_0^2 > \frac{1}{2}\,R_1^2$ then $|p|_x > R$, $v = \nabla_p H(x,p) = p$ and $H(x,p) = \frac{1}{2}\,|p|_x^2$. Hence $|v|_x = |p|_x \geq R_0$. If $|p|_x \geq R_0 > R$ then $H(x,p) = \frac{1}{2}\,|p|_x^2 \geq \frac{1}{2}\,R_0^2$. If $|v|_x = |\nabla_p H(x,p)|_x \geq R_0 > R_1$ then $|p|_x > R$, $v = \nabla_p H(x,p) = p$ and $H(x,p) = \frac{1}{2}\,|p|_x^2 = \frac{1}{2}\,|v|_x^2 \geq \frac{1}{2}\,R_0^2$. □

We start by estimating the difference between (y_n, p_n) and an orbit of the hamiltonian flow.

Lemma 3.15 *Given $t_0 \in [0, T_n]$, let $(x(t), q(t))$ be the solution of the hamiltonian equations*

$$\dot{x} = \nabla_p H(x,q), \quad \dot{q} = -\nabla_x H(x,q),$$

with initial conditions $x(t_0) = y_n(t_0)$, $q(t_0) = p_n(t_0)$.

There is $n_0 > 0$ such that if $n > n_0$, $t_0 \in A_n$ and $|p_n(t_0)| \leq R_0$, then for all $t \in [0, T_n]$,

$$d_M(x(t), y_n(t)) < 1,$$
$$d_{TM}\big[(x(t), q(t)), (y_n(t), p_n(t))\big] \leq |\rho_n(t)|_{y_n(t)} + e^{-T_n^2}. \tag{50}$$

Remark 3.16 The bounds in Lemma 3.15 are actually made for z_t defined in (52) instead of the distance between $(x(t), q(t))$ and $(y_n(t), p_n(t))$.

Proof We only prove the estimates for $t > t_0$. The case $t < t_0$ is similar.

Recall that using Lemma 3.3, we are assuming that y_n is C^∞. For all $t \in [0, T]$, let $\gamma_t : [0,1] \to M$ be a geodesic joining $x(t)$ to $y_n(t)$ such that $\gamma_{t_0}(s) \equiv y_n(t_0)$ for all $s \in [0,1]$ and that $f(s,t) := \gamma_t(s)$ is C^∞. Let

$$e_t := |\dot{\gamma}_t| = \text{length}(\gamma_t) \geq d_M(x(t), y_n(t)).$$

Let I_n be the maximal interval in $[0, T_n]$ containing t_0 such that $|e_t| < 1$ for all $t \in I_n$.

We first prove that there is $B = B(L, R_0) > 0$ such that for all n, if $|p_n(t_0)| \le R_0$ and $t \in I_n$, then

$$d_{TM}\big[(x(t), q(t)), (y_n(t), p_n(t))\big] \le |\rho_n(t)|_{y_n(t)} + \big(1 + e^{B(t-t_0)}\big)\ |\rho_n(t_0)|_{y_n(t_0)} \\ + B\, e^{B(t-t_0)}\ \|\rho_n\|_{\mathcal{L}^2([0,T_n])}\,. \tag{51}$$

Let $A_t(s_2, s_1) : T_{\gamma_t(s_1)}M \to T_{\gamma_t(s_2)}M$ be the parallel transport along γ_t. Let

$$z_t := e_t + |p_n(t) - A_t(1,0)\cdot q(t)|_{y_n(t)}\,. \tag{52}$$

Let $\Lambda_t : [0,2] \to TM$ be the curve defined by

$$\Lambda_t(s) = \begin{cases} (\gamma_t(s), A_t(s,0)\cdot q(t)) & \text{if } 0 \le s \le 1, \\ (y_n(t), (s-1)\, p_n(t) + (2-s)\, A_t(1,0)\cdot q(t)) & \text{if } 1 \le s \le 2. \end{cases} \tag{53}$$

Then

$$\begin{aligned} &d_{TM}((x(t), q(t)), (y_n(t), p_n(t))) \\ &\qquad \le \text{length}(\Lambda_t) \\ &\qquad \le \int_0^1 |\dot\gamma_t|\ ds + \int_1^2 |p_n(t) - A_t(1,0)\cdot q(t)|_{y_n(t)}\ ds \\ &\qquad \le z_t\,. \end{aligned}$$

Recall that $f(s,t) = \gamma_t(s)$. Then

$$e_t^2 = \int_0^1 e_t^2\, ds = \int_0^1 |\dot\gamma_t|^2\ ds = \int_0^1 \left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right\rangle_{f(s,t)} ds.$$

Since $\frac{D}{ds}\frac{\partial f}{\partial s} = \frac{D}{ds}\dot\gamma_t = 0$, we have that

$$\begin{aligned} \frac{1}{2}\,\frac{d\,e_t^2}{dt} &= \frac{1}{2}\int_0^1 \frac{d}{dt}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right\rangle_{f(s,t)} ds = \int_0^1 \frac{d}{ds}\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right\rangle_{f(s,t)} ds \\ &= \left\langle \frac{\partial f}{\partial t}(1,t), \frac{\partial f}{\partial s}(1,t)\right\rangle_{y_n(t)} - \left\langle \frac{\partial f}{\partial t}(0,t), \frac{\partial f}{\partial s}(0,t)\right\rangle_{x(t)} \\ &= \langle \dot y_n(t), \dot\gamma_t(1)\rangle_{y_n(t)} - \langle A_t(1,0)\cdot \dot x(t),\ A_t(1,0)\cdot\dot\gamma_t(0)\rangle_{y_n(t)} \\ &= \langle \dot y_n(t) - A_t(1,0)\cdot\dot x(t), \dot\gamma_t(1)\rangle_{y_n(t)} \\ &\le |\dot y_n(t) - A_t(1,0)\cdot\dot x(t)|_{y_n(t)}\ e_t. \end{aligned} \tag{54}$$

Since $p_n = \nabla_v L(y_n, \dot y_n)$, $\dot y_n = \nabla_p H(y_n, p_n)$. Since $\nabla_p H(y,p) = p$ when $|p|_y > R_0$, $(x,p) \mapsto \nabla_p H(x,p)$ has bounded derivative on TM. Let $K_1 > 1$ be a bound for its derivative. Then

$$\begin{aligned} |\dot y_n - A_t(1,0)\cdot\dot x| &= \big|\nabla_p H(y_n, p_n) - A_t(1,0)\cdot\nabla_p H(x,q)\big| \\ &\le \int_0^1 \left|\frac{D}{ds}[A_t(1, \min\{2s,1\})\cdot\nabla_p H(\Lambda_t(2s))]\right|\ ds \\ &= \int_0^1 \left|\frac{D}{ds}\ \nabla_p H(\Lambda_t(2s))\right|\ ds \\ &\le K_1\ \text{length}(\Lambda_t) = K_1\ z_t. \end{aligned} \tag{55}$$

Thus, from (54),

$$\frac{1}{2}\,\frac{d\,e_t^2}{dt} \le K_1\, z_t\, e_t.$$

Since $e_{t_0} = 0$,

$$e_t = \int_{t_0}^{t} \frac{1}{2}\,\frac{1}{e_s}\,\frac{d\,e_s^2}{ds}\,ds \le \int_{t_0}^{t} K_1\, z_s\, ds. \tag{56}$$

Let $T(t_2, t_1) : T_{y_n(t_1)}M \to T_{y_n(t_2)}M$ be the parallel transport along $y_n(t)$. Since $p_n(t_0) = q(t_0)$, we have that

$$\begin{aligned}
p_n(\tau) - A_\tau(1,0)\cdot q(\tau) &= \int_{t_0}^{\tau} T(\tau,t)\left[\frac{D}{dt}p_n(t) - \frac{D}{dt}A_t(1,0)\cdot q(t)\right]dt,\\
&= \int_{t_0}^{\tau} T(\tau,t)\left[\frac{D}{dt}p_n + \nabla_x H(y_n,p_n)\right]dt\\
&\quad + \int_{t_0}^{\tau} T(\tau,t)\left[A_t(1,0)\cdot\nabla_x H(x,q) - \nabla_x H(y_n,p_n)\right]dt \qquad (57)\\
&\quad + \int_{t_0}^{\tau} T(\tau,t)\left[-A_t(1,0)\cdot\nabla_x H(x,q) - \frac{D}{dt}A_t(1,0)\cdot q(t)\right]dt.
\end{aligned}$$

Since $\nabla_x H(x,p) = 0$ if $|p|_x > R_0$, the function $\nabla_x H$ has bounded derivative on TM. Then, as in (55),

$$\text{the norm of the second term in (57)} \le \int_{t_0}^{\tau} K_2\, z_t\, dt,$$

where K_2 is a bound for the derivative of $\nabla_x H$. We estimate the third term. Since (x,q) is a solution of the hamiltonian equations,

$$\frac{D}{dt}\,q = -\nabla_x H(x,q).$$

Let $F(s,t) := A_t(s,0)\cdot \frac{D}{dt}\,q(t) - \frac{D}{dt}\,A_t(s,0)\cdot q(t) \in T_{f(s,t)}M$, then $F(0,t) \equiv 0$ and

$$\begin{aligned}
\frac{D}{ds}F(s,t) &= \frac{D}{ds}\left[A_t(s,0)\cdot\frac{D}{dt}\,q(t) - \frac{D}{dt}\,A_t(s,0)\cdot q(t)\right]\\
&= 0 - \frac{D}{ds}\,\frac{D}{dt}\,A_t(s,0)\cdot q(t)\\
&= -\frac{D}{dt}\,\frac{D}{ds}\left[A_t(s,0)\cdot q(t)\right] + R\left(\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right)\left[A_t(s,0)\cdot q(t)\right]\\
&= R\left(\dot\gamma_t(s),\frac{\partial f}{\partial t}\right)\left[A_t(s,0)\cdot q(t)\right],
\end{aligned}$$

where R is the curvature tensor. Let $K_3 > 1$ be such that $|R(u,v)w|_x \le K_3\,|u|_x\,|v|_x\,|w|_x$ for all $x \in M$, $u, v, w \in T_xM$.

Observe that $J(s) = \frac{\partial f}{\partial t}(s,t)$ is a Jacobi field along the geodesic γ_t with $J(0) = \dot{x}(t)$, $J(1) = \dot{y}_n(t)$ and that if $t \in I_n$ then $|\dot{\gamma}_t| < 1$. By Lemma 3.4 there is $K_4 > 1$ such that

$$\left|\frac{\partial f}{\partial t}\right|_{f(s,t)} \le K_4\,[|\dot{x}(t)| + |\dot{y}_n(t)|] \quad \text{for all } (s,t) \in [0,1] \times I_n.$$

Then, using (55),

$$\begin{aligned}\left|\frac{D}{ds}F(s,t)\right| &\le K_3\, e_t\, K_4\,[|\dot{x}(t)| + |\dot{y}_n(t)|]\; |q(t)| \\ &\le K_3\, e_t\, K_4\,[|\dot{x}(t)| + |\dot{y}_n(t) - A_t(1,0)\cdot \dot{x}(t)| + |\dot{x}(t)|]\; |q(t)| \\ &\le K_3\, K_4\, e_t\,[2\,|\dot{x}(t)| + K_1\, z_t]\; |q(t)|\,.\end{aligned}$$

Since, by hypothesis, $|q(t_0)| \le R_0$, by Lemma 3.14, $E(x,\dot{x}) = H(x,q) \le \frac{1}{2}R_0^2$ and hence, by Lemma 3.14, $|\dot{x}(t)| \le R_0$ and $|q(t)| \le R_0$ for all t. Let $K_5 := 2\,K_1 K_3 K_4$. If $t \in I_n$ then $|e_t| \le 1$ and hence

$$\begin{aligned}|F(1,t)|_{y_n(t)} &= \left|0 + \int_0^1 A_t(1,s)\cdot \frac{D}{ds}F(s,t)\, ds\right|_{y_n(t)} \\ &\le \int_0^1 \left[K_5\, R_0^2\, e_t + K_5\, R_0\, z_t\right] ds \\ &\le 2\,K_5\, R_0^2\, z_t \qquad \text{for all } t \in I_n.\end{aligned}$$

Thus, when $\tau \in I_n$,

$$|\text{third term in (57)}| \le \int_{t_0}^{\tau} |F(1,t)|_{y_n(t)}\; dt \le 2\,K_5\, R_0^2 \int_{t_0}^{\tau} z_t\, dt.$$

From (52), (56), (57) and (45) when $\tau \in I_n$, we get that

$$\begin{aligned} z_\tau &\le K_1 \int_{t_0}^{\tau} z_t\; dt + \left|\int_{t_0}^{\tau} \left[T(\tau,t)\cdot \frac{D}{dt}\rho_n(t)\right] dt\right|_{y_n(\tau)} \\ &\quad + K_2 \int_{t_0}^{\tau} z_t\; dt + 2\,K_5\, R_0^2 \int_{t_0}^{\tau} z_t\; dt \\ z_\tau &\le |\rho_n(\tau) - T(\tau,t_0)\cdot \rho_n(t_0)|_{y_n(\tau)} + B\int_{t_0}^{\tau} z_t\; dt \\ z_\tau &\le |\rho_n(\tau)|_{y_n(\tau)} + |\rho_n(t_0)|_{y_n(t_0)} + B\int_{t_0}^{\tau} z_t\; dt \quad \text{when } \tau \in I_n, \end{aligned} \tag{58}$$

where $B := \max\{1, K_1 + K_2 + 2\,K_5\, R_0^2\}$. Let $u(\tau) := \int_{t_0}^{\tau} z_t\; dt$. Then, using (59), we have that

$$\begin{aligned}\frac{d}{dt}\big(e^{-B(t-t_0)}\,u(t)\big) &= e^{-B(t-t_0)}(z_t - B\,u(t)) \\ &\le e^{-B(t-t_0)}\big(\,|\rho_n(t)|_{y_n(t)} + |\rho_n(t_0)|_{y_n(t_0)}\,\big) \quad \text{for } t \in I_n.\end{aligned}$$

Since $u(t_0) = 0$,

$$
\begin{aligned}
u(\tau) &\le e^{B(\tau-t_0)} \int_{t_0}^{\tau} e^{-B(t-t_0)} \, |\rho_n(t)|_{y_n(t)} \; dt \\
&\quad + e^{B(\tau-t_0)} \int_{t_0}^{\tau} e^{-B(t-t_0)} \, |\rho_n(t_0)|_{y_n(t_0)} \; dt \\
&\le \frac{e^{B(\tau-t_0)}}{\sqrt{2B}} \; \|\rho_n\|_{\mathcal{L}^2([0,T_n])} + \frac{e^{B(\tau-t_0)}}{B} \; |\rho_n(t_0)|_{y_n(t_0)} \quad \text{when } \tau \in I_n.
\end{aligned}
$$

Then from (59), if $\tau \in I_n$,

$$
\begin{aligned}
z_\tau &\le |\rho_n(\tau)|_{y_n(\tau)} + |\rho_n(t_0)|_{y_n(t_0)} + B\, u(\tau) \\
&\le |\rho_n(\tau)|_{y_n(\tau)} + |\rho_n(t_0)|_{y_n(t_0)} \\
&\quad + \sqrt{\frac{B}{2}}\, e^{B(\tau-t_0)} \; \|\rho_n\|_{\mathcal{L}^2([0,T_n])} + e^{B(\tau-t_0)} \; |\rho_n(t_0)|_{y_n(t_0)}.
\end{aligned}
$$

Since $B > 1$, $\sqrt{\frac{B}{2}} < B$. This completes the proof of (51).

Since $\lim_n T_n = +\infty$, by (46), there exists $n_0 > 0$ such that if $n > n_0$ then

$$
\begin{aligned}
B\, e^{BT_n} \; \|\rho_n\|_{\mathcal{L}^2([0,T_n])} \le B\, e^{BT_n} \, \frac{1}{n}\, e^{-2T_n^2} < \frac{1}{2}\, e^{-T_n^2}, \\
(1 + e^{BT_n})\, e^{-\frac{3}{2}T_n^2} < \frac{1}{2}\, e^{-T_n^2}
\end{aligned}
$$

and

$$
K_1 \left(\frac{1}{n} \sqrt{T_n}\, e^{-2T_n^2} + T_n\, e^{-T_n^2} \right) < 1.
$$

If $t_0 \in A_n$ then $|\rho_n(t_0)| < e^{-\frac{3}{2}T_n^2}$. Thus if $n > n_0$ each of the last two terms in (51) is bounded by $\frac{1}{2}\, e^{-T_n^2}$. This validates (50) when $t \in I_n$.

It remains to prove that if $t_0 \in A_n$ then $I_n = [0, T_n]$. Let $I_n = [a, b]$. From (56) and (50) we have that for $\tau \in I_n$,

$$
\begin{aligned}
d_M(x(t), y_n(t)) \le e_t &\le \int_{t_0}^{\tau} K_1\, z_t \; dt \le K_1 \int_{t_0}^{\tau} |\rho_n(t)| \; dt + K_1 \int_{t_0}^{\tau} e^{-T_n^2} \; dt, \\
&\le K_1 \; \|\rho_n\|_{\mathcal{L}^2([0,T_n])} \; \sqrt{|\tau - t_0|} + K_1\, |\tau - t_0|\, e^{-T_n^2} \\
&\le K_1 \left(\frac{1}{n}\, e^{-2T_n^2} \sqrt{T_n} + T_n\, e^{-T_n^2} \right) \\
&< 1.
\end{aligned}
$$

If $b < T_n$, since $t \mapsto e_t$ is continuous, I_n could be extended. Then $b = T_n$. Similarly, $a = 0$. □

Lemma 3.17 *If $t \in A_n$ and n is large enough then*

$$
|H(y_n(t), p_n(t)) - k| < \frac{3}{n} \quad \textit{and} \quad |p_n(t)| \le R_0.
$$

Proof
Claim 1. $A_n \cap \{t \in [0, T_n] \mid |p_n(t)| < R_0\} \neq \emptyset$.

Proof Let $d_1 > |k|$ be from (48). Since $\|d_{(x_n,T_n)}\mathcal{A}_k\| < \frac{1}{n}$,

$$-\left.\frac{\partial \mathcal{A}_k}{\partial T}\right|_{(x_n,T_n)} = \frac{1}{T_n}\int_0^{T_n}\big[H(y_n, p_n) - k\big]\,dt < \frac{1}{n}.$$

Then

$$\begin{aligned}\frac{1}{2}\,R_0^2\cdot m([|p_n| \geq R_0]) &\leq \int_0^{T_n}\frac{1}{2}\,|p_n(t)|^2\,dt \leq \int_0^{T_n}\big[H(y_n, p_n) + d_1\big]\,dt\\ &\leq \left[k + d_1 + \frac{1}{n}\right]T_n.\\ m([|p_n| \geq R_0]) &\leq \frac{2\left[k + d_1 + \frac{1}{n}\right]}{R_0^2}\,T_n\\ &\leq \frac{1}{10}\,T_n, \quad \text{from (49).}\end{aligned}$$

Therefore, using (47),

$$\begin{aligned}m(A_n \cap [|p_n| < R_0]) &= T_n - m\big(A_n^c \cup [|p_n| \geq R_0]\big)\\ &\geq T_n - \frac{1}{n} - \frac{1}{10}\,T_n\\ &> 0.\end{aligned}$$

This proves Claim 1.

□

Claim 2. *There exist $K = K(R_0) > 0$ and $n_1 > 0$ such that*

$$|p_n(t)|_{y_n(t)} \leq K(R_0)\,[|\rho_n(t)|_{y_n(t)} + 1], \quad \forall\, t \in [0, T_n], \quad \forall\, n > n_1.$$

Proof By Claim 1 there exists $t_0 \in A_n$ such that $|p_n(t_0)| < R_0$. Let $(x(t), q(t))$ be the solution of the hamiltonian equations with initial conditions $x(t_0) = y_n(t_0)$, $q(t_0) = p_n(t_0)$. By Lemma 3.15, $d_M\big(x(t), y_n(t)\big) < 1$ for all $t \in [0, T_n]$. Given $t \in [0, T_n]$, let $\gamma_t : [0, 1] \to M$ be the minimizing geodesic joining $x(t)$ to $y_n(t)$. Let $\Lambda_t : [0, 2] \to TM$ be defined by (53), let $e_t := d_M(x(t), y_n(t))$ and let $z_t := e_t + |p_n(t) - A_t(1, 0)\cdot q(t)|$ be as in Lemma 3.15. Then

$$\begin{aligned}&H(y_n(t), p_n(t)) - H(y_n(t_0), p_n(t_0))\\ &\quad = H(y_n(t), p_n(t)) - H(x(t), q(t))\\ &\quad = \int_0^1 \big\langle \nabla_x H(\Lambda_t(s))\,,\ \dot{\gamma}_t(s)\big\rangle_{\gamma_t(s)}\,ds\\ &\qquad + \int_1^2 \langle \nabla_p H(\Lambda_t(s))\,,\ p_n(t) - A_t(1, 0)\cdot q(t)\rangle_{y_n(t)}\,ds.\end{aligned}$$

Since $\nabla_x H(x,p) = 0$ when $|p|_x > R_0$, there is $d_2 > 0$ such that

$$|\nabla_x H(x,p)|_x < d_2 \qquad \text{for all } (x,p) \in TM.$$

Since $\nabla_p H(x,p) = p$ when $|p|_x > R_0$, there is $d_3 > 0$ such that

$$\big|\nabla_p H(x,p)\big|_x \le |p|_x + d_3 \qquad \text{for all } (x,p) \in TM.$$

Then

$$\begin{aligned}
&|H(y_n(t), p_n(t)) - H(y_n(t_0), p_n(t_0))| \\
&\quad \le \int_0^1 d_2\, e_t \; ds + \int_1^2 \big[|p_n(t)|_{y_n(t)} + |q(t)|_{x(t)} + d_3\big]\, z_t \; ds.
\end{aligned}$$

Since $|p_n(t_0)| < R_0$, by Lemma 3.14, $H(y_n(t_0), p_n(t_0)) = H(x(t), q(t)) < \frac{1}{2} R_0^2$ and $|q(t)| < R_0$ for all t. Let $d_4 := \max\{d_2, d_3\}$. Then

$$|H(y_n(t), p_n(t)) - H(y_n(t_0), p_n(t_0))| \le [|p_n(t)| + R_0 + 2\, d_4]\; z_t. \tag{59}$$

Suppose first that $|p_n(t)| > R_0$. Then $H(y_n(t), p_n(t)) = \frac{1}{2}\, |p_n(t)|^2$. In this case we have that

$$\frac{1}{2}\, |p_n(t)|^2 \le H_0 + |p_n(t)|\; z_t + [R_0 + 2\, d_4]\, z_t$$

where $H_0 := H(y_n(t_0), p_n(t_0))$. Using that $H_0 < \frac{1}{2} R_0^2$, we get

$$\begin{aligned}
\frac{1}{2}\, [|p_n(t)| - z_t]^2 &\le H_0 + [R_0 + 2\, d_4]\, z_t + \frac{1}{2}\, z_t^2 \\
&\le \frac{1}{2}\, R_0^2 + R_0\, z_t + \frac{1}{2}\, z_t^2 + 2\, d_4\, z_t \\
&\le \frac{1}{2}\, [R_0 + z_t]^2 + 2\, d_4\, z_t.
\end{aligned}$$

$$|p_n(t)| \le z_t + \sqrt{[R_0 + z_t]^2 + 4\, d_4\, z_t}\;. \tag{60}$$

The other case is when $|p_n(t)| \le R_0$. Since the right hand side in (60) is $\ge R_0$, the inequality (60) is valid for all $t \in [0, T_n]$. Using the identity $(a+b)^2 \le 3\,(a^2+b^2)$, we have that

$$\begin{aligned}
|p_n(t)|^2 &\le 3\, z_t^2 + 3\, [R_0 + z_t]^2 + 12\, d_4\, z_t \\
&\le d_5 \left[\, z_t^2 + R_0^2 \,\right] && \text{for some constant } d_5 > 0, \\
&\le d_6(R_0) \left[\, z_t + \frac{1}{2} \,\right]^2 && \text{for some } d_6(R_0) > 0, \\
&\le d_6(R_0) \left[\, |\rho_n(t)| + e^{-T_n^2} + \frac{1}{2} \,\right]^2 && \text{using Remark 3.16 and (50)}, \\
&\le d_6(R_0)\, [|\rho_n(t)| + 1]^2 && \text{if } n \text{ is large enough.}
\end{aligned}$$

Now let $K(R_0) := \sqrt{d_6(R_0)}$.

□

Claim 3 *There is $n_2 > 0$ such that if $n > n_2$ and $t_1, t_2 \in A_n$ then*

$$|H(y_n(t_1), p_n(t_1)) - H(y_n(t_2), p_n(t_2))| < \frac{1}{n}.$$

Proof By Claim 1 and the triangle inequality, it is enough to prove that if $t_0 \in A_n$, $|p_n(t_0)| < R_0$ and $t_1 \in A_n$ then

$$|H(y_n(t_1), p_n(t_1)) - H(y_n(t_0), p_n(t_0))| < \frac{1}{2n}.$$

Let $(x(t), q(t))$ be the solution of the hamiltonian equations with initial conditions $x(t_0) = y_n(t_0)$, $q(t_0) = p_n(t_0)$ and let z_t be as in Claim 2.

If $t_1 \in A_n$, $n > n_0$, then $|\rho_n(t_1)| < e^{-\frac{3}{2}T_n^2}$ and by Remark 3.16 and (50),

$$z_{t_1} \le |\rho_n(t_1)| + e^{-T_n^2} < 2\, e^{-T_n^2}.$$

From (59) and Claim 2, we get that

$$\begin{aligned}
|H(y_n(t_1), p_n(t_1)) - H(y_n(t_0), p_n(t_0))| &\le [K(R_0)\,(|\rho_n(t_1)| + 1) + R_0 + 2\,d_4]\; z_{t_1}\\
&\le \left[K(R_0)\left(e^{-\frac{3}{2}T_n^2} + 1\right) + R_0 + 2\,d_4\right] 2\,e^{-T_n^2}\\
&< \frac{1}{2n}
\end{aligned}$$

if n is large enough. □

We now finish the proof of Lemma 3.17. Let $d_7 > 0$ be such that

$$\left|H(x, p) - \frac{1}{2}\,|p|_x^2\right| \le d_7 \quad \text{for all } (x, p) \in TM.$$

Then, using Claim 2, and the inequality $(a + b)^2 \le 3(a^2 + b^2)$,

$$\begin{aligned}
&\left|\int_{A_n^c} H\right|\\
&\quad\le \int_{A_n^c} \left(\frac{1}{2}\,|p_n(t)|^2_{y_n(t)} + d_7\right)\,dt\\
&\quad\le \frac{1}{2}\,K(R_0)^2 \int_{A_n^c} (|\rho_n(t)| + 1)^2\,dt + d_7\, m\big(A_n^c\big)\\
&\quad\le \frac{3}{2}\,K(R_0)^2\;\|\rho_n\|^2_{\mathcal{L}^2} + \left[\frac{3}{2}\,K(R_0)^2 + d_7\right] m\big(A_n^c\big)\\
&\quad\le \frac{3}{2}\,K(R_0)^2\,\frac{1}{n^2}\,e^{-4T_n^2} + \left[\frac{3}{2}\,K(R_0)^2 + d_7\right]\frac{1}{n^2}\,e^{-T_n^2}, \qquad \text{using (46) and (47),}\\
&\quad\le \frac{1}{n}, \qquad\qquad \text{if } n \text{ is large enough.}
\end{aligned}$$

Since $\|d_{(x_n,T_n)}\mathcal{A}_k\| < \frac{1}{n}$,

$$\left|\frac{\partial \mathcal{A}_k}{\partial T}\bigg|_{(x_n,T_n)}\right| = \frac{1}{T_n}\left|\int_0^{T_n}[H(y_n,\, p_n)-k]\,dt\right| < \frac{1}{n},$$

$$\left|\int_0^{T_n}[H(y_n,\, p_n)-k]\,dt\right| \le \frac{T_n}{n}.$$

Therefore

$$\begin{aligned}\left|\int_{A_n} H\ dt - k\, m(A_n)\right| &\le \left|\int_{A_n^c} H\right| + |k|\ m(A_n^c) + \frac{T_n}{n}\\ &\le \frac{1}{n} + \frac{|k|}{n^2}\, e^{-T_n^2} + \frac{T_n}{n}.\end{aligned} \tag{61}$$

Let $t \in A_n$. By Claim 3, we have that

$$\left|\int_{A_n} H\, dt - m(A_n)\, H(y_n(t),\, p_n(t))\right| \le \frac{1}{n}\, m(A_n). \tag{62}$$

Adding (61) and (62) we get that

$$|H(y_n(t),\, p_n(t)) - k|\ \ m(A_n) \le \frac{1}{n}\, m(A_n) + \frac{1}{n} + \frac{|k|\, e^{-T_n^2}}{n^2} + \frac{T_n}{n}$$

$$\begin{aligned}|H(y_n(t),\, p_n(t)) - k| &\le \frac{1}{n} + \frac{1}{n\, m(A_n)} + \frac{|k|\, e^{-T_n^2}}{n^2\, m(A_n)} + \frac{T_n}{n\left(T_n - \frac{1}{n^2}\, e^{-T_n^2}\right)},\\ &\le \frac{3}{n} \qquad \text{if } n \text{ is large enough.}\end{aligned}$$

Since by (49), $|H(y_n(t),\, p_n(t))| \le |k|+1 < \frac{1}{2}\, R_0^2$, by Lemma 3.14, $|p_n(t)| < R_0$ if $t \in A_n$ and n is large enough. ☐

Let ν_n be the Borel probability measure defined by

$$\int f\ d\nu_n = \frac{1}{m(A_n)}\ \int_{A_n} f\big(y_n(t),\, p_n(t)\big)\ dt$$

for any continuous function $f : TM \to \mathbb{R}$. By Lemma 3.17 we have that

$$\operatorname{supp}(\nu_n) \subseteq H^{-1}\left(\left[k - \frac{3}{n},\, k + \frac{3}{n}\right]\right) \subseteq H^{-1}([k-1,\, k+1]).$$

Since $H^{-1}([k-1,\, k+1])$ is compact, there exists a convergent subsequence ν_{n_i} in the weak* topology. Let

$$\nu := \lim_i \nu_{n_i}.$$

Then

$$\operatorname{supp}(\nu) \subseteq H^{-1}\{k\}.$$

Lemma 3.18 *We can assume that ν is supported on a connected component of $H^{-1}\{k\}$.*

Proof If k is a singular value of H then $H^{-1}\{k\}$ contains a singularity of the Hamiltonian flow. In that case a Dirac measure supported on the corresponding singularity of the Lagrangian flow satisfies the thesis of Proposition 3.13.

If k is a regular value of H then there is $\varepsilon > 0$ such that each of the finitely many connected components of $H^{-1}(]k-\varepsilon, k+\varepsilon[)$ contains exactly one connected component of $H^{-1}\{k\}$. Since the measures ν_n are supported on the images of the connected curves y_n and $\text{supp}(\nu) \subseteq H^{-1}(]k-\varepsilon, k+\varepsilon[)$ for n large, we can take the convergent subsequence ν_{n_i} in a single connected component of $H^{-1}(]k-\varepsilon, k+\varepsilon[)$. □

Lemma 3.19 *The probability ν is invariant under the hamiltonian flow.*

Proof Given $0 < s < 1$ let

$$\begin{aligned} D_n(s) :&= \left\{ t \in [0, T_n] \mid |\rho_n(t)| < e^{-\frac{3}{2}T_n^2},\ |\rho_n(t+s)| < e^{-\frac{3}{2}T_n^2} \right\}, \\ &= A_n \cap (A_n - s). \end{aligned}$$

Then $m(D_n^c \setminus [T_n - s, T_n]) \le m([0, T_n - s] \cap \{A_n^c \cup (A_n - s)^c\}) \le \frac{2}{n^2}\, e^{-T_n^2}$, and

$$m(D_n) \ge T_n - s - \tfrac{2}{n^2}\, e^{-T_n^2} \ge T_n - 2.$$

Let $\psi_t : TM \hookleftarrow$ be the hamiltonian flow. Let $F : TM \to \mathbb{R}$ be a continuous function with compact support.

Given $t \in D_n \subset A_n$, by Lemma 3.17, $|p_n(t)| \le R_0$. Observe that $t + s \in A_n$. By Lemma 3.15 we have that

$$\begin{aligned} d_{TM}[\psi_s(y_n(t), p_n(t)), (y_n(t+s), p_n(t+s))] &\le |\rho_n(t+s)| + e^{-T_n^2} \\ &\le e^{-\frac{3}{2}T_n^2} + e^{-T_n^2} \le \frac{1}{n}, \quad \text{if } n \text{ is large.} \end{aligned}$$

Since F is uniformly continuous then

$$\mathcal{O}\big(F, \tfrac{1}{n}\big) := \sup_{d(z,w) < \frac{1}{n}} |F(z) - F(w)| \xrightarrow{n \to \infty} 0.$$

Since by (47) $m(A_n) \ge T_n - 1$, $m(D_n^c) \le 2$ and $\frac{m(D_n)}{m(A_n)} \le 1$, we have that

$$\begin{aligned} \left| \int F\ d(\psi_s^* \nu_n) - \int F\ d\nu_n \right| &\le 2\ \|F\|_\infty \frac{m(D_n^c)}{m(A_n)} + \frac{1}{m(A_n)} \int_{D_n} |F \circ \psi_s - F|_{(y_n, p_n)} \\ &\le \frac{4\ \|F\|_\infty}{T_n - 1} + \mathcal{O}\big(F, \tfrac{1}{n}\big) \xrightarrow{n} 0. \end{aligned}$$

Hence, for all $0 < s < 1$,

$$\int F\ d(\psi_s^* \nu) = \int F\ d\nu. \qquad \square$$

Let $\mu = \mathfrak{L}_*(\nu)$ be the push forward of ν under the Legendre transform $\mathfrak{L} : T^*M \to TM$, $\mathfrak{L}(x, p) = \nabla_p H(x, p)$.

Lemma 3.20 *The homology class $\rho(\mu) \in H_1(M, \mathbb{R})$ of μ is zero.*

Proof If we are working on $\Omega_M(q_0, q_1)$, let $\gamma = \gamma_n$ be a minimizing geodesic joining the two common endpoints $\gamma(1) = q_0 = y_n(0)$, $\gamma(0) = q_1 = y_n(T_n)$ of all y_n. If we are working on Λ_M, let γ_n be the constant curve $\gamma_n(t) \equiv y_n(0) = y_n(T_n)$, $t \in [0, 1]$.

Let μ_{n_i} be the probability measure defined by

$$\begin{aligned}\int_{TM} f\, d\mu_{n_i} := \frac{1}{T_{n_i}+1}\Bigg[&\int_0^{T_{n_i}} f\left(y_{n_i}(s), \dot{y}_{n_i}(s)\right)\, ds \\ &+ \int_0^1 f\left(\gamma_{n_i}(s), \dot{\gamma}_{n_i}(s)\right)\, ds \Bigg]\end{aligned}$$

for any continuous function $f \in \mathcal{F}$ with quadratic growth:

$$\mathcal{F} := \left\{ f \in C^0(TM, \mathbb{R}) \;\middle|\; \sup_{v \in TM} \frac{|f(v)|}{1+|v|^2} < +\infty \right\}.$$

We show that for any $f \in \mathcal{F}$,

$$\lim_i \int f\, d\mu_{n_i} = \int f\, d\mu = \lim_i \int f\, d(\mathfrak{L}_*\nu_{n_i}) = \int f\, d(\mathcal{L}_*\nu). \tag{63}$$

We have that

$$\int f\, d\mu_{n_i} = \frac{m(A_{n_i})}{T_{n_i}+1}\int f\, d(\mathfrak{L}_*\nu_{n_i}) + \frac{1}{T_{n_i}+1}\oint_{\gamma_{n_i}} f + \frac{1}{T_{n_i}+1}\int_{A^c_{n_i}} f \circ \mathfrak{L}.$$

Observe that either γ_n is a constant curve or γ_n does not depend on n. Then

$$\begin{aligned}\lim_i \int f\, d\mu_{n_i} &= \lim_i \int f\, d(\mathfrak{L}_*\nu_{n_i}) + 0 + \lim_i \frac{1}{T_{n_i}+1}\int_{A^c_{n_i}} f \circ \mathfrak{L} \\ &= \int f\, d\mu + \lim_i \frac{1}{T_{n_i}+1}\int_{A^c_{n_i}} f \circ \mathfrak{L}.\end{aligned}$$

Let $\|f\|_{\mathcal{F}} := \sup_{v \in TM} \frac{|f(v)|}{1+|v|^2}$. Then

$$\int_{A^c_n} f \circ \mathfrak{L} \le \|f\|_{\mathcal{F}} \int_{A^c_n} \left[1+|v|^2\right] \circ \mathfrak{L}.$$

Let $d_3 > 0$ be such that

$$|\mathfrak{L}(x, p)|_x = \left|\nabla_p H(x, p)\right|_x \le |p|_x + d_3.$$

Then, using Claim 2 in Lemma 3.17 and the identity $(a+b)^2 \le 3\,(a^2+b^2)$,

$$\begin{aligned}
&\int_{A_{n_i}^c} [1+|v|^2]\circ \mathfrak{L} \\
&\quad\le \int_{A_{n_i}^c} 1 + \big[|p_{n_i}(t)|_{y_{n_i}(t)} + d_3\big]^2 \\
&\quad\le \big(1+3\,d_3^2\big)\, m\big(A_{n_i}^c\big) + 3\,K(R_0)^2 \int_{A_{n_i}^c} [|\rho_{n_i}(t)|+1]^2\, dt \\
&\quad\le \big(1+3\,d_3^2\big)\, m\big(A_{n_i}^c\big) + 3\,K(R_0)^2 \big[3\,\left\|\rho_{n_i}\right\|_{\mathcal{L}^2}^2 + 3\,m\big(A_{n_i}^c\big)\big] \\
&\quad\le \big(1+3\,d_3^2\big)\cdot 2 + 3\,K(R_0)^2 \cdot \left[3\cdot \frac{e^{-4T_{n_i}^2}}{{n_i}^2} + 3\cdot 2\right] \qquad \text{by (47) and (46).}
\end{aligned}$$

So that

$$\lim_i \frac{1}{T_{n_i}+1} \int_{A_{n_i}^c} f\circ\mathfrak{L} = 0,$$

and hence (63) holds.

Let η_n be the closed curve $\eta_n = y_n * \gamma_n$ and let $[\eta_n] \in H_1(M,\mathbb{Z})$ be its homology class. Since all the y_n's are in the same free homotopy class, $\alpha = [\eta_n]$ is constant in n. Let ω be a closed (bounded) 1-form and $[\omega] \in H^1(M,\mathbb{R})$ its cohomology class. Observe that ω has linear growth, in particular $\omega \in \mathcal{F}$. Then

$$\begin{aligned}
\int_{TM} \omega\, d\mu &= \lim_i \int_{TM} \omega\, d\mu_{n_i} = \lim_i \frac{1}{T_{n_i}} \oint_{\eta_{n_i}} \omega \\
&= \lim_i \frac{1}{T_{n_i}} \langle [\omega], \alpha\rangle = 0.
\end{aligned}$$ □

Finally, we prove that the $L+k$ action of μ is zero. Since L is quadratic at infinity then $L \in \mathcal{F}$. By (63), we have that

$$\begin{aligned}
\int_{TM} [L+k]\, d\mu &= \lim_i \int_{TM} [L+k]\, d\mu_{n_i} \\
&= \lim_i \frac{A_{L+k}(y_{n_i}) + A_{L+k}(\gamma_{n_i})}{T_{n_i}+1} \\
&= \lim_i \frac{\mathcal{A}_k(x_{n_i}, T_{n_i}) + A_{L+k}(\gamma_{n_i})}{T_{n_i}+1} = 0.
\end{aligned}$$

This finishes the proof of Proposition 3.13. □

4 Energy levels satisfying the Palais-Smale condition

In this section we prove Corollary B. Let $\widetilde{L} = L : T\widetilde{M} \to \mathbb{R}$ be the lift of L to the universal cover $\widetilde{M}$ of M.

Lemma 4.1 *Identify* $S^1 = [0,1]/_{0\equiv 1}$.

Let $\sigma \in [S^1, M]$ *be a free homotopy class of closed curves in* M. *If* $k \ge c_u(L)$ *then*

$$\inf\{\mathcal{A}_k(x,T) \mid x \in \sigma,\ T > 0\} > -\infty.$$

Proof Fix $y \in \sigma$. Since x and y are freely homotopic, there are points $p \in y([0,1])$, $q \in x([0,1])$ and a curve $z : [0,1] \to M$, with $z(0) = p$, $z(1) = q$ such that $z * x * z^{-1} * y^{-1}$ is homotopic to a point. We can assume that $x(0) = q$ and $y(0) = p$. Then there are lifts $\tilde{x}$, $\tilde{y}$, $\tilde{z}_0$, $\tilde{z}_1$ of x, y, z such that $\tilde{z}_0 * \tilde{x} * \tilde{z}_1^{-1} * \tilde{y}^{-1}$ is a closed curve in $\widetilde{M}$. Let φ be the deck transformation of the covering $\widetilde{M} \to M$ such that $\varphi(\tilde{z}_0) = \tilde{z}_1$. Let $\tilde{x}_n := \varphi^n(\tilde{x})$, $\tilde{y}_n := \varphi^n(\tilde{y})$ and $\tilde{z}_n := \varphi^n(\tilde{z})$. Since the curves $\tilde{z}_n * \tilde{x}_n * \tilde{z}_{n+1}^{-1} * \tilde{y}_n^{-1} = \varphi^n(\tilde{z}_0 * \tilde{x} * \tilde{z}_1^{-1} * \tilde{y}^{-1})$ are closed in $\widetilde{M}$, the curves $\tilde{z}_n * \tilde{x}_n * \tilde{z}_{n+1}^{-1}$ and $\tilde{y}_n$ have the same endpoints. Hence the curve $\tilde{w} := \tilde{z}_0 * (\tilde{x} * \tilde{x}_1 \cdots * \tilde{x}_n) * \tilde{z}_{n+1}^{-1} * (\tilde{y}_n^{-1} * \cdots * \tilde{y}_1^{-1} * \tilde{y}^{-1})$ is closed in $\widetilde{M}$. Given $T > 0$ let $S := 1 + (n+1)\,T + 1 + (n+1)$ and $\tilde{\eta}(t) := \tilde{w}(t/S)$. Since $k \ge c_u(L)$,

$$A_{\widetilde{L}+k}(\tilde{\eta}) = \mathcal{A}_k(z,1) + (n+1)\,\mathcal{A}_k(x,T) + \mathcal{A}_k(z^{-1},1) + (n+1)\,\mathcal{A}_k(y^{-1},1) \ge 0.$$

Dividing by $n+1$,

$$\frac{1}{n+1}\,\mathcal{A}_k(z,1) + \mathcal{A}_k(x,T) + \frac{1}{n+1}\mathcal{A}_k(z^{-1},1) \ge -\mathcal{A}_k(y^{-1},1).$$

Letting $n \to +\infty$ we get that for all $T > 0$ and $x \in \sigma$

$$\mathcal{A}_k(x,T) \ge -\mathcal{A}_k(y^{-1},1). \qquad \square$$

Proof of Corollary B Let Λ_1 be a connected component of $\Omega_M(q_0,q_1)$ or Λ_M. Let (x_n,T_n) be a sequence in Λ_1 such that

$$|\mathcal{A}_k(x_n,T_n)| < A_1 \quad \text{and} \quad \left\| d_{(x_n,T_n)}\mathcal{A}_k \right\| < \frac{1}{n}.$$

Claim *If the energy level* $E^{-1}\{k\}$ *does not contain singularities of the Euler-Lagrange flow and* T_n *is bounded from above, then there is a convergent subsequence of* (x_n,T_n).

Proof By Propositions 3.8.(ii) and (iii), $\liminf_n T_n > 0$. Since T_n is bounded, by Proposition 3.12, (x_n,T_n) has a convergent subsequence.

Suppose that $k > c_u$. If $(x_n,T_n) \in \Lambda_1 \subset \Omega_M(q_0,q_1)$ let $(z_n,S_n) = (x_n * x_0^{-1}, T_n + 1) \in \Lambda_M$. If $(x_n,T_n) \in \Lambda_1 \subset \Lambda_M$ let $(z_n,S_n) = (x_n,T_n)$. By Lemma 4.1, $B := \inf_n \mathcal{A}_{c_u}(z_n,S_n)$ is finite and so

$$\begin{aligned} A_1 + \max\{0, \mathcal{A}_k(x_0^{-1},1)\} &\ge \mathcal{A}_k(z_n,S_n) \\ &= \mathcal{A}_{c_u}(z_n,S_n) + (k-c_u)\,S_n \ge B + (k-c_u)\,S_n. \end{aligned}$$

Hence S_n is bounded and so T_n is bounded. Since $k > c_u \ge e_0$, the energy level $E^{-1}\{k\}$ does not contain singularities of the Euler-Lagrange flow. By the claim, (x_n,T_n) has a convergent subsequence.

Now assume that $\Lambda_1 \subset \Omega_M(q_0, q_1)$. Since all the curves x_n have the same homotopy class with fixed endpoints, there are lifts $\tilde{q}_0,\ \tilde{q}_1 \in \widetilde{M}$ of q_0, q_1 and lifts $\tilde{x}_n$ of x_n such that for all n, $(\tilde{x}_n, T_n) \in \Omega_{\widetilde{M}}(\tilde{q}_0, \tilde{q}_1)$.

Suppose that $h_{c_u} \equiv +\infty$. Since

$$A_1 \geq \mathcal{A}_{c_u}(x_n, T_n) \geq \Phi_{c_u}(\tilde{q}_0, \tilde{q}_1; T_n)$$

and

$$h_{c_u}(\tilde{q}_0, \tilde{q}_1) = \liminf_{T \to +\infty} \Phi_{c_u}(\tilde{q}_0, \tilde{q}_1; T) = +\infty,$$

the sequence T_n is bounded. If $E^{-1}\{c_u\}$ contains a singularity $(q_2, 0)$ of the Euler-Lagrange flow, then $L(q_2, 0) + c_u = 0$ and

$$h_{c_u}(q_2, q_2) \leq \liminf_n \int_0^n [L(q_2, 0) + c_u]\ dt = 0.$$

This contradicts $h_{c_u} \equiv +\infty$. By the claim, (x_n, T_n) has a convergent subsequence.

If $h_{c_u} \not\equiv +\infty$, the proof of Theorem C in [7] applies to our action functional $\mathcal{A}_{c_u}$ with our riemannian metric, showing that $\mathcal{A}_{c_u}$ does not satisfy the Palais-Smale condition. Indeed, the Palais-Smale sequence $(\tilde{x}_n, T_n)$ obtained from there has $\lim_n T_n = +\infty$ and is made with solutions of the Euler-Lagrange equation joining any two given points $\tilde{q}_0, \tilde{q}_1$ in the universal cover $\widetilde{M}$. Take their projections $x_n := \pi \circ \tilde{x}_n$. Then the curves x_n are in the same homotopy class. Since the curves x_n are solutions of the Euler-Lagrange equation, $\frac{\partial}{\partial x}\mathcal{A}_{c_u}(x_n, T_n) = 0$ and $k - E(x_n, \frac{\dot{x}_n}{T_n})$ is constant. The theorem proves that

$$\left\| d_{(x_n, T_n)} \mathcal{A}_{c_u} \right\| = \left| \frac{\partial \mathcal{A}_{c_u}}{\partial T}(x_n, T_n) \right| = \left| k - E\left(x_n, \frac{\dot{x}_n}{T_n} \right) \right| \xrightarrow{n} 0. \qquad \square$$

5 The mountain pass geometry

In this section we show that a small closed curve of a given length ℓ contained in the the projection of the energy level $E^{-1}\{k\}$ has positive $(L + k)$ action bounded away from zero. This gives a mountain pass geometry when we consider families of curves going from a constant curve, with arbitrarily small action, to a curve with negative action[6] .

In the case of closed curves in Λ_M without a basepoint, we need that $k > e_0(L)$, because otherwise a family of curves could move by constant curves until it leaves the projection $\pi(E^{-1}\{k\})$, where it already becomes negative, without passing through a curve of length ℓ inside $\pi(E^{-1}\{k\})$.

[6] This resembles the phrase by Taĭmanov in [33]: "constant curves are local minimizers of the action" for magnetic flows. But in our case the constant curves are not critical points for $\mathcal{A}_k$, because they don't have energy k, and also the gradient flow of $-\mathcal{A}_k$ is not complete on a constant curve (x_0, T) because T reaches zero at a finite gradient flow time.

Lemma 5.1 *Let θ_x be a 1-form on M. Let $x_0 \in M$ and $x_0 \in V \subset M$ be a neighbourhood of x_0. Then there exists an open ball $U \subseteq V$ centered at x_0 in M and $b > 0$ such that if γ is a closed curve in U then*

$$\left| \int_\gamma \theta_x \right| \leq b \cdot \text{length}(\gamma)^2.$$

Proof Shrinking V if necessary and using a local chart, we can assume that V is a closed ball in $\mathbb{R}^m$ with the euclidean metric. Moreover, we can assume that $\gamma(0) = 0 \in V \subset \mathbb{R}^m$. Let $b > 0$ be such that $|d_x\theta(u, v)| \leq b\, |u|\, |v|$ for all $u,\ v \in \mathbb{R}^m$, $x \in V$. Let $\gamma : [0, T] \to V$ be a closed curve. Let $F : [0, 1] \times [0, T] \to V$ be defined by $F(s, t) := s\, \gamma(t)$. Then

$$\begin{aligned}
\left| \int_\gamma \theta_x \right| = \left| \int_F d_x\theta \right| &\leq \int_0^1 \int_0^T b \left| \frac{\partial F}{\partial s} \right| \left| \frac{\partial F}{\partial t} \right| \, dt\, ds \\
&= b \int_0^1 \int_0^T |\gamma(t)| \cdot s\ |\dot{\gamma}(t)| \ dt\, ds \\
&\leq b \int_0^1 \int_0^T \ell(\gamma) \cdot s\ |\dot{\gamma}(t)| \ dt\, ds \\
&\leq b \int_0^1 \ell(\gamma) \cdot s\ \ell(\gamma)\, ds \leq b \cdot \ell(\gamma)^2,
\end{aligned}$$

where $\ell(\gamma)$ is the length of γ. Now let U be an open ball for the Riemannian metric centered at x_0 and contained in V. □

Proposition C

1. *Let $x_0 \in M$ and $k > E(x_0, 0)$. Then there exists $c > 0$ such that if $\Gamma : [0, 1] \to \Omega_M(x_0, x_0)$ is a path joining a constant loop $\Gamma(0) = x_0 : [0, T] \to \{x_0\} \subset M$ (with any $T > 0$) to any closed loop $\Gamma(1) \in \Omega_M(x_0, x_0)$ with negative $(L+k)$-action, $A_{L+k}(\Gamma(1)) < 0$, then*

$$\sup_{s \in [0,1]} A_{L+k}(\Gamma(s)) > c > 0.$$

2. *Let $k > e_0(L)$. Then there exists $c > 0$ such that if $\Gamma : [0, 1] \to \Lambda_M$ is a path joining any constant curve $\Gamma(0) = x_0 : [0, T] \to \{x_0\} \subset M$ to any closed curve $\Gamma(1)$ with negative $(L + k)$-action, $A_{L+k}(\Gamma(1)) < 0$, then*

$$\sup_{s \in [0,1]} A_{L+k}(\Gamma(s)) > c > 0.$$

Proof (1). Let $d_1 \in \mathbb{R}$ be such that

$$E(x_0, 0) = -\psi(x_0) < d_1 < k.$$

Let V be a neighbourhood of x_0 such that

$$\inf_{x \in V} \psi(x) \geq -d_1.$$

Write, as in Lemma 3.1,

$$L(x,v) \geq \frac{1}{2}\, a\, |v|_x^2 + \theta_x(v) + \psi(x),$$

where $\theta_x(v) := L_v(x,0)\cdot v$, $\psi(x) := L(x,0)$ and $a := \inf_v\{v\cdot L_{vv}(x,v)\cdot v\}/|v|_x^2 > 0$.

Let $U \subseteq V$ be an open ball centered at x_0 given by Lemma 5.1 for (x_0, V). Let

$$0 < \ell_0 < \min\left\{\frac{1}{2}\ \mathrm{diam}(U),\ \sqrt{\frac{a\,(k-d_1)}{2\,b^2}}\right\}. \tag{64}$$

Claim *There exists* $0 < s_0 < 1$ *such that* $\mathrm{length}(\Gamma(s_0)) = \ell_0$.

Proof Suppose that $\Gamma(s_1) \not\subset U$ for some $s_1 \in [0,1]$. Since $s \mapsto \mathrm{length}(\Gamma(s))$ is continuous, $\mathrm{length}(\Gamma(0)) = 0$ and since $\mathrm{length}(\Gamma(s_1)) \geq d(x_0, U^c) \geq \frac{1}{2}\,\mathrm{diam}(U) > \ell_0$, there exists $0 < s_0 < s_1$ such that $\mathrm{length}(\Gamma(s_0)) = \ell_0 < \frac{1}{2}\,\mathrm{diam}(U)$.

Now assume that $\Gamma(s) \subset U$ for all $s \in [0,1]$. Write $\gamma_1 := \Gamma(1) : [0, T_1] \to M$ and $\ell_1 = \mathrm{length}(\gamma_1)$. By Lemma 5.1 we have that

$$\begin{aligned}
0 > A_{L+k}(\gamma_1) &\geq \frac{1}{2}\int_0^{T_1} a\,|\dot\gamma_1|^2\,dt - \left|\int_{\gamma_1}\theta_x\right| + \int_0^{T_1}\psi(\gamma_1(t))\,dt + k\,T_1 \\
&\geq \frac{a}{2}\int_0^{T_1}|\dot\gamma_1|^2\ dt - b\,\ell_1^2 + (k-d_1)\,T_1.
\end{aligned}$$

By the Cauchy-Schwartz inequality,

$$T_1\ \int_0^{T_1}|\dot\gamma_1|^2\ dt \geq \left(\int_0^{T_1}|\dot\gamma_1|\ dt\right)^2 = \ell_1^2.$$

Hence

$$0 > A_{L+k}(\gamma_1) \geq \left(\frac{a}{2\,T_1} - b\right)\ell_1^2 + (k-d_1)\,T_1. \tag{65}$$

Since $(k-d_1) > 0$ and $T_1 > 0$, we have that $\frac{a}{2T_1} - b < 0$, i.e.

$$T_1 > \frac{a}{2\,b}.$$

From (65), we have that

$$\ell_1^2 > \frac{(k-d_1)\,T_1}{b - \frac{a}{2T_1}} > \frac{(k-d_1)\,T_1}{b} > \frac{a\,(k-d_1)}{2\,b^2} > \ell_0^2.$$

Since $\mathrm{length}(\Gamma(0)) = 0$, there is $s_0 \in]0,1[$ with $\mathrm{length}(\Gamma(s_0)) = \ell_0$. $\square$

Since $\operatorname{length}(\Gamma(s_0)) = \ell_0 < \frac{1}{2}\operatorname{diam} U$ and U is an open ball centered at x_0 and $\Gamma(s_0) \in \Omega_M(x_0, x_0)$, we have that $\Gamma(s_0) \subset U$. In particular, the right estimate in (65) holds for $\Gamma(s_0)$. Let

$$f(t) := \left(\frac{a}{2\,t} - b\right)\,\ell_0^2 + (k - d_1)\,t.$$

If $\Gamma(s_0) : [0, T_0] \to M$, then

$$A_{L+k}(\Gamma(s_0)) \geq f(T_0) \geq \min_{t\in\mathbb{R}^+} f(t) = \ell_0[\sqrt{2a\,(k-d_1)} - b\,\ell_0] =: c > 0,$$

because

$$\ell_0 < \sqrt{\frac{a\,(k-d_1)}{2\,b^2}} < \frac{\sqrt{2\,a\,(k-d_1)}}{b}.$$

(2). Since $k > e_0(L)$,

$$k > \sup_{x\in M} E(x, 0) = e_0(L). \tag{66}$$

Let $U_1, \ldots, U_N$ be a finite cover of M by open balls given by Lemma 5.1 with corresponding constants $b_i = b_i(U_i) > 0$. Let $r_0 > 0$ be such that any ball of radius r_0 in M is contained in one U_i. Write $b = \max_{1\leq i\leq N} b_i$ and let

$$0 < \ell_0 < \min\left\{r_0\,,\ \sqrt{\frac{a\,(k-e_0)}{2\,b^2}}\right\}. \tag{67}$$

Let $\Gamma : [0, 1] \to \Lambda_M$ be a path joining a constant curve $\Gamma(0) = x_0 : [0, T] \to \{x_0\} \subset M$ to a closed curve $\Gamma(1)$ with negative $(L + k)$-action, $A_{L+k}(\Gamma(1)) < 0$.

Claim *There is* $s_0 \in [0, 1]$ *such that* $\operatorname{length}(\Gamma(s_0)) = \ell_0$.

Proof Suppose that $\operatorname{length}(\Gamma(s)) < \ell_0$ for all $s \in [0, 1]$. Since $\ell_0 < r_0$, for each $s \in [0, 1]$, $\Gamma(s)$ is contained in some U_i. From (66) we have that $\psi(x) = -E(x, 0) \geq -e_0$ for all $x \in M$. Let $\ell_1 := \operatorname{length}(\Gamma(1))$. Then the same argument as in item (1) proves that

$$\ell_1^2 > \frac{a\,(k-e_0)}{2\,b^2} > \ell_0^2.$$

Therefore there is $s_0 \in [0, 1]$ such that $\operatorname{length}(\Gamma(s_0)) = \ell_0$. □

Let

$$g(t) := \left(\frac{a}{2\,t} - b\right)\,\ell_0^2 + (k - e_0)\,t.$$

Since $\operatorname{length}(\Gamma(s_0)) = \ell_0 < r_0$, $\Gamma(s_0)$ is contained in some U_i and we can apply Lemma 5.1. Therefore, if $\Gamma(s_0) : [0, T_0] \to M$, then

$$A_{L+k}(\Gamma(s_0)) \geq g(T_0) \geq \min_{t\in\mathbb{R}^+} g(t) = \ell_0[\sqrt{2a\,(k-e_0)} - b\,\ell_0] =: c > 0,$$

because

$$\ell_0 < \sqrt{\frac{a\,(k-e_0)}{2\,b^2}} < \frac{\sqrt{2\,a\,(k-e_0)}}{b}.$$ □

6 Some results on Morse theory

Let X be an open set in a Riemannian manifold and $f : X \to \mathbb{R}$ be a C^2 map. Observe that if the vector field $-\nabla f$ is not globally Lipschitz, the gradient flow ψ_t of $-f$ is a priori only a local flow. Given $p \in X$, define

$$\alpha(p) := \sup\{a > 0 \mid s \mapsto \psi_s(p) \text{ is defined on } s \in [0, a]\}.$$

We say that the flow ψ_t of $-\nabla f$ is *relatively complete on* $[a \le f \le b]$ if for $a \le f(p) \le b$, either $\alpha(p) = +\infty$ or $f(\psi_\beta(p)) \le a$ for some $0 \le \beta < \alpha(p)$.

We say that a function $\tau : X \to [0, +\infty[$ is an *admissible time* if τ is differentiable and $0 \le \tau(x) < \alpha(x)$ for all $x \in X$. Given an admissible time τ and a subset $F \subset X$, define

$$F_\tau := \left\{\psi_{\tau(x)}(x) \mid x \in F\right\}.$$

Given a closed subset $B \subseteq X$, we say that the function f satisfies the *Palais-Smale condition restricted to B at level c*, $(PS)_{c,B}$, if any sequence $\{x_n\} \subset B$ with $\lim_n \|df(x_n)\| = 0$ and $\lim_n f(x_n) = c$ has a convergent subsequence.

Given a closed subset $K \subset X$ and $x \in X$, define

$$d(x, K) := \inf_{k \in K} d(x, k).$$

In the case when K is empty, define $d(x, \emptyset) := \infty$.

Given $c \in \mathbb{R}$, $\delta > 0$ and a closed subset $B \subseteq X$, define

$$\begin{aligned} K_{c,B} &:= \{x \in B \mid f(x) = c,\ df(x) = 0\}, \\ W_{c,\delta,B} &:= \{x \in B \mid d(x, K_{c,B}) < \delta\}, \\ V_{c,\delta,B} &:= \{x \in B \mid \|df(x)\| < \delta,\ |f(x) - c| < \delta\}. \end{aligned}$$

Lemma 6.1 *Let X be a Riemannian manifold and $f : X \to \mathbb{R}$ a C^1 function.*

If f satisfies the Palais-Smale condition $(PS)_{c,B}$ at level c restricted to B, then

i. *$K_{c,B}$ is compact.*
ii. *The family $\{W_{c,\delta,B}\}_{\delta>0}$ is a fundamental system of neighbourhoods of $K_{c,B}$ relative to B.*
iii. *The family $\{V_{c,\delta,B}\}_{\delta>0}$ is a fundamental system of neighbourhoods of $K_{c,B}$ relative to B.*

Proof

(i) By $(PS)_{c,B}$ any sequence in $K_{c,B}$ has a convergent subsequence. Since df is continuous and B is closed, the limit is also in $K_{c,B}$.

(ii) Suppose item (ii) is false. Then there is a relative neighbourhood U of $K_{c,B}$ with $K_{c,B} \subset U \subset B$, and a sequence $x_n \in W_{c,1/n,B} \cap U^c$. Then there is a sequence $y_n \in K_{c,B}$ such that $d(x_n, y_n) \le \frac{2}{n}$. Since $K_{c,B}$ is compact there is a convergent subsequence $z = \lim_k y_{n_k} \in K_{c,B}$. Also, $\lim_k x_{n_k} = z \in K_{c,B}$. This contradicts $x_n \notin U$ for all n.

(iii) Suppose item (iii) is false. Then there is a relative neighbourhood U of $K_{c,B}$ with $K_{c,B} \subset U \subset B$ and a sequence $x_n \in V_{c,1/n,B} \cap U^c$. By $(PS)_{c,B}$ there is a convergent subsequence $z = \lim_k x_{n_k}$. Since $f \in C^1$, $z \in K_{c,B}$. This contradicts $x_n \notin U$ for all n. $\square$

Lemma 6.2 *Let X be a Riemannian manifold and $B \subset A \subset X$ closed subsets such that A contains the ε_1-neighbourhood of B:*

$$\{x \in X \mid d(x, B) < \varepsilon_1\} \subset A.$$

Let $f : X \to \mathbb{R}$ be a C^2 function. Let $c \in \mathbb{R}$ be such that

i. f satisfies the Palais-Smale condition $(PS)_{c,A}$ at level c restricted to A.
ii. The flow of $-\nabla f$ is relatively complete on $[|f - c| \le \varepsilon_2]$ for some $\varepsilon_2 > 0$.

Given any neighbourhood N of $K_{c,A}$ relative to A and $\varepsilon_3 > 0$, there are $0 < \varepsilon < \delta < \varepsilon_3$, such that for all $0 < \lambda \le \varepsilon$ there is an admissible time $\tau(x)$ such that

$$F_\tau \subseteq N \cup [\, f \le c - \lambda\,] \quad \textit{and} \quad \tau(x) = 0 \textit{ on } [\,|f - c| \ge \delta\,],$$

where $F = ([\, f \le c + \varepsilon\,] \cap B) \cup [f \le c - \lambda]$.

Proof By Lemma 6.1, there are $0 < \delta,\ \rho,\ \eta < \min\{\varepsilon_2, \varepsilon_3, 1\}$ such that

$$V_{c,\delta,A} \subset W_{c,\rho,A} \subset W_{c,2\rho,A} \subset V_{c,\eta,A} \subset N \subset A.$$

Let

$$\varepsilon := \frac{1}{2} \min \left\{ \frac{\varepsilon_1^2}{2},\ \varepsilon_2,\ \varepsilon_3,\ \frac{\delta^2}{2},\ \frac{\rho\,\delta}{2} \right\}.$$

Let $h : X \to [0, 1]$ be a smooth function such that

$$h(x) = \begin{cases} 0 & \text{if } x \in \ V_{c,\frac{\delta}{2},A} \ \cup [|f(x) - c| > \delta], \\ 1 & \text{if } x \in (V_{c,\delta,A})^c \cap \left[|f(x) - c| < \dfrac{\delta}{2} \right]. \end{cases}$$

Since f is C^2, the vector field $Y(x) := -h(x)\,\nabla f(x)$ is locally Lipschitz. Since $\delta < \varepsilon_2$, by hypothesis (ii), the flow η_s of Y is complete.

Let ψ_t be the flow of $-\nabla f$. Define $\tau(x)$ by $\psi_{\tau(x)}(x) = \eta_1(x)$. We show that $\tau(x)$ is an admissible time. Write $\eta_s(x) = \psi_{t(s)}(x)$, then

$$Y(\eta_s(x)) = -h(\eta_s(x))\ \nabla f(\eta_s(x)) = -\frac{dt}{ds}\ \nabla f(\eta_s(x)),$$

so that $\frac{dt}{ds} = h(\eta_s(x))$ and

$$\tau(x) = \int_0^1 h(\eta_s(x))\, ds \le 1.$$

Therefore $\tau(x)$ is finite and differentiable.

Let $x \in F$. If $x \notin B$ then $f(x) \le c - \lambda$ and hence $f(\eta_1(x)) \le f(x) \le c - \lambda$.

Let $x \in F \cap B$. We can assume $|f(\eta_s(x)) - c| < \frac{\delta}{2}$ for all $s \in [0, 1]$. For, if not, since $s \mapsto f(\eta_s(x))$ is non-increasing and at $s = 0$, $f(x) \le c + \varepsilon < c + \frac{\delta}{2}$, then we already have that

$$f(\psi_{\tau(x)}(x)) = f(\eta_1(x)) = \inf_{s\in[0,1]} f(\eta_s(x)) \le c - \frac{\delta}{2} \le c - \varepsilon.$$

We can also assume that

$$\eta_s(x) \in A \quad \text{for all } s \in [0,1]. \tag{68}$$

For, if not, since $x = \eta_0(x) \in F \cap B$ and A contains the ε_1-neighbourhood of B,

$$\begin{aligned}\varepsilon_1 \le \text{length}\left[\eta_{[0,1]}(x)\right] &= \int_0^1 \|h(\eta_s(x))\ \nabla f(\eta_s(x))\|\ ds \\ &\le \left[\int_0^1 |h|^2\ \|\nabla f\|^2\ ds\right]^{\frac{1}{2}} \le \left[\int_0^1 h\ \|\nabla f\|^2\ ds\right]^{\frac{1}{2}},\end{aligned}$$

and so

$$\begin{aligned}f(\eta_1(x)) &= f(x) + \int_0^1 \langle \nabla f,\ Y\rangle\ ds \le c + \varepsilon - \int_0^1 h\ \|\nabla f\|^2\ ds \\ &\le c + \varepsilon - \varepsilon_1^2 < c - \varepsilon.\end{aligned}$$

Suppose that $x \in F \cap B$ and $\eta_s(x) \notin V_{c,\delta,A}$ for all $s \in [0,1]$. Then $h(\eta_s(x)) = 1$ for all $s \in [0,1]$. By (68), $\eta_s(x) \in A \cap (V_{c,\delta,A})^c$ for all $s \in [0,1]$, and so $\|\nabla f(\eta_s(x))\| \ge \delta$ for all $s \in [0,1]$. Then,

$$\begin{aligned}f(\eta_1(x)) &\le c + \varepsilon - \int_0^1 h(\eta_s(x))\ \|\nabla f(\eta_s(x))\|^2\ ds \\ &\le c + \varepsilon - \delta^2 \le c - \varepsilon.\end{aligned}$$

Now suppose that $x \in F \cap B$, $\eta_{s_0}(x) \in V_{c,\delta,A}$ for some $s_0 \in [0,1]$ and $\eta_1(x) \notin N$. Let

$$\begin{aligned}s_1 &:= \inf\{\, s > s_0 \mid \eta_s(x) \notin V_{c,\delta,A}\},\\ s_2 &:= \inf\{\, s > s_1 \mid \eta_s(x) \notin V_{c,\eta,A}\} \le 1.\end{aligned}$$

By (68), the image of $[0,1] \ni s \mapsto \eta_s(x)$ is in A. Since the segment $[s_1, s_2] \mapsto \eta_s(x)$ crosses the annulus of width ρ: $W_{c,2\rho,A} \setminus W_{c,\rho,A}$, inside of A, we have that

$$\begin{aligned}\rho \le \text{length}[\eta_{[s_1,s_2]}] &\le \int_{s_1}^{s_2} h(\eta_s(x))\ \|\nabla f(\eta_s(x))\|\ ds \\ &\le \frac{1}{\delta}\int_{s_1}^{s_2} h\ \|\nabla f\|^2\ ds, \quad \text{because } \eta_{]s_1,s_2[}(x) \subset A \cap (V_{c,\delta,A})^c.\end{aligned}$$

Since $s_2 \le 1$,

$$\begin{aligned}f(\eta_1(x)) &\le c + \varepsilon - \int_0^1 h\ \|\nabla f\|^2\ ds \\ &\le c + \varepsilon - \rho\,\delta \le c - \varepsilon.\end{aligned}$$

Therefore

$$F_\tau = \eta_1(F) \subseteq N \cup [f \le c - \varepsilon]. \qquad \square$$

Given a function $f : X \to \mathbb{R}$ on a topological space X, we say that $x \in X$ is a *strict local minimizer* of f if there is a neighbourhood V of x in X such that $f(y) > f(x)$ for all $y \in V \setminus \{x\}$.

Let $\mathcal{F}$ be a family of subsets $F \subset X$. We say that $\mathcal{F}$ is *forward invariant* if $F_\tau \in \mathcal{F}$ for all $F \in \mathcal{F}$ and any admissible time τ. Define

$$c(f, \mathcal{F}) = \inf_{F \in \mathcal{F}} \sup_{x \in F} f(x).$$

Proposition 6.3 *Let X be a Riemannian manifold and $f : X \to \mathbb{R}$ a C^2 function. Let $\mathcal{F}$ be a family of subsets of X. Suppose that*

i. The subsets $F \in \mathcal{F}$ are connected.
ii. $\mathcal{F}$ is a forward invariant family.
iii. $c := c(f, \mathcal{F}) \in \mathbb{R}$.
iv. The flow of $-\nabla f$ is relatively complete on $[c - \varepsilon_2 \le f \le c + \varepsilon_2]$ for some $\varepsilon_2 > 0$.
v. There is a closed subset $B \subset X$ such that:
$\forall \varepsilon > 0$, $\exists \lambda \in]0, \varepsilon[$, $\exists F \in \mathcal{F}$ such that $F \subset B \cup [f \le c - \lambda]$ and $F \subset [f \le c + \varepsilon]$.
vi. There is $\varepsilon_1 > 0$ and a closed subset $A \subset X$ which contains the ε_1-neighbourhood of B such that f satisfies the Palais-Smale condition $(PS)_{c,A}$, restricted to A at level c.

Then $K_{c,A} \neq \emptyset$, i.e. f has a critical point $\overline{x}$ in A with $f(\overline{x}) = c$.

Moreover, if

$$\sup_{F \in \mathcal{F}} \inf_{x \in F} f(x) < c, \tag{69}$$

then there is a point in $K_{c,A}$ which is not a strict local minimizer.

Remark 6.4 It is enough to consider admissible times $\tau(x)$ such that $\tau(x) = 0$ if $f(x) \le c - \delta$ for some $\delta > 0$. For, the value of $c(f, \mathcal{F})$ does not change and the proof of Proposition 6.3 only uses this kind of admissible times.

Proof Suppose that $K_{c,A}$ is empty. Let ε, $\delta > 0$ be given by Lemma 6.2 for $N = \emptyset$ and ε_1, ε_2, and $\varepsilon_3 = 1$. By hypothesis (v), there is $F \in \mathcal{F}$ such that $F \subset B \cup [f \le c - \lambda]$ and $F \subset [f \le c + \varepsilon]$. By Lemma 6.2 there is an admissible time τ, with $\tau = 0$ on $[f \le c - \delta]$, such that $F_\tau \subset [f \le c - \lambda]$. This contradicts the definition of $c(f, \mathcal{F})$.

Now suppose that $K_{c,A}$ consists entirely of strict local minimizers of f and that inequality (69) holds. Let $\varepsilon_0 > 0$ be such that

$$\sup_{F \in \mathcal{F}} \inf_{x \in F} f(x) < c - 2\,\varepsilon_0. \tag{70}$$

For each $x \in K_{c,A}$ let $N(x)$ be a neighbourhood of x such that $f(y) > f(x)$ for all $y \in N(x) \setminus \{x\}$. Let

$$N_0 := \bigcup_{x \in K_{c,A}} N(x).$$

Let $N := A \cap N_0$.

Let $0 < \varepsilon < \delta < \varepsilon_0$ be given by Lemma 6.2 for c, B, A and N_0. By hypothesis (v) there are $\lambda \in]0, \varepsilon[$ and $F \in \mathcal{F}$ such that $F \subset B \cup [f \le c - \lambda]$ and $F \subset [f \le c + \varepsilon]$. By Lemma 6.2, there is an admissible time τ such that $\tau|_{[f \le c-\delta]} \equiv 0$ and

$$F_\tau \subseteq N \cup [f \le c - \lambda] \subseteq N_0 \cup [f \le c - \lambda].$$

By definition of N_0, the sets N_0 and $[f \le c - \lambda]$ are disjoint, and so $N \cup [f \le c - \lambda]$ is disconnected. By hypothesis (i) and (ii), $F_\tau \in \mathcal{F}$ is connected, so that either $F_\tau \subset N$ or $F_\tau \subset [f \le c - \lambda]$. Since $\lambda \le \varepsilon < \varepsilon_0$ and the value of f decreases under the flow of $-\nabla f$, by (70), $F_\tau \cap [f \le c - \lambda] \ne \emptyset$. Hence $F_\tau \subset [f \le c - \lambda]$. This contradicts the definition of $c(f, \mathcal{F})$. □

We shall use the following "mountain pass" theorem.

Corollary 6.5 *Let X be a C^2 Riemannian manifold and $f : X \to \mathbb{R}$ a C^2 function. Let $p, q \in X$ and*

$$c := \inf_{\gamma \in \Gamma} \sup_{s \in [0,1]} f(\gamma(s)),$$

where $\Gamma := \{\gamma : [0, 1] \to X \mid \gamma \in C^0,\ \gamma(0) = p,\ \gamma(1) = q\}$.

Suppose that

i. $c \in \mathbb{R}$.

ii. The flow of $-\nabla f$ is relatively complete on $[c - \varepsilon_2 \le f \le c + \varepsilon_2]$ for some $\varepsilon_2 > 0$.

iii. $\max\{f(p),\ f(q)\} < c$.

iv. There are closed subsets $B \subset A \subset X$ such that

(a) f satisfies the Palais-Smale condition $(PS)_{c,A}$, restricted to A, at level c.

(b) For some $\varepsilon_1 > 0$, A contains the ε_1-neighbourhood of B.

(c) For all $\varepsilon > 0$, there exist $\lambda_\varepsilon \in]0, \varepsilon[$ and $\gamma_\varepsilon \in \Gamma$ such that

$$\gamma_\varepsilon([0, 1]) \subset (B \cup [f \le c - \lambda_\varepsilon]) \cap [f \le c + \varepsilon].$$

Then c is a critical value of f. Moreover the set

$$K_{c,A} := \{x \in X \mid x \in A,\ df(x) = 0,\ f(x) = c\}$$

contains a point which is not a strict local minimizer.

Proof Let $\mathcal{F} := \{\gamma([0, 1]) \mid \gamma \in C^0([0, 1], X),\ \gamma(0) = p,\ \gamma(1) = q\}$. Let $\delta > 0$ be such that $\max\{f(p), f(q)\} < c - 2\delta$. Then $c = c(f, \mathcal{F})$ and the family $\mathcal{F}$ is forward invariant when we consider only admissible times $\tau(x)$ such that $\tau(x) = 0$ when $f(x) \le c - \delta$. Then Proposition 6.3 and Remark 6.4 prove that $K_{c,A} \ne \emptyset$ and that $K_{c,A}$ contains a point which is not a strict local minimizer. □

Applying Proposition 6.3 to the family of subsets $F = \{p\}$, $p \in X$, and $B = A = X$, we obtain

Corollary 6.6 *Let X be a C^2 Riemannian manifold and $f : X \to \mathbb{R}$ a C^2 function. Suppose that*

i. $c := \inf_{x \in X} f(x) > -\infty$.

ii. The flow of $-\nabla f$ *is relatively complete on* $[c-\varepsilon \le f \le c+\varepsilon]$ *for some* $\varepsilon > 0$.

iii. f *satisfies the Palais-Smale condition* $(PS)_c$ *at level* c.

Then c *is a critical value of* f.

Now we concentrate on the relative completeness condition for the action functional $\mathcal{A}_k$. For completeness we present the following statement:

Lemma 6.7 *Suppose that* $f : X \to \mathbb{R}$ *is* C^2, ψ_t *is the gradient flow of* $-f$ *and the subset* $[a \le f \le b] \subset X$ *is complete. Then the flow* ψ_t *is relatively complete on* $[a \le f \le b]$.

This lemma can be found in [7, Lemma 22], but its proof is similar to the first part of the proof of Lemma 6.9 below. Recall from Remark 3.11 that the spaces Λ_M and $\Omega_M(q_0, q_1)$ are not complete with our riemannian metric. Also the gradient flow of $-\mathcal{A}_k$ is not complete as the following example shows.

Example 6.8 The gradient flow of $-\mathcal{A}_k$ is not complete.

Let $(q_0, 0)$ be a fixed point of the Euler-Lagrange flow, let $x : [0,1] \to \{q_0\}$ be the constant curve, let $T > 0$ and let $y(t) = x(t/T)$. Since x is a solution of the Euler-Lagrange equation, from (7) or (16), we have that the partial derivative $(\partial_x \mathcal{A}_k)(x, T) = 0$. But

$$\left.\frac{\partial \mathcal{A}_k}{\partial T}\right|_{(x,T)} = \int_0^1 [k - E(q_0, 0)]\; dt = k - E(q_0, 0).$$

Suppose that $k > E(q_0, 0)$. Let $a := k - E(q_0, 0) > 0$. Since the metric on the $\mathbb{R}^+$ factor of $\mathcal{H}^1(M) \times \mathbb{R}^+$ is the euclidean metric, the gradient flow Ψ_s of $-\mathcal{A}_k$ at (x, T) has differential equations $\frac{dx}{ds} = 0$ and $\frac{dT}{ds} = -a$. Then

$$\Psi_s(x, T) = (x, T - as),$$

which leaves the space $\mathcal{H}^1(M) \times \mathbb{R}^+$ through $(x, 0)$ in finite time.

One could change the riemannian metric (2) in such a way that $\mathcal{H}^1(M) \times \{0\}$ lies “at infinity,” for example replacing the first term in (2), (4) by $\alpha\beta/T^2$. In that case the Hilbert manifolds Λ_M and $\Omega_M(q_0, q_1)$ become complete and the gradient flow of $-\mathcal{A}_k$ becomes relatively complete by Lemma 6.7. Indeed, in the example above we would have

$$\left.\frac{\partial \mathcal{A}_k}{\partial T}\right|_{(x,T)} \cdot \alpha = a\,\alpha = \frac{1}{T^2}\,(a\,\alpha\,T^2) = \langle a\,T^2, \alpha\rangle_T.$$

Thus the projection $\nabla_T \mathcal{A}_k$ to the $\mathbb{R}^+$ component of $\mathcal{H}^1(M) \times \mathbb{R}^+$ of the gradient of $\mathcal{A}_k$ is $a\,T^2$. The differential equations for Ψ_s become $\frac{dx}{ds} = 0$, $\frac{dT}{ds} = -a\,T^2$ and $\Psi_s(x, T) = (x, (T^{-1} + as)^{-1})$.

Nevertheless, such a change in the riemannian metric would make Proposition 3.8. and Theorem A false. Indeed, in Proposition 3.8 items (i), (ii) and (iv) would still hold but not item (iii) because inequality (31) would have a factor T_n on the right hand side. In the example above, the sequence $(x_n, T_n) = \Psi_n(x, a^{-1}) = (x, \frac{1}{a(n+1)})$, with $T_0 = a^{-1}$, would satisfy

$$\mathcal{A}_k(x_n, T_n) = T_n[L(q_0, 0) + k] \xrightarrow{n} 0,$$

$$\left\| d_{(x_n,T_n)}\mathcal{A}_k \right\| = \frac{|\,\partial_T \mathcal{A}_k(x_n, T_n)\cdot\alpha\,|}{\|(0,\alpha)\|_{(x_n,T_n)}} = \frac{\left|\alpha\cdot\int_0^1 [k - E(q_0,0)]\,ds\right|}{|\alpha|\,/\,T_n} = a\cdot T_n \xrightarrow{n} 0,$$

because $y_n(t) := x_n(t/T_n) \equiv q_0$ is a solution of the Euler-Lagrange equation for all n and hence, by equation (16), $\partial_x \mathcal{A}_k(x_n, T_n) = 0$. Therefore (x_n, T_n) is a Palais-Smale sequence, but it is non-convergent because

$$\begin{aligned}
&\text{distance}[(x_n, T_n), (x_0, T_0)] = \text{dist}[(x, T_n), (x, T_0)] \\
&\quad\geq \int_0^1 \|(0, f'(s))\|_{(x, f(s))}\, ds = \int_0^1 \frac{|f'(s)|}{|f(s)|}\, ds = \int_{T_n}^{T_0} \frac{1}{u}\, du \xrightarrow{n} +\infty,
\end{aligned}$$

where $f : [0, 1] \to [T_n, T_0]$ is a diffeomorphism. But the energy $E(x_n, \frac{\dot{x}_n}{T_n}) = E(q_0, 0) \equiv k - a$ is not converging to k.

Also Theorem A becomes false. For, in the example above we can take $M \neq \mathbb{T}^2$ and a lagrangian of the form $L(x, v) = \frac{1}{2}\,|v|_x^2 + \phi(x)$, with $\phi(x)$ non-constant, a regular energy level $E^{-1}\{k\}$ with

$$-\max\phi(M) < k > e_0(L) = -\min\phi(M)$$

and $q_0 \in M$ a minimum of ϕ. Then $(q_0, 0)$ is a singularity of the Euler-Lagrange flow with energy $e_0(L)$, so that $a = k - E(q_0, 0) > 0$. The example above gives a non-convergent Palais-Smale sequence. Thus $E^{-1}\{k\}$ does not satisfy the Palais-Smale condition for the metric defined above.

But the energy level $E^{-1}\{k\}$ contains no invariant probability as those in Theorem A. Indeed, in Proposition C.2 below we prove that such regular energy levels for L are of contact type. Suppose that μ is such an invariant probability in $E^{-1}\{k\}$, with $A_{L+k}(\mu) = 0$ and $\rho(\mu) = 0$. Since $M \neq \mathbb{T}^2$, by Lemma 10.1, $\pi_* : H_1(E^{-1}\{k\}, \mathbb{R}) \to H_1(M, \mathbb{R})$ is injective. If $\mathcal{S}(\mu)$ is the asymptotic cycle of μ (see Sect. 10), then $\pi_*(\mathcal{S}(\mu)) = \rho(\mu) = 0$. Therefore $\mathcal{S}(\mu) = 0$. Since $A_{L+k}(\mu) = 0$ this contradicts Corollary C.4. $\square$

Lemma 6.9 *For all $k \in \mathbb{R}$, if $0 \notin [a, b] \subset \mathbb{R}$, then the gradient flow of $-\mathcal{A}_k$ on Λ_M or $\Omega_M(q_0, q_1)$ is relatively complete on $[a \leq \mathcal{A}_k \leq b]$.*

Proof Write $f = \mathcal{A}_k : \Lambda_M \to \mathbb{R}$ [resp. $f = \mathcal{A}_k : \Omega_M(q_0, q_1) \to \mathbb{R}$] and let ψ_t be the flow of $Y = -\nabla f$. Then

$$f(\psi_{t_1}(p)) - f(\psi_{t_2}(p)) = -\int_{t_1}^{t_2} \nabla f(\psi_s(p))\cdot Y(\psi_s(p))\, ds = \int_{t_1}^{t_2} \|Y(\psi_s(p))\|^2\, ds. \tag{71}$$

Moreover, using the Cauchy-Schwartz inequality, we have that

$$d(\psi_{t_1}(p), \psi_{t_2}(p))^2 \leq \left[\int_{t_1}^{t_2} \|Y(\psi_s(p))\|\, ds\right]^2 \leq |t_2 - t_1| \int_{t_1}^{t_2} \|Y(\psi_s(p))\|^2\, ds.$$

Thus

$$d(\psi_{t_1}(p), \psi_{t_2}(p))^2 \leq |t_2 - t_1| \; |f(\psi_{t_1}(p)) - f(\psi_{t_2}(p))|. \tag{72}$$

Let $I = [0, \alpha[$ a maximal interval of definition of $t \mapsto \psi_t(p)$. Suppose that $a \leq f(\psi_t(p)) \leq b$ for $0 \leq t < \alpha < \infty$. By inequality (72), for any sequence $s_n \uparrow \alpha$ we have that $n \mapsto \psi_{s_n}(p) = (x_{s_n}, T(s_n))$ is a Cauchy sequence in $\Lambda_M \cap [a \leq f \leq b]$ (resp. in $\Omega_M(q_0, q_1) \cap [a \leq f \leq b]$). Then

$$T_0 = \lim_{s \uparrow \alpha} T(s) \in [0 + \infty[\quad \text{exists.}$$

If $0 < T_0 < +\infty$, since all such $\{\psi_{s_n}(p)\}_n$ are Cauchy sequences, $q = \lim_{s \uparrow \alpha} \psi_s(p) = \psi_\alpha(p)$ exists. Since f is C^2, we can extend the solution $t \mapsto \psi_t(p)$ at $t = \alpha$. This contradicts the definition of α.

If $T_0 = 0$, then there is a sequence $s_n \uparrow \alpha$ such that

$$\frac{d}{ds} T(s_n) \leq 0.$$

Since L is quadratic at infinity, there exist constants $a_0, a_1, b_0, b_1, c_0, c_1 > 0$ such that

$$\begin{aligned} c_0 \, |v|_x^2 - c_1 \leq L(x, v) + k &\leq b_0 \, |v|_x^2 + b_1, \\ E(x, v) &\geq a_0 \, |v|_x^2 - a_1. \end{aligned} \tag{73}$$

Write $T_n := T(s_n)$, $y_n(t) := x_{s_n}(t/T_n)$. Then

$$\begin{aligned} 0 \geq \frac{d}{ds} T(s_n) = -\frac{\partial \mathcal{A}_k}{\partial T} &= -k + \frac{1}{T_n} \int_0^{T_n} E(y_n, \dot{y}_n) \, dt \\ &\geq -k - a_1 + \frac{a_0}{T_n} \int_0^{T_n} |\dot{y}_n|^2 \; dt. \end{aligned}$$

Since $\lim_n T_n = 0$, this implies that

$$\lim_n \int_0^{T_n} |\dot{y}_n|^2 \; dt = 0.$$

Also, from (73)

$$c_0 \int_0^{T_n} |\dot{y}_n|^2 \; dt - c_1 \, T_n \leq \mathcal{A}_k(x_{s_n}, T_n) \leq b_0 \int_0^{T_n} |\dot{y}_n|^2 \, dt + b_1 \, T_n.$$

Hence $\lim_n \mathcal{A}_k(x_{s_n}, T_n) = 0$. This contradicts the hypothesis $\mathcal{A}_k(x_{s_n}, T_n) \in [a, b] \not\ni 0$. □

Corollary 6.10 *For all $k \in \mathbb{R}$, if $q_0, q_1 \in M$, $q_0 \neq q_1$ and $b \in \mathbb{R}$, then the gradient flow of $-\mathcal{A}_k$ on $\Omega_M(q_0, q_1)$ is relatively complete on $[\mathcal{A}_k \leq b]$.*

Proof By Lemma 3.6, if $q_0 \neq q_1$ then

$$\inf\{T > 0 \,|\, (x, T) \in \Omega_M(q_0, q_1) \cap [\mathcal{A}_k \leq b]\} > 0.$$

Therefore the case $T_0 = 0$ in the proof of Lemma 6.9 does not happen. □

Corollary 6.11 *Let Λ_1 be a connected component of Λ_M or $\Omega_M(q_0, q_0)$ with non-trivial homotopy class. For all $k, b \in \mathbb{R}$, the gradient flow of $-\mathcal{A}_k$ on Λ_1 is relatively complete on $[\mathcal{A}_k \le b\,]$.*

Proof By Lemma 3.7, the case $T_0 = 0$ in the proof of Lemma 6.9 does not happen. □

7 Generic Palais-Smale condition for the Mountain-Pass geometry

Proposition 7.1 *Given $\gamma_0,\ \gamma_1 \in \Lambda_M$ [resp. $\Omega_M(q_0, q_1)$] let*

$$\begin{aligned}\mathcal{C}(\gamma_0, \gamma_1) := \{\Gamma : [0, 1] \to \Lambda_M\ [\textit{resp.}\ \Omega_M(q_0, q_1)]\ \mid \\ \Gamma\ \textit{is continuous}, \Gamma(0) = \gamma_0,\ \Gamma(1) = \gamma_1\}.\end{aligned}$$

For $k \in \mathbb{R}$, let

$$c(k) := \inf_{\Gamma\in\mathcal{C}(\gamma_0,\gamma_1)}\ \max_{s\in[0,1]}\ \mathcal{A}_k(\Gamma(s)).$$

Suppose that for some $k_0 \in \mathbb{R}$ we have that $\lim_{k\to k_0^+} c(k) \neq 0$ and

$$c(k_0) > \max\{\mathcal{A}_{k_0}(\gamma_0),\ \mathcal{A}_{k_0}(\gamma_1)\}.$$

Then there exists $\varepsilon > 0$ such that for Lebesgue almost every $k \in]k_0, k_0 + \varepsilon[$, $c(k)$ is a critical value for $\mathcal{A}_k$ on Λ_M [resp. $\Omega_M(q_0, q_1)$], with a critical point which is not a strict local minimizer.

Proof Observe that for all $k \in \mathbb{R}$, the number $c(k)$ is finite and $c(k) \ge \max\{\mathcal{A}_k(\gamma_0),\ \mathcal{A}_k(\gamma_1)\}$. Since for all $\gamma \in \Lambda_M$ [resp. $\Omega_M(q_0, q_1)$] the function $k \mapsto \mathcal{A}_k(\gamma)$ is non-decreasing, $k \mapsto c(k)$ is non-decreasing. By the continuity of $\mathcal{A}_k$ in k, the functions $k \mapsto \mathcal{A}_k(\gamma_0)$ and $k \mapsto \mathcal{A}_k(\gamma_1)$ are continuous. Let $\varepsilon > 0$ be such that[7]

$$\max\{\mathcal{A}_k(\gamma_0),\ \mathcal{A}_k(\gamma_1)\} < c(k_0) \le c(k) \neq 0 \qquad \text{for all } k_0 < k < k_0 + \varepsilon. \tag{74}$$

Write $I_\varepsilon :=]k_0, k_0 + \varepsilon[$. Since the function $c : I_\varepsilon \to \mathbb{R}$ is non-decreasing, by Lebesgue's Theorem there is a total measure subset of I_ε where $c(\cdot)$ is locally Lipschitz, i.e. the subset

$$\mathcal{K} := \{k \in I_\varepsilon | \exists M > 0, \exists \delta_0 > 0, \forall |\delta| < \delta_0 : |c(k + \delta) - c(k)| < M|\delta|\}$$

has total Lebesgue measure in I_ε.

Now fix $\underline{k} \in \mathcal{K}$ and a sequence $k_n \ge \underline{k}$ with $\lim_n k_n = \underline{k}$. By the definition of ε in (74) the functionals $\mathcal{A}_{k_n}$, $\mathcal{A}_{\underline{k}}$ both show a mountain pass geometry with the same set of paths $\mathcal{C}(\gamma_0, \gamma_1)$.

Let $\Gamma_n \in \mathcal{C}(\gamma_0, \gamma_1)$ be a path such that

$$\max_{s\in[0,1]} \mathcal{A}_{k_n}(\Gamma_n(s)) \le c(k_n) + (k_n - \underline{k}). \tag{75}$$

[7] Observe that the function $k \mapsto c(k)$ may be discontinuous, in particular, we allow k_0 to be a discontinuity point for $k \mapsto c(k)$.

Let $M = M(\underline{k}) > 0$ be given by the property $\underline{k} \in \mathcal{K}$. Let $B \subset A \subset \Lambda_M$ [resp. $\Omega_M(q_0, q_1)$] be the closed subsets defined by

$$B := \{(x, T) \in \Lambda_M \text{ [resp. } \Omega_M(q_0, q_1)] \mid T \leq M + 2\},$$
$$A := \{(x, T) \in \Lambda_M \text{ [resp. } \Omega_M(q_0, q_1)] \mid T \leq M + 3\}.$$

Then A contains the $\frac{1}{2}$-neighbourhood of B in Λ_M [resp. $\Omega_M(q_0, q_1)$]. Since from (74), $c(\underline{k}) \neq 0$, by Propositions 3.12 and 3.8.(i), the functional $\mathcal{A}_{\underline{k}}$ satisfies the Palais-Smale condition restricted to A, at level $c(\underline{k})$.

By the choice $\underline{k} \in \mathcal{K}$, the function $k \mapsto c(k)$ is continuous at $\underline{k}$. Since $k \mapsto \mathcal{A}_k(\gamma)$ is increasing,

$$\max_{s\in[0,1]} \mathcal{A}_{\underline{k}}(\Gamma_n(s)) \leq \max_{s\in[0,1]} \mathcal{A}_{k_n}(\Gamma_n(s)) \leq c(k_n) + (k_n - \underline{k}) \xrightarrow{n} c(\underline{k}).$$

If $s \in [0, 1]$ is such that

$$\mathcal{A}_{\underline{k}}(\Gamma_n(s)) > c(\underline{k}) - (k_n - \underline{k}),$$

then $\Gamma_n(s) = (x, T)$ with

$$T = \frac{\mathcal{A}_{k_n}(\Gamma_n(s)) - \mathcal{A}_{\underline{k}}(\Gamma_n(s))}{k_n - \underline{k}} \leq \frac{c(k_n) - c(\underline{k})}{k_n - \underline{k}} + 2 \leq M(\underline{k}) + 2,$$

if n is large enough.

Given $\delta > 0$, let n be so large that

$$c(k_n) - c(\underline{k}) + (k_n - \underline{k}) < \delta,$$
$$0 < \lambda_n := (k_n - \underline{k}) < \delta.$$

Then

$$\Gamma_n([0, 1]) \subset \big(B \cup [\mathcal{A}_{\underline{k}} \leq c(\underline{k}) - \lambda_n]\big) \cap [\mathcal{A}_{\underline{k}} \leq c(\underline{k}) + \delta].$$

Since $c(\underline{k}) \neq 0$, by Lemma 6.9 the gradient flow of $-\mathcal{A}_k$ is relatively complete on $[c(\underline{k}) - \varepsilon, c(\underline{k}) + \varepsilon]$ for some $\varepsilon > 0$. Now Corollary 6.5 implies that $\mathcal{A}_{\underline{k}}$ has a critical point in A which is not a strict local minimizer. □

8 The displacement energy

Write $I := [0, 1]$. Given a subset $A \subseteq T^*M$ let $\mathcal{H}_c(I \times A)$ be the set of smooth functions $H : I \times T^*M \to \mathbb{R}$ whose support is compact and contained in $I \times A$. To such $H \in \mathcal{H}_c(I \times A)$ we associate its Hamiltonian vector field X_{H_t}, defined by $\omega(X_{H_t}, \cdot) = -dH_t(\cdot)$, where $\omega = dp \wedge dx$, and its corresponding Hamiltonian flow h_t. The set of functions in $\mathcal{H}_c(I \times A)$ which do not depend on $t \in I$ is denoted by $\mathcal{H}_c(A)$.

We say that $F \in \mathcal{H}_c(A)$ is *slow* if all non-constant contractible (in T^*M) periodic orbits of its hamiltonian flow f_t have period > 1. Define the π_1-sensitive Hofer-Zehnder capacity of A by

$$c^\circ_{HZ}(A, T^*M, \omega) := \sup \{\max F \mid F \in \mathcal{H}_c(\text{int}\, A) \ \text{ is slow}\}.$$

The equivalence of this definition of the Hofer-Zehnder capacity with the original definition in [13] is proven in Theorem 2.9 of [11].

Given $H \in \mathcal{H}_c(I \times A)$ define its norm $\|H\|$ as

$$\|H\| := \int_0^1 \left(\sup_{z\in A} H(t,z) - \inf_{z\in A} H(t,z) \right) \, dt.$$

The displacement energy $e(A, T^*M, \omega)$ of a *compact* subset $A \subseteq T^*M$ is defined as

$$e(A, T^*M, \omega) := \inf\{\|H\| \mid H \in \mathcal{H}_c(I \times T^*M),\ h_1(A) \cap A = \emptyset\},$$

where h_1 is the time 1 map of the hamiltonian flow of H.

Lemma 8.1 *Given an open subset $U \subset M$ there is a smooth function $\phi : M \to \mathbb{R}$ whose critical points are all in U.*

Proof Let $f : M \to \mathbb{R}$ be a Morse function. Its set of critical points $C(f) = \{x_1, \dots, x_N\}$ is finite. Let $\{\gamma_i\}_{i=1}^N$ be a collection of disjoint smooth curves $\gamma_i : [0,1] \to M$ such that $\gamma_i(0) = x_i$ and $\gamma_i(1) \in U$. Let $\{B_i\}_{i=1}^N$ be a collection of disjoint tubular neighbourhoods of the curves γ_i. For each i, let h_i be a smooth diffeomorphism of M with support in B_i such that $h_i(\gamma_i(1)) = x_i$. Now let $\phi = f \circ h_1 \circ \cdots \circ h_N$. □

Proposition 8.2 *If $k < e_0(L)$ then $e([H \le k], T^*M, \omega) < +\infty$.*

Proof Write $N := \pi([H \le k])$. Since $k < e_0(L)$ we have that $U := M \setminus N$ is a non-empty open subset of M. Let $\phi : M \to \mathbb{R}$ be a smooth function such that all its critical points are in U. Consider the Hamiltonian $G(x,p) := -\phi(x)$. Its Hamiltonian equations are

$$\begin{aligned} \dot{x} &= \ \ \nabla_p G = \ 0, \\ \dot{p} &= -\nabla_x G = d_x\phi. \end{aligned}$$

Its Hamiltonian flow is $g_t(x,p) = (x,\ p + t\, d_x\phi)$.

Since N is compact, $\min_{x\in N} |d_x\phi| > 0$. Since $[H \le k]$ is compact, there is $T > 0$ such that

$$g_T([H \le k]) \cap [H \le k] = \emptyset.$$

The set $W := \bigcup_{t\in[0,T]} g_t([H \le k])$ is compact. Let $\lambda : T^*M \to [0, +\infty[$ be a smooth function with compact support such that $\lambda|_W \equiv T$. Let $F := \lambda\, G : T^*M \to \mathbb{R}$. Then F has compact support and its Hamiltonian flow satisfies

$$f_s(x,p) = g_{sT}(x,p) \quad \text{when } H(x,p) \le k \quad \text{and} \quad 0 \le s \le 1.$$

Therefore $f_1([H \le k]) \cap [H \le k] = \emptyset$, and hence $e([H \le k], T^*M, \omega) \le \|F\| < +\infty$. □

Corollary 8.3 *For Lebesgue almost every $k < e_0(L)$ the energy level $[E = k]$ has a periodic orbit which is contractible in M.*

Proof By Theorem 1.3 in [28] (also [10]), $c^\circ_{HZ}(A, T^*M, \omega) \le 4\, e(A, T^*M, \omega)$. From Proposition 8.2, we get that $c^\circ_{HZ}([E \le k], T^*M, \omega) < +\infty$ for all $k < e_0(L)$. A standard argument using the Hofer-Zehnder capacity [13, p. 118–119] shows that almost all energy levels $[E = k]$, $k < e_0(L)$ have a periodic orbit which is contractible in T^*M but possibly non-contractible in $[E \le k]$. Since T^*M retracts to the zero section $M \times \{0\}$, the projection of the closed orbit to M is contractible in M. □

9 Loops, closed orbits and conjugate points

Proof of Theorem D (a) We first prove that for all $k > c_u(L)$, $E^{-1}\{k\}$ contains a periodic orbit. By Corollary B, $\mathcal{A}_k$ satisfies the Palais-Smale condition.

Suppose first that $\pi_1(M) \neq 0$. Let $\sigma \in [S^1, M]$ be a non-trivial free homotopy class. Let Λ_σ be the connected component of Λ_M corresponding to σ. By Lemma 4.1, $\mathcal{A}_k$ is bounded below on Λ_σ. Since σ is non-trivial, by Corollary 6.11 the gradient flow of $-\mathcal{A}_k$ is relatively complete on $[\mathcal{A}_k \le b] \cap \Lambda_\sigma$ for any $b \in \mathbb{R}$. By Corollary 6.6 there is a minimizer of $\mathcal{A}_k$ on Λ_σ.

If $\pi_1(M) = 0$, then $c_u(L) = c_0(L) = c(L)$ and $k > c(L)$. Since M is closed, there is some non-trivial homotopy group $\pi_\ell(M) \neq 0$. Choose a non-trivial free homotopy class $0 \neq \sigma \in [S^\ell, M]$. A map $f : S^\ell \to M$ with homotopy class σ can be seen as a family F of closed curves in M (see e.g. [16, page 37]). Let $\mathcal{F}$ the set of all such families corresponding to the homotopy class σ. Clearly $\mathcal{F}$ is a forward invariant family. Since the homotopy class σ is non-trivial, (cf. [16, Th. 2.1.8, page 37]):

$$\inf_{F \in \mathcal{F}} \max_{(x,T) \in F} \text{length}(x) =: a > 0.$$

By inequality (9) there are $a_1,\ b > 0$ such that $L(x, v) > a_1 |v|_x - 2b$ for all $(x, v) \in TM$. We can assume that $b \gg k$. If $(x, T) \in \Lambda_M$ is a closed curve with length $\ell \ge a$, bounded action $\mathcal{A}_k(x, T) \le \alpha$ and speed $|v|$, then

$$\ell^2 = \left[\int_0^T |v| \right]^2 \le T \int_0^T |v|^2,$$

$$\alpha \ge \mathcal{A}_k(x, T) \ge a_1 \int_0^T |v|^2 - 2b\, T + k\, T \ge a_1 \frac{\ell^2}{T} - (2b - k)\, T.$$

Hence $(2b - k)\, T^2 + \alpha\, T - a_1 \ell^2 \ge 0$. Since $T \ge 0$ and $\ell^2 \ge a^2$, we have that

$$T \ge \frac{-\alpha + \sqrt{\alpha^2 + 4\,(2b - k)\, a_1 a^2}}{2\,(2b - k)} =: d > 0.$$

And then

$$\begin{aligned} \mathcal{A}_k(x, T) &= \mathcal{A}_{c(L)}(x, T) + [k - c(L)]\, T \\ &\ge 0 + [k - c(L)]\, d > 0. \end{aligned}$$

Thus

$$c(\mathcal{F}) := \inf_{F \in \mathcal{F}} \sup_{(x,T) \in F} \mathcal{A}_k(x, T) \ge [k - c(L)]\, d > 0.$$

Since $c(\mathcal{F}) \neq 0$, Corollary B, Lemma 6.9 and Proposition 6.3 with $B = A = X = \Lambda_M$ imply that there is a critical point on Λ_M for $\mathcal{A}_k$.

By definition of $c_u(L)$, if $e_0(L) < k < c_u(L)$ then there is a closed curve $(x_1, T_1) \in \Lambda_M$ homotopic to a point, such that $\mathcal{A}_k(x_1, T_1) < 0$. Then Proposition C.(2) and Proposition 7.1 imply that for almost every $k \in]e_0, c_u[$ there is a critical point for $\mathcal{A}_k$ in Λ_M with $c(k) > 0$: i.e. a periodic orbit with trivial homotopy class and positive $(L + k)$-action which is not a strict local minimizer.

The case $k < e_0(L)$ is proven in Corollary 8.3. The closed orbit obtained in Corollary 8.3 could be a singularity of the Euler-Lagrange flow. But in that case k is a critical value of the energy function. By Sard's theorem that can only happen on a set of measure zero of values of k.

(b) For item (b) and $k > c_u(L)$ the proofs are similar to those of item (a) working on $\Omega_M(q_0, q_0)$. Namely, if $\pi_1(M, q_0) \neq 0$ one finds a minimizing loop in a non-trivial homotopy class. If $\pi_1(M, q_0) = 0$ we decompose a map $(S^\ell, \text{N.Pole}) \to (M, q_0)$ in a non-trivial homotopy class of $\pi_\ell(M, q_0)$ into a family of closed loops in $\Omega_M(q_0, q_0)$.

For $E(q_0, 0) < k < c_u(L)$, item (b) follows from Proposition C.(1) and Proposition 7.1 similarly to item (a).

(c) If $k < c_u$ and the Palais-Smale condition holds, the proof is similar to items (a) and (b), but now using Corollary 6.5, with $B = A = X$ instead of Proposition 7.1. □

Now we will prove Theorem E. Let $H : T^*M \to \mathbb{R}$ be the hamiltonian associated to L and ψ_t its hamiltonian flow. Recall that two points θ_1, θ_2 are said *conjugate* if there is $T \in \mathbb{R} \setminus \{0\}$ such that

$$\theta_2 = \psi_T(\theta_1) \quad \text{and} \quad d_{\theta_1}\psi_T(V(\theta_1)) \cap V(\theta_2) \neq \{0\},$$

where $V(\theta) := \ker d_\theta \pi \subset T_\theta(TM)$ is the *vertical subspace* and $\pi : TM \to M$ is the projection. This definition coincides with the one given in Sect. 1.3 because the Legendre transform $\mathcal{L}(x, v) = (x, L_v)$ maps the vertical subspace of $T_{v_x}TM$ to the vertical subspace of $T_{\mathcal{L}(v_x)}T^*M$.

Proposition 9.1 *Suppose that the forward orbit of* (x_0, v_0) *has no conjugate points. Let* $\gamma : [\varepsilon, T] \to M$ *be the solution* $\gamma(t) = \pi(\varphi_t(x_0, v_0))$. *Let* $x_{\varepsilon,T}(s) := \gamma(\varepsilon + s(T - \varepsilon))$, $s \in [0, 1]$ *and* $k := E(x_0, v_0)$. *Then for all* $T > \varepsilon > 0$ *the solution* $(x_{\varepsilon,T}, T - \varepsilon)$ *is a strict local minimizer of the free time* $(L + k)$*-action* $\mathcal{A}_k$ *on* $\Omega_M(\gamma(\varepsilon), \gamma(T))$.

In [7, p. 663] we gave an example of an orbit segment $(x, T = \pi)$ which has no conjugate points and which is not a local minimizer of $\mathcal{A}_k$ on $\Omega_M(x(0), x(1))$. But in that example the forward orbit of $(x(0), \frac{\dot{x}(0)}{T})$ has a conjugate point at time $t = 2\pi$ (it is the same lagrangian as in Example A.3, p. 949 in [5]).

Proof Let $H : T^*M \to \mathbb{R}$ be the hamiltonian associated to L, ψ_t its hamiltonian flow and X its hamiltonian vector field. Let $\omega = dp \wedge dx$ be the canonical symplectic form on T^*M. Given $\theta \in H^{-1}\{k\}$, let

$$\Sigma := H^{-1}\{k\}, \quad \Sigma_{\pi(\theta)} := T^*_{\pi(\theta)}M \cap \Sigma, \quad V(\theta) := \ker d_\theta\pi = T_\theta(T^*_{\pi(\theta)}M) \quad \text{and}$$
$$\Lambda_{\pi(\theta)} := \cup_{t>0}\, \psi_t(\Sigma_{\pi(\theta)}).$$

Then $T_\theta \Sigma_{\pi(\theta)} = V(\theta) \cap T_\theta \Sigma$ and

$$\begin{aligned} T_\theta \Lambda_{\pi(\theta)} &= (V(\theta) \cap T_\theta \Sigma) \oplus \langle X(\theta) \rangle =: W(\theta), \\ T_{\psi_t(\theta)} \Lambda_{\pi(\theta)} &= d\psi_t\,(W(\theta))\,. \end{aligned}$$

By definition $i_X \omega = -dH$, hence $i_X \omega|_{T\Sigma} \equiv 0$. Since the vertical subspace $V(\theta)$ is lagrangian, we get that $\Lambda_{\pi(\theta)}$ is an invariant lagrangian submanifold of T^*M inside the energy level Σ. Since $V = \ker d\pi$, the kernel of the projection $d\pi|_{\Lambda_{\pi(\theta)}}$ restricted to $\Lambda_{\pi(\theta)}$ is

$$\ker d_{\psi_t(\theta)}\pi|_{\Lambda_{\pi(\theta)}} = V(\psi_t(\theta)) \cap d\psi_t(W(\theta)).$$

By Proposition 1.16 and Remark 1.17 in [5], if the *whole* forward orbit of θ has no conjugate points then $V(\psi_t(\theta)) \cap d\psi_t(W(\theta)) = \{0\}$ for $t > 0$, and hence the derivative of the projection $d_{\psi_t(\theta)}\pi|_{\Lambda_{\pi(\theta)}}$ is injective along the forward orbit $\psi_t(\theta)$, $t > 0$. If $T > \varepsilon > 0$, then the projection $\pi|_{\Lambda_{\pi(\theta)}}$ is an immersion in a small tubular neighbourhood $N \subset \Lambda_{\pi(\theta)} \subset H^{-1}\{k\}$ of the compact orbit segment $\psi_{[\varepsilon,T]}(\theta)$.

Now fix $\theta_0 := L_v(x_0, v_0)$. Observe that if the tubular neighbourhood N is small enough, then N is either contractible or N is homeomorphic to a solid torus and the orbit of θ_0 is periodic with period smaller than or equal to $T - \varepsilon$.

If $(x, p) \in N$ we have that

$$k = H(x, p) = \sup_{v \in T_x M} \; p \cdot v - L(x, v).$$

Since $N \subset H^{-1}\{k\}$, for any curve $(z(s), q(s))$ inside N,

$$q(s) \cdot \dot z(s) \le L(z, \dot z) + k,$$

with strict inequality if $L_v(z, \dot z) \neq q \in N$.

Now let $(y, S) \in \Omega_M(\gamma(\varepsilon), \gamma(T))$ be a curve near $(x_{\varepsilon,T}, T - \varepsilon)$ in the metric of $\mathcal{H}^1(M) \times \mathbb{R}^+$. Since the time parameters S, $T - \varepsilon$, are bounded, by Lemma 2.3, if (y, S) is sufficiently near $(x_{\varepsilon,T}, T - \varepsilon)$ then the Hausdorff distance $d_H(y([0, 1]), x_{\varepsilon,T}([0, 1]))$ is small. In particular, y is homotopic to $x_{\varepsilon,T}$ with fixed endpoints and $y([0, 1]) \subset \pi(N)$. Let $z(t) := y(tS)$ and let $(z(s), q(s))$ be the lift of z to N with $q(0) = \psi_\varepsilon(\theta_0)$. Then (z, q) is homotopic in N to the orbit segment $\psi_{[\varepsilon,T]}(\theta_0)$ with fixed endpoints. Since N is a lagrangian submanifold of T^*M, the Liouville 1-form $p\,dx$ is closed on N. Then

$$\begin{aligned} \mathcal{A}_k(x_{\varepsilon,T}, T - \varepsilon) &= \oint_\gamma (L + k) = \oint_{\psi_t(\theta_0)} p\,dx \\ &= \oint_{(z,q)} p\,dx \le \oint_z (L + k) = \mathcal{A}_k(y, S), \end{aligned} \tag{76}$$

with strict inequality if $q(s) \neq L_v(z, \dot z)$ on a set of positive measure. Thus $\gamma|_{[\varepsilon,T]}$ is a local minimum of the $(L + k)$-action.

We now show that $\gamma|_{[\varepsilon,T]}$ is a strict local minimum of the $(L + k)$-action. Let $\mathcal{L} : TM \to T^*M$ be the Legendre transform $\mathcal{L}(x, v) = L_v(x, v)$. Observe that the hamiltonian vector field X satisfies

$$d\pi \circ X(x, p) = \mathcal{L}^{-1}(x, p) \qquad \text{for all } (x, p) \in T^*M.$$

Suppose that (76) is an equality. Then $q(s) = L_v(z, \dot z) \in N$ for almost every $s \in [0, S]$. Therefore

$$\dot z = \mathcal{L}^{-1}(z, q) = d\pi(X(z, q)) \quad \text{for almost every } s \in [0, S]. \tag{77}$$

Since $z(s)$ is continuous, its lift $q(s)$ is continuous. Hence, from (77), its derivative $\dot z$ is continuous. Then equation (77) says that the curve z is an orbit of the projection of the hamiltonian vector field on N. Since X is tangent to N, N is ψ_t-invariant and the lift (z, q) is unique, we have that (z, q) must be an orbit of X. Since $z(0) = \gamma(\varepsilon)$ and $(z(0), q(0)) = \psi_\varepsilon(\theta_0)$, we have that $z(t) = \gamma(t)$ for all $t \in [\varepsilon, T]$. Since $z(S) = x_{\varepsilon,T}(1) = \gamma(T)$, either $S = T - \varepsilon$ or θ_0 is a periodic point and $|S - (T - \varepsilon)|$ is a multiple of its period. Since we are assuming that $|S - (T - \varepsilon)|$ is small, $S = T - \varepsilon$. Therefore (76) is a strict inequality unless $(z, S) \equiv (x_{\varepsilon,T}, T - \varepsilon)$.

□

Lemma 9.2 *If $H : T^*M \to \mathbb{R}$ is a convex superlinear Hamiltonian, then the set of energy levels $k \in \mathbb{R}$ such that $H^{-1}\{k\}$ has an orbit with a conjugate point is open in* $[\min H, +\infty[$.

Proof Backgorund information for this proof can be found in [4]. Let $\omega = dp \wedge dx$ be the canonical symplectic form on T^*M and let

$$\mathbb{G} := \{E \subset T_\theta T^*M \;\big|\; \theta \in T^*M,\ \dim E = \dim M,\ \omega|_E \equiv 0\}$$

be the *Lagrangian Grassmannian bundle* over T^*M. The *Maslov cycle* associated to the vertical subbundle $\mathbb{V} := \ker d\pi \subset T(T^*M)$ is

$$\Lambda := \{E \in \mathbb{G} \;\big|\; E \cap \mathbb{V} \neq \{0\}\}.$$

The Maslov cycle Λ is a compact stratified submanifold of $\mathbb{G}$ of codimension 1 which defines a homological cycle $[\Lambda] \in H_1(\mathbb{G}, \mathbb{Z})$ because it has no codimension 2 strata. Moreover, it is transversally orientable.

Given a continuous curve $\Gamma : [a, b] \to \mathbb{G}$ whose endpoints are in $\mathbb{G} \setminus \Lambda$, the *Maslov index* $\mathfrak{M}(\Gamma)$ of Γ is the oriented intersection number of Γ with Λ. Then $\mathfrak{M}(\Gamma)$ is a homotopy invariant of Γ under homotopies with endpoints in $\mathbb{G} \setminus \Lambda$.

Given $\theta \in T^*M$, let $\Gamma_\theta : \mathbb{R} \to \mathbb{G}$ be the curve $\Gamma_\theta(t) = d_\theta\psi(\mathbb{V}(\theta))$, where ψ is the Hamiltonian flow of H. The convexity of H implies that if the curve Γ_θ intersects the Maslov cycle Λ, then the intersection is positively oriented. Moreover, the set $\{t \in \mathbb{R} \mid \Gamma_\theta(t) \in \Lambda\}$ is discrete on $\mathbb{R}$. Also, if $T > 0$ and $\Gamma_\theta(T) \notin \Lambda$, then for $\varepsilon > 0$ sufficiently small

$$\mathfrak{M}\left(\Gamma_\theta|_{[\varepsilon,T]}\right) = \sum_{\substack{\Gamma_\theta(t)\in\Lambda \\ 0<t<T}} \dim[\Gamma_\theta(t) \cap \mathbb{V}(\psi_t(\theta))]. \tag{78}$$

In particular, the orbit segment $\psi_{[0,T]}(\theta)$ has a conjugate point if and only if the Maslov index in (78) is non-zero.

The homotopy invariance of the Maslov index and the continuity of ψ_t and $d\psi_t$ on H and θ imply the Lemma. □

Proof of Theorem E Observe that the convexity of L implies[8] that $\min_{v\in T_xM} E(x,v) = E(x,0)$. If $k > e_m(L)$ then there is $x_0 \in M$ such that $k > E(x_0,0)$. Then Theorem D.(b) says that for almost every $k \in]e_m(L), c_u(L)[$ there is an orbit segment with energy k which is not a strict local minimizer of the action functional $\mathcal{A}_k$. Then Proposition 9.1 implies that the forward orbit of the initial point of such an orbit segment must have a conjugate point. Lemma 9.2 proves that having a conjugate point is an open condition.

If for a specific $k \in]e_m(L), c_u(L)[$ the energy level $E^{-1}\{k\}$ satisfies the Palais-Smale condition then the same argument, now using Theorem D.(c) and Proposition 9.1, implies that the energy level k has conjugate points. □

10 Contact type energy levels

Fix $k \in \mathbb{R}$ and let $\Sigma := H^{-1}\{k\}$ be the energy level. Let X be the hamiltonian vector field for H and ψ_t be its flow. Let $\pi : T^*M \to M$ be the projection, $\omega = dp \wedge dx$ the canonical symplectic form on T^*M and $\Theta = p\,dx$ the Liouville 1-form on T^*M.

Given a ψ_t-invariant Borel probability measure ν supported on Σ, the *Schwartzman asymptotic cycle* $\mathcal{S}(\nu) \in H_1(\Sigma,\mathbb{R}) \approx H^1(\Sigma,\mathbb{R})^*$ of ν is defined by

$$\langle \mathcal{S}(\nu), [\eta]\rangle = \int_\Sigma \eta(X)\, d\nu$$

for every closed 1-form η on Σ. The map $(\pi|_\Sigma)_* : H_1(\Sigma,\mathbb{R}) \to H_1(M,\mathbb{R})$ maps $\mathcal{S}(\nu)$ to the homology class $\rho(\nu)$ of ν.

Proposition F *If Σ is of contact type and $\pi_* : H_1(\Sigma,\mathbb{R}) \to H_1(M,\mathbb{R})$ is injective, then $\mathcal{A}_k$ satisfies the Palais-Smale condition.*

Proof Let $\Theta = p\,dx$ be the Liouville 1-form. Observe that on the energy level Σ:

$$\Theta(X) = p \cdot H_p = v \cdot L_v = L + k.$$

Suppose that $\mathcal{A}_k$ does not satisfy the Palais-Smale condition. Let μ be the measure given by Theorem A and let $\nu = \mathcal{L}_*(\mu)$ be its push-forward under the Legendre transform $\mathcal{L}(x,v) = L_v(x,v)$. Let λ be a contact-type form on Σ. Since $\lambda(X) \neq 0$, $\lambda(X)$ has a fixed sign on each connected component of $H^{-1}\{k\}$, in particular in the support of ν. Since $d\lambda = \omega = d\Theta$, the form $\eta := \lambda - \Theta$ is closed on Σ. Since $\pi_*(\mathcal{S}(\nu)) = \rho(\nu) = \rho(\mu) = 0$ and π_* is injective, $\mathcal{S}(\nu) = 0$. Then

$$\begin{aligned} A_{L+k}(\mu) &= \int_\Sigma \Theta(X)\, d\nu + 0 = \int_\Sigma \Theta(X)\, d\nu + \langle \mathcal{S}(\nu), [\eta]\rangle \\ &= \int_\Sigma (\Theta + \eta)(X)\, d\nu = \int_\Sigma \lambda(X)\, d\nu \neq 0. \end{aligned} \tag{79}$$

This contradicts Theorem A. □

Since for $k > e_0$ we have that $\pi(\Sigma) = M$, the next Lemma 10.1 and Proposition F prove Corollary G.

[8] See e.g. Lemma 3.1.

Lemma 10.1 *If* $\dim M \geq 2$ *and either*

- $M \neq \mathbb{T}^2$ *or*
- $M = \mathbb{T}^2$ *and* $k \leq e_0$,

then $\pi_* : H_1(\Sigma, \mathbb{R}) \to H_1(\pi(\Sigma), \mathbb{R})$ *is an isomorphism.*

In the following proof we shall use the lagrangian[9] version $\Sigma = E^{-1}\{k\}$. Its intersections with the fibers of TM, $\Sigma \cap T_x M$ are, if non-empty, convex subsets containing $(x, 0)$ in its interior which are either homeomorphic to a sphere S^{n-1} or to a point, when $E(x, 0) = k$.

Proof Suppose first that $k > e_0(L)$ and $M \neq \mathbb{T}^2$.

Since $k > e_0(L)$, the energy level $\Sigma := E^{-1}\{k\}$ is isomorphic to the unit tangent bundle of M with the projection $\pi : \Sigma \to M$. If M is orientable, the Lemma follows from an argument using the Gysin exact sequence, e.g. [26, Lemma 1.45].

If M is not orientable and $n = \dim M \geq 3$, from the exact homotopy sequence of the fiber bundle $\pi : \Sigma \to M$:

$$0 = \pi_1(S^{n-1}) \xrightarrow{i_*} \pi_1(\Sigma) \xrightarrow{\pi_*} \pi_1(M) \longrightarrow \pi_0(S^{n-1}) = 0,$$

we get that $\pi_* : \pi_1(\Sigma) \to \pi_1(M)$ is an isomorphism. This implies that $\pi_* : H_1(\Sigma, \mathbb{R}) \to H_1(M, \mathbb{R})$ is an isomorphism.

If M is not orientable and $\dim M = 2$, from the homotopy sequence above we get an isomorphism $f : \pi_1(\Sigma)/_{\mathrm{im}\, i_*} \to \pi_1(M)$. Let $h : \pi_1(M) \to H_1(M, \mathbb{Z})/_{\mathrm{Torsion}}$ and $k : \pi_1(\Sigma) \to H_1(\Sigma, \mathbb{Z})/_{\mathrm{Torsion}}$ be the natural homomorphisms. We show that $\mathrm{im}\, i_* \subset \ker k$ and therefore, k induces a homomorphism $k_1 : \pi_1(\Sigma)/_{\mathrm{im}\, i_*} \to H_1(\Sigma, \mathbb{Z})/_{\mathrm{Torsion}}$. Indeed, the fiber $F = S^1$ lies inside a Klein bottle K inside Σ, which is $\pi^{-1}(\gamma)$, where γ is a closed curve containing the base point $\pi(F)$ along which M is non-orientable. Then the generator 1_F of the fundamental group of the fiber $\pi_1(F) = \mathbb{Z}$ is represented by a cycle which is homologous in Σ to itself with inverse orientation. Hence its image in homology is torsion and thus $k(i_*(1_F)) = 0 \in H_1(\Sigma, \mathbb{Z})/_{\mathrm{Torsion}}$.

The following diagram commutes. There k_1 and h are surjective and f is an isomorphism.

$$\begin{array}{ccc} \pi_1(\Sigma)/_{\mathrm{im}\, i_*} & \underset{\approx}{\xrightarrow{f}} & \pi_1(M) \\ k_1 \downarrow & & \downarrow h \\ H_1(\Sigma, \mathbb{Z})/_{\mathrm{Torsion}} & \xrightarrow{\pi_*} & H_1(M, \mathbb{Z})/_{\mathrm{Torsion}} \end{array}$$

Then π_* is surjective. Suppose that $\pi_*(a) = 0$. Let $b \in \pi_1(\Sigma)/_{\mathrm{im}\, i_*}$ be such that $k_1(b) = a$ and let $c = f(b)$. Then $h(c) = 0$. Hence some power c^m is in the commutator subgroup of $\pi_1(M)$. Since f is an isomorphism, b^m is in the commutator subgroup of $\pi_1(\Sigma)/_{\mathrm{im}\, i_*}$. Therefore $a = k_1(b) = 0$. Thus π_* is injective. Hence π_* is an isomorphism and also in real coefficients $\pi_* : H_1(\Sigma, \mathbb{R}) \to H_1(M, \mathbb{R})$ is an isomorphism.

Now assume that $k \leq e_0(L)$. Let $B := \pi(\Sigma)$ and let $E := T^1_B M$ be the restriction of the unit tangent bundle to B. Let $\equiv$ be the equivalence relation on E defined by $(x, v) \sim (y, w)$ iff either $E(x, 0) \neq 0$ and $(x, v) = (y, w)$ or

[9] The hamiltonian version, $\Sigma = H^{-1}\{k\}$, may not contain the zero section in its interior

$E(x,0)=k$ and $x=y$. Then the energy level Σ is homeomorphic to $E/\sim$, i.e. the one point compactification of the fibers over the points x with $E(x,0)=k$.

We can assume that B is connected, for the connected components of Σ are in 1-1 correspondence with the connected components of B under the projection π.

We can also assume that there is $b_1 \in B$ such that $E(b_1,0) \neq k$. For, if not, then $E^{-1}\{k\} = \{(x,0)\,|\,x \in B\}$ and the Lemma becomes trivial.

Let $p: E \to B$ be the restriction of the projection of the unit tangent bundle and $f: E \to \Sigma = E/\sim$ the canonical projection. Then $p = \pi \circ f$. The homotopy exact sequence of the fibering $S^{n-1} \hookrightarrow E \to B$ gives

$$\pi_1(S^{n-1}) \xrightarrow{i_*} \pi_1(E,e_1) \xrightarrow{p_*} \pi_1(B,b_1) \longrightarrow \pi_0(S^{n-1}) = 0.$$

Then p_* is an epimorphism and it induces an isomorphism $g: \pi_1(E)/_{\operatorname{im} i_*} \to \pi_1(B)$. We see that $\operatorname{im} i_* \subset \ker f_*$, so that f_* induces a homomorphism $\hat{f}_*: \pi_1(E)/_{\operatorname{im} i_*} \to \pi_1(\Sigma)$. Indeed, if $n \geq 3$ then $\pi_1(S^{n-1}) = 0$ and so $\operatorname{im} i_* = 0$. If $n=2$, let 1_F be a generator of the fundamental group of the fiber $\pi_1(S^1) = \mathbb{Z}$. Since $k \leq e_0(L)$ there is a point $x_1 \in B$ such that $E(x_1,0)=k$. Let λ be a curve in B joining b_1 to x_1. Since $E(b_1,0) \neq k$, by a suitable choice of x_1 and λ we can assume that $E(\lambda(t),0) \neq k$ for $t \in [0,1[$. The fiber bundle E over the interval λ is trivial $E|_\lambda \approx S^1 \times [0,1]$. Observe that the inverse image $\pi^{-1}(\lambda) \subset \Sigma$ has the topology of a cylinder with one of its boundary circles compactified to a point. Hence it is homeomorphic to a 2-disc, and the class 1_F is represented by its boundary circle. Hence $f_*(i_*(1_F)) = 0 \in \pi_1(\Sigma)$.

We prove that $\pi_*: \pi_1(\Sigma) \to \pi_1(B)$ is an isomorphism. This implies the Lemma. Since $g = \pi_* \circ \hat{f}_*$, it follows that $\hat{f}_*$ is injective and π_* is surjective. In order to prove that π_* is injective it is enough to prove that $\hat{f}_*$ is onto. Since f_* and $\hat{f}_*$ have the same image, it is enough to prove that f_* is surjective.

Let $\sigma_1 = f(e_1) \in \Sigma$. Let $\Gamma: (S^1,1) \to (\Sigma,\sigma_1)$ be a loop in Σ based at σ_1. We want a preimage under f_* of the homotopy class of Γ. If $\Gamma(s) \neq 0$ for all $s \in S^1$, such preimage is the homotopy class of $\Psi_E(s) = \frac{\Gamma(s)}{\|\Gamma(s)\|}$. In general, the problem is that such definition of Ψ_E may have no continuous extension to the s where $\Gamma(s)=0$. Assume now that there is $s_0 \in S^1$ such that $\Gamma(s_0)=0$.

Let $\gamma = \pi \circ \Gamma : (S^1,1) \to (B,b_1)$ be the projection of Γ. Let $C := \{x \in M \,|\, E(x,0)=k\}$ and $D := \{t \in S^1 \,|\, \gamma(t) \in C\}$. Then D is a compact subset of S^1 and its complement is a union of open intervals I_i. Choose any continuous loop $\Lambda_E: (S^1,1) \to (E,e_1)$ such that $p \circ \Lambda_E = \gamma$.

The pullback of the sphere bundle E along each open segment $\gamma(I_i)$ is trivial $(\gamma|_{I_i})^*E \approx I_i \times S^{n-1}$. Then the inverse image $\pi^{-1}\{\gamma(\overline{I_i})\} \subset \Sigma$ of the *closed* segment $\gamma(\overline{I_i})$ has the topology of a cylinder $S^{n-1} \times [0,1]$ with its boundary spheres $S^{n-1} \times \{0\}$ and $S^{n-1} \times \{1\}$ compactified to two points $\{A,B\}$ or to a single point $A=B$.

Both segments $f \circ \Lambda_E|_{\overline{I_i}}$ and $\Gamma|_{\overline{I_i}}$ must have the same endpoints A and B. Hence they are homotopic with fixed endpoints inside $\pi^{-1}\{\gamma(\overline{I_i})\} \subset \Sigma$. Join all these homotopies for each interval I_i to obtain a homotopy in Σ between $f \circ \Lambda_E$ and Γ. Therefore $f_*([\Lambda_E]) = [\Gamma] \in \pi_1(\Sigma)$. □

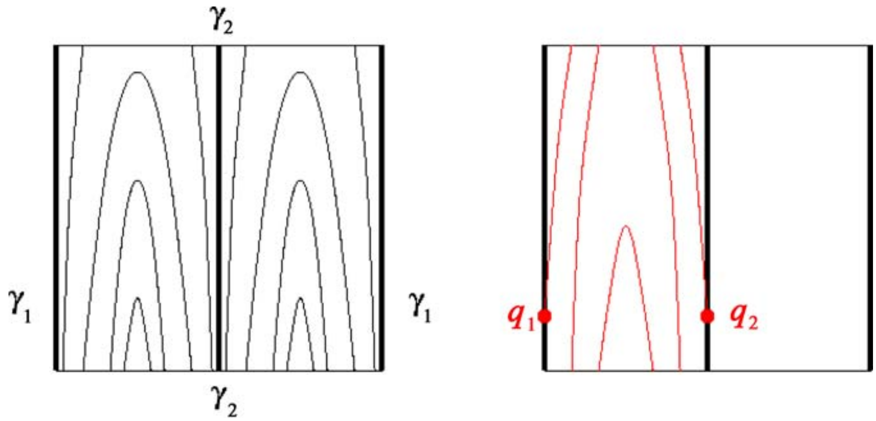

Fig. 1 The left figure shows the flow lines of the vector field X in the example. The right figure shows a curve $(x_n, T_n) \in \Omega_M(q_1, q_2)$ in an unbounded Palais-Smale sequence. The probability measures associated to (x_n, T_n) converge to the non-ergodic measure $\frac{1}{2}\,\mu_1 + \frac{1}{2}\,\mu_2$, where μ_1 and μ_2 are the invariant probabilities for the Euler-Lagrange flow corresponding to the periodic solutions γ_1 and γ_2.

Appendix A: A non ergodic measure in Theorem A

Consider the flat metric on the 2-torus $\mathbb{T}^2$. Let X be a vector field with norm 1 on $\mathbb{T}^2$ whose orbits form a Reeb foliation. Let $L : T\mathbb{T}^2 \to \mathbb{R}$ be the lagrangian

$$L(x, v) := \frac{1}{2}\,|v - X(x)|^2 .$$

Its Euler-Lagrange flow is the same as the exact magnetic flow with lagrangian

$$L(x, v) - \frac{1}{2} = \frac{1}{2}\,|v|^2 - \eta_x(v),$$

where $\eta_x(v) = \langle X(x), v\rangle$. It is easy to see from the definition of critical value that $c(L) = 0$. The vector field X has two closed orbits γ_1 and γ_2 with *opposite* homology classes. Since $L \geq 0$ the Euler-Lagrange flow has only two ergodic invariant measures μ_1, μ_2, with zero L-action, corresponding to the periodic orbits $\Gamma_i = (\gamma_i, \dot{\gamma}_i)$. The unique invariant probability μ with $A_L(\mu) = 0$ and zero homology class is $\mu = \frac{1}{2}\,\mu_1 + \frac{1}{2}\,\mu_2$. It follows that $c_u(L) = c_0(L) = c(L) = 0$. Let $\mathbb{L} : T\mathbb{R}^2 \to \mathbb{R}$ be the lift of L to the universal cover $\mathbb{R}^2$ of $\mathbb{T}^2$.

It is easy to see that the Peierls barrier for $\mathbb{L}$ is finite, because one can join two points $q_0, q_1 \in \mathbb{R}^2$ by curves with bounded action which spend long time on a lift of γ_1 and come back on a lift of γ_2. Since the action of γ_1 and γ_2 is zero, the total action spent on them is bounded. By Corollary B, there is an unbounded Palais-Smale sequence (x_n, T_n) with $\lim_n T_n = +\infty$. Nevertheless, there is no *ergodic* invariant probability in TM with zero action and zero homology class.

Appendix B: Energy levels of non-contact type

The following theorem is not explicitly stated in [25].

Theorem B.1 (G. Paternain) *Suppose that* $\dim M \geq 2$.

If $k > c_0(L)$ *then* $H^{-1}\{k\}$ *is of contact type.*

If $M \neq \mathbb{T}^2$ *and* $c_u(L) < k \leq c_0(L)$ *then* $H^{-1}\{k\}$ *is not of contact type.*

There is an example in [8] of a Lagrangian on $\mathbb{T}^2$ for which the energy level $E = c_0(L)$ is of contact type. In Proposition C.2 below we prove that for a non-magnetic lagrangian every regular energy level is of contact type.

As an application (cf. [25, Th. 1.1]), if M admits a metric with negative curvature and if the Lagrangian flow on an energy level $H^{-1}\{k\}$ with $c_u(L) < k < c_0(L)$ is Anosov, then the strong stable and unstable subbundles E^{ss}, E^{uu} can not be C^1. For if they were C^1, the form λ defined by $\lambda(X) \equiv 1$ and $\lambda|_{E^{ss}\oplus E^{uu}} \equiv 0$ is a contact form on $H^{-1}\{k\}$ (cf. U. Hamendstädt [12], G. Paternain [25, Th. 5.5]). Examples of such Anosov energy levels appear in G. Paternain & M. Paternain [27].

If Σ is a regular energy level, the *Liouville measure* m on Σ is the smooth measure induced by the volume form $i_Y\omega^n$, where Y is a vector field on T^*M such that $\omega(Y, X) \equiv 1$ on Σ. It is invariant under the hamiltonian flow because $L_X(i_Y\omega^n) = 0$. We choose the orientation on Σ that makes m a positive measure.

Lemma B.2 *If λ is a 1-form on Σ such that $d\lambda = \omega$ and $\lambda(X) \neq 0$, then $\lambda(X) > 0$ on Σ.*

Proof Let ξ be a 1-form on Σ. We have that

$$\begin{aligned}
\xi(X)\, i_Y\omega^n &= i_X(\xi \wedge i_Y\omega^n) + \xi \wedge (i_X\, i_Y\, \omega^n) \\
&= 0 + \xi \wedge i_X[n\,(i_Y\,\omega) \wedge \omega^{n-1}] && \text{because } \dim\Sigma = 2n-1 \\
&= \xi \wedge (n\,\omega(Y,X)\,\omega^{n-1} - 0) && \text{because on } \Sigma,\ i_X\omega = -dH \equiv 0 \\
&= n\,\xi \wedge \omega^{n-1}.
\end{aligned} \tag{80}$$

We show first that the asymptotic cycle (cf. Sect. 10) of the Liouville measure m is zero. Indeed, let $\Theta := p\ dx$ be the Liouville 1-form on T^*M and $\tau := \Theta \wedge \omega^{n-2}$. If η is a closed 1-form on Σ then $\eta \wedge \omega^{n-1} = \eta \wedge d\tau = d(\eta \wedge \tau)$. Hence by (80),

$$\int_\Sigma \eta(X)\ dm = \int_\Sigma \eta(X)\ i_Y\omega^n = n \int_\Sigma \eta \wedge \omega^{n-1} = n \int_\Sigma d(\eta \wedge \tau) = 0.$$

Since $\lambda(X) \neq 0$ on Σ, it is enough to prove that for any connected component $\mathcal{N}$ of Σ we have $\int_{\mathcal{N}} \lambda(X)\ dm > 0$. Since $d\lambda = \omega = d\Theta$, the form $\eta = \lambda - \Theta$ is closed on Σ. Then

$$\int_{\mathcal{N}} \lambda(X)\ dm = \langle \mathcal{S}(m), [\eta] \rangle + \int_{\mathcal{N}} \Theta(X)\ dm = \int_{\mathcal{N}} \Theta(X)\ dm.$$

From (80) we have that

$$\Theta(X)\ i_Y\omega^n\big|_\Sigma = n\ \Theta \wedge \omega^{n-1}\big|_\Sigma.$$

Let $\mathcal{W}$ be the fiberwise convex hull of $\mathcal{N}$ in T^*M. Then $\partial\mathcal{W} = \mathcal{N}$. By Stokes Theorem,

$$\int_{\mathcal{N}} \Theta(X)\ dm = \int_{\partial\mathcal{W}} \Theta(X)\, i_Y\omega^n = \int_{\partial\mathcal{W}} n\ \Theta \wedge \omega^{n-1} = n \int_{\mathcal{W}} \omega^n.$$

We prove that the last integral is positive.[10] Since $dH(Y) = i_Y(-i_X\omega) = 1$, the convexity of H implies that Y is an *outwards* pointing vector in $\Sigma = \partial\mathcal{W}$. A basis $(v_1, \ldots, v_{2n-1})$ of $T\Sigma$ is positively oriented iff $i_Y\omega^n(v_1, \ldots, v_{2n-1}) = \omega^n(Y, v_1, \ldots, v_{2n-1}) > 0$, and Stokes theorem uses $(Y, v_1, \ldots, v_{2n-1})$ as a positively oriented basis for $T\mathcal{W}$. □

Remark B.3 Lemma B.2 also says that the Liouville measure has always positive $(L+k)$-action, for in the energy level $E^{-1}\{k\}$ we have that $\Theta(X) = v \cdot L_v = L + k$.

[10] Alternatively, the integral is equal to the $(L + k)$ action of the Liouville measure m_k on $E^{-1}\{k\}$. If $\ell > c_0(L)$ by (1), $A_{L+\ell}(m_\ell) > 0$, and one can show that the orientation on $E^{-1}\{\ell\}$ defined by the Liouville measure varies continuously with ℓ.

Proof of Theorem B.1 From (1), there is a closed 1-form η on M such that $c_0(L) = c(L - \eta)$. The Hamiltonian of $L - \eta$ is $\mathbb{H}(x, p) = H(x, p + \eta)$. If $k > c_0(L)$, by Theorem A in [6] there is a smooth function $u : M \to \mathbb{R}$ such that $\mathbb{H}(x, d_x u) < k$ for all $x \in M$. From the definition (3) of the Hamiltonian $\mathbb{H}$ we have that

$$L(x, v) - \eta_x(v) - d_x u(v) + k > 0 \quad \text{for all } (x, v) \in TM.$$

Let $\Theta = p\,dx$ be the Liouville 1-form on T^*M. Let $\lambda := \Theta - \eta \circ d\pi - du \circ d\pi$, where $\pi : T^*M \to M$ is the projection. Since η is closed, $d\lambda = d\Theta = \omega$. On $H^{-1}\{k\}$ we have that

$$\Theta(X) = p \cdot H_p = L(x, v) + k,$$

where $v = H_p(x, p)$. Since $X = (H_p, *)$, on $H^{-1}\{k\}$ we have that

$$\lambda(X) = L(x, v) + k - \eta_x(v) - d_x u(v) > 0,$$

where $v = H_p(x, p)$, proving the first claim.

Now assume that $c_u(L) < k < c_0(L)$. Let $\widehat{L}$ be the lift of L to the abelian cover $\hat{M}$. Since $k < c_0(L) = c(\widehat{L})$, there exists a closed curve $\widehat{\gamma}$ in $\hat{M}$ with negative $(L + k)$-action. Observe that the projection γ of $\widehat{\gamma}$ to M has trivial homology class. The homotopy class of γ can not be trivial because if it where, its lift to the universal cover would be closed and since $k > c_u(L)$ its $(L + k)$-action would be non-negative. Let σ be the free homotopy class of γ and let

$$\Lambda_\sigma := \{(x, T) \in \Lambda_M \mid x \in \sigma,\ T > 0\}.$$

Since $k > c_u(L)$, by Lemma 4.1,

$$-\infty < \inf_{(x,T)\in\Lambda_\sigma} \mathcal{A}_k(x, T) \le A_{\widehat{L}+k}(\widehat{\gamma}) < 0.$$

Since $k > c_u(L)$, by Corollary B, $\mathcal{A}_k$ satisfies the Palais-Smale condition. By Corollary 6.11 the gradient flow of $-\mathcal{A}_k$ is relatively complete on $[\mathcal{A}_k \le 0]$. Then by Corollary 6.6 there is a minimizer (x, T) of $\mathcal{A}_k$ on Λ_σ. The curve $y(t) := x(t/T)$ is a periodic orbit of the Euler-Lagrange flow with negative $(L + k)$-action and energy k. Let μ be the invariant probability measure supported on the periodic orbit $(y, \dot{y})$ and let $\nu = \mathcal{L}_*(\mu)$, $\mathcal{L}(x, v) = (x, L_v(x, v))$. Since the homology class corresponding to σ is trivial, $\rho(\mu) = 0$. Since $k > c_u \ge e_0$, $\pi(H^{-1}\{k\}) = M$. By Lemma 10.1, since $\pi_*(\mathcal{S}(\nu)) = \rho(\mu) = 0$ and $M \ne \mathbb{T}^2$, $\mathcal{S}(\nu) = 0$. If λ is a contact-type form, since $\text{supp}(\mu) \subset E^{-1}\{k\}$, the same calculation as in (79) gives

$$\int \lambda(X)\, d\nu = A_{L+k}(\mu) < 0. \tag{81}$$

This contradicts Lemma B.2.

When $k = c_0(L)$, by (1) there is an invariant probability μ such that $\rho(\mu) = 0$ and $A_{L+c_0}(\mu) = 0$. The same argument as in (81) shows that $H^{-1}\{c_0\}$ is not of contact type. □

Appendix C: Non-magnetic lagrangians

The following result was suggested by R. Mañé in [17].

Theorem C.1 *If L is a convex superlinear lagrangian and the 1-form $\theta_x := L_v(x, 0)$ is closed, then every energy level contains a closed orbit.*

Proof Since we are looking for closed orbits on specified energy levels, by [7, prop. 18] we can assume that L is Riemannian at infinity. Since the 1-form θ_x is closed, the lagrangian $\mathbb{L} = L - \theta_x$ has the same Euler-Lagrange flow as L. Replacing L by $\mathbb{L}$ we can also assume that $\theta_x \equiv 0$. If $k \ge e_0(L)$, by Lemma 3.1,

$$L(x, v) + k \ge \frac{1}{2}\, a_0\, |v|_x^2 + [k - E(x, 0)] \ge 0.$$

Therefore $c(L) = e_0(L) = c_u(L)$. By Theorem D, when $k > e_0 = c(L)$ the energy level $E^{-1}\{k\}$ has (non-trivial) closed orbits and at $k = e_0$ it has a singularity.

By Proposition 8.2 and by the Frauenfelder-Schlenk theorem [10], for $k < e_0$ the set $[E \leq k]$ has finite Hofer-Zehnder capacity. Then [13, Th. 5, p. 123] it has a closed orbit on every contact type energy level $E^{-1}\{k\}$ with $k < e_0$.

Singular energy levels have a singularity of the Euler-Lagrange flow. Then Proposition C.2 finishes the proof. $\square$

Proposition C.2 *If L is a convex superlinear lagrangian and the 1-form $\theta_x := L_v(x, 0)$ is closed, then every regular energy level is of contact type.*

We shall use the following

Proposition C.3 *[McDuff [19], also [8, sec. 2 & app. B]] Suppose that the flow of a vector field X on a compact orientable manifold Σ does not admit a global cross section. Let Θ be a smooth 1-form on Σ. Then the following are equivalent:*

(1) $\int \Theta(X)\, d\mu \neq 0$ for every invariant Borel probability with zero asymptotic cycle.

(2) There exists a smooth closed 1-form φ on Σ such that $\Theta(X) + \varphi(X)$ never vanishes.

Corollary C.4 *A compact energy level $E^{-1}\{k\}$ of a convex lagrangian is of contact type if and only if $A_{L+k}(\mu) > 0$ for every Borel invariant probability on $E^{-1}\{k\}$ with zero asymptotic cycle.*

Proof Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian of L. Let $\Theta = p\,dx$ be the Liouville form on T^*M. Observe that on $\Sigma = H^{-1}\{k\}$ we have that

$$\Theta(X) = p \cdot H_p = v \cdot L_v = L + k.$$

Suppose that Σ is of contact type and let λ be a contact type form on Σ, i.e. $d\lambda = \omega$ and $\lambda(X) \neq 0$. By Lemma B.2, $\lambda(X) > 0$ on Σ. Since $d\Theta = \omega$, the 1-form $\varphi := \lambda - \Theta$ is closed on Σ. Let $\mathcal{L}(x, v) := (x, L_v(x, v))$. If μ is an invariant probability on $E^{-1}\{k\}$ with asymptotic cycle $\mathcal{S}(\mu) = 0$, then $\mathcal{S}(\mathcal{L}_*(\mu)) = 0$ and

$$A_{L+k}(\mu) = \int_\Sigma \Theta(X)\, d\mathcal{L}_*(\mu) = \int_\Sigma \Theta(X)\, d\mathcal{L}_*(\mu) + \langle \mathcal{S}(\mathcal{L}_*(\mu)), \varphi\rangle = \int_\Sigma \lambda(X)\, d\mu > 0.$$

Conversely, suppose that $A_{L+k}(\mu) > 0$ for every invariant probability on $E^{-1}\{k\}$ with $\mathcal{S}(\mu) = 0$. If ν is an invariant probability on $\Sigma = H^{-1}\{k\}$ with $\mathcal{S}(\nu) = 0$, then $\mathcal{S}(\mathcal{L}_*^{-1}(\nu)) = 0$ and

$$\int_\Sigma \Theta(X)\, d\nu = A_{L+k}(\mathcal{L}_*^{-1}(\nu)) > 0.$$

Since the Liouville measure has zero asymptotic cycle (see the proof of Lemma B.2) the energy level $\Sigma = H^{-1}\{k\}$ has no global cross section. For, by Poincaré duality, such a cross section would give a closed 1-form η such that $\langle \mathcal{S}(\mu), \eta\rangle = \int_\Sigma \eta(X)\, d\mu > 0$ for every invariant probability μ.

By Proposition C.3 there is a closed 1-form φ on Σ such that $\Theta(X) + \varphi(X) \neq 0$. Let $\lambda = \Theta + \varphi$. Then $\lambda(X) \neq 0$ and $d\lambda = d\Theta = \omega$. $\square$

Proof of Proposition C.2 The convexity of L implies that

$$\frac{d}{ds}\left[v \cdot L_v(x, sv)\right] = v \cdot L_{vv}(x, sv) \cdot v > 0 \quad \text{if } v \neq 0.$$

Let k be a regular value of the energy function. Then $\mu([v = 0]) = 0$ for all invariant probablities μ in $E^{-1}\{k\}$. If $E(x, v) = k$ and $v \neq 0$, we have that

$$L(x, v) + k = v \cdot L_v(x, v) = \theta_x(v) + \int_0^1 v \cdot L_{vv}(x, sv) \cdot v\, ds > \theta_x(v).$$

If μ is an invariant probability on $E^{-1}\{k\}$ with asymptotic cycle $\mathcal{S}(\mu) = 0$ then

$$A_{L+k}(\mu) > \int_{E^{-1}\{k\}} \theta\, d\mu = \langle \mathcal{S}(\mu), \pi^*\theta\rangle = 0,$$

because $\pi^*\theta$ is closed. By Corollary C.4, $E^{-1}\{k\}$ is of contact type. $\square$

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