POSITIVE TOPOLOGICAL ENTROPY FOR GENERIC GEODESIC FLOWS

Gonzalo Contreras
CIMAT

GEODESIC FLOW

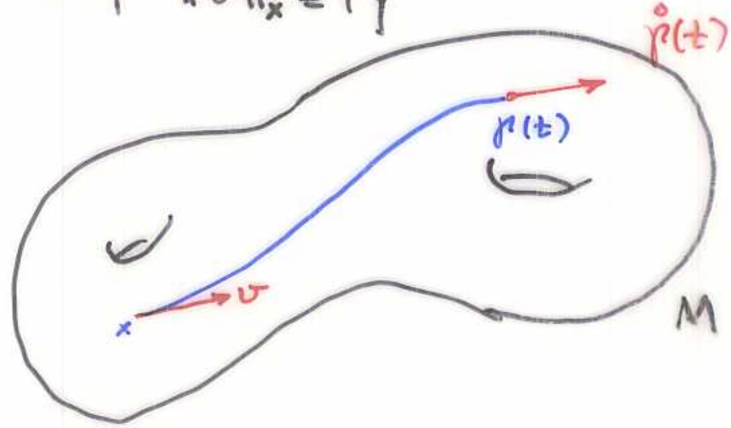
M closed C^∞ manifold [compact, connected, $\partial M = \phi$]

$g = \langle\ ,\ \rangle_x$ C^∞ riemannian metric on M.

Unit tangent bundle = sphere bundle of (M,g)

$$SM = \{ (x,v) \in TM \mid \|v\|_x = 1 \}$$

$$\pi : SM \longrightarrow M$$
$$(x,v) \longmapsto x$$



$(x,v) \in SM$

$\gamma : \mathbb{R} \longrightarrow M$

geodesic s.t. $\gamma(0) = x$, $\dot\gamma(0) = v$

"locally length minimizing curve with $|\dot\gamma| \equiv 1$"

Geodesic Flow

$$\phi_t : SM \longrightarrow SM$$
$$(x,v) \longmapsto (\gamma(t), \dot\gamma(t))$$

TOPOLOGICAL ENTROPY

① Measures the "complexity" of the orbit structure of the flow.

Measures the difficulty in predicting the position of an orbit given an approximation of its initial state.

Dynamic Ball: $\theta \in SM$, $\epsilon, T > 0$

$$\mathbb{B}(\theta, \epsilon, T) = \{ w \in SM \mid d(\phi_t \theta, \phi_t w) \leq \epsilon, \ \forall t \in [0, T] \}$$

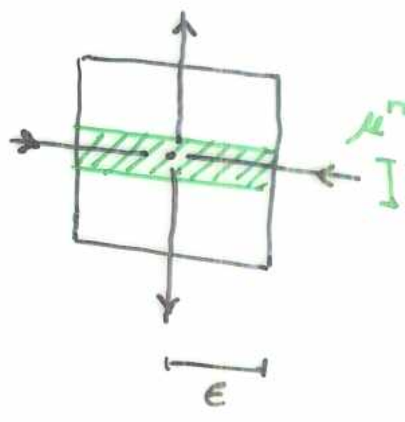
↑ points whose orbit stay near the orbit of θ for times in $[0, T]$

$$N_\epsilon(T) := \min \{ \# \mathcal{C} \mid \mathcal{C} \text{ cover of } SM \text{ by } (\epsilon, T)\text{-dyn. balls} \}$$

$$h_{top}(g) := \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log N_\epsilon(T).$$

$$N_\epsilon(T) \sim e^{h_{top} \cdot T}$$

If $h_{top} > 0$ some dynamic balls must contract exponentially at least in one direction



(2) For C^∞ riemannian metrics

Mañé

$$h_{top}(g) = \lim_{T\to\infty} \frac{1}{T} \log \int_{M\times M} n_T(x,y)\, dx\, dy$$

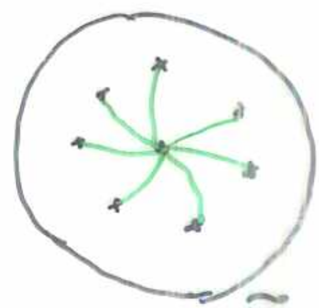
$n_T(x,y) := \#\{$ geod. arcs $x \longrightarrow y$ of length $\leq T\}$

$h_{top} > 0 \implies$ positive measure of (x,y) s.t. $n_T(x,y)$ is exponentially large.

TOPOLOGY $\implies$ Some manifolds have always $h_{top}(g) > 0$

- Dinaburg : $\pi_1(M)$ exponential growth $\implies h_{top} > 0$

[# dyn balls growths expo]



$\widetilde{M}$

Also if $\lim_{R\to\infty} \frac{1}{R} \log \mathrm{Vol}(\widetilde{B}(x,R)) > 0$ "volume entropy"

- Paternain - Petean : If $H_*(\text{Loop Space}, x)$ growths exponentially $\implies h_{top} > 0$.

GEOMETRY

sectional curvatures $K < 0 \implies \phi_t$ Anosov $\implies h_{top} > 0$

$K > 0$ not clear.

If the geod. flow ϕ_t contains a "horseshoe"
= a non-trivial hyperbolic basic set
$\Longrightarrow$ $h_{top}(g) > 0$.

$\exists$ hyperbolic periodic orbit with transversal homoclinic point. $\Longleftrightarrow$ $\exists$ horseshoe.

$\mathcal{R}^2(M) :=$ C^∞ riemannian metrics on M with the C^2 topology

THEOREM

$\dim M \geq 2$

$\exists\ \mathcal{U} \subset \mathcal{R}^2(M)$ open and dense s.t.

$g \in \mathcal{U} \Longrightarrow \phi_t^g$ has a horseshoe.

Previous Work:

- Proved for $\dim M = 2$ Paternain & C. JDG 2002
- $\dim M = 2$ & C^∞ topology Knieper & Weiss JDG 2002

Application:

A. Delshams, R. de la Llave, T. Seara:

Initial system that allows Arnold's diffusion by perturbation with generic non-autonomous potentials.

mp_arc

Comparison with other systems:

1. General Hamiltonian Systems

S. Newhouse: (M^{2n}, ω) closed symplectic manifold

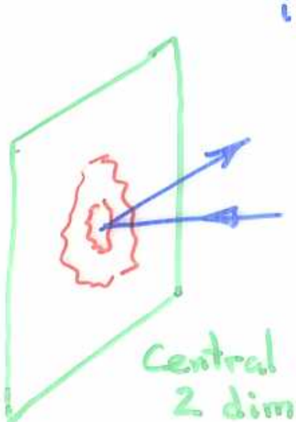
$\exists\, \mathcal{R} \subset C^2(M, \mathbb{R})$ residual s.t.

$H \in \mathcal{R} \Rightarrow$ Hamiltonian flow of H → Anosov; → Has a generic 1-elliptic periodic orbit

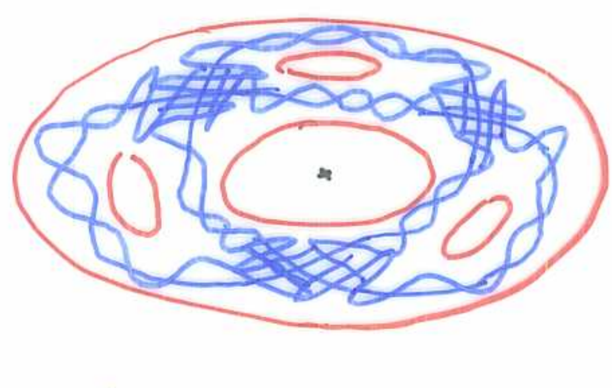
1-elliptic = 2 (elliptic) eigenvalues of modulus 1
1 eigenvalue $\lambda = 1$ (direct. of Ham. vect. field)
1 eigenvalue $\lambda = 1$ ($\pitchfork$ direct. to energy level)
$2n-4$ hyperbolic eigenvalues.

In this case:

Poincaré map restricted to energy level is Twist map $\times$ normally hyperbolic.



Central mfld
2 dim

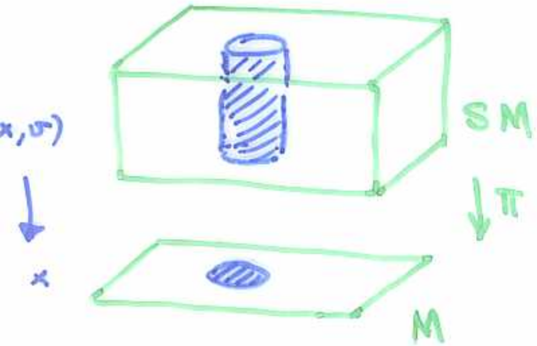


$\Longrightarrow$ homoclinic orbits

Newhouse thm uses the closing lemma.

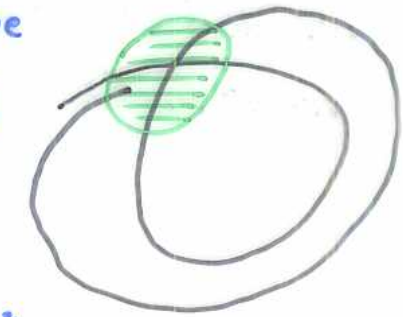
Closing Lemma is not known for geodesic flows.

reason : Proof uses local perturbations.



Perturbations of riemannian metrics $g_{ij}(x)$ are never local in phase space = SM.

"the orbit to close could have passed through the cylinder before comming back"



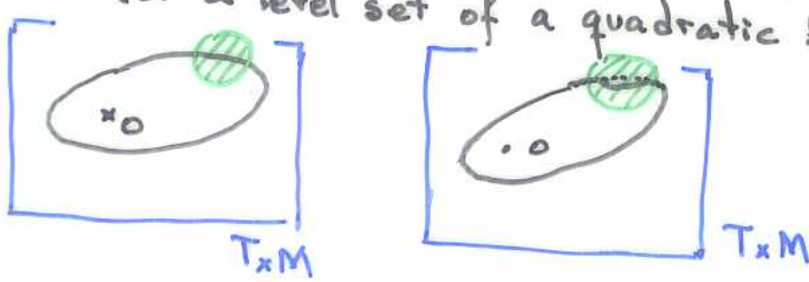
Newhouse theorem for geodesic flows is only known for $M = S^2$ or $\mathbb{RP}^2$

ETDS 2004.

General Finsler Metrics

= norm $\| \ \|_x$ on tangent spaces $T_x M$

unit sphere does not need to be symmetric (or a level set of a quadratic form)



- Closing Lemma holds
- Newhouse theorem should hold.

INGREDIENTS OF THE PROOF

① Kupka-Smale Theorem (for Geod. Flows)

M^{n+1}

$J^k_s(n) = \left\{ \begin{array}{l} \text{k-Jets of symplectic} \\ \text{diffeos } f : (\mathbb{R}^{2n}, 0) \circlearrowleft \end{array} \right\}$

$Q \subset J^k_s(n)$ is invariant iff

$$\sigma Q \sigma^{-1} = Q \quad \forall \sigma \in J^k_s(n)$$

$\mathcal{R}^r(M) = C^\infty$ riem. metrics on M with C^r topology

Theorem

If $Q \subset J^k_s(n)$ is open, dense and invariant

$\Rightarrow \forall r \geq k+1 \;\; \exists \mathcal{G} \subset \mathcal{R}^r(M)$ residual s.t.

(a) [Anosov, Klingenberg-Takens]
Poincaré maps of all periodic orbits of ϕ^g are in Q.

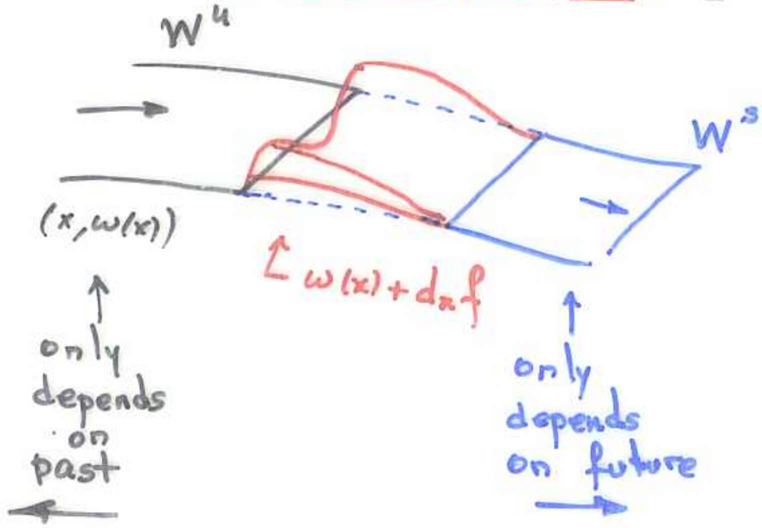
(b) All heteroclinic intersections are transversal.

OBS:

(a) Also holds for Q residual and invariant.

(b): Donnay for $n=2$, Petroll $n>2$ show how to perturb a single non-transverse intersection.
But perhaps this is not enough.

Simple Proof of (b) [$\pitchfork$ inters.]:



- W^s is Lagrangian in (T^*M, ω_0)
- Choose place where is locally a lagrangian graph
- Deform to another lagrangian graph (by adding a $d_x f$)

  $\omega_0 = dp \wedge dx$ fixed cannonical sympl. form.
- Change metric s.t. $H(\text{new } W^s) = 1$

$\Rightarrow$ New W^s is invariant.

② Elliptic Fixed Points

Symplectic Diffeomorphism $F : (\mathbb{R}^{2n}, 0) \circlearrowleft$

will be Poincaré map of closed orbit.

elliptic periodic point := non-hyperbolic.

If q-elliptic $\Rightarrow \exists$ $2q$-dim. central manifold which is normally hyperbolic.

We choose $Q \subset J^3_s(n)$: 3-Jets of sympl. C^∞ diffeos $F : (\mathbb{R}^{2n}, 0) \circlearrowleft$

s.t. map restricted to central manifold is "weakly monotonous" twist map.

i.e. (a) Elliptic eigenvalues $\rho_1, \dots, \rho_q, \bar{\rho}_1, \dots, \bar{\rho}_q$ are 4-elementary:

$$1 \le \sum_{i=1}^{q} |\nu_i| \le 4 \implies \prod_{i=1}^{q} \rho_i^{\nu_i} \ne 1 .$$

(b) Birkhoff normal form

$$Z_k = e^{2\pi i \phi_k} + f_k(z)$$

$$\phi_k = a_k + \sum_{l=1}^{q} \beta_{kl} |z_l|^2$$

satisfies $\det[\beta_{kl}] \ne 0$.

Using techniques of Moser, Herman, M.C. Arnaud

Theorem: If $F : (\mathbb{R}^{2n}, 0) \circlearrowleft$ germ of sympl. diffeo

s.t. (a) F is Q-Kupka-Smale.

(b) 0 is elliptic fixed point.

$\Rightarrow$ F has a 1-elliptic periodic point.

In particular, F has a $\pitchfork$ homoclinic orbit.

③ **Rademacher Theorem**

$\exists\ \mathcal{D} \subset \mathcal{R}^\infty(M)$ Residual set s.t.

$g \in \mathcal{D} \Rightarrow (M,g)$ has infinitely many prime closed geodesics.

Moreover, one can take

$\mathcal{D}$ = bumpy metrics = eigenvalues of Poincaré maps are not roots of 1.

④ **Theory of Dominated Splittings** [Mañé]

"If one can not perturb in C^2 topology to create an elliptic periodic orbit

$\Rightarrow$ closure of hyperbolic per. orbits is uniformly hyperbolic."

[$\Rightarrow$ (Spectral Decomposition Thm) contains a horseshoe]

Theory of Dominated Splittings

$Sp(n) :=$ symplectic linear isom. of $\mathbb{R}^{2n}$

sequence $\xi : \mathbb{Z} \to Sp(n)$ is periodic if
$\exists m \quad \xi_{i+m} = \xi_i \quad \forall i \in \mathbb{Z}$

will be time 1 Poincaré map

A Periodic sequence ξ is hyperbolic if
$\prod_{i=1}^{m} \xi_i$ is hyperbolic.

Family of periodic sequences $\xi = \{\xi^\alpha\}_{\alpha \in \mathcal{a}}$
is bounded if $\exists B > 0 \quad \|\xi_i^\alpha\| < B \quad \forall i \in \mathbb{Z}, \forall \alpha \in \mathcal{a}$
is hyperbolic if ξ^α is hyp. $\forall \alpha \in \mathcal{a}$.

Families $\xi = \{\xi^\alpha\}_{\alpha \in \mathcal{a}}$, $\eta = \{\eta^\alpha\}_{\alpha \in \mathcal{a}}$ are
periodically equivalent iff $\forall \alpha$ ξ^α, η^α have same periods.

Families ξ, η period. equiv. define

$$\|\xi - \eta\| := \sup \{ \|\xi_n^\alpha - \eta_n^\alpha\| : \alpha \in \mathcal{a}, n \in \mathbb{Z} \}$$

This determines how to perturb:
up to a fixed amount in each
time 1 - Poincaré map.

$\Rightarrow$ Following theorem would be usefull
only in C^1-topology of flow
$= C^2$-topology of metric (or Hamiltonian)

Family ξ is stably hyperbolic iff
$\exists \epsilon > 0$ s.t. If η family period. equiv. to ξ
$\|\eta - \xi\| < \epsilon \implies \eta$ is hyperbolic.

Family ξ is uniformly hyperbolic iff
$\exists M > 0$ s.t.

$$\Big\| \prod_{i=0}^{M} \xi^{\alpha}_{i+j}\Big|_{E^s_j(\xi^\alpha)} \Big\| < 1/2 \,, \quad \Big\| \Big(\prod_{i=0}^{M} \xi^{\alpha}_{i+j}\Big|_{E^u_j(\xi^\alpha)} \Big)^{-1} \Big\| < 1/2$$

$$\forall \alpha \in \mathcal{a}, \quad \forall j \in \mathbb{Z}.$$

Theorem

ξ Bounded periodic family is stably hyperbolic $\implies \xi$ uniformly hyperbolic.

Remark:

- Families in $Sp(n)$: stably hyp $\Rightarrow$ unif. hyp.
- Families in $GL(\mathbb{R}^n)$: stably hyp $\Rightarrow$ dominated splitting

i.e.

$$\Big\| \prod^{M} \xi^{\alpha}_{i+j}\Big|_{E^s} \Big\| \cdot \Big\| \Big(\prod^{M} \xi^{\alpha}_{i+j}\Big|_{E^u} \Big)^{-1} \Big\| < 1/2$$

⑥ Perturbation Lemma: "Franks Lemma".

Example: statement for Diffeos $f: M \to M$.

$\exists \epsilon_0 > 0 \quad \forall \epsilon \in [0, \epsilon_0] \quad \exists \delta > 0$ s.t. if

$\mathcal{F} = \{x_1, \dots, x_N\} \subset M$ any finite set

U any neighbourhood of $\mathcal{F}$

$A_i \in L(T_{x_i} M, T_{f(x_i)} M)$ "candidate for Df"

$\| Df(x_i) - A_i \| < \epsilon$

$\Rightarrow \quad \exists g \in \mathrm{Diff}(M)$ s.t.

$$g|_{M \setminus U} = f|_{M \setminus U}$$

$$g(x_i) = f(x_i) \quad \forall x_i \in \mathcal{F}$$

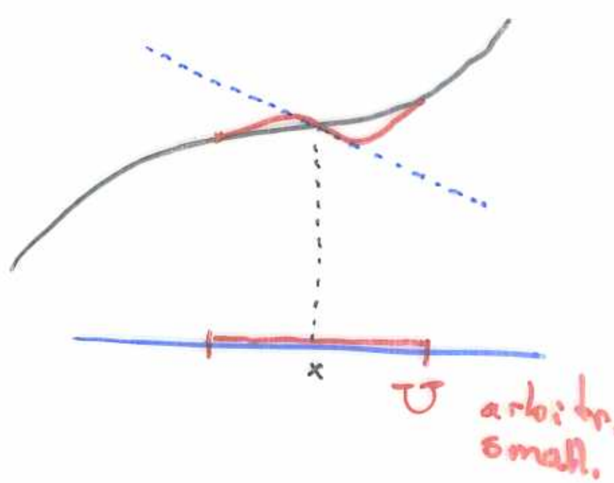
$$Dg(x_i) = A_i$$

$$\| f - g \|_{C^1} < \delta$$

arbitrarily small support

no size problem as in closing lemma.

Example: dimension 1



Analogous for geodesic flows:

realize any perturbation in $Sp(n)$ of a fixed distance of the derivative of the Poincaré map of any geodesic segment of length 1

- fixing the geodesic
- With support in an arbitrarily narrow strip U
- Outside small neighb. of given finitely many transversal segments

By a metric which is C^2 close.

U

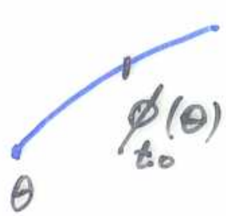
The perturbation is done on nbhd of one point.

Following result allowed to pass from dim 2 to dim $n \geq 2$

Theorem $\exists\, \mathcal{G} \subset \mathcal{R}^\infty(M)$ residual s.t. $\forall g \in \mathcal{G}$ $\forall \theta \in SM$ $\exists\, t_0 \in [0, 1/2]$ s.t. sectional curvature matrix

$$K_{ij}(\theta) = \langle R(\theta, e_i)\theta, e_j \rangle$$

has no repeated eigenvalues.



The Perturbation Lemma

Derivative of the geodesic flow

$$d\phi_t(J(0), \dot{J}(0)) = (J(t), \dot{J}(t))$$

$J(t)$ = Jacobi field orthogonal to the geodesic $\gamma = \pi \circ \phi_t(\theta)$.

Jacobi Equation: $\ddot{J} + K(t)\, J = 0$

$$K(u,v) = \langle R(u,\dot{\gamma})v, \dot{\gamma} \rangle$$

① Can change the Jacobi eq. at will

Use Fermi coordinates:

$e_0 = \dot{\gamma}$, $e_1, \dots, e_n$ = parallel transp. of orthonormal basis along γ.

$$F(t = x_0, x_1, \dots, x_n) = \exp\left(\sum_{i=1}^{n} x_i\, e_i(t) \right)$$

exp for a fixed initial metric g_0

Our general perturbation of the metric g^0 is:

$$g_{00}(t,x) = [g^0(t,x)]_{00} + \sum_{i,j=1}^{n} \alpha_{ij}(t,x)\, x_i\, x_j$$

$$g_{ij}(t,x) = [g^0(t,x)]_{ij} \qquad \text{if } (i,j) \neq (0,0)$$

This perturbation:

(1) Preserves the geodesic γ.

(2) Preserves the metric along γ.
(orthogonal vect. fields along γ are still $\perp$)

(3) Changes the curvature along γ by

$$K(t) = K_0(t) - \alpha(t,x)$$

(4) If the perturbation term is

$$x^* \alpha\, x = \varphi(x)\; x^* P(t)\, x$$

$\leftarrow$ bump funct. in $x_1, \dots, x_n$

and $\mathrm{supp}(\varphi)$ is sufficiently small

$$\Rightarrow \quad \| x^* \alpha\, x \|_{C^2} \sim \| P(t) \|_{C^0}$$

We will use

$$x^{*} \alpha\, x = h(t)\, \varphi(x)\, x^{*} P(t)\, x$$

$\varphi(x)$ = bump function in $x_1, \dots, x_n$

$h(t)$ = approximation of characteristic function of $[0,1] \setminus F^{-1}(\mathcal{T}_i)$

i.e. $0 \le h(t) \le 1$, $\operatorname{supp}(h) \cap \left(\begin{smallmatrix}\text{intersecting}\\ \text{points}\end{smallmatrix}\right) = \emptyset$

$$\int_0^1 h \geq 1 - \epsilon$$

[only $\|h\|_{C^0} \lesssim 1$ counts if $\operatorname{supp}(\varphi)$ small]

$$P(t) = a\,\delta(t) + b\,\delta'(t) + c\,\delta''(t) + d\,\delta'''(t)$$

$a, b, c, d \in \mathrm{Sym}(n \times n) =: S(n)$ $\quad d_{ii} = 0 \quad S^{*}(n)$

$\delta(t)$ = approximation of Dirac δ at some point τ near $1/2$ where $K(t)$ has no repeated eigenvalues:

$$\min_{i \neq j} |\lambda_i - \lambda_j| > \eta = \eta(\mathcal{U}) > 0.$$

← nbhd of g_0.

(2) Estimate the perturbation in the solutions of the Jacobi equation.

$$\ddot{J} + K(t)\,J = 0$$

$$\begin{bmatrix} J \\ \dot{J} \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix}}_{\mathbb{A}} \begin{bmatrix} J \\ \dot{J} \end{bmatrix}$$

$$\mathring{X} = \mathbb{A} X, \quad X \in \mathbb{R}^{n \times n}$$

$$X(0) = I \implies X(t) = d\phi_t \leftarrow \text{Fundamental solution}$$

OBS:

- Can only perturb on K not on whole matrix $\mathbb{A}$
- Only perturbations $K \longmapsto K + \alpha$ ← symmetric matrices

[because it was $x^* \alpha x$]

The solutions X are symplectic linear maps

$$Sp(n) = \{ X \in \mathbb{R}^{n \times n} \mid X^* J X = J \}, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

$$T_X\, Sp(n) = \{ XY \mid Y^* J + J Y = 0 \}$$

$$= \left\{ XY \;\middle|\; Y = \begin{bmatrix} \beta & \gamma \\ \alpha & -\beta^* \end{bmatrix} \quad \begin{matrix} \alpha, \gamma \text{ symmetric} \\ \beta \text{ arbitrary} \end{matrix} \right\}$$

Strategy:

Think on 1-parameter family of metrics $s \mapsto g_s$

$$s \longmapsto K_s(t) = K(t) + s\,\alpha(t)$$

$$s \longmapsto X_s(t) = d\phi_t^{g_s}$$

$\alpha(t) = \alpha(t, E)$ $\qquad E = (a,b,c,d) \in S(n)^3 \times S^{*}(n)$

$S(n)$: sym$(n \times n)$ $\qquad S^{*}(n)$: sym$(n \times n)$, diag $\equiv 0$

same dim as $T_X Sp(n)$

Take the derivative

$$Z_s = \frac{dX_s}{ds} = \frac{d}{ds}\left(d\phi_t^{s}\right)$$

Prove that

$$\| Z_s(1) \| \geq k \| E \| \sim k \, \| x^{*} \alpha x \|_{C^2}$$

with $k = k(\mathcal{U})$

uniform for every geod. segment of length 1 and $\forall g \in \mathcal{U}$

$\Rightarrow$ $\{ d\phi_1^{g} \mid g \in \mathcal{U} \}$ covers a neighbourhood of the original linearized Poincaré map $d\phi_1^{g_0}$ of size depending only on the C^2 norm of the perturbation

Derivative of the Jacobi equation

$$\mathring{X}_s = \mathbb{A}_s X_s$$

$$Z = \frac{dX_s}{ds}, \quad \mathbb{A}_s = \mathbb{A} + s\mathbb{B}, \quad \mathbb{B} = \begin{bmatrix} 0 & 0 \\ I(t) & 0 \end{bmatrix}$$

$$\mathring{Z} = \mathbb{A} Z + \mathbb{B} \qquad I(t) = a\delta(t) + b\delta'(t) + \cdots$$

"variation of parameters": $Z = XY$

$$X\mathring{Y} = \mathbb{B} X$$

$$Y(t) = \int_0^t X^{-1} \mathbb{B} X$$

$$Z(1) = X(1) \underbrace{\int_0^1 X^{-1} \mathbb{B}(t) X \, dt}$$

want this to cover $\begin{bmatrix} \beta & \gamma \\ \alpha & -\beta^* \end{bmatrix}$ ← arbitrary

Integrating by parts:

$$\int_0^1 X^{-1} \mathbb{B}(t) X \, dt \approx$$

$$\approx X_\tau^{-1} \left\{ \begin{bmatrix} a & \\ & \end{bmatrix} + \begin{bmatrix} b & \\ & -b \end{bmatrix} + \begin{bmatrix} & -2c \\ -(Kc+cK) & \end{bmatrix} + \begin{bmatrix} -Kd-3dK & \\ & 3Kd+dK \end{bmatrix} \right\} X_\tau$$

↑ symmetric, not arbitrary

To solve

$$b - (Kd + 3dK) = \beta$$

↑ symm ↑ sym ↑ arbitrary

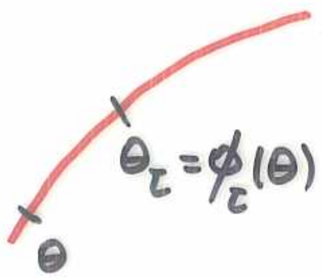
is equiv. to solve $Ke - eK = f$

↑ sym ↑ antisym

may not have solution unless K has no repeated eigenvalues

A generic condition on the curvature

Theorem $\exists\, \mathcal{G} \subset \mathcal{R}^\infty(M)$ residual s.t. $\forall g \in \mathcal{G}$ $\forall \theta \in SM$ $\exists\, \tau \in [0, 1/2]$ s.t. the Jacobi matrix



$$K_{ij}(\theta_\tau) = \langle R(\theta_\tau, e_i)\theta_\tau, e_j \rangle$$

has no repeated eigenvalue.

- Why need this and not just a preliminary perturbation?

preliminary perturb. to separate the eigenvalues $\longrightarrow$ Franks lemma depends on amount of separation of e.v.'s

$$g_0 \longrightarrow g_1$$

Strategy: Use a transversality argument.

Know: can perturb Jacobi matrix (curvature) at will.

$$\Sigma = \{ A \in \underset{\substack{\shortparallel \\ S(n)}}{\mathrm{Sym}(n\times n)} \mid A \text{ has repeated eigenvalues} \}$$

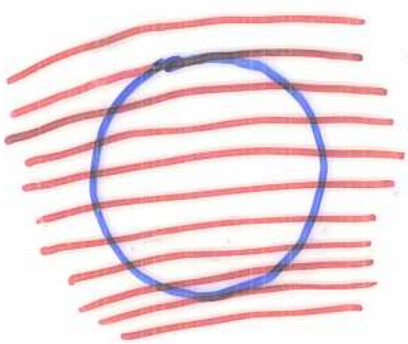
it is an algebraic set <u>with</u> singularities

$$A \in \Sigma \iff \det P_A(A) = \prod (\lambda_i - \lambda_j)^2 = 0.$$

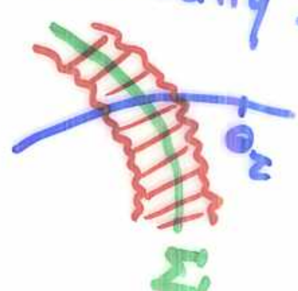
$$P_A(x) = \det [xI - A].$$

Enough to show that geodesic vector field "crosses Σ transversally".

<u>Example</u>:



- Flow in $\mathbb{R}^2$ without sing.
- $\Sigma = S^1$



- Can ask that a chosen orbit segment is $\pitchfork S^1$ <u>but not all</u>.

- Can ask that tangency is not of order 2.

If Σ were a smooth manifold:

$J^k\Sigma$ = k-jets of curves inside Σ.

$$\dim J^k\Sigma = (k+1)\dim\Sigma$$

coefs Taylor series $t \mapsto a_0 + a_1 t + \dots + a_k t^k$
in local chart $\qquad a_i \in \mathbb{R}^\sigma$, $\sigma = \dim\Sigma$

$$\dim J^k S(n) = (k+1)\dim S(n)$$

$$\operatorname{codim}_{S(n)} \Sigma = r \geq 1$$

$$\operatorname{codim}_{J^k S(n)} J^k\Sigma = (k+1)r \longrightarrow \infty$$

when $k \to \infty$

$$F : \mathcal{R}^\infty(M) \times SM \times]0,1[\longrightarrow J^k S(n)$$

$$(g, \theta, \tau) \longmapsto J^k_\tau\, K(g, \phi_t\theta)$$

Jacobi matrix

k-jet at $t = \tau$

If $F \pitchfork J^k\Sigma$

$\Rightarrow \exists$ residual $\mathcal{G} \subset \mathcal{R}^\infty(M)$ s.t.

$$g \in \mathcal{G} \Rightarrow F(g,\cdot,\cdot) \pitchfork J^k\Sigma.$$

k large $\Rightarrow$ $\operatorname{codim} J^k\Sigma > \dim(SM \times]0,1[)$

$\pitchfork$ $\Rightarrow$ no intersection

+ compacity argument $\Rightarrow$ required bounds on eigenvalues

use $$\min_{\theta \in SM} \max_{t \in [0,1]} \prod |\lambda_i - \lambda_j|^2 > 0$$ when $\pitchfork$

But Σ has singularities

Algebraic Jet space

$\mathcal{L}_k(\Sigma)$ = polynoms $a_0 + a_1 t + \dots + a_k t^k = p(t)$
s.t. $f \circ p(t) \equiv 0 \pmod{t^{k+1}}$

Arc space

$\mathcal{L}_\infty(\Sigma)$ = formal power series $p(t)$
s.t. $f \circ p \equiv 0$

$\pi_k : \mathcal{L}_\infty(\Sigma) \to \mathcal{L}_k(\Sigma)$ truncation

$\mathcal{L}_k(\Sigma)$ is an algebraic variety.

$\pi_k(\mathcal{L}_\infty(\Sigma)) \subset \mathcal{L}_k(\Sigma)$ is a finite union of algebraic subsets.

(it is "constructible")

$J^k\Sigma$ = k-jets of C^∞ curves in Σ

$\Longrightarrow$ $$J^k\Sigma \subset \pi_k(\mathcal{L}_\infty(\Sigma)) \subset \mathcal{L}_k(\Sigma).$$

Denef & Loeser:

$$\dim \pi_k(\mathcal{L}_\infty(\Sigma)) \leq (k+1)\dim\Sigma.$$

(same bound as in smooth case).