WEAK SOLUTIONS OF THE HAMILTON-JACOBI EQUATION FOR TIME PERIODIC LAGRANGIANS

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ABSTRACT. In this work we prove the existence of Fathi's weak KAM solutions for periodic Lagrangians and give a construction of all of them.

0. INTRODUCTION AND STATEMENT OF RESULTS

Let M be a closed connected manifold, TM its tangent bundle. Let $L : TM \times \mathbb{R} \to \mathbb{R}$ be a C^∞ Lagrangian. We will assume for the Lagrangian the hypothesis of Mather's seminal paper [9]. The Lagrangian L should be:

(1) *Convex.* The Lagrangian L restricted to T_xM, in linear coordinates should have positive definite Hessian.

(2) *Superlinear.* For some Riemannian metric we have

$$\lim_{|v|\to\infty} \frac{L(x,v,t)}{|v|} = \infty,$$

uniformly on x and t.

(3) *Periodic.* The Lagrangian should be periodic in time, i.e.

$$L(x,v,t+1) = L(x,v,t),$$

for all x, v, t.

(4) *Complete.* The Euler Lagrange flow associated to the Lagrangian should be complete.

Let $\mathcal{M}(L)$ be the set of probabilities on the Borel σ-algebra of $TM \times \mathbb{S}^1$ that have compact support and are invariant under the Euler-Lagrange flow ϕ_t.

The *action* of $\mu \in \mathcal{M}(L)$ is defined by

$$A_L(\mu) = \int L\, d\mu.$$

Mather defined the function $\alpha : H^1(M,\mathbb{R}) \to \mathbb{R}$ as

$$\alpha([\omega]) = -\min\left\{ \int (L-\omega)\, d\mu :\ \mu \in \mathcal{M}(L) \right\}. \tag{1}$$

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For any k in $\mathbb{R}$ define the $(L+k)$-action of an absolutely continuous curve $\gamma : [a,b] \to M$ as

$$A_{L+k}(\gamma) = \int_a^b (L+k)(\gamma(\tau), \dot\gamma(\tau), \tau)\, d\tau$$

For t in $\mathbb{R}$ we denote by $[t]$ the corresponding point in $\mathbb{S}^1$. For any pair of points $(x,[s]),(y,[t])$ on $M \times \mathbb{S}^1$ and n a non negative integer, define $\mathcal{C}((x,[s]),(y,[t]);n)$ as the set of absolutely continuous curves $\gamma : [a,b] \to M$ with $\gamma(a) = x$ and $\gamma(b) = y$ such that $[a] = [s]$ and $[b] = [t]$, and the integer part of $b-a$ is n.

Let Φ_k^n be the real function defined on $M \times \mathbb{S}^1 \times M \times \mathbb{S}^1$ as

$$\Phi_k^n((x,[s]),(y,[t])) = \min_{\gamma \in \mathcal{C}((x,[s]),(y,[t]);n)} \{A_{L+k}(\gamma)\}.$$

so that $\Phi_k^n = \Phi_0^n + kn$.

Then the *action functional* is defined by

$$\Phi_k = \inf_n \Phi_k^n,$$

and the *Extended Peierls barrier* by

$$h_k = \liminf_{n\to\infty} \Phi_k^n.$$

Thus $\Phi_k \leq h_k$.

A curve $\gamma : [a,b] \to M$ will be called *closed* if $\gamma(a) = \gamma(b)$ and $b-a$ is an integer. In analogy to the autonomous case [8], [2], there is a critical value $c(L)$ given by the following proposition:

1. **Proposition.**

(1) *If $k < c(L)$, then $\Phi_k((x,[s]),(y,[t])) = -\infty$, for all $(x,[s]),(y,[t])$ on $M \times \mathbb{S}^1$*

(2)

$$c(L) = \min\{k : \int_\gamma L + k \geq 0 \text{ for all closed curves } \gamma\}$$

(3) *If $k \geq c(L)$, then $\Phi_k((x,[s]),(y,[t])) > -\infty$ for all $(x,[s]),(y,[t])$ on $M \times \mathbb{S}^1$.*

(4) *In terms of Mather's α function we have*

$$c(L) = -\min\Big\{\int L\,d\mu : \mu \text{ is an invariant probability}\Big\} \tag{2}$$

$$= \alpha(0) \tag{3}$$

Invariant probabilities realizing the infimum above are called minimizing measures.

From now on, set $c = c(L)$.

In contrast with the autonomous case, the action potential Φ_c may fail to be continuous and to satisfy the triangle inequality. However, for the extended Peierls barrier we shall prove the following:

2. **Proposition.**

(1) *If* $k < c$, $h_k \equiv -\infty$.
(2) *If* $k > c$, $h_k \equiv \infty$.
(3) h_c *is finite.*
(4) $h_c((x,[s]),(z,[\tau])) \leq h_c((x,[s]),(y,[t])) + \Phi_c((y,[t]),(z,[\tau]))$.
(5) h_c *is Lipschitz.*

Let $H(x,p,t)$ be the Hamiltonian associated to the Lagrangian;

$$H : T^*M \times \mathbb{R} \to \mathbb{R}$$

$$H(x,p,t) = \max_{v \in T_xM} pv - L(x,v,t) \tag{4}$$

In [4] the critical value or $\alpha(0)$ for the autonomous case is characterized as

$$\begin{aligned} c(L) &= \inf_{f \in C^\infty(M,\mathbb{R})} \sup_{x \in M} H(x, d_x f) \\ &= \inf\{k \in \mathbb{R} : \text{ there exists } f \in C^\infty(M,\mathbb{R}) \text{ such that } H(df) < k\}, \end{aligned}$$

This can be restated in physical terms, by saying that $c(L)$ is either the infimum of the values of $k \in \mathbb{R}$ for which there is an exact Lagrangian graph with energy less than k, or the infimum of the values of $k \in \mathbb{R}$ for which there exist smooth solutions of the Hamilton-Jacobi inequality $H(df) < k$.

The second interpretation has a natural generalization. We will prove in section 2 the following

3. **Theorem.** *The critical value,* $c(L)$ *or* $\alpha(0)$ *is characterized as the infimum of* k *such that there exists a subsolution* $f : M \times \mathbb{S}^1 \to \mathbb{R}$ *of the Hamilton Jacobi equation,*

$$d_t f + H(x, d_x f, t) \leq k.$$

We can recover the previous interpretation by using the autonomous Hamiltonian $\mathbb{H}(x,p,t,e) = H(x,p,t) + e$ on $T^*(M \times \mathbb{S}^1)$. Then $df = (d_x f, d_t f)$ is an exact Lagrangian graph and $c(L) = \inf_u \sup_{(x,t)} \mathbb{H}(d_{(x,t)}u)$. The results in [4] can not be directly applied to this case because the Hamiltonian $\mathbb{H}$ does not come from a Lagrangian.

The other values of Mather's alpha function can be similarly characterized by recalling that $\alpha([\omega]) = c(L - \omega)$ and that the Hamiltonian of $L - \omega$ is $(x,p,t) \mapsto H(x, p + \omega, t)$.

In corollary 14 we observe that differentiable solutions may only exist when $k = c(L)$.

4. **Definition.** Following Fathi [6] we say that $u : M \times \mathbb{S}^1 \to \mathbb{R}$ is a forward weak KAM solution if

(1) u is $L + c$ dominated, i.e.
$$u(y, [t]) - u(x, [s]) \leq \Phi_c((x, [s]), (y, [t])).$$
We use the notation $u \prec L + c$.

(2) For every $(x, [s]) \in M \times \mathbb{R}$ there exists a curve $\gamma : (s, \infty) \to M$ such that $u(\gamma(t), [t]) - u(x, [s]) = A_{L+c}(\gamma|_{[s,t]})$, in that case we say that γ *realizes* u.

Similarly $u : M \times \mathbb{S}^1 \to \mathbb{R}$ is a backward weak KAM solution if it is dominated and for every $(x, [s]) \in M \times \mathbb{S}^1$ there exists a curve $\gamma : (-\infty, s) \to M$ such that $u(x, [s]) - u(\gamma(t), [t]) = A_{L+c}(\gamma|_{[t,s]}))$

Let $\mathcal{S}^-$ (resp. $\mathcal{S}^+$) be the set of *backward* (resp. *forward*) weak KAM solutions.

A point $(x, v, [s]) \in TM \times \mathbb{S}^1$ is a positive (resp. negative) *semistatic* point if the solution $\gamma = \gamma_{(x,v,s)}$ of the Euler-Lagrange equation with initial conditions $(x, v, [s])$, satisfies for all t
$$A_{L+c}(\gamma|_{[s,t]}) = \Phi_c((x, [s]), (\gamma(t), [t]));$$
(resp. $A_{L+c}(\gamma|_{[t,s]}) = \Phi_c((\gamma(t), [t]), (x, [s]))$ for all t).

A point $(x, v, [s]) \in TM \times \mathbb{S}^1$ is a *static* point if it is positive semistatic and
$$A_{L+c}(\gamma|_{[s,t]}) = -\Phi_c((\gamma(t), [t]), (x, [s])).$$
It turns out that if a point is static then its whole orbit under the Euler-Lagrange flow is static.

We denote by Σ^+ the set of positive semistatic points.

For a forward weak KAM solution u we define its *forward basin* as
$$\Gamma_0^+(u) = \{(x, v, [s]) \in \Sigma^+ : \\ u(\gamma_{(x,v,s)}(t), [t]) - u(x, [s]) = \Phi_c((x, [s]), (\gamma_{(x,v,s)}(t), [t]))\, \forall t > s\};$$
and define its *cut locus* by $\pi\big(\Gamma_0^+(u) \setminus \Gamma^+(u)\big)$, where $\pi : TM \times \mathbb{S}^1 \to M \times \mathbb{S}^1$ is the proyection,
$$\Gamma^+(u) = \bigcup_{t>0} \phi_t\big(\Gamma_0^+(u)\big),$$
and ϕ_t is the Euler-Lagrange flow. It is easy to see that the sets Σ^+ and $\Gamma_0^+(u)$ are positively invariant and so $\Gamma^+(u) \subset \Gamma_0^+(u)$. Similarly, define the backward basins $\Gamma_0^-(u)$, $\Gamma^-(u)$ for $u \in \mathcal{S}^-$.

The relevance of weak KAM solutions is that they have several properties, including those given by the following theorem.

5. **Theorem.** *If $u : M \times \mathbb{S}^1 \to \mathbb{R}$ is a weak KAM solution then*

(1) *u is Lipschitz and satisfies the Hamilton Jacobi equation*
$$H(x, d_x u, t) + d_t u = c$$

at any point of differentiability. Moreover, $d_x u$ and $\dot\gamma$ are Legendre conjugate.

(2) *Graph property: $\pi : \Gamma^+(u) \to M \times \mathbb{S}^1$ is injective and its inverse is Lipschitz.*

(3) *u is differentiable on $\pi(\Gamma^+(u))$.*

Observe that since a weak KAM solution u is Lipschitz, by Rademacher's theorem it is differentiable Lebesgue almost everywhere.

Define the *Aubry set* $\mathcal{A}$ as

$$\mathcal{A} := \{\, (x,[t]) \in M \times \mathbb{S}^1 \mid h_c\big((x,[t]),(x,[t])\big) = 0 \,\}.$$

We define an equivalence relation on $\mathcal{A}$ by $(x,[s]) \sim (y,[t])$ if and only if

$$\Phi_c((x,[s]),(y,[t])) + \Phi_c((y,[t]),(x,[s])) = 0.$$

The equivalence classes of this relation are called *static classes*.

Let $\mathfrak{A}$ be the set of static classes. For each static class $\Gamma \in \mathfrak{A}$ choose a point $(p,[s]) \in \Gamma$ and let $\mathbb{A}$ be the set of such points.

6. **Remark.** Observe that by item 4 of proposition 2, if $(p,[\tau]) \in \mathcal{A}$ then

$$h_c((p,[\tau]),(x,[t])) = \Phi_c((p,[\tau]),(x,[t])).$$

7. **Theorem.** *The map $\{f : \mathbb{A} \to \mathbb{R} \mid f \text{ dominated } \} \longrightarrow \mathcal{S}^-$*

$$f \longmapsto u_f(x,[t]) = \min_{(p,[s])\in\mathbb{A}} f(p,[s]) + h_c((p,[s]),(x,[t])),$$

and the map $\{f : \mathbb{A} \to \mathbb{R} \mid f \text{ dominated } \} \longrightarrow \mathcal{S}^+$

$$f \longmapsto v_f(x,[t]) = \max_{(p,[s])\in\mathbb{A}} f(p,[s]) - h_c((x,[t]),(p,[s])),$$

are bijections.

1. The Peierls barrier

We will be using the following lemma due to Mather [9]. We say that an absolutely continuous curve $\gamma : [a,b] \to M$ is a *minimizer* if $A_L(\gamma) \le A_L(\eta)$ for any absolutely continuous curve $\eta : [a,b] \to M$ with $\eta(a) = \gamma(a)$ and $\eta(b) = \gamma(b)$. It turns out that a minimizer is a solution of the Euler-Lagrange equation $\frac{d}{dt} L_v = L_x$.

8. **Lemma.** *There is $A > 0$ such that if $b - a \ge 1$ and $\gamma : [a,b] \to M$ is a minimizer, then $|\dot\gamma(t)| \le A$ for $t \in [a,b]$.*

The proof of most of Propositions 1 and 2 follow standard arguments. We only give the proof of the Lipschitz continuity of h_c.

9. **Lemma.** *Given $(z,[\sigma]) \in M \times \mathbb{S}^1$ define*

$$u(x,[t]) := h_c((z,[\sigma];x,[t]), \quad v(x,[t]) := -h_c((x,[t];z,[\sigma]).$$

Then $u \in \mathcal{S}^-$ and $v \in \mathcal{S}^+$.

Proof: By item 4 of proposition 2, $h(z,[\sigma]),(x,[t])) \le h_c((z,[\sigma]),(y,[s]))+\Phi_c((y,[s]),(x,[t]))$ for all $(y,[s]),(x,[t]) \in M\times\mathbb{S}^1$. Thus $u \prec L+c$.

Given $(x,[t])\in M\times\mathbb{S}^1$ choose sequences $n_k\to+\infty$, $n_k\in\mathbb{Z}$ and $(x,v_k)\in T_xM$ such that

$$h_c((z,[\sigma]),(x,[t])) = \lim_k A_{L+c}\big(\gamma_k|_{[\sigma-n_k,t]}\big),$$

where $\gamma_k(s)=\pi\,\varphi_{s-t}(x,v_k,t)$ is the solution of the Euler-Lagrange equation such that $(\gamma_k(t),\dot\gamma_k(t))=(x,v_k)$. By lemma 8, the norm $\|v_k\|$ is uniformly bounded. Choose a convergent subsequence $v_k\to w$. Let $\eta(s):=\pi\,\varphi_{s-t}(x,w,t)$, then for any fixed $s<0$,

$$\begin{aligned}
h_c((z,[\sigma]),(x,[t])) &\le h_c((z,[\sigma]),(\eta(s),[s])) + A_{L+c}\big(\eta|_{[s,t]}\big)\\
&= \lim_k h_c((z,[\sigma]),(\gamma_k(s),[s])) + A_{L+c}\big(\gamma_k|_{[s,t]}\big)\\
&\le \lim_k A_{L+c}\big(\gamma_k|_{[\sigma-n_k,s]}\big) + A_{L+c}\big(\gamma_k|_{[s,t]}\big)\\
&= h_c((z,[\sigma]),(x,[t])).
\end{aligned}$$

So that $u(x,[t])-u(\eta(s),[s]) = A_{L+c}\big(\eta|_{[s,t]}\big)$ for all $s<0$.

□

For autonomous lagrangians, dominated functions are Lipschitz. In contrast, for time periodic lagrangians the action potential is dominated but it is not continuous at $\big((x,s),(x,s)\big)$ when (x,s) is not in the Aubry set. Nevertheless, we have the following:

10**. Lemma.** *If $u: M\times\mathbb{S}^1\to\mathbb{R}$ is a weak KAM solution (i.e. $u\in\mathcal{S}^+\cup\mathcal{S}^-$) then it is Lipschitz. Moreover the Lipschitz constant does not depend on u.*

Proof: Assume that $u\in\mathcal{S}^-$, the case $u\in\mathcal{S}^+$ is similar. Let $(x,[t_0]),(y,[s_0])\in M\times\mathbb{S}^1$ be nearby points with $|s_0-t_0|<\frac14$. Let $\gamma:[0,1]\to M$ be a length minimizing geodesic joining x to y and let $\tau(r)=t_0+r\,(s_0-t_0)$, $r\in[0,1]$. Fix $\delta>1$ and let $z:[t_0-\delta,t_0]\to M$ be such that

$$u(x,[t_0]) = u(z(t),[t]) + \int_t^{t_0} L(z,\dot z)+c\,dt \qquad \text{for all } t_0-\delta<t\le t_0. \tag{5}$$

For $r\in[0,1]$, let $\eta(r,t)$, $t\in[t_0-\delta,\tau(r)]$, be a locally minimizing solution of (E-L) such that $\eta(r,t_0-\delta)=z(t_0-\delta)$ and $\eta(r,\tau(r))=\gamma(r)$.

Then

$$u(\gamma(r),[\tau(r)]) \le u(z(t_0-\delta),[t_0-\delta]) + \int_{t_0-\delta}^{\tau(r)} L\big(\eta,\tfrac{\partial\eta}{\partial t},t\big)+c\ \ dt.$$

with equality for $r=0$. Substracting the equality (5) at $r=0$, we get that

$$u(\gamma(r),[\tau(r)]) - u(x,[t_0]) \le \int_{t_0-\delta}^{\tau(r)} \big(L+c\big)\,dt - A_{L+c}(z|_{[t_0-\delta,t_0]}). \tag{6}$$

Observe that this formula holds either for $s_0 \le t_0$ or $t_0 \le s_0$. As we shall see below, formula (6) implies that $u(y,s)-u(x,t) \le K\,\big(|s-t|+d(x,y)\big)$ for some fixed $K>0$. Then changing the roles of s and t we get that u is Lipschitz.

Indeed, differentiating the right hand side and integrating by parts, we have

$$\frac{d}{dr}\int_{t_0-\delta}^{t_0+r\,(s_0-t_0)} L\big(\eta,\tfrac{\partial\eta}{\partial t},t\big)+c\;\;dt =$$

$$= \Big[L\big(\eta,\tfrac{\partial\eta}{\partial t},t\big)\big|_{(r,\tau(r))}+c\Big]\,(s_0-t_0)+\int_{t_0-\delta}^{\tau(r)} L_x\,\tfrac{\partial\eta}{\partial r}+L_v\,\tfrac{\partial^2\eta}{\partial t\partial r}$$

$$= \Big[L\big(\eta,\tfrac{\partial\eta}{\partial t},t\big)\big|_{(r,\tau(r))}+c\Big]\,(s_0-t_0)+\frac{\partial L}{\partial v}\big(\eta,\tfrac{\partial\eta}{\partial t},t)\big)\Big|_{(r,\tau(r))}\cdot\frac{\partial\eta}{\partial r}\Big|_{(r,\tau(r))}$$

Observe that since u is dominated the realizing curve z must be a minimizer. By lemma 8, $\|\dot z\|$ is uniformly bounded. By the continuity of the solutions of (E-L) with respect to initial values, $\big\|\frac{\partial\eta}{\partial t}\big\|$ is uniformly bounded. Hence there is a uniform constant $K>0$ (independent of $z(t),x,y,[s_0],[t_0],u$) such that

$$\big|L\big(\eta,\tfrac{\partial\eta}{\partial t},t\big)+c\big|\le K \qquad\text{and}\qquad \big\|\tfrac{\partial L}{\partial v}(\eta,\tfrac{\partial\eta}{\partial t},t)\big\|<K.$$

Since $\displaystyle\frac{\partial\eta}{\partial r}\Big|_{(r,\tau(r))}=\dot\gamma(r)$, we get that

$$\frac{d}{dr}\left[\int_{t_0-\delta}^{\tau(r)}[L+c]\;\;-A_{L+c}(z)\right]\le K\,|s_0-t_0|+K\;\|\dot\gamma\|\,.$$

The value of the right hand side of (6) is 0 at $r=0$. Integrating this inequality,

$$u(y,[s_0])-u(x,[t_0])\le K\;\big[|s_0-t_0|+d(x,y)\big].$$

Interchanging the roles of $(x,[t_0])$ and $(y,[s_0])$ we obtain that the function u is Lipschitz.

□

Combining lemmas 9 and 10 we get that the functions $f,g : M\times\mathbb{S}^1\to\mathbb{R}$, $f(y,[t])=h_c((x,[s]),(y,[t]))$ and $g(x,[s])=h_c((x,[s]),(y,[t]))$ are Lipschitz. This implies that h_c is Lipschitz.

2. Subsolutions of the Hamilton Jacobi equation

Following the same ideas as in [4], one obtains

11. **Lemma.** *If k is a real number such that there exists a function f in $C^1(M\times\mathbb{S}^1)$ subsolution of the Hamilton Jacobi equation*

$$H(x,d_xf)+d_tf\le k$$

Then $k\ge c(L)$.

12. **Lemma.** *Let $k \geq c(L)$. If $f : M \times \mathbb{S}^1 \to \mathbb{R}$ is differentiable at $(x, [t]) \in M \times \mathbb{S}^1$ and satisfies*

$$f(y, [t_2]) - f(x, [t_1]) \leq \Phi_k(x, [t_1], y, [t_2])$$

for all y in a neighbourhood of x, then $H(x, d_x f) + d_t f \leq k$.

13. **Proposition.** *For any $k > c(L)$ there exists $f \in C^\infty(M \times \mathbb{S}^1, \mathbb{R})$ such that $H(x, d_x f, t) + d_t f < k$.*

We give a proof of the following fact

14. **Corollary.** *If u is a $C^{1+\mathrm{Lip}}$ global solution of the Hamilton-Jacobi equation $u_t + H(x, u_x, t) = k$, then $k = c(L)$ and u is a weak KAM solution in $\mathcal{S}^- \cap \mathcal{S}^+$.*

Proof: By lemma 11, $k \geq c(L)$. Let $\mathcal{L}_t(x, v) = L_v(x, v, t)$ be the conjugate moment associated to L and let $\xi(x, t)$ be the vector field defined by
$\xi(x, t) = \mathcal{L}_t^{-1}(u_x) \in T_x M$. Then the vector field $(\xi, 1)$ in $M \times \mathbb{S}^1$ is Lipschitz. Let ρ_t be the flow of $(\xi, 1)$ in $M \times \mathbb{S}^1$. From the Hamilton-Jacobi equation we get that

$$d_{(x,[t])} u \cdot (v, 1) = u_x(x, t) \cdot v + u_t \cdot 1 \leq L(x, v, [t]) + k. \tag{7}$$

and that

$$d_{(x,[t])} u \cdot (\xi(x, t), 1) = L(x, \xi(x, t), t) + k \qquad \text{for all } (x, [t]) \in M \times \mathbb{S}^1. \tag{8}$$

Integrating equation (7) along absolutely continuous curves $(\gamma(t), [t])$ in $M \times \mathbb{S}^1$ from $(x, [s])$ to $(y, [t])$, we get that

$$u(y, [t]) - u(x, [s]) \leq \inf_\gamma \oint_\gamma (L + k) = \Phi_k\big((x, [s]), (y, [t])\big).$$

So that $u \prec L + k$.

Also, integrating equation (8), we get that the orbits of ρ_t realize u in the sense of the definition of a weak KAM solution. In particular, the orbits of ρ are global minimizers of the $(L + k)$-action, and thus they are solutions of the Euler-Lagrange equation.

It remains to prove that $k = c(L)$. Let ν be an invariant Borel probability for ρ_t and let μ be its lift to $TM \times \mathbb{S}^1$ using the vectorfield ξ. Then μ is an invariant probability of the Lagrangian flow and, by equation (8),

$$\int (L + k)\, d\mu = \int du \;\; d\mu = 0.$$

This implies that $k \leq c(L)$. Thus $k = c(L)$ and also μ is a minimizing measure. $\square$

3. Weak KAM solutions

Proof of theorem 5:

We first prove item 1. By lemma 10 we have that u is Lipschitz and hence it is differentiable almost everywhere. Let $(x,[t])$ be a point of differentiability, then by lemma 12 we have

$$H(x,d_xu,t)+u_t\le c.$$

Moreover let $\gamma:[t,\infty)\to M$ be such that

$$u(\gamma(s),[s])-u(x,[t])=A_{L+c}(\gamma|_{[t,s]}),$$

$$\lim_{s\to t}\frac{u(\gamma(s),[s])-u(x,[t])}{s-t}=\lim_{s\to t}\frac{1}{s-t}\int_t^s(L+c)(\gamma(s),\dot\gamma(s),[s])ds,$$

so

$$d_xu(x,[t])\dot\gamma+d_tu(x,[t])=L(x,\dot\gamma,t)+c.$$

Therefore

$$c=d_xu\,\dot\gamma-L+d_tu\le H(x,d_xu,t)+d_tu\le c.$$

So u is a solution of the Hamilton Jacobi Equation and d_xu and $\dot\gamma$ are related by the Legendre transformation of L.

□

Proof of the Graph Property:

We need the following lemma due to Mather, a proof of which can be found in [9].

15. **Lemma.** *Given $A>0$ there exists $K>0$ $\varepsilon_1>0$ and $\delta>0$ with the following property: if $|v_i|<A$, $(p_i,v_i,[t_i])\in TM\times\mathbb{S}^1$, $i=1,2$ satisfy $d((p_1,[t_1]),(p_2,[t_2]))<\delta$ and $d((p_1,v_1,[t_1]),(p_2,v_2,[t_2]))\ge K^{-1}d((p_1,[t_1]),(p_2,[t_2]))$ then, if $a\in\mathbb{R}$ and $x_i:\mathbb{R}\to M$, $i=1,2$, are the solutions of L with $x_i(t_i)=p_i$, $\dot x_i(p_i)=v_i$, there exist solutions $\gamma_i:[t_i-\varepsilon,t_i+\varepsilon]\to M$ of L with $0<\varepsilon<\varepsilon_1$, satisfying*

$$\begin{aligned}\gamma_1(t_1-\varepsilon)=&x_1(t_1-\varepsilon)\quad,\quad \gamma_1(t_2+\varepsilon)=x_2(t_2+\varepsilon)\,,\\ \gamma_2(t_2-\varepsilon)=&x_2(t_2-\varepsilon)\quad,\quad \gamma_2(t_1+\varepsilon)=x_1(t_1+\varepsilon)\,,\\ S_L(x_1|_{[t_1-\varepsilon,t_2+\varepsilon]})&+S_L(x_2|_{[t_2-\varepsilon,t_1+\varepsilon]})>S_L(\gamma_1)+S_L(\gamma_2)\end{aligned}$$

We now prove the graph property. Let $(p_1,v_1,[t_1])$, $(p_2,v_2,[t_2])\in\Gamma^+(u)$ and suppose that $K\,d((v_1,[t_1]),(v_2,[t_2]))>d((p_1,[t_1]),(p_2,[t_2]))$, where K is from lemma 15 and the A that we input on lemma 15 is from lemma 8. Let $y_i^+=x_i(t_i+\varepsilon)$, $i=1,2$, and $y_i^-=x_i(t_i-\varepsilon)$ for ε small, then

$$u(y_1^+,[t_1+\varepsilon])-u(y_1^-,[t_1-\varepsilon])=\Phi_c((y_1^-,[t_1-\varepsilon]),(y_1^+,[t_1+\varepsilon]))\tag{9}$$

$$u(y_2^+,[t_2+\varepsilon])-u(y_2^-,[t_2-\varepsilon])=\Phi_c((y_2^-,[t_2-\varepsilon]),(y_2^+,[t_2+\varepsilon]))\tag{10}$$

Then using that $u \prec L + c$ and lemma 15 , we get that

$$\begin{aligned}(11)\quad u(y_2^+,[t_2+\varepsilon]) - u(y_1^-,[t_1-\varepsilon]) + u(y_1^+,[t_1+\varepsilon]) - u(y_2^-,[t_2-\varepsilon]) \\ \le S_{L+c}(\gamma_1) + S_{L+c}(\gamma_2) \\ < S_{L+c}(x_1|_{[t_1-\varepsilon,t_2+\varepsilon]} + S_{L+c}(x_2|_{[t_2-\varepsilon,t_1+\varepsilon]}) \\ = \Phi_c((y_1^-,[t_1-\varepsilon]),(y_1^+,[t_1+\varepsilon]) + \Phi_c((y_2^-,[t_2-\varepsilon]),(y_2^+,[t_2+\varepsilon]).\end{aligned}$$

Which is a contradiction with the sum of (9) and (10). □

Proof of item 3:

Let $(x,[s])$ in $\pi\Gamma^+(u)$, let $(\sigma(\tau),[\alpha(\tau)])$ be a curve on $M\times\mathbb{S}^1$ with $\sigma(0)=x, \alpha(0)=s$. Let γ_s be the curve such that

$$u(\gamma_s(t),[t]) - u(\gamma_s(s-\delta),[s-\delta]) = A_{L+c}(\gamma_s|_{[s-\delta,t]}).$$

Since $(x,[s])$ is in $\pi\Gamma^+(u)$ we can make a backwards variation (γ_τ) of the solution γ_s. That is, $\gamma_\tau : [s-\delta,\alpha(\tau)] \to M$ is a solutions of the Euler -Lagrange equation joining the points $p=\gamma_s(s-\delta)$ and $\sigma(\tau)$.

Since u is dominated we have

$$\begin{aligned} u(\sigma(\tau),[\alpha(\tau)]) - u(x,[s]) &= u(\sigma(\tau),[\alpha(\tau)]) - u(p,[s-\delta]) - (u(x,[s]) - u(p,[s-\delta])) \\ &\le A_{L+c}(\gamma_\tau|_{[s-\delta,\alpha(\tau)]}) - A_{L+c}(\gamma_s|_{[s-\delta,s]}) \\ &= A_{L+c}(\gamma_\tau|_{[s-\delta,s]}) - A_{L+c}(\gamma_s|_{[s-\delta,s]}) + A_{L+c}(\gamma_\tau|_{[s,\alpha(\tau)]})\end{aligned}$$

Dividing by $\tau - s$ and taking limits as τ tends to s and using the fact that γ_τ is a solution of the Euler-Lagrange equation, we obtain

$$\limsup_{\tau\to s} \frac{u(\sigma(\tau),[\alpha(\tau)]) - u(x,[s])}{\tau - s} \le L_v(\dot\gamma_s,s)\cdot\sigma'(0) + L + c(\dot\gamma_s,s)\alpha'(0)$$

Similarly we can make a forward variation to get

$$\liminf_{\tau\to s} \frac{u(\sigma(\tau),[\alpha(\tau)]) - u(x,[s])}{\tau - s} \ge L_v(\dot\gamma_s,s)\cdot\sigma'(0) + L + c(\dot\gamma_s,s)\alpha'(0)$$

□

Proof of theorem 7:

Let $u \in \mathcal{S}^-$, since u is dominated, then

$$u(x,[t]) \le \min_{(y,[r])} u(y,[r]) + \Phi_c((y,[r]),(x,[t])). \tag{12}$$

Let $\gamma :]-\infty,t] \to M$ be such that for all $s \le t$,

$$u(x,[t]) - u(\gamma(s),[s]) = A_{L+c}(\gamma|_{[s,t]}).$$

Then $\gamma(s)$ is semistatic and the minimum in (12) is realized at every point $(\gamma(s),[s])$ with $s<t$. Choose a convergent sequence $(\gamma(s_n),[s_n])\to(p,[\tau])\in M\times\mathbb{S}^1$, with $s_n\to-\infty$. Then by lemma 16 below, $(p,[\tau])$ is in the Pierls set. Therefore, using the continuity of Φ_c at $(p,[\tau])$ (see lemma 17 below) and (12), we have that

$$\begin{aligned} u(x,[t]) &= u(p,[\tau])+\Phi_c((p,[\tau]),(x,[t])) \\ &= \min_{(q,[\sigma])\in\mathcal{A}} u(q,[\sigma])+\Phi_c((q,[\sigma]),(x,[t])). \end{aligned} \tag{13}$$

We show now that it is enough to choose one point on each static class to achieve the minimum on (13). Suppose that $(p,[\tau])$ and $(q,[\sigma])$ are in the same static class. Then

$$\begin{aligned} \Phi_c((q,[\sigma]),(x,[t])) &\le \Phi_c((q,[\sigma]),(p,[\tau]))+\Phi_c((p,[\tau]),(x,[t])) \\ &\le \Phi_c((q,[\sigma]),(p,[\tau]))+\Phi_c((p,[\tau]),(q,[\sigma]))+\Phi_c((q,[\sigma]),(x,[t])) \\ &= \Phi_c((q,[\sigma]),(x,[t])). \end{aligned}$$

So that $\Phi_c((q,[\sigma]),(x,[t]))=\Phi_c((q,[\sigma]),(p,[\tau]))+\Phi_c((p,[\tau]),(x,[t]))$. Moreover,

$$\begin{aligned} u(p,[\tau]) &\le u(q,[\sigma]))+\Phi_c((q,[\sigma],p,[\tau])) \\ &\le u(p,[\tau])+\Phi_c((p,[\tau]),(q,[\sigma]))+\Phi_c((q,[\sigma]),(p,[\tau])) \\ &= u(p,[\tau]). \end{aligned}$$

So that $u(q,[\sigma])+\Phi_c((q,[\sigma]),(p,[\tau]))=u(p,[\tau])$. Thus

$$\begin{aligned} u(q,[\sigma])+\Phi_c((q,[\sigma]),(x,[t])) &= u(q,[\sigma])+\Phi_c((q,[\sigma]),(p,[\tau]))+\Phi_c((p,[\tau]),(x,[t])) \\ &= u(p,[\tau])+\Phi_c((p,[\tau]),(x,[t])). \end{aligned}$$

So that $u=u_f$, with $f=u|_{\mathbb{A}}$.

Observe that by definition, if $f:\mathbb{A}\to\mathbb{R}$ is dominated, then $u_f|_{\mathbb{A}}\equiv f$. This implies that the map $\{f \text{ dominated}\}\mapsto u_f$ is injective.

Finally, it remains to prove that if $f:\mathbb{A}\to\mathbb{R}$ then $u_f\in\mathcal{S}^-$. This follows from lemma 9 and lemma 18 below. ☐

16. **Lemma.** *If $\gamma:]-\infty,t_0]\to M$ is semistatic and $s_n\to-\infty$ is such that $\lim_n\big(\gamma(s_n),[s_n]\big)=(p,[\tau])$ exists. Then $(p,[\tau])$ is in the Aubry set.*

Proof: Let $\varepsilon>0$ be small. Chose $n_0>0$ such that for $n>n_0$, we have

$$|s_n-\tau \mod 1|<\tfrac{\varepsilon}{2} \qquad , \qquad d(\gamma(s_n),p)<\tfrac{\varepsilon}{2}.$$

Let $\lambda_n^-:[\tau,s_n+\varepsilon \mod 1]\to M$ be a minimizer with $\lambda_n^-(\tau)=p$, $\lambda_n^-(s_n+\varepsilon \mod 1)=\gamma(s_n+\varepsilon)$. By lemma 8, $\|\dot\gamma\|$ is uniformly bounded. By the same argument, using the first variation formula, as in proposition 2.c,

$$A_{L+c}(\lambda_n^-)\le K_1\left[\,d\big(\gamma(s_n),p\big)+|s_n+\varepsilon-\tau \mod 1|\,\right]\le 3\,\varepsilon\,K_1.$$

Let $\lambda_n^+ : [s_n - \varepsilon \mod 1, \tau] \to M$ be a minimizer with $\lambda_n^+(s_n - \varepsilon) = \gamma(s_n - \varepsilon)$, $\lambda_n^+(\tau) = p$. Similarly,

$$A_{L+c}(\lambda_n^+) \leq 3\,\varepsilon\, K_1.$$

We have that

$$\begin{aligned} h_c\big((p,[\tau]),(p,[\tau])\big) &\leq \liminf_{N\to\infty} A_{L+c}(\lambda_N^-) + A_{L+c}\big(\gamma|_{[s_N+\varepsilon, s_n-\varepsilon]}\big) + A_{L+c}(\lambda_n^+) \\ &\leq 6\,\varepsilon\, K_1 + \liminf_N A_{L+c}\big(\gamma|_{[s_N+\varepsilon, s_n-\varepsilon]}\big). \qquad (14) \end{aligned}$$

Adding the action of γ on the intervals with endpoints $s_N - \varepsilon < s_N + \varepsilon < s_n - \varepsilon < s_n + \varepsilon$ and using that γ is semistatic on $[s_N - \varepsilon, s_n + \varepsilon]$, we have that

$$\begin{aligned} A_{L+c}\big(\gamma|_{[s_N+\varepsilon, s_n-\varepsilon]}\big) &= \Phi_c\big((\gamma(s_N-\varepsilon), s_N-\varepsilon)\,,\,(\gamma(s_n+\varepsilon), s_n+\varepsilon)\big) \\ &\quad - A_{L+c}\big(\gamma|_{[s_N-\varepsilon, s_N+\varepsilon]}\big) - A_{L+c}\big(\gamma|_{[s_n-\varepsilon, s_n+\varepsilon]}\big). \qquad (15) \end{aligned}$$

Comparing Φ_c with the action of a minimal length geodesic, parameterized by the *small* interval $I = [s_N - \varepsilon \mod 1,\ s_n + \varepsilon \mod 1]$ of length $\varepsilon \leq \ell(I) \leq 3\varepsilon$, with

$$\begin{aligned} \text{speed} &\leq \tfrac{1}{\varepsilon}\, d\big(\gamma(s_N-\varepsilon), \gamma(s_n+\varepsilon)\big) \leq \tfrac{1}{\varepsilon}\, \big[\, \varepsilon\, \|\dot\gamma\| + d\big(\gamma(s_N), \gamma(s_n)\big) + \varepsilon\, \|\dot\gamma\|\, \big] \\ &\leq 2\, \|\dot\gamma\| + 1; \end{aligned}$$

we have that

$$\Phi_c\big((\gamma(s_N-\varepsilon), s_N-\varepsilon)\,,\,(\gamma(s_n+\varepsilon), s_n+\varepsilon)\big) \leq \ell(I)\, \Big[\max_{|v|\leq 2\,\|\dot\gamma\|+1} L + c \Big] \leq 3\,\varepsilon\, K_2.$$

The two actions in (15) are bounded by $2\,(2\varepsilon \cdot K_2)$. Thus, from (15),

$$A_{L+c}\big(\gamma|_{[s_N+\varepsilon, s_n-\varepsilon]}\big) \leq 7\,\varepsilon\, K_2.$$

From (14),

$$0 \leq h_c\big((p,[\tau]),(p,[\tau])\big) \leq 6\,\varepsilon\, K_1 + 7\,\varepsilon\, K_2.$$

Now let $\varepsilon \to 0$. □

17. **Lemma.** *If* $\lim_n (y_n, [s_n]) = (p, [\tau]) \in \mathcal{A}$ *then for all* $(x, [t]) \in M \times \mathbb{S}^1$,

$$\lim_n \Phi_c\big((y_n,[s_n]),(x,[t])\big) = \Phi_c\big((p,[\tau]),(x,[t])\big) = h_c\big((p,[\tau]),(x,[t])\big).$$

Proof: Recall that by remark 6, $h_c((p,[\tau]),(x,[t])) = \Phi_c((p,[\tau]),(x,[t]))$. By item 4 of proposition 2,

$$\begin{aligned} \Phi_c((p,[\tau]),(x,[t])) &= h_c((p,[\tau]),(x,[t])) \\ &\leq h_c((p,[\tau]),(y_n,[s_n])) + \Phi_c((y_n,[s_n]),(x,[t])) \qquad (16) \\ &\leq h_c((p,[\tau]),(y_n,[s_n])) + h_c((y_n,[s_n]),(x,[t])) \\ &\leq h_c((p,[\tau]),(y_n,[s_n])) + h_c((y_n,[s_n]),(p,[\tau])) + \Phi_c((p,[\tau]),(x,[t])) \qquad (17) \end{aligned}$$

Using that h_c is continuous, taking $\lim_n$ on inequalities (16) and (17), we get that $\lim_n \Phi_c((y_n,[s_n]),(x,[t])) = \Phi_c((p,[\tau]),(x,[t]))$.

□

18. **Lemma.**
If $\mathcal{U} \subset \mathcal{S}^-$, *let* $\mathfrak{u}(x,[t]) := \inf_{u\in\mathcal{U}} u(x,[t])$ *then either* $\mathfrak{u} \equiv -\infty$ *or* $\mathfrak{u} \in \mathcal{S}^-$.
If $\mathcal{V} \subset \mathcal{S}^+$, *let* $\mathfrak{v}(x,[t]) := \sup_{v\in\mathcal{V}} v(x,[t])$ *then either* $\mathfrak{v} \equiv +\infty$ *or* $\mathfrak{v} \in \mathcal{S}^+$.

Proof: Since $u \prec L + c$ for all $u \in \mathcal{U}$, for all $(x,[s]),(y,[t]) \in M \times \mathbb{S}^1$,

$$u(y,[t]) \le u(x,[s]) + \Phi_c((x,[s]),(y,[t])), \qquad \text{for all } u \in \mathcal{U},$$

$$\min_{u\in\mathcal{U}} u(y,[t]) = \mathfrak{u}(y,[t]) \le u(x,[s]) + \Phi_c((x,[s]),(y,[t])), \qquad \text{for all } u \in \mathcal{U},$$

$$\mathfrak{u}(y,[t]) \le \mathfrak{u}(x,[s]) + \Phi_c((x,[s]),(y,[t])). \tag{18}$$

Now fix $(x,[t]) \in M \times \mathbb{S}^1$ and fix a sequence $u_k \in \mathcal{U}$ such that $\mathfrak{u}(x,[t]) = \lim_k u_k(x,[t])$. Let $(x,v_k,[t]) \in \Gamma^-(u_k)$. By lemma 8, $\|v_k\|$ is uniformly bounded. We can assume that $v_k \to w$. Let $\gamma_{v_k}(s) := \pi\,\varphi_{s-t}(x,v_k,t)$ and $\gamma_w(s) := \pi\,\varphi_{s-t}(x,w,t)$. Then

$$u_k(x,t) = u_k(\gamma_{v_k}(s),[s]) + A_{L+c}\big(\gamma_{v_k}|_{[s,t]}\big), \qquad \text{for all } s < t,$$

Since $\gamma_{v_k} \xrightarrow{C^1} \gamma_w$ uniformly on bounded intervals, using that by lemma 10 all the u_k's have the same Lipschitz constant, taking the lim inf on k we get that

$$\mathfrak{u}(x,t) \ge \mathfrak{u}(\gamma_w(s),[s]) + A_{L+c}\big(\gamma_w|_{[s,t]}\big), \qquad \text{for all } s < t, \tag{19}$$

The domination condition (18) implies that (19) is an equality. □

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