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Action potential and weak KAM solutions

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Abstract. For convex superlinear lagrangians on a compact manifold M we characterize the Peierls barrier and the weak KAM solutions of the Hamilton-Jacobi equation, as defined by A. Fathi [9], in terms of their values at each static class and the action potential defined by R. Mañé [14]. When the manifold M is non-compact, we construct weak KAM solutions similarly to Busemann functions in riemannian geometry. We construct a compactification of $M/_{d_c}$ by extending the Aubry set using these Busemann weak KAM solutions and characterize the set of weak KAM solutions using this extended Aubry set.

Introduction

In this paper we intend to relate the weak KAM solutions of the Hamilton-Jacobi equation of a convex superlinear lagrangian as introduced by A. Fathi in [9] and the action potential as defined by R. Mañé in [14].

Let M be a boundaryless n-dimensional complete riemannian manifold. An (autonomous) Lagrangian on M is a smooth function $L : TM \to \mathbb{R}$ satisfying the following conditions:

(a) *Convexity:* The Hessian $\frac{\partial^2 L}{\partial v_i \partial v_j}(x, v)$, calculated in linear coordinates on the fiber T_xM, is uniformly positive definite for all $(x, v) \in TM$, i.e. there is $A > 0$ such that

$$w \cdot L_{vv}(x, v) \cdot w \geq A\, |w|^2 \quad \text{for all } (x, v) \in TM \text{ and } w \in T_xM.$$

(b) *Superlinearity:*

$$\lim_{|v|\to+\infty} \frac{L(x, v)}{|v|} = +\infty, \quad \text{uniformly on } x \in M,$$

equivalently, for all $A \in \mathbb{R}$ there is $B \in \mathbb{R}$ such that

$$L(x, v) \geq A\, |v| - B \quad \text{for all } (x, v) \in TM.$$

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(c) *Boundedness:* For all $r > 0$,

$$\ell(r) = \sup_{\substack{(x,v)\in TM,\\ |v|<r}} L(x,v) < +\infty. \tag{1}$$

$$g(r) = \sup_{\substack{|w|=1\\ |(x,v)|\le r}} w \cdot L_{vv}(x,v) \cdot w < +\infty. \tag{2}$$

Using standard properties of convex functions one can see that condition (c) is equivalent to say that the hamiltonian associated to L is convex and superlinear.

The *Euler-Lagrange equation* associated to a lagrangian L is (in local coordinates)

$$\frac{d}{dt}\frac{\partial L}{\partial v}(x,\dot{x}) = \frac{\partial L}{\partial x}(x,\dot{x}). \tag{E-L}$$

The condition (c) implies that the Euler-Lagrange equation (E-L) defines a complete flow φ_t on TM (Proposition 1.2), called the *Euler-Lagrange flow*, by setting $\varphi_t(x_0,v_0) = \big(x_v(t),\dot{x}_v(t)\big)$, where $x_v : \mathbb{R} \to M$ is the solution of (E-L) with $x_v(0) = x_0$ and $\dot{x}_v(0) = v_0$.

The action $A_L(\gamma)$ of an absolutely continuous curve $\gamma : [0,T] \to M$ is defined as

$$A_L(\gamma) = \int_0^T L(\gamma(t),\dot{\gamma}(t))\, dt.$$

Given $x, y \in M$ and $T > 0$, let $\mathcal{C}_T(x,y)$ be the set of absolutely continuous curves $\gamma : [0,T] \to M$ with $\gamma(0) = x$ and $\gamma(T) = y$. Given $k \in \mathbb{R}$, the *action potential* $\Phi_k : M \times M \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$\Phi_k(x,y) = \inf\{\, A_{L+k}(\gamma) \,|\, \gamma \in \cup_{T>0}\mathcal{C}_T(x,y)\,\}.$$

R. Mañé defines in [14] the *critical value* $c(L)$ of the lagrangian L as

$$c(L) = \sup\{\, k \in \mathbb{R} \,|\, A_{L+k}(\gamma) < 0 \text{ for some } \gamma \in \cup_{x\in M} \cup_{T>0} \mathcal{C}_T(x,x)\,\}.$$

The critical value $c(L)$ can also be characterized in terms of minimizing measures, Tonelli's theorem on fixed energy levels [2], weak KAM solutions [9], lagrangian graphs and Finsler metrics [4] and the Palais-Smale condition for Morse theory [5].

It turns out [2], that $\Phi_k \equiv -\infty$ for $k < c(L)$ and $\Phi_k > -\infty$ for $k \ge c(L)$. Moreover,

$$d_k(x,y) := \Phi_k(x,y) + \Phi_k(y,x)$$

is a metric on M for $k > c(L)$ and a pseudo-metric for $k = c(L)$ (i.e. perhaps $d_{c(L)}(x,y) = 0$ for some $x \ne y$). The action potential always satisfies a triangle inequality $\Phi_k(x,z) \le \Phi_k(x,y) + \Phi_k(y,z)$, and for $k \ge c(L)$, Φ_k is Lipschitz and $\Phi_k(x,x) = 0$ for all $x \in M$.

J. Mather defines in [16] the Peierls barrier as

$$h(x,y) := \liminf_{T\to+\infty} \Phi_c(x,y;T),$$

where $x, y \in M$ and $\Phi_c(x, y; T)$ is the finite action potential

$$\Phi_k(x, y; T) := \inf_{\gamma \in \mathcal{C}_T(x,y)} A_{L+k}(\gamma)$$

for $k = c(L)$. Some properties of the Peierls barrier are given in Proposition 2.1. The *Aubry set* is defined as

$$\mathcal{A} := \{\, x \in M \mid h(x, x) = 0 \,\}.$$

We begin by characterizing the Peierls barrier h when M is compact.

Proposition 0.1 *If M is compact, then*

$$h(x, y) = \inf_{p \in \mathcal{A}} \big[\Phi_c(x, p) + \Phi_c(p, y)\big].$$

This resembles the characterization of h given by Fathi [10] in terms of weak KAM solutions and suggest a characterization of weak KAM solutions in terms of the action potential.

A function $u : B \subseteq M \to \mathbb{R}$ is said to be *dominated* by $L + k$ ($u \prec L + k$) if

$$u(y) - u(x) \le \Phi_k(x, y) \quad \text{for all } x, y \in B.$$

Dominated functions are Lipschitz with the same Lipschitz constant as Φ_k (Lemma 3.1).

Following Fathi [9], a forward *weak KAM solution* is a function $u : M \to \mathbb{R}$ such that

(a) $u \prec L + c$, $c = c(L)$.
(b) For all $x \in M$ there exists an absolutely continuous curve $\gamma : [0, +\infty[\to M$ such that $\gamma(0) = x$ and

$$u(\gamma(t)) - u(x) = A_{L+c}(\gamma|_{[0,t]})$$

for all $t > 0$.

The set of such functions u is denoted $\mathfrak{S}^+$. A backwards weak KAM solution is a function $u : M \to \mathbb{R}$ which satisfies (a) and

(c) For all $x \in M$ there exists an absolutely continuous curve $\gamma :]-\infty, 0[\to M$ such that $\gamma(0) = x$ and

$$u(x) - u(\gamma(t)) = A_{L+c}(\gamma|_{[t,0]})$$

for all $t < 0$.

We denote by $\mathfrak{S}^-$ the set of backwards weak KAM solutions. The curves γ apearing in (b) and (c) are said to *realize* u.

These functions are called solutions because at any differentiability point x they satisfy the Hamilton-Jacobi equation

$$H(x, d_x u) = c(L),$$

where $H : T^*M \to \mathbb{R}$ is the hamiltonian associated to L,

$$H(x,p) := \sup\{\, p \cdot v - L(x,v) \,|\, v \in T_xM \,\}.$$

Since they are Lipschitz, by Rademacher's Theorem [8], they are differentiable almost everywhere.

The pseudo-metric d_c defines an equivalence relation on M by

$$x \overset{d_c}{\sim} y \iff d_c(x,y) = 0.$$

This relation is non-trivial, $d_c(x,y) = 0$ with $x \neq y$, only if $x, y \in \mathcal{A}$. The Aubry set $\mathcal{A}$ is then partitioned into the equivalence classes of d_c which we call *static classes*. Let $\boldsymbol{\Gamma} := \mathcal{A}/d_c$ be the set of static classes. For each static class $\Gamma \in \boldsymbol{\Gamma}$ choose $p_\Gamma \in \Gamma$, and let $\mathbb{A} := \{\, p_\Gamma \,|\, \Gamma \in \boldsymbol{\Gamma} \,\}$. When M is compact, we prove the following characterization of weak KAM solutions

Theorem 0.2 *If M is compact, the maps*

$$\begin{gathered}
\{\, f : \mathbb{A} \to \mathbb{R} \,|\, f \prec L + c \,\} \to \mathfrak{S}^-, \\
f \longmapsto u_f(x) := \inf_{p \in \mathcal{A}} u(p) + \Phi_c(p,x);
\end{gathered}$$

$$\begin{gathered}
\{\, g : \mathbb{A} \to \mathbb{R} \,|\, g \prec L + c \,\} \to \mathfrak{S}^+, \\
g \longmapsto u_g(x) := \inf_{p \in \mathcal{A}} u(p) - \Phi_c(x,p).
\end{gathered}$$

are bijections.

So that a weak KAM solution is determined by its values at one point of each static class. In particular, we get in Corollary 4.7 that if there is only one static class then there is a unique weak KAM solution in $\mathfrak{S}^-$ (resp. $\mathfrak{S}^+$) modulo an additive constant. This is the case for generic lagrangians on a compact manifold [6].

Next we comment some methods to construct weak KAM solutions in the non-compact case. We observe in Lemma 4.1, that $\mathfrak{S}^-$ is closed under minima. The basic example given by Theorem 0.2 is $u_-(x) = \Phi_c(p,x) \in \mathfrak{S}^-$ with $p \in \mathcal{A}$. This is also true in the non-compact case but it may happen that $\mathcal{A} = \varnothing$. Replacing $p \in \mathcal{A}$ by $z \in M$, we have that $u_-(x) = h(z,x) \in \mathfrak{S}^-$, where h is the Peierls barrier. This is more general, in fact it is used in Proposition 4.2 to prove Theorem 0.2, but there are lagrangians for which the Peierls barrier is infinite.

Another method, originally presented in [4], resembles the construction of Busemann functions in riemannian geometry. An absolutely continuous curve $\gamma : [a,b] \to M$ is said *semistatic* if

$$A_{L+c}(\gamma|_{[s,t]}) = \Phi_c(\gamma(s),\gamma(t)) \qquad \text{for} \quad a \leq s < t \leq b.$$

Equivalently, by the triangle inequality for Φ_c, if it holds only for $s = a$ and $t = b$. Examples of semistatic curves are given in conditions (b) and (c) for u above. Semistatic curves are solutions of the Euler-Lagrange equation. The curve γ is said *static* if it is semistatic and $d_c(\gamma(a),\gamma(b)) = 0$. This is equivalent to

$$A_{L+c}(\gamma|_{[s,t]}) = -\Phi_c(\gamma(t),\gamma(s)) \qquad \text{for} \quad a \leq s < t \leq b.$$

Moreover [3], this implies that the whole orbit $t \mapsto \pi(\varphi_t(\dot\gamma(a)))$ is static. For $v \in TM$, write $x_v(t) := \pi(\varphi_t(v))$. Set

$$\begin{aligned} \Sigma^+(L) &:= \{\, v \in TM \mid x_v : [0,+\infty[\to M \text{ is semistatic} \,\}, \\ \Sigma^-(L) &:= \{\, v \in TM \mid x_v :]-\infty,0] \to M \text{ is semistatic} \,\}, \\ \widehat{\Sigma}(L) &:= \{\, v \in TM \mid x_v : \mathbb{R} \to M \text{ is static} \,\}. \end{aligned}$$

These are closed invariant subsets included in the energy level $E = c(L)$, [14,2]. The Aubry set can be characterized as $\mathcal{A} = \pi(\widehat{\Sigma}(L))$. It is easy to see [14,2] that the α-limits (resp. ω-limits) of vectors in Σ^- (resp. Σ^+) are contained in $\widehat{\Sigma}$. In [4] we proved that always $\Sigma^+ \neq \varnothing$ and $\Sigma^- \neq 0$. When M is compact, this implies that $\widehat{\Sigma} \neq \varnothing$. When M is non-compact, it may happen that $\widehat{\Sigma} = \varnothing$.

The following proposition constructs a kind of weak KAM solutions that we shall call *Busemann* weak KAM solutions.

Proposition 0.3 [4]

1. *If $w \in \Sigma^-(L)$ and $\gamma(t) = x_w(t)$, then*

$$\begin{aligned} u_w(x) &= \inf_{t<0} \left[\Phi_c(\gamma(t),x) - \Phi_c(\gamma(t),\gamma(0))\right] \\ &= \lim_{t\to-\infty} \left[\Phi_c(\gamma(t),x) - \Phi_c(\gamma(t),\gamma(0))\right] \end{aligned}$$

   is in $\mathfrak{S}^-$.
2. *If $w \in \Sigma^+(L)$ and $\gamma(t) = x_w(t)$, then*

$$\begin{aligned} u_w(x) &= \sup_{t>0} \left[\Phi_c(\gamma(0),\gamma(t)) - \Phi_c(x,\gamma(t))\right] \\ &= \lim_{t\to+\infty} \left[\Phi_c(\gamma(0),\gamma(t)) - \Phi_c(x,\gamma(t))\right] \end{aligned}$$

   is in $\mathfrak{S}^+$.

Then the existence of weak KAM solutions implies, from properties (b) and (c) above, that $\pi(\Sigma^+) = M = \pi(\Sigma^-)$.

A. Fathi suggested that using weak KAM solutions a similar construction to the sphere at infinity for manifolds of non-positive curvature could be made. This is in fact possible and gives more examples of weak KAM solutions than Proposition 0.3. It is done as follows, let $\mathcal{F} := C^0(M,\mathbb{R})/\sim$, where $f \sim g$ iff $f - g$ is constant. The functions $M \ni x \mapsto \Phi_c(z,x)$ and $M \ni x \mapsto \Phi_c(x,z)$ for fixed $x \in M$, are dominated by $L + c$, hence they form an equicontinuous family. Define $\mathfrak{M}^+$ (resp. $\mathfrak{M}^-$) as the closure, under the uniform convergence topology on compact subsets, of the set of equivalence classes of $-\Phi_c(x,z)$ (resp. $\Phi_c(z,x)$) for all $z \in M$. Then the sets $\mathfrak{M}^+$ and $\mathfrak{M}^-$ are compact and there are embeddings $M/d_c \hookrightarrow \mathfrak{M}^-$, $M/d_c \hookrightarrow \mathfrak{M}^+$, given by $M \ni z \mapsto [\Phi_c(z,x)] \in \mathcal{F}$ and $M \ni z \mapsto [-\Phi_c(x,z)]$. The limits in Proposition 0.3 show that the classes of Busemann weak KAM solutions are subsets $\mathfrak{B}^- \subseteq \mathfrak{M}^-$, $\mathfrak{B}^+ \subseteq \mathfrak{M}^+$ of $\mathfrak{M}$. When M is compact, $\mathfrak{M}^- \approx M/d_c$, $\mathfrak{M}^+ = M/d_c$ and the classes that contain weak KAM solutions are represented by the static classes $\mathcal{A}/d_c$, which also correspond to Busemann functions. In the non-compact case we define the *extended static set* as

$$\mathfrak{A}^{\mp} := \mathfrak{M}^{\mp} \setminus (M - \mathcal{A}) = \mathcal{A}/d_c \cup (\mathfrak{M}^{\mp} - M/d_c).$$

We prove in Proposition 5.6 that $\mathfrak{A}^- \subset \mathfrak{S}^-$ and $\mathfrak{A}^+ \subset \mathfrak{S}^+$. Observe that Proposition 0.3 implies that the α-limits (resp. ω-limits) of vectors in Σ^- (resp. Σ^+) are in $\mathfrak{B}^-$ (resp. $\mathfrak{B}^+$). In Example 5.4 we show a lagrangian where $\mathfrak{A}^{\mp}$ contains some non-Busemann weak KAM solutions. In particular, the extended static classes in $\mathfrak{A}^{\mp} \setminus \mathfrak{B}^{\mp}$ are not α-limits (resp. ω-limits) of semistatic orbits.

Using Lemma 4.1, other weak KAM solutions in $\mathfrak{S}^-$ can be obtained by taking minima of functions in $\mathfrak{A}^-$. In fact we characterize all weak KAM solutions in $\mathfrak{S}^{\mp}$ as minima of Busemann functions in $\mathfrak{B}^{\mp}$ in the following theorem.

For each extended static class $\alpha \in \mathfrak{B}^-$ choose a point $q_\alpha \in M$ such that the map $\mathfrak{B}^- \ni \alpha \mapsto q_\alpha$ is injective and there exists exactly one semistatic vector $v \in \Sigma^-$ with $\pi(v) = q_\alpha$ and α-limit $\pi(\alpha\text{-}\lim(v)) = \alpha$. This can be done because the static classes in $\mathfrak{B}^-$ are projections of α-limits of semistatic vectors in Σ^- and by the graph property 3.6.4. Let

$$\mathbb{A} := \{\, q_\alpha \mid \alpha \in \mathfrak{B}^- \,\}. \tag{3}$$

Define $b_{\alpha,q_\alpha} \in \mathfrak{B}^-$ by

$$b_{\alpha,q_\alpha}(x) = \lim_{t\to-\infty} \Phi_c(x_v(t),x) - \Phi_c(x_v(t),q_\alpha),$$

where v is the unique vector in $\Sigma^- \cap T_{q_\alpha}M$ with $\pi(\alpha\text{-}\lim(v)) = \alpha$. We say that a function $f : \mathbb{A} \to \mathbb{R}$ is *strictly dominated* if

$$f(q_\alpha) < f(q_\beta) + b_{\beta,q_\beta}(q_\alpha) \qquad \text{for all } q_\alpha \neq q_\beta \in \mathbb{A}.$$

Then we have

Theorem 0.4 *The map*

$$\{\, f : \mathbb{A} \to \mathbb{R} \mid f \textit{ is strictly dominated} \,\} \to \{\, u \in \mathfrak{S}^- \mid u|_{\mathbb{A}} \textit{ is strictly dominated} \,\},$$

given by $f \mapsto u_f$, where

$$u_f(x) = \inf_{\alpha\in\mathfrak{B}^-} f(q_\alpha) + b_{\alpha,q_\alpha}(x),$$

is bijective. In fact $u_f(q_\alpha) = f(q_\alpha)$ for all $\alpha \in \mathbb{A}$.

For $u \in \mathfrak{S}^-$ define its backwards *basin* by

$$\Gamma^-(u) := \{\, v \in \Sigma^- \mid u(x_v(0)) - u(x_v(t)) = \Phi_c(x_v(t),x_v(0)) \;\; \forall t < 0 \,\},$$
$$\Gamma_0^-(u) := \bigcup_{t<0} \varphi_t(\Gamma^-(u));$$

and define the *endpoints* of u by

$$\mathfrak{B}^-(u) := \{\, \alpha \in \mathfrak{B}^- \mid \exists v \in \Gamma^-(u),\ \alpha\text{-lim}(v) = \alpha \,\}.$$

The set $\pi\big(\Gamma^-(u)\setminus\Gamma_0^-(u)\big)$ is called (backward) *cut locus* of u.

By the graph property in Theorem 4.9, $\pi^{-1}\mathcal{A}\cap\Sigma^- = \widehat{\Sigma}$. So that $\mathcal{A}/d_c \subseteq \mathfrak{B}^-(u)$. In particular, when M is compact, $\mathcal{A}/d_c = \mathfrak{B}^- = \mathfrak{A}^- = \mathfrak{B}^-(u)$ for all $u\in\mathfrak{S}^-$. When M is not compact, it may happen that $\mathfrak{B}^-(u)\neq\mathfrak{B}^-$. This usually holds for Busemann functions because they are *"directed towards a single static class"*. In Sect. §6 we show that the horocycle flow is an example in which $\mathcal{A}=\varnothing$, $\mathfrak{B}^-\approx S^1$ and there are differentiable Busemann functions $b_\alpha\in\mathfrak{S}^-$ with $\mathfrak{B}^-(b_\alpha)=\{\alpha\}\subsetneq\mathfrak{B}^-$.

For $\alpha\in\mathfrak{B}^-$, define the *immediate basin* of the extended static class α by

$$\begin{aligned}\Lambda^-(\alpha) :&= \{\, v\in\Sigma^- \mid \alpha\text{-lim}(v)=\alpha\,\},\\ &= \{\, w\in\Sigma^- \mid b_w\in\alpha\in\mathfrak{B}^-\,\};\end{aligned}$$

where b_w is the Busemann function defined in Proposition 0.3.1.

By analogy to Theorem 0.2, the values q_α in Theorem 0.4 are used to *"fix the value of u_f at the infinite point α"*, which could be infinite. We prove in Theorem 0.4 that for all $q_\alpha\in\mathbb{A}$, $\Gamma^-(u_f)\cap T_{q_\alpha}M=\{v_\alpha\}$, where v_α is the unique semistatic vector $v_\alpha\in\Sigma^-$ with α-lim$(v_\alpha)=\alpha=\Lambda^-(\alpha)\cap T_{q_\alpha}M$. This implies that the set $\mathbb{A}$ in (3) and not only the values $f(q_\alpha)$ or the classes α has a strong association to the weak KAM solution u_f in Theorem 0.4.

Nevertheless, varying the set $\mathbb{A}$, we can characterize all the weak KAM solutions in $\mathfrak{S}^-$. Observe that in the following theorem the set $\mathbb{A}(u)$ contains one point for each extended static class in $\mathfrak{B}^-(u)\subset\mathfrak{B}^-$.

Theorem 0.5 *Given $u\in\mathfrak{S}^-$, for all $\alpha\in\mathfrak{B}^-(u)$ choose $q_\alpha\in\pi[\Lambda_0^-(\alpha)\cap\Gamma_0^-(u)]$, and let $\mathbb{A}(u):=\{q_\alpha \mid \alpha\in\mathfrak{B}^-(u)\,\}$. Then*

$$u(x)=\inf_{q_\alpha\in\mathbb{A}(u)} u(q_\alpha)+b_{\alpha,q_\alpha}(x) \qquad \textit{for all } x\in M.$$

1 Preliminary results

1.1 Lagrangians on non-compact manifolds

The *energy function* of the lagrangian L is $E:TM\to\mathbb{R}$, defined by

$$E(x,v)=\frac{\partial L}{\partial v}(x,v)\cdot v - L(x,v). \tag{4}$$

Observe that if $x(t)$ is a solution of the Euler-Lagrange equation (E-L), then

$$\tfrac{d}{dt}\,E(x,\dot x)=\left(\tfrac{d}{dt}L_v-L_x\right)\cdot\dot x=0.$$

Hence $E:TM\to\mathbb{R}$ is an integral for the lagrangian flow φ_t and its level sets, called *energy levels* are invariant under φ_t. Moreover, the convexity implies that

$$\tfrac{d}{ds}\,E(x,sv)|_{s=1}=v\cdot L_{vv}(x,v)\cdot v>0.$$

Thus

$$\min_{v\in T_xM} E(x,v)=E(x,0)=-L(x,0).$$

Write

$$e_0 := \max_{x\in M} E(x,0) = -\min_{x\in M} L(x,0) > -\infty, \tag{5}$$

by the superlinearity, then

$$e_0 = \min\{\, k \in \mathbb{R} \mid \pi : E^{-1}\{k\} \to M \text{ is surjective} \,\}.$$

By the uniform convexity, and the boundedness condition,

$$A := \inf_{\substack{(x,v)\in TM \\ |w|=1}} w \cdot L_{vv}(x,v) \cdot w > 0,$$

and then using (1) and (2),

$$\begin{aligned} E(x,v) &= E(x,0) + \int_0^{|v|} \tfrac{d}{ds}\, E\big(x, s\tfrac{v}{|v|}\big)\, ds \\ &\ge -\ell(0) + A\,|v|. \end{aligned} \tag{6}$$

Similarly,

$$E(x,v) \le e_0 + g(|v|)\, |v|. \tag{7}$$

Hence

Remark 1.1 If $k \in \mathbb{R}$ and $K \subseteq M$ is compact, then $E^{-1}\{k\} \cap T_K M$ is compact.

Proposition 1.2 *The Euler-Lagrange flow is complete.*

Proof. Suppose that $]\alpha, \beta[$ is the maximal interval of definition of $t \mapsto \varphi_t(v)$, and $-\infty < \alpha$ or $\beta < +\infty$. Let $k = E(v)$. Since $E(\varphi_t(v)) \equiv k$, by (6), there is $a > 0$ such that $0 \le |\varphi_t(v)| \le a$ for $\alpha \le t \le \beta$. Since $\varphi_t(v)$ is of the form $(\gamma(t), \dot\gamma(t))$, then $\varphi_t(v)$ remains in the interior of the compact set

$$Q := \left\{\, (y,w) \in TM \;\middle|\; d(y,x) \le a\,\big[|\beta-\alpha| + 1\big],\;\; |v| \le a+1 \,\right\},$$

where $x = \pi(v)$. The Euler-Lagrange vector field is uniformly Lipschitz on Q. Then by the theory of ordinary differential equations, we can extend the interval of definition $]\alpha, \beta[$ of $t \mapsto \varphi_t(v)$. $\square$

Given $x,\ y \in M$ and $T > 0$, let

$$\mathcal{C}_T(x,y) := \left\{\, \gamma \in \mathcal{C}^{ac}([0,T], M) \mid \gamma(0) = x,\ \gamma(T) = y \,\right\}.$$

We say that $\gamma \in \mathcal{C}_T(x,y)$ is a *Tonelli minimizer* if

$$A_L(\gamma) = \min_{\eta\in\mathcal{C}_T(x,y)} A_L(\eta)\,.$$

Tonelli's Theorem 1.3 *[15]*
For all $x,\ y \in M$ and $T > 0$ there exists a Tonelli minimizer on $\mathcal{C}_T(x,y)$.

The only difference in the proof of this theorem when M is non-compact is the following proposition. For a proof in the compact case see [15], [13], [12] or [3].

Proposition 1.4 *For all $c \in \mathbb{R}$ and $T > 0$ there is $R > 0$ such that for all $x \in M$, if $\gamma : [0,T] \to M$ is absolutely continuous with $\gamma(0) = x$ and $A_L(\gamma) \le c$, then $d(\gamma(t), x) \le R$ for all $t \in [0,T]$.*

Proof. Adding a constant we may assume that $L \ge 0$. There is $B > 0$ such that $L(x,v) \ge |v| - B$ for all $(x,v) \in TM$. Then for $0 \le s \le t \le T$, we have that

$$d(x_s, x_t) \le \int_s^t |\dot{x}| \le BT + \int_s^t L(x,\dot{x}) \le BT + c.$$

□

The following lemma, due to Mather [15] for Tonelli minimizers in the non-autonomous case, will be very useful. In the autonomous case its proof is very simple.

Lemma 1.5 *For $C > 0$ there exists $A = A(C) > 0$ such that if $x, y \in M$ and $\gamma \in \mathcal{C}_T(x,y)$ is a solution of the Euler-Lagrange equation with $A_L(\gamma) \le C\,T$, then $|\dot{\gamma}(t)| < A$ for all $t \in [0,T]$.*

Proof. By the superlinearity there is $D > 0$ such that $L(x,v) \ge |v| - D$ for all $(x,v) \in TM$. Since $A_L(\gamma) \le C\,T$, the mean value theorem implies that there is $t_0 \in]0,T[$ such that

$$|\dot{\gamma}(t_0)| \le D + C.$$

The conservation of the energy and the uniform bounds (7) and (6) imply that there is $A = A(C) > 0$ such that $|\dot{\gamma}| \le A$. □

Lemma 1.6 *There exists $A > 0$ such that if $x, y \in M$ and $\gamma \in \mathcal{C}_T(x,y)$ is a solution of the Euler-Lagrange equation with*

$$A_{L+c}(\gamma) \le \Phi_c(x,y) + d_M(x,y),$$

then

(a) $T > \frac{1}{A}\, d_M(x,y)$.
(b) $|\dot{\gamma}(t)| < A$ *for all* $t \in [0,T]$.

Proof. Let $\eta : [0, d(x,y)] \to M$ be a minimal geodesic with $|\dot{\eta}| \equiv 1$. Let $\ell(r)$ be from (1) and $D = \ell(1) + c + 2$. From the superlinearity condition there is $B > 0$ such that

$$L(x,v) + c > D\,|v| - B, \qquad \forall (x,v) \in TM.$$

Then

$$
\begin{aligned}
&(8) \qquad & [\ell(1) + c]\; d(x,y) &\ge A_{L+c}(\eta) \ge \Phi_c(x,y) \\
&(9) \qquad & &\ge A_{L+c}(\gamma) - d(x,y) \\
& & &\ge \int_0^T \big(D\,|\dot{\gamma}| - B\,\big)\,dt - d(x,y) \\
& & &\ge D\; d(x,y) - B\,T - d(x,y).
\end{aligned}
$$

Hence

$$T \geq \tfrac{D-\ell-c-1}{B}\, d(x,y) \geq \tfrac{1}{B}\, d(x,y).$$

From (8) and (9), we get that

$$\begin{aligned} A_L(\gamma) &\leq \big[\, \ell(1)+c+1 \,\big]\, d(x,y) - c\,T, \\ &\leq \big\{\, B\,[\,\ell(1)+c+1\,] - c \,\big\}\, T. \end{aligned}$$

Then Lemma 1.5 completes the proof. □

2 The Peierls barrier

For $T>0$ and $x,\ y\in M$ define

$$h_T(x,y) = \Phi_c(x,y;T) := \inf_{\gamma\in\mathcal{C}_T(x,y)} A_{L+c}(\gamma).$$

So that the curves which realize $h_T(x,y)$ are the Tonelli minimizers on $\mathcal{C}_T(x,y)$. Define the *Peierls barrier* as

$$h(x,y) := \liminf_{T\to+\infty} h_T(x,y).$$

Fathi [11] proves that when M is compact the functions h_T converge uniformly to h. The difference between the action potential and the Peierls barrier is that in the Peierls barrier the curves must be defined on large time intervals. Clearly

$$h(x,y) \geq \Phi_c(x,y).$$

Proposition 2.1 *If $h : M\times M\to\mathbb{R}$ is finite, then*

1. *h is Lipschitz.*
2. *$\forall\ x,\ y\in M$, $h(x,y)\geq\Phi_c(x,y)$, in particular $h(x,x)\geq 0$, $\forall\ x\in M$.*
3. *$h(x,z)\leq h(x,y)+h(y,z)$, $\qquad \forall\, x,\ y,\ z\in M$.*
4. *$h(x,y)\leq \Phi_c(x,p)+h(p,q)+\Phi_c(q,y)$, $\qquad \forall\ x,\ y,\ p,\ q\in M$.*
5. *$h(x,x)=0 \iff x\in\pi(\widehat{\Sigma})=\mathcal{A}$.*
6. *If $p\in\mathcal{A}$, then $\Phi_c(p,x)=h(p,x)$ and $\Phi_c(x,p)=h(x,p)$ for all $x\in M$.*
7. *If $\widehat{\Sigma}\neq\varnothing$, $h(x,y)\leq\inf_{p\in\pi(\widehat{\Sigma})}\Phi_c(x,p)+\Phi_c(p,y)$.*

Proof. Item 2 is trivial. Observe that for all S, $T>0$ and $y\in M$,

$$h_{T+S}(x,z) \leq h_T(x,y) + h_S(y,z).$$

Taking $\liminf_{T\to+\infty}$ we get that

$$h(x,z)\leq h(x,y)+h_S(y,z), \qquad \text{for all } S>0.$$

Taking $\liminf_{S\to+\infty}$, we obtain item *3*.

1. Taking the infimum on $S>0$, we get that

$$\begin{aligned} h(x,z) &\leq h(x,y)+\Phi_c(y,z) \qquad \forall\, x,\ y,\ z\in M. \\ &\leq h(x,y) + A\, d_M(y,z), \end{aligned}$$

where A is a Lipschitz constant for Φ_c. Changing the roles of x, y, z, we obtain that h is Lipschitz.

4. Observe that

$$\inf_{S>T} h_S(x,y) \le \Phi_c(x,p) + h_T(p,q) + \Phi_c(q,x).$$

Taking $\liminf_{T\to+\infty}$ we get item 4.

5. We first prove that if $p \in \mathcal{A} = \pi(\widehat{\Sigma})$, then $h(p,p) = 0$. Take $v \in \widehat{\Sigma}$ such that $\pi(v) = p$ and $y \in \pi(\omega\text{-limit}(v))$. Let $\gamma(t) := \pi\,\varphi_t(v)$ and choose $t_n \uparrow +\infty$ such that $\gamma(t_n) \to y$. Then

$$\begin{aligned} 0 \le h(p,p) &\le h(p,y) + \Phi_c(y,p) \\ &\le \lim_n A_{L+c}(\gamma|_{[0,t_n]}) + \Phi_c(y,p) \\ &\le \lim_n -\Phi_c(\gamma(t_n),p) + \Phi_c(y,p) = 0. \end{aligned}$$

Conversely, if $h(x,x) = 0$, then there exists a sequence of Tonelli minimizers $\gamma_n \in \mathcal{C}(x,x;T_n)$ with $T_n \to +\infty$ and $A_{L+c}(\gamma_n) \xrightarrow{n} 0$. By Lemma 1.6, $|\dot\gamma|$ is uniformly bounded. Let v be an accumulation point of $\dot\gamma_n(0)$ and $\eta(t) := \pi\,\varphi_t(v)$. Then if $\dot\gamma_{n_k}(0) \xrightarrow{k} v$, for any $s > 0$ we have that

$$\begin{aligned} 0 &\le \Phi_c(x, \pi\,\varphi_s\, v) + \Phi_c(\pi\,\varphi_s\, v, x) \\ &\le A_{L+c}\big(\eta|_{[0,s]}\big) + \Phi_c(\pi\,\varphi_s\, v, x) \\ &\le \lim_k A_{L+c}\big(\gamma_{n_k}|_{[0,s]}\big) + A_{L+c}\big(\gamma_{n_k}|_{[s,T_n]}\big) \\ &= 0. \end{aligned}$$

Thus $v \in \widehat{\Sigma}$.

6. By items 2, 4 and 5, we have that

$$\Phi_c(p,x) \le h(p,x) \le h(p,p) + \Phi_c(p,x) = \Phi_c(p,x).$$

The equality $\Phi_c(x,p) = h(p,x)$ is similar.

7. Using items 4 and 5, we get that

$$h(x,y) \le \inf_{p\in\pi(\widehat{\Sigma})} \big[\, \Phi_c(x,p) + 0 + \Phi_c(p,y)\,\big]. \qquad \square$$

Proposition 0.1 *If M is compact, then*

$$h(x,y) = \inf_{p\in\pi(\widehat{\Sigma})} \big[\Phi_c(x,p) + \Phi_c(p,y)\big].$$

We shall use the following characterization of minimizing measures. A minimizing measure is an invariant measure under the Euler-Lagrange flow μ, such that its *action* satisfies

$$A_L(\mu) := \int_{TM} L\; d\mu = -c(L).$$

It is proven in [14, 2] that an invariant measure μ is minimizing if and only if it is supported on $\widehat{\Sigma}$.

Proof.

Using items 4 and 5 of Proposition 2.1, we get that

$$h(x,y) \leq \inf_{p\in\pi(\widehat{\Sigma})} \big[\,\Phi_c(x,p) + 0 + \Phi_c(p,y)\,\big].$$

In particular $h(x,y) < +\infty$ for all $x,\, y \in M$. Now let $\gamma_n \in \mathcal{C}_{T_n}(x,y)$ with $T_n \to +\infty$ and $A_{L+c}(\gamma_n) \to h(x,y) < +\infty$. Then $\frac{1}{T} A_{L+c}(\gamma_n) \to 0$. Let μ be a weak limit of a subsequence of the measures μ_{γ_n}:

$$\int f\; d\mu_{\gamma_n} := \frac{1}{T_n} \int_0^{T_n} f(\gamma_n(t), \dot\gamma_n(t))\; dt.$$

Then μ is minimizing. Let $q \in \pi\big(\mathrm{supp}(\mu)\big)$ and $q_n \in \gamma_n([0,T_n])$ be such that $\lim_n q_n = q$. Then,

$$\begin{aligned}\Phi_c(x,q) + \Phi_c(q,y) &\leq \Phi_c(x,q_n) + \Phi_c(q_n,y) + 2A\, d(q_n,q)\\ &\leq A_{L+c}(\gamma_n) + 2A\, d(q_n,q).\end{aligned}$$

Letting $n \to \infty$, we get that

$$\Phi_c(x,q) + \Phi_c(q,y) \leq h(x,y).$$

□

3 The Hamilton-Jacobi equation

For an autonomous hamiltonian $H : T^*M \to \mathbb{R}$, the Hamilton-Jacobi equation is

(H-J) $$H(x, d_x u) = k,$$

where $u : U \subseteq M \to \mathbb{R}$. Here we are interested on *global* solutions of (H-J), i.e. $u : M \to \mathbb{R}$ satisfying (H-J).

It may not be possible to obtain a smooth global solution of (H-J). Instead, for certain values of k, we shall find *weak* solutions of (H-J), which are Lipschitz. By Rademacher's Theorem [8], a Lipschitz function is Lebesgue almost everywhere differentiable, so that (H-J) makes sense at a.e. point.

The results of this section are due to A. Fathi [9] and are restated here for completeness in the non-compact case. We say that a function u is *dominated* by $L+k$, and write $u \prec L+k$. if

$$u(y) - u(x) \leq \Phi_k(x,y) \quad \text{ for all } x,y \in M.$$

Lemma 3.1

1. *If $u \prec L+k$, then u is Lipschitz with the same Lipschitz constant as Φ_c. In particular, a family of dominated functions is equicontinuous.*
2. *If $u \prec L+k$ then $H(x,d_xu) \leq k$ at any differentiability point x of u.*

Proof.

1. We have that $u(y)-u(x) \le \Phi_c(x,y) \le A\ d_M(x,y)$, where A is a Lipschitz constant for Φ_c. Changing the roles of x and y, we get that u is Lipschitz.

2. We have that

$$u(y) - u(x) \le \int_\gamma L(\gamma, \dot\gamma) + k$$

for all curves $\gamma \in \mathcal{C}(x,y)$. This implies that

$$d_x u \cdot v \le L(x,v) + k$$

for all $v \in T_xM$ when u is differentiable at $x \in M$. Since

$$H(x, d_x u) = \sup\{\, d_x u \cdot v - L(x,v) \,|\, v \in T_xM \,\},$$

then $H(x, d_x u) \le k$. □

The following proposition shows that we actually get a solution of H-J if there are (semistatic) curves which realize a dominated function u.

Proposition 3.2 *If $u \prec L+k$, $x \in M$ and there exists $\gamma :]-\varepsilon, \varepsilon[\to \mathbb{R}$ such that $\gamma(0) = x$, and*

(10) $$u(\gamma(t)) - u(\gamma(s)) = A_{L+k}(\gamma|_{[s,t]}) \quad \textit{for} \quad -\varepsilon < s \le t < \varepsilon,$$

then u is differentiable at x, $d_x u = L_v(x, \dot\gamma(0))$ and $H(x, d_x u) = k$.

Remarks 3.3

1. Equation $d_x u = L_v(x, \dot\gamma(0))$ means that the tangent vector $(x, \dot\gamma(0))$ of any a.c. curve γ realizing u is sent by the Legendre transform to $d_x u$.
2. In particular, since the functions $u(x) = \Phi_k(p,x)$ (resp. $v(x) = -\Phi_c(x,p)$) are dominated, then they are differentiable at any point which is not at the (backward) (resp. forward) $(L+k)$-cut locus of p.
3. Observe that the energy $E(x, \dot\gamma(0)) = H(x, d_x u)$. In Proposition 0.3, if $w \in \Sigma^{\mp}$, then $w \in \Gamma^{\mp}(u_w)$. Thus we obtain that $\Sigma \subset E^{-1}\{c\}$, i.e. that the semistatic orbits have energy $c(L)$.

Proof. Let $w \in T_xM$ and let $\eta(s,t)$ be a variation of γ fixing the endpoints $\gamma(-\varepsilon)$, $\gamma(\varepsilon)$ such that $\eta(0,t) = \gamma(t)$ and $\frac{\partial}{\partial s}\eta(0,0) = w$. Define

$$\mathbb{A}(s) := \int_{-\varepsilon}^{0} L(\tfrac{\partial}{\partial t}\eta(s,t)) + k \ \ dt.$$

Then, integrating by parts and using the Euler-Lagrange equation (E-L),

$$\mathbb{A}'(0) = L_v\ \xi|_{-\varepsilon}^{0} + \int_{-\varepsilon}^{0} \left[L_x - \tfrac{d}{dt}L_v\right]\ \xi\ dt = L_v(x, \dot\gamma(0)) \cdot w\,,$$

where $\xi(t) := \frac{\partial}{\partial s}\eta(0,t)$. Also

$$\begin{aligned}\tfrac{1}{s}\big[\,u(\eta(s,0)) - u(x)\,\big] &= \tfrac{1}{s}\big[u(\eta(s,0)) - u(\gamma(-\varepsilon)) + u(\gamma(-\varepsilon)) - u(\gamma(0))\,\big] \\ &\le \tfrac{1}{s}\,[\,\mathbb{A}(s) - \mathbb{A}(0)\,]\ ,\end{aligned}$$

where we used that $u \prec L+k$ and (10). Hence

$$\limsup_{s\to 0} \tfrac{1}{s}\,\big[\,u(\eta(s,0)) - u(x)\,\big] \le \mathbb{A}'(0). \tag{11}$$

Similarly, if $\mathcal{B}(s) := A_{L+k}\big(\eta(s,\cdot)|_{[0,\varepsilon]}\big)$, then

$$u(\gamma(\varepsilon)) - u(\eta(s,0)) - u(\gamma(\varepsilon)) + u(x) \le \mathcal{B}(s) - \mathcal{B}(0),$$

$$\limsup_{s\to 0} \tfrac{1}{s}\,\big[\,u(x) - u(\eta(s,0))\,\big] \le \mathcal{B}'(0) = -L_v(x,\dot\gamma(0))\cdot w.$$

Hence

$$\liminf_{s\to 0} \tfrac{1}{s}\,\big[\,u(\eta(s,0)) - u(x)\,\big] \ge L_v(x,\dot\gamma(0))\cdot w. \tag{12}$$

From (11) and (12) we get that u is differentiable at x and $d_x u = L_v(x,\dot\gamma(0))$.

Finally, since $u \prec L+k$, by Lemma 3.1, $H(x,d_x u) \le k$. Since for $0<t<\varepsilon$,

$$u(\gamma(t)) - u(\gamma(0)) = A_{L+k}(\gamma|_{[0,t]}) = \int_0^t \big[\,L(\gamma(s),\dot\gamma(s)) + k\,\big]\,ds,$$

then

$$d_x u\cdot\dot\gamma(0) = L(\gamma(0),\dot\gamma(0)) + k.$$

Hence

$$H(x,d_x u) = \sup_{v\in T_xM}\{\,d_x u\cdot v - L(x,v)\,\} \ge k.$$

□

Definition 3.4 A function $u_- : M \to \mathbb{R}$ is a *backward weak KAM* solution of (H-J) if

1. $u_- \prec L+c$.
2. For all $y\in M$ there is $\gamma\in\mathcal{C}^{ac}(]-\infty,0],M)$ such that $\gamma(0)=y$ and

$$u_-(\gamma(-t)) = u_-(y) + A_{L+c}(\gamma|_{[-t,0]}) \qquad \text{for all} \quad t\ge 0.$$

A function $u_+ : M\to\mathbb{R}$ is a *forward weak KAM* solution of (H-J) if

1. $u_+ \prec L+c$.
2. For all $y\in M$ there is $\gamma\in\mathcal{C}^{ac}([0,+\infty[,M)$ such that $\gamma(0)=y$ and

$$u_+(\gamma(t)) = u_+(y) + A_{L+c}(\gamma|_{[0,t]}) \qquad \text{for all} \quad t\ge 0.$$

We say that the curves γ above *realize* u.

Remark 3.5 From the domination condition it follows that u is Lipschitz and that the curve γ is semistatic. From Proposition 3.2, at an interior point x of such a curves γ, u is differentiable and $H(x, d_x u) = c$. Moreover, the last argument in Proposition 3.2 shows that if u is differentiable at an endpoint of a curve γ, then $H(x, d_x u) = c$. By Rademacher's Theorem [8], u is differentiable at (Lebesgue) almost every point in M. So that u is indeed a weak solution of the Hamilton-Jacobi equation for $k = c(L)$.

Given a dominated function $u \prec L + c$ define the sets

$$\Gamma_0^+(u) := \{\, v \in \Sigma^+ \mid u(x_v(t)) - u(x_v(0)) = \Phi_c(x_v(0), x_v(t)),\ \forall t > 0 \,\},$$

$$\Gamma_0^-(u) := \{\, v \in \Sigma^- \mid u(x_v(0)) - u(x_v(t)) = \Phi_c(x_v(t), x_v(0)),\ \forall t < 0 \,\},$$

$$\Gamma^+(u) := \bigcup_{t>0} \phi_t\big(\Gamma_0^+(u)\big) \quad , \qquad \Gamma^-(u) := \bigcup_{t<0} \phi_t\big(\Gamma_0^-(u)\big),$$

where $x_v(t) = \pi\,\phi_t(v)$. We call $\Gamma^+(u)$ (resp. $\Gamma^-(u)$) the *basin* of u and $\pi\big(\Gamma_0^+(u) \setminus \Gamma^+(u)\big)$ (resp. $\pi\big(\Gamma_0^-(u) \setminus \Gamma^-(u)\big)$) the *cut locus* of u.

Theorem 3.6 (Fathi [9]) *If $u \in \mathfrak{S}^+$ (resp. $u \in \mathfrak{S}^-$) is a weak KAM solution, then*

1. *u is Lipschitz and hence differentiable (Lebesgue)-almost everywhere. Also $H(x, d_x u) = c(L)$ at any differentiability point x.*
2. *$u \prec L + c$.*
3. **Covering Property:** *$\pi(\Gamma_0^+(u)) = M$.*
4. **Graph Property:** *$\pi : \Gamma^+(u) \to M$ is injective and its inverse is Lipschitz, with Lipschitz constant depending only on L.*
5. **Smoothness Property:** *u is differentiable on $\Gamma^+(u)$ and its derivative $d_x u$ is the image of $(\pi|_{\Gamma^+(u)})^{-1}(x)$ under the Legendre transform $\mathcal{L}$ of L. In particular, the energy of $\Gamma_0^+(u)$ is $c(L)$.*

Proof. Items 2 and 3 are the definition of $u \in \mathfrak{S}^+$. Item 1 follows from Proposition 3.1.1 and Remark 3.5. Item 5 follows from Remarks 3.3.1 and 3.3.3.

A proof of the following lemma can be found in [15] or [13].

Mather's Crossing Lemma 3.7 *[15]*

Given $A > 0$ there exist $K > 0$, $\varepsilon_1 > 0$ and $\delta > 0$ with the following property: if $|v_i| < A$, $(p_i, v_i) \in TM$, $i = 1, 2$ satisfy $d(p_1, p_2) < \delta$ and $d((p_1, v_1), (p_2, v_2)) \geq K^{-1} d(p_1, p_2)$ then, if $a \in \mathbb{R}$ and $x_i : \mathbb{R} \to M$, $i = 1, 2$, are the solutions of L with $x_i(a) = p_i$, $\dot{x}_i(a) = v_i$, there exist solutions $\gamma_i : [a - \varepsilon, a + \varepsilon] \to M$ of L with $0 < \varepsilon < \varepsilon_1$, satisfying

$$\begin{aligned} \gamma_1(a-\varepsilon) &= x_1(a-\varepsilon) \quad , \quad \gamma_1(a+\varepsilon) = x_2(a+\varepsilon)\,, \\ \gamma_2(a-\varepsilon) &= x_2(a-\varepsilon) \quad , \quad \gamma_2(a+\varepsilon) = x_1(a+\varepsilon)\,, \\ S_L(x_1|_{[a-\varepsilon, a+\varepsilon]}) &+ S_L(x_2|_{[a-\varepsilon, a+\varepsilon]}) > S_L(\gamma_1) + S_L(\gamma_2) \end{aligned}$$

We prove item 4. Let (z_1, v_1), $(z_2, v_2) \in \Gamma^+(u)$ and suppose that $d_{TM}(v_1, v_2) > K\, d_M(z_1, z_2)$, where K is from Lemma 3.7 and the A that we input on Lemma 3.7 is from Lemma 1.6. Let $0 < \varepsilon < \varepsilon_1$, (with ε_1 from Lemma 3.7) be such that $\varphi_{-\varepsilon}(z_i, v_i) \in \Gamma^+(u)$. Let $x_i = x_{v_i}(-\varepsilon)$, $y_i = x_{v_i}(\varepsilon)$, $i = 1, 2$, then $u(y_i) = u(x_i) + \Phi_c(x_i, y_i)$, $i = 1, 2$. Then Lemma 3.7 implies that

$$\Phi_c(x_1, y_2) + \Phi_c(x_2, y_1) < \Phi_c(x_1, y_1) + \Phi_c(x_2, y_2).$$

Adding $u(y_1) + u(y_2)$ and using that $u \prec L + c$, we get that

$$\begin{aligned} u(x_1) + u(x_2) &\le \Phi_c(x_1, y_2) + u(y_2) + \Phi_c(x_2, y_1) + u(y_1) \\ &< \Phi_c(x_1, y_1) + u(y_1) + \Phi_c(x_2, y_2) + u(y_2) \\ &= u(x_1) + u(x_2), \end{aligned}$$

which is a contradiction. This proves item 4.

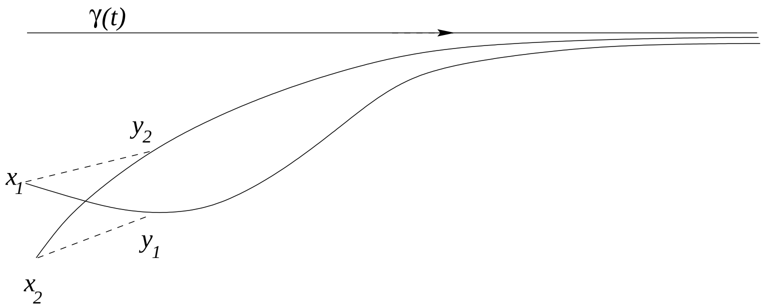

Fig. 1. Graph property

□

4 Construction of weak KAM solutions

In this section we present three ways to construct weak KAM solutions: when the Aubry set is non-empty (in Remark 4.3.4), when the Peierls barrier is finite (in Proposition 4.2), and the general case (in Proposition 0.3).

We begin by observing that

Lemma 4.1

1. *If* $\mathcal{U} \subseteq \mathfrak{S}^-$ *is such that* $v(x) := \inf_{u\in\mathcal{U}} u(x) > -\infty$, *for all* $x \in M$*; then* $v \in \mathfrak{S}^-$.
2. *If* $\mathcal{U} \subseteq \mathfrak{S}^+$ *is such that* $v(x) := \sup_{u\in\mathcal{U}} u(x) < +\infty$, *for all* $x \in M$*; then* $v \in \mathfrak{S}^+$.

Proof. We only prove item 1. Since $u \prec L + c$ for all $u \in \mathcal{U}$, then

$$v(y) = \inf_{u\in\mathcal{U}} u(y) \le \inf_{u\in\mathcal{U}} u(x) + \Phi_c(x, y) = v(x) + \Phi_c(x, y). \tag{13}$$

Thus $v \prec L + c$.

Let $x \in M$ and choose $u_n \in \mathcal{U}$ such that $u_n(x) \to v(x)$. Choose $w_n \in \Gamma^-(u_n) \cap T_xM$. Since by Lemma 1.6 $|w_n| < A$, we can assume that $w_n \to w \in T_xM$. By Lemma 3.1.1, all the functions $u \in \mathcal{U}$ have the same Lipschitz constant K as Φ_c. For $t < 0$, we have that

$$\begin{aligned} v(x_w(t)) &\le \liminf_n u_n(x_{w_n}(t)) + K\, d_M(x_w(t), x_{w_n}(t)) \\ &= \liminf_n u_n(x) - \Phi_c(x_w(t), x) + K\, d_M(x_w(t), x_{w_n}(t)) \\ &= v(x) - \Phi_c(x_w(t), x) \le v(x_w(t)), \qquad \text{because } v \prec L + c. \end{aligned}$$

Hence $w \in \Gamma^-(v)$. □

Proposition 4.2 *If $h_c < +\infty$ and $f : M \to \mathbb{R}$ is a continuous function. Suppose that*

$$\begin{aligned} v_-(x) &:= \inf_{z\in M} f(z) + h_c(z, x) > -\infty, \\ v_+(x) &:= \sup_{z\in M} f(z) - h_c(x, z) > -\infty. \end{aligned}$$

Then $v_- \in \mathfrak{S}^-$ and $v_+ \in \mathfrak{S}^+$.

Proof. We only prove that $v_- \in \mathfrak{S}^-$. By Lemma 4.1 it is enough to prove that the functions $u(x) \mapsto h_c(z, x)$ are in $\mathfrak{S}^-$ for all $z \in M$.

By Proposition 2.1.4, $u \prec L + c$. Now fix $x \in M$. Choose Tonelli minimizers $\gamma_n : [T_n, 0] \to M$ such that $\gamma_n \in \mathcal{C}(z, x)$, $T_n < -n$ and

$$A_{L+c}\big(\gamma_n|_{[T_n,0]}\big) \le h_c(z, x) + \tfrac{1}{n}.$$

By Lemma 1.6, $|\dot\gamma_n(0)| < A$ for all n. We can assume that $\dot\gamma_n(0) \xrightarrow{n} w \in T_xM$. If $-n \le s \le 0$, then $s > T_n$ and

$$\begin{aligned} A_{L+c}\big(\gamma_n|_{[T_n,s]}\big) &+ \Phi_c(\gamma_n(s), x) \le \\ &\le A_{L+c}\big(\gamma_n|_{[T_n,s]}\big) + A_{L+c}\big(\gamma_n|_{[s,0]}\big) \\ &\le h_c(z, x) + \tfrac{1}{n} \\ &\le h_c(z, \gamma_n(s)) + \Phi_c(\gamma_n(s), x) + \tfrac{1}{n}, \qquad \text{for } -n \le s < 0. \end{aligned}$$

Taking $\liminf_{n\to\infty}$, we get that

$$h_c(z, x_w(s)) + A_{L+c}\big(x_w|_{[s,0]}\big) = h_c(z, x).$$

Hence $w \in \Gamma^-(u)$. □

Remarks 4.3

1. Observe that, since $\Phi_c(x, x) = 0$,

$$u \prec L + c \iff u(x) = \inf_{z\in M} u(z) + \Phi_c(z, x).$$

2. Item 4.3.1 implies that the function h_c in Proposition 4.2 can not be replaced by Φ_c. In fact, the function $u_z(x) = \Phi_c(z,x)$ satisfies $u_z \prec L + c$, but in general $u \notin \mathfrak{S}^-$, if z is not properly chosen.
3. For any $z \in M$ the function $u_z(x) = h_c(z,x) \in \mathfrak{S}^-$ and $v_z(x) := -h_c(x,z) \in \mathfrak{S}^+$.
4. If $p \in \mathcal{A}$ then $u_p(x) := \Phi_c(p,x) \in \mathfrak{S}^-$, because

$$\Phi_c(p,x) \le h_c(p,x) \le h_c(p,p) + \Phi_c(p,x) \le \Phi_c(p,x).$$

Similarly, $v_p(x) := -\Phi_c(x,p) \in \mathfrak{S}^+$.

Corollary 4.4 *If M is compact and $u : M \to \mathbb{R}$ is continuous, then*

1. $$u \in \mathfrak{S}^- \iff u(x) = \min_{p\in\mathcal{A}} u(p) + \Phi_c(p,x).$$
2. $$u \in \mathfrak{S}^+ \iff u(x) = \min_{p\in\mathcal{A}} u(p) - \Phi_c(x,p).$$

Proof. We only prove item 1. Observe that if $u \prec L + c$, then

$$\begin{aligned} v(x) :&= \min_{z\in M}\ u(z) + h_c(z,x) \\ &= \min_{z\in M}\min_{p\in\mathcal{A}}\ u(z) + \Phi_c(z,p) + \Phi_c(p,x) \\ &= \min_{p\in\mathcal{A}}\ u(p) + \Phi_c(p,x) =: w(x). \end{aligned}$$

If $u = w$, then $u \prec L + c$, because

$$u(y) = \min_{p\in\mathcal{A}} u(p) + \Phi_c(p,y) \le \min_{p\in\mathcal{A}} u(p) + \Phi_c(p,x) + \Phi_c(x,y) = u(x) + \Phi_c(x,y).$$

Then $u = v \in \mathfrak{S}^-$ by Proposition 4.2.

Now suppose that $u \in \mathfrak{S}^-$. Since $u \prec L + c$ then $u \le w$. Let $x \in M$ and choose $v \in \Sigma^-$ such that

$$u(x) - u(\gamma(t)) = A_{L+c}(\gamma|_{[t,0]}) = \Phi_c(\gamma(t),x) \quad \text{for } t < 0. \tag{14}$$

Choose $p \in \pi[\alpha\text{-lim}(v)] \subset \mathcal{A}$, and $t_n \to -\infty$ such that $\gamma(t_n) \xrightarrow{n} p$. Using $t = t_n$ on equation (14), we have that

$$u(x) = u(p) + \Phi_c(p,x) \ge w(x).$$

Thus $u = w$. □

Remarks 4.5

1. If M is compact, $u, v \in \mathfrak{S}^-$ and $u|_{\mathcal{A}} = v|_{\mathcal{A}}$, then $u = v$.
2. Observe that if $u \prec L + c$ and $d_c(p,q) = 0$ then $u(q) = u(p) + \Phi_c(q,p)$, because

$$u(q) \le u(p) + \Phi_c(p,q) \le u(q) + \Phi_c(q,p) + \Phi_c(p,q) = u(q).$$

3. By item 2, if M is compact, the values of $u \in \mathfrak{S}^-$ on only one point of each static class determine u.

Let $\boldsymbol{\Gamma} = \mathcal{A}/d_c$ be the set of static classes of L. For each $\gamma \in \boldsymbol{\Gamma}$ choose $p_\Gamma \in \boldsymbol{\Gamma}$ and let $\mathbb{A} = \{ P_\Gamma \,|\, \Gamma \in \boldsymbol{\Gamma} \}$. We say that a function $f : \mathbb{A} \to \mathbb{R}$ is *dominated* ($f \prec L + c$) if $\quad f(p) \le f(q) + \Phi_c(q,p), \quad$ for all p, $q \in \mathbb{A}$.

Corollary 4.6 *If M is compact, the map $\{ f : \mathbb{A} \to \mathbb{R} \,|\, f \prec L + c \} \to \mathfrak{S}^-$,*

$$f \longmapsto u_f(x) := \inf_{p \in \mathcal{A}} f(p) + \Phi_c(p,x),$$

is a bijection.

Proof. By Remark 4.5.3, the map is surjective. The injectivity follows from

$$u_f(p) = \min_{q \in \mathbb{A}} f(q) + \Phi_c(q,p) = f(p) \qquad \forall p \in \mathbb{A},$$

because f is dominated. □

Corollary 4.7 *If M is compact and there is only one static class, then $\mathfrak{S}^-$ (resp. $\mathfrak{S}^+$) is unitary modulo an additive constant.*

This characterization of weak KAM solutions allows us to recover the following theorem: We say that two weak KAM solutions $u_- \in \mathfrak{S}^-$ and $u_+ \in \mathfrak{S}^+$ are *conjugate* if $u_- = u_+$ on $\mathcal{A}$ and denote it by $u_- \sim u_+$.

Corollary 4.8 (Fathi [10]) *If M is compact, then*

$$h(x,y) = \sup_{\substack{u_\mp \in \mathfrak{S}^\mp \\ u_- \sim u_+}} \{ u_-(y) - u_+(x) \}.$$

Proof. If $u_+ \sim u_-$ and $p \in \mathcal{A}$, from the domination we get that

$$\begin{aligned} u_+(p) &\le u_+(x) + \Phi_c(x,p), \\ u_-(y) &\le u_-(p) + \Phi_c(p,y). \end{aligned}$$

Adding these equations and using that $u_+(p) = u_-(p)$, we get that

$$u_-(y) - u_+(x) \le \Phi_c(x,p) + \Phi_c(p,y).$$

Taking $\inf_{p \in \mathcal{A}}$ and then $\sup_{u_+ \sim u_-}$ we obtain

$$\sup_{u_+ \sim u_-} \big\{ u_-(y) - u_+(x) \big\} \le h(x,y).$$

On the other hand, let $u_+(z) := -h(z,y)$ and

$$\begin{aligned} &(15) & u_-(z) :&= \min_{q \in \mathcal{A}} \{ u_+(q) + \Phi_c(q,z) \} \\ && &= \min_{q \in \mathcal{A}} \big\{ -h(q,y) + \Phi_c(q,z) \big\} \\ &(16) & &= \min_{q \in \mathcal{A}} \big\{ -\Phi_c(q,y) + \Phi_c(q,z) \big\} \end{aligned}$$

From Remark 4.3.3 and Corollary 4.6, $u_\pm \in \mathfrak{S}^\pm$. Since u_+ is dominated, from (15) we get that $u_+ \sim u_-$. From (16), $u_-(y) = 0$ and hence $u_-(y) - u_+(x) = h(x,y)$. □

When $h_c = +\infty$, we use another method to obtain weak KAM solutions, resembling the constructions of Busemann functions in riemannian geometry. In [4] we proved that $\Sigma^+ \neq \varnothing$ and $\Sigma^- \neq \varnothing$ even when M is non-compact. We call the functions of Proposition 0.3 *weak KAM Busemann* functions.

Proposition 0.3

1. *If* $w \in \Sigma^-(L)$ *and* $\gamma(t) = x_w(t)$*, then*
$$\begin{aligned} u_w(x) &= \inf_{t<0} \left[\Phi_c(\gamma(t), x) - \Phi_c(\gamma(t), \gamma(0))\right] \\ &= \lim_{t\to-\infty} \left[\Phi_c(\gamma(t), x) - \Phi_c(\gamma(t), \gamma(0))\right] \end{aligned}$$
is in $\mathfrak{S}^-$.
2. *If* $w \in \Sigma^+(L)$ *and* $\gamma(t) = x_w(t)$*, then*
$$\begin{aligned} u_w(x) &= \sup_{t>0} \left[\Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t))\right] \\ &= \lim_{t\to+\infty} \left[\Phi_c(\gamma(0), \gamma(t)) - \Phi_c(x, \gamma(t))\right] \end{aligned}$$
is in $\mathfrak{S}^+$.

Item 2 of Proposition 0.3 was proved in [4]. For completeness, we show here the proof of item 1. To prove Proposition 0.3, we shall need the following graph property. For $v \in TM$, write $x_v(t) = \pi\,\varphi_t(v)$. Given $\varepsilon > 0$, let

$$\Sigma^\varepsilon := \{w \in TM \mid x_w : [0, \varepsilon) \to M \text{ or } x_w : (-\varepsilon, 0] \to M \text{ is semistatic }\}.$$

Theorem 4.9 (Mañé) [14] *For all* $p \in \pi(\widehat{\Sigma})$ *there exists a unique* $\xi(p) \in T_pM$ *such that* $(p, \xi(p)) \in \Sigma^\varepsilon$*, in particular* $(p, \xi(p)) \in \widehat{\Sigma}$ *and* $\widehat{\Sigma} = $ *graph*(ξ).

Moreover, the map $\xi : \pi(\widehat{\Sigma}) \to \Sigma$ *is Lipschitz.*

Proof of Proposition 0.3. We only prove item 1. We start by showing that the function $\delta(t) = \Phi_c(\gamma(t), x) - \Phi_c(\gamma(t), 0)$ is increasing. If $s < t$, then

$$\begin{aligned} \delta(t) - \delta(s) &= \Phi_c(\gamma(t), x) - \Phi_c(\gamma(s), x) + \big[\,\Phi_c(\gamma(s), \gamma(0)) - \Phi_c(\gamma(t), \gamma(0))\,\big] \\ &= \Phi_c(\gamma(t), x) - \Phi_c(\gamma(s), x) + \Phi_c(\gamma(s), \gamma(t)) \\ &\geq 0, \end{aligned}$$

where the last inequality follows from the triangle inequality applied to the triple $(\gamma(s), \gamma(t), x)$. By the triangle inequality, $\delta(t) \leq \Phi_c(\gamma(0), x)$, hence $\lim_{t\downarrow-\infty} \delta(t) = \inf_{t<0} \delta(t)$ and this limit is finite.
Since

$$\begin{aligned} u(y) &= \inf_{t<0} \Phi_c(\gamma(t), y) - \Phi_c(\gamma(t), \gamma(0)) \\ &\leq \inf_{t<0} \Phi_c(\gamma(t), x) + \Phi_c(x, y) - \Phi_c(\gamma(t), \gamma(0)) \\ &= u(x) + \Phi_c(x, y), \end{aligned}$$

then $u \prec L + c$.

Suppose that $x \in \mathcal{A} \neq \varnothing$. Let $(x, v) \in \widehat{\Sigma}$ and $t < 0$. Let $p = x_v(t)$ and $y \in M$. Since $d_c(x, p) = 0$, then

$$\begin{aligned} \Phi_c(y, x) &= \Phi(y, x) + \Phi_c(x, p) + \Phi_c(p, x) \\ &\geq \Phi_c(y, p) + \Phi_c(p, x) \geq \Phi_c(y, x). \end{aligned}$$

Hence $\Phi_c(y, x) = \Phi_c(y, p) + \Phi_c(p, x)$. For $y = \gamma(s)$ (and $p = x_v(t)$), we have that

$$\begin{aligned} u(x) - u(x_v(t)) &= \lim_{s \to +\infty} [\Phi_c(\gamma(s), x) - \Phi_c(\gamma(s), x_v(t))] = \Phi_c(x_v(t), x) \\ &= A_{L+c}\left(x_v|_{[t,0]}\right). \end{aligned}$$

Now let $x \in M \setminus \mathcal{A}$ and choose $y_n : [T_n, 0] \to M$ a Tonelli minimizer such that $y_n(T_n) = \gamma(\text{-}n)$, $y_n(0) = x$ and

$$A_{L+c}(y_n|_{[T_n,0]}) \leq \Phi_c(\gamma(\text{-}n), x) + \tfrac{1}{n}.$$

This implies that

$$\text{(17)} \qquad A_{L+c}(y_n|_{[s,t]}) \leq \Phi_c(y_n(s), y_n(t)) + \tfrac{1}{n}, \qquad \text{for } T_n \leq s < t \leq 0.$$

By Lemma 1.6, $|\dot{y}_n| < A$. We can assume that $\dot{y}_n(0) \to v \in T_x M$. Then

$$\text{(18)} \qquad A_{L+c}(x_v|_{[t,0]}) = \Phi_c(\gamma(t), x) \qquad \text{for } \liminf_n T_n \leq t \leq 0.$$

We prove below that $\lim_n T_n = -\infty$. Then $v \in \Sigma^-(L)$. Observe that for $T_n \leq s \leq 0$ we have that

$$\begin{aligned} \Phi_c(\gamma(\text{-}n), x) &\leq \Phi_c(\gamma(\text{-}n), y_n(s)) + \Phi_c(y_n(s), x) \leq A_{L+c}(y_n|_{[T_n,0]}) \\ &\leq \Phi_c(\gamma(\text{-}n), x) + \tfrac{1}{n}. \end{aligned}$$

Since $y \mapsto \Phi_c(z, y)$ is uniformly Lipschitz, we obtain that

$$\begin{aligned} u(x) &= \lim_n \Phi_c(\gamma(\text{-}n), x) - \Phi_c(\gamma(\text{-}n), \gamma(0)) \\ &= \lim_n \Phi_c(\gamma(\text{-}n), x_v(s)) + \Phi_c(x_v(s), x) - \Phi_c(\gamma(\text{-}n), \gamma(0)) \\ &= u(x_v(s)) + \Phi_c(x_v(s), x) \qquad \text{for all } s < 0. \\ &= u(x_v(s)) + A_{L+c}(x_v|_{[s,0]}) \qquad \text{because } v \in \Sigma^-. \end{aligned}$$

Now we prove that $\lim_n T_n = -\infty$. Suppose, for simplicity, that $\lim_n T_n = T_0 > -\infty$. Since $\dot{y}_n(0) \to v$, then $y_n|_{[T_n,0]} \xrightarrow{C^1} x_v|_{[T_0,0]}$ and hence $\gamma(\text{-}n) = y_n(T_n) \to x_v(T_0) =: p$. Since by Lemma 1.6 $|\dot{\gamma}|$ is bounded, we can assume that $\lim_n \dot{\gamma}(\text{-}n) = (p, w_1)$. Then $w_1 \in \alpha\text{-lim}(\dot{\gamma}) \subseteq \widehat{\Sigma}$. From (18), $\dot{x}_v(T_0) \in \Sigma^\varepsilon$. Since $\pi(w_1) = x_v(T_0) = p$, then Lemma 4.9 implies that $\dot{x}_v(T_0) \in \widehat{\Sigma}$. Since $\widehat{\Sigma}$ is invariant, then $v \in \widehat{\Sigma}$ and hence $x = \pi(v) \in \pi(\widehat{\Sigma}) = \mathcal{A}$. This contradicts the hypothesis $x \in M \setminus \mathcal{A}$.

□

5 The extended static classes

The method in Proposition 0.3 resembles the construction of Busemann functions in complete manifolds of non-positive curvature. In that case, Ballmann, Gromov and Schroeder [1] proved that the manifold can be compactified adjoining the sphere at infinity that can be defined in terms of Busemann functions.

Here we emulate that construction to obtain a compactification of the manifold $M/_{d_c}$ that identifies the points in the Aubry set which are in the same static class and adjoins what we call the *extended Aubry set* $\mathfrak{A}^{\mp}$. By definition of Busemann function, the extended static classes in $\mathfrak{B}^{\mp}$ correspond to the α-limits (resp. ω-limits) of semistatic orbits in the compactification. But as we shall see in Example 5.4 the classes in $\mathfrak{A}^{\mp} \setminus \mathfrak{B}^{\mp}$ do not correspond to α or ω limits of orbits in TM.

On $C^0(M, \mathbb{R})$ we use the topology of uniform convergence on compact subsets. Consider the equivalence relation on $C^0(M, \mathbb{R})$ defined by $f \sim g$ if $f - g$ is constant. Let $\mathcal{F} := C^0(M, \mathbb{R})/_\sim$ with the quotient topology.

Let $\mathfrak{M}^-$ be the closure in $\mathcal{F}$ of $\{\, f(x) = \Phi_c(z, x) \,|\, z \in M \,\}/_\sim$ and $\mathfrak{M}^+$ the closure in $\mathcal{F}$ of $\{\, g(x) = \Phi_c(x, z) \,|\, z \in M \,\}$. Fix a point $0 \in M$. We can identify

$$\mathcal{F} \approx \{\, f \in C^0(M, \mathbb{R}) \,|\, f(0) = 0 \,\}.$$

Lemma 5.1 *$\mathfrak{M}^-$ and $\mathfrak{M}^+$ are compact.*

Proof. Observe that the functions in $\mathfrak{M}^-$ and $\mathfrak{M}^+$ are dominated. By Lemma 3.1.1 the families $\mathfrak{M}^-$ and $\mathfrak{M}^+$ are equicontinuous. Since M is separable by Arzelá-Ascoli theorem $\mathfrak{M}^-$ and $\mathfrak{M}^+$ are compact in the topology of uniform convergence on compact subsets. □

Then $\mathfrak{M}^-$ is the closure of the classes of the functions

$$f_z(x) := \Phi_c(z, x) - \Phi_c(z, 0), \qquad \forall\, x \in M$$

and $\mathfrak{M}^+$ is the closure of the classes of

$$g_z(x) := \Phi_c(x, z) - \Phi_c(0, z), \qquad \forall\, x \in M.$$

Lemma 5.2

1. *If $f_w(x) = f_z(x)$ for all $x \in M$, then $d_c(w, z) = 0$.*
2. *If $g_w(x) = g_z(x)$ for all $x \in M$, then $d_c(w, z) = 0$.*

Proof. We only prove item 1. Suppose that $f_z = f_w$. From $f_z(z) = f_w(z)$ we get that

$$\Phi_c(w, z) = \Phi_c(w, 0) - \Phi_c(z, 0),$$

and from $f_z(w) = f_z(w)$ we get

$$\Phi_c(z, w) = -\Phi_c(w, 0) + \Phi_c(z, 0).$$

Adding these equations we get that $d_c(z, w) = 0$.

Conversely, if $d_c(z,w)=0$ and $x\in M$, then

$$\Phi_c(w,x)\le d_c(w,z)+\Phi_c(z,x)=\Phi_c(z,x)-\Phi_c(z,w)\le\Phi_c(w,x).$$

Thus $\Phi_c(w,x)=\Phi_c(w,z)+\Phi_c(z,x)$ for all $x\in M$. This implies that $f_z=f_w$. □

Then we have embeddings $M/_{d_c}\hookrightarrow\mathfrak{M}^-$, by $z\mapsto[f_z]\in\mathcal{F}$ and $M/_{d_c}\hookrightarrow\mathfrak{M}^+$ by $z\mapsto[g_z]\in\mathcal{F}$, where $M/_{d_c}$ is the quotient space under the equivalence relation $x\equiv y$ if $d_c(x,y)=0$. Let $\mathfrak{B}^-$ be the functions defined in Proposition 0.3.1 and $\mathfrak{B}^+$ those of 0.3.2. Let $\boldsymbol{\mathfrak{B}}^+=\mathfrak{B}^+/_\sim$ and $\boldsymbol{\mathfrak{B}}^-=\mathfrak{B}^-/_\sim$.

Remark 5.3 By Proposition 0.3, if $p\in\mathcal{A}\neq\varnothing$ then $u_-(x):=\Phi_c(p,x)\in\boldsymbol{\mathfrak{B}}^-$ and $u_+(x):=-\Phi_c(x,p)\in\boldsymbol{\mathfrak{B}}^+$ (modulo an additive constant).

Observe that $d_c(z,w)=0$ if and only if $z=w$ or $z,w\in\mathcal{A}$ and they are in the same static class. Under the identifications $M\hookrightarrow\mathfrak{M}^\mp$ we have that $\boldsymbol{\mathfrak{B}}^\mp\cup(M\setminus\mathcal{A})\subseteq\mathfrak{M}^\mp$ respectively. But this inclusion may be strict as the following example shows:

Example 5.4 $\boldsymbol{\mathfrak{B}}^-\cup(M\setminus\mathcal{A})\neq\mathfrak{M}^-$.

Let $M=\mathbb{R}$ and $L(x,v):=\frac{1}{2}v^2-\cos(2\pi x)$, corresponding to the universal cover of the simple pendulum lagrangian. Then $c(L)=1$, and the static orbits are the fixed points $(2k+1,0)\in T\mathbb{R}$, $k\in\mathbb{Z}$. Moreover, $H(x,p)=\frac{1}{2}p^2+\cos(2\pi x)$ and the Hamilton-Jacobi equation $H(x,d_xu)=c(L)$ gives $d_xu=\pm2\sqrt{1-\cos(2\pi x)}$. The function

$$u(x)=\int_0^x 2\sqrt{1-\cos(2\pi s)}\,ds,$$

with $d_xu\equiv+2\sqrt{1-\cos(2\pi x)}$, is in $\mathfrak{S}^-$, is the limit of $u_n(x):=\Phi_c(-n,x)-\Phi_c(-n,0)$ but it is not a Busemann function associated to a semistatic orbit γ because if $\gamma(-\infty)=2k+1\in\mathbb{Z}$ is the α-limit of γ, then the Busemann function b_γ associated to γ satisfies

$$d_xb_\gamma=\begin{cases}+2\sqrt{1-\cos(2\pi x)} & \text{if } x\ge\gamma(-\infty),\\ -2\sqrt{1-\cos(2\pi x)} & \text{if } x\le\gamma(-\infty).\end{cases}\tag{19}$$

Similarly a function $v:\mathbb{R}\to\mathbb{R}$ with $d_xv\equiv-2\sqrt{1-\cos(2\pi x)}$ is in $\mathfrak{S}^+$ but it is not a Busemann function.

Observe that in the Busemann function in (19), at the point $y=\gamma(-\infty)+3$ the semistatic orbit $\eta(t)$ with $\dot\eta(0)=\Gamma^-(u)\cap T_yM$ has α-limit $\eta(-\infty)=\gamma(-\infty)+2\neq\gamma(-\infty)$. Moreover, the Busemann function b_η associated to η satisfies

$$d_xb_\eta=\begin{cases}+2\sqrt{1-\cos(2\pi x)} & \text{if } x\ge\gamma(-\infty)+2,\\ -2\sqrt{1-\cos(2\pi x)} & \text{if } x\le\gamma(-\infty)+2;\end{cases}$$

so that $b_\eta \neq b_\gamma$. In fact, there is no semistatic orbit passing through y with α-limit $\gamma(-\infty)$. This implies that the Busemann functions can not be parametrized just by a (semistatic) vector based on a unique point $0 \in M$ as in the riemannian case. In particular, it may not be possible to choose a single point $q_\alpha \equiv 0 \in M,\ \forall \alpha \in \mathfrak{B}^-$ in the construction for Theorem 0.4. □

The functions in $\mathfrak{B}^-$ and $\mathfrak{B}^+$ are special among the weak KAM solutions. They are "directed" towards a single static class and they are the most regular in the following sense:

Lemma 5.5

1. *If $w \in \Sigma^-$ and $u_w \in \mathfrak{B}^-$ is as in Proposition 0.3.1, then*

$$u_w(x) = \max\{\, u(x) \,|\, u \in \mathfrak{S}^-,\ u(\pi(w)) = 0,\ w \in \Gamma^-(u) \,\}.$$

2. *If $w \in \Sigma^+$ and $u_w \in \mathfrak{B}^+$ is as in Proposition 0.3.2, then*

$$u_w(y) = \min\{\, u(y) \,|\, u \in \mathfrak{S}^+,\ u(\pi(w)) = 0,\ w \in \Gamma^+(u) \,\}.$$

By the Remark 5.3, this also holds for the functions $u_-(x) = \Phi_c(p,x)$ and $u_+(x) = -\Phi_c(x,p)$ (modulo an additive constant), for any $p \in \mathcal{A}$.

Proof. We prove item 1. Let $x := \pi(w)$ and $v \in \mathfrak{S}^-$ with $v(x) = u_w(x) = 0$ and $w \in \Gamma^-(v)$. Let $x_w(t) = \pi(\Phi_t(w))$. Since $v \prec L + c$ and $w \in \Gamma^-(v)$, then for $t < 0$, we have that

$$\begin{aligned} v(y) &\leq v(x_w(t)) + \Phi_c(x_w(t), y) \\ &= v(x) - \Phi_c(x_w(t), x) + \Phi_c(x_w(t), y). \end{aligned}$$

Since $v(x) = u_w(x) = 0$, letting $t \downarrow -\infty$, we get that $v(y) \leq u_w(y)$ for all $y \in M$. On the other hand, u_w is in the set of such u's, so that the maximum is realized by u_w. □

Define

$$\mathfrak{A}^- := \mathfrak{M}^- \setminus \big[(M - \mathcal{A})/_{d_c}\big] \quad , \qquad \mathfrak{A}^+ := \mathfrak{M}^+ \setminus \big[(M - \mathcal{A})/_{d_c}\big].$$

Proposition 5.6 *The functions in $\mathfrak{A}^-$ and $\mathfrak{A}^+$ are weak KAM solutions.*

Proof. Let $u \in \mathfrak{M}^- \setminus (M \setminus \mathcal{A})/_{d_c}$. Since u is dominated, we only have to prove the condition 3.4.2. Adding a constant, we can assume that $u(0) = 0$. Then there is a sequence $z_n \in M$ such that $u(x) = \lim_n \Phi_c(z_n, x) - \Phi_c(z_n, 0)$. Let $x \in M$ and let $\gamma_n \in \mathcal{C}_{T_n}(z_n, x)$ be a Tonelli minimizer such that $T_n < 0$, $\gamma_n(0) = x$, $\gamma_n(T_n) = z_n$ and $A_{L+c}(\gamma_n) \leq \Phi_c(z_n, x) + \frac{1}{n}$. In particular

$$\Phi_c(z_n, \gamma_n(t)) + A_{L+c}\big(\gamma_n|_{[t,0]}\big) \leq A_{L+c}(\gamma_n) \leq \Phi_c(z_n, x) + \tfrac{1}{n}, \quad \forall\ T_n \leq t \leq 0.$$

Since $u \in \mathfrak{A}^-$, then we can assume that either $d_M(z_n, x) \to \infty$ or $z_n \to p \in \mathcal{A}$. Since by Lemma 1.6 $|\dot{\gamma}_n| < A$ and $h_c(p,p) = 0$ for $p \in \mathcal{A}$, in either case we can assume that $T_n \to -\infty$.

We can assume that $\dot\gamma_n(0)\to v\in T_xM$. Then for $t\le 0$,

$$
\begin{aligned}
u(x_v(0))-u(x_v(t)) &= \lim_n \Phi_c(z_n,x)-\Phi_c(z_n,x_v(t))\\
&= \lim_n \Phi_c(z_n,x)-\Phi_c(z_n,\gamma_n(t))+K\,d_c(\gamma_n(t),x_v(t))\\
&\ge \lim_n A_{L+c}\big(\gamma_n|_{[t,0]}\big)-\tfrac{1}{n}\\
&\ge A_{L+c}\big(x_v|_{[t,0]}\big),
\end{aligned}
$$

where K is a Lipschitz constant for Φ_c.

□

For $p\in\mathfrak{B}$ and $z\in M$ let $x\mapsto b_{p,z}(x)$ be the function in the class $p\in\mathfrak{B}$ such that $b_{p,z}(z)=0$, i.e.

$$
b_{p,z}(x)=\lim_{y\to p}\Phi_c(y,x)-\Phi_c(y,z).
$$

We now give a characterization of weak KAM solutions similar to that of Corollary 4.6. For each $\alpha\in\mathfrak{B}^-$ choose $q_\alpha\in M$ such that there is a unique semistatic vector $v\in\Sigma^-$ such that $\pi(v)=q$ and the α-limit of v is in the static class α. This can be done by the graph property 3.6.4. Moreover, choose them such that the map $\mathfrak{B}^-\ni\alpha\mapsto q_\alpha\in M$ is injective. Let $\mathbb{A}:=\{\,q_\alpha\,|\,\alpha\in\mathfrak{B}^-\,\}$. We say that a function $f:\mathbb{A}\to M$ is *strictly dominated* if

$$
f(q_\alpha)<f(q_\beta)+b_{\beta,q_\beta}(q_\alpha)
$$

for all $\alpha\neq\beta$ in $\mathfrak{B}^-$. And we say that f is *dominated* if $f(q_\alpha)\le f(q_\beta)+b_{\beta,q_\beta}(q_\alpha)$ for all $\alpha\neq\beta$ in $\mathfrak{B}^-$.

0.4. Theorem. *The map* $\{\,f:\mathbb{A}\to\mathbb{R}\,|\,f$ *strictly dominated*$\}\to\{u\in\mathfrak{S}^-\,|\,u|_{\mathbb{A}}$ *strictly dominated*$\}$, $f\mapsto u_f$, *given by*

$$
u_f(x):=\inf_{\alpha\in\mathfrak{B}^-} f(q_\alpha)+b_{\alpha,q_\alpha}(x),
$$

is a bijection.

Proof. We first prove that u_f is bounded below. The domination condition implies that $u_f(q_\alpha)=f(q_\alpha)$ for all $\alpha\in\mathfrak{B}^-$. Then the same argument as in formula (13), shows that $u_f\prec L+c$. Fix $\alpha\in\mathfrak{B}^-$, then for all $x\in M$,

$$
u_f(x)\ge u_f(q_\alpha)-\Phi_c(q_\alpha,x)=f(q_\alpha)-\Phi_c(q_\alpha,x)>-\infty. \tag{20}
$$

Since $u_f>-\infty$ and it is an infimum of weak KAM solutions, from Lemma 4.1 we get that $u_f\in\mathfrak{S}^-$. Since $u_f(q_\alpha)=f(q_\alpha)$ for all $\alpha\in\mathfrak{B}^-$, the map $f\mapsto u_f$ is injective.

We now prove the surjectivity. Suppose that $u\in\mathfrak{S}^-$ and $u|_{\mathbb{A}}$ is strictly dominated. Let

$$
v(x):=\inf_{\alpha\in\mathfrak{B}^-} u(q_\alpha)+b_{\alpha,q_\alpha}(x).
$$

Observe that the domination condition implies that

$$v(q_\alpha) = u(q_\alpha) \quad \text{for all } \alpha \in \mathfrak{B}^-. \tag{21}$$

Given $x \in M$, let $\theta \in \Gamma^-(u) \cap T_x M$ and let $\alpha \in \mathfrak{B}^-$ be the α-limit of θ. Then,

$$u(x) = u(x_\theta(s)) + \Phi_c(x_\theta(s), x) \qquad \text{for all } s < 0.$$

Since u is dominated, $u(q_\alpha) \le u(x_\theta(s)) + \Phi_c(x_\theta(s), q_\alpha)$. Hence

$$u(x) \ge u(q_\alpha) - \Phi_c(x_\theta(s), q_\alpha) + \Phi_c(x_\theta(s), x) \qquad \text{for all } s < 0.$$

Taking the limit when $s \to -\infty$, we get that

$$u(x) \ge v(x) \qquad \text{for all } x \in M. \tag{22}$$

Now we prove that $u = v$ on the projection of the backward orbits of vectors in $\Gamma^-(u)$ ending at the points q_α, $\alpha \in \mathfrak{B}^-$. Let $\xi \in \Gamma^-(u) \cap T_{q_\alpha} M$ and let $\beta \in \mathfrak{B}^-$ be the α-limit of ξ. From the definition of $v(x)$ for all $\varepsilon > 0$ and $s < 0$ there exists $\gamma = \gamma(s, \varepsilon) \in \mathfrak{B}^-$ such that

$$v(x_\xi(s)) \ge v(q_\gamma) + b_{\gamma, q_\gamma}(x_\xi(s)) - \varepsilon.$$

Since $\xi \in \Gamma^-(u) \cap T_{q_\alpha} M$, then for $s < 0$,

(23)

$$\begin{aligned}
u(q_\alpha) &= u(x_\xi(s)) + \Phi_c(x_\xi(s), q_\alpha) \\
\text{(24)} \quad &\ge v(x_\xi(s)) + \Phi_c(x_\xi(s), q_\alpha) \qquad \text{by (22)} \\
&\ge v(q_\gamma) + b_{\gamma, q_\gamma}(x_\xi(s)) - \varepsilon + \Phi_c(x_\xi(s), q_\alpha) \\
&= \lim_{t \to -\infty} v(q_\gamma) + \Phi_c(x_\xi(t), x_\xi(s)) + \Phi_c(x_\xi(s), q_\alpha) - \Phi_c(x_\xi(t), q_\gamma) - \varepsilon \\
&\ge v(q_\gamma) + \lim_{t \to -\infty} \Phi_c(x_\xi(t), q_\alpha) - \Phi_c(x_\xi(t), q_\gamma) - \varepsilon \\
&\ge v(q_\gamma) + b_{\gamma, q_\gamma}(q_\alpha) - \varepsilon \\
\text{(25)} \quad &\ge v(q_\alpha) - \varepsilon \\
&= u(q_\alpha) - \varepsilon. \qquad \text{by (21).}
\end{aligned}$$

Letting $\varepsilon \downarrow 0$, from the equality between (24) and (25) we get that

$$v(q_\alpha) = v(x_\xi(t)) + \Phi_c(x_\xi(t), q_\alpha) \qquad \text{for all } t < 0. \tag{26}$$

But then

$$v(q_\beta) \le v(x_\xi(t)) + \Phi_c(x_\xi(t), q_\beta) = v(q_\alpha) - \Phi_c(x_\xi(t), q_\alpha) + \Phi_c(x_\xi(t), q_\beta).$$

Equivalently

$$v(q_\alpha) \ge v(q_\beta) + \Phi_c(x_\xi(t), q_\alpha) - \Phi_c(x_\xi(t), q_\beta).$$

Taking the limit when $t \to -\infty$, we get that $v(q_\alpha) \geq v(q_\beta) + b_{\beta,q_\beta}(q_\alpha)$. This contradicts the strict domination, hence $\beta = \alpha$. Then, from the equality between (23) and (24), we have that

$$(27) \quad u(x_\xi(t)) = v(x_\xi(t)) \quad \text{for all } t < 0, \text{ and } \xi \in \Sigma^- \cap T_{q_\alpha}M, \ \alpha\text{-lim}(\xi) = \alpha.$$

Now let $x \in M$ and $\alpha \in \mathfrak{B}^-$. Let $\xi \in \Sigma^- \cap T_{q_\alpha}M$ with α-lim$(\xi) = \alpha$. Then for $t < 0$,

$$\begin{aligned} u(x) &\leq u(x_\xi(t)) + \Phi_c(x_\xi(t), x) \\ &= v(x_\xi(t)) + \Phi_c(x_\xi(t), x) && \text{by (27)} \\ &= v(q_\alpha) - \Phi_c(x_\xi(t), q_\alpha) + \Phi_c(x_\xi(t), x), && \text{by (26).} \end{aligned}$$

Letting $t \to -\infty$, we have that

$$u(x) \leq v(q_\alpha) + b_{\alpha,q_\alpha}(x) = u(q_\alpha) + b_{\alpha,q_\alpha}(x).$$

Since $\alpha \in \mathfrak{B}^-$ is arbitrary, from the definition of v we get that $u \leq v$. □

Theorem 5.7 *Given* $u \in \mathfrak{S}^-$*, for all* $\alpha \in \mathfrak{B}^-(u)$ *choose* $q_\alpha \in \pi[\Lambda_0^-(\alpha) \cap \Gamma_0^-(u)]$*, and let* $\mathbb{A}(u) := \{q_\alpha \mid \alpha \in \mathfrak{B}^-(u)\}$*. Then*

$$u(x) = \inf_{q_\alpha \in \mathbb{A}(u)} u(q_\alpha) + b_{\alpha,q_\alpha}(x) \qquad \textit{for all } x \in M.$$

Proof. Let $u \in \mathfrak{S}^-$. For all $\alpha \in \mathfrak{B}^-(u)$, choose $q_\alpha \in \pi(\Lambda_0^-(\alpha) \cap \Gamma_0^-(u))$. Let $\mathbb{A}(u) := \{\, q_\alpha \,|\alpha \in \mathfrak{B}^-(u)\,\}$. We show that $u|_{\mathbb{A}(u)}$ is dominated. Let $\alpha, \beta \in \mathfrak{B}^-(u)$ and let $\theta \in T_{q_\beta}M \cap \Lambda^-(\beta) \cap \Gamma^-(u)$. Then for $t < 0$,

$$\begin{aligned} u(q_\alpha) &\leq u(x_\theta(t)) + \Phi_c(x_\theta(t), q_\alpha) \\ &= u(q_\beta) - \Phi_c(x_\theta(t), q_\beta) + \Phi_c(x_\theta(t), q_\alpha). \end{aligned}$$

Letting $t \to -\infty$, we get that $u(q_\alpha) \leq u(q_\beta) + b_{\beta,q_\beta}(q_\alpha)$, for all $\alpha, \beta \in \mathfrak{B}^-(u)$.

Let

$$(28) \qquad v(x) := \inf_{q_\alpha \in \mathbb{A}(u)} u(q_\alpha) + b_{\alpha,q_\alpha}(x).$$

The same arguments as in equation (20) show that $v > -\infty$ and by Lemma (4.1) $v \in \mathfrak{S}^-$.

Given $x \in M$, let $\theta \in \Gamma^-(u) \cap T_xM$ and let $\alpha \in \mathfrak{B}^-(u)$ be the α-limit of θ. Then,

$$\begin{aligned} &u(x) = u(x_\theta(s)) + \Phi_c(x_\theta(s), x) && \text{for all } s < 0, \\ &u(x) \geq u(q_\alpha) - \Phi_c(x_\theta(s), q_\alpha) + \Phi_c(x_\theta(s), x) && \text{because } u \text{ is dominated.} \end{aligned}$$

Since $\alpha \in \mathfrak{B}^-(u)$, taking the limit when $s \to -\infty$, we get that

$$(29) \qquad u(x) \geq v(x) \qquad \text{for all } x \in M.$$

Now let $x \in M$ and $q_\alpha \in \mathbb{A}(u)$. Let $\xi \in \Lambda(\alpha) \cap \Gamma^-(u) \cap T_{q_\alpha}M$. Then for $s < 0$,

$$\begin{aligned} u(x) &\leq u(x_\xi(s)) + \Phi_c(x_\xi(s), x) \\ &= u(q_\alpha) - \Phi_c(x_\xi(s), q_\alpha) + \Phi_c(x_\xi(s), x), \quad \text{because } \xi \in \Gamma^-(u) \cap T_{q_\alpha}M. \end{aligned}$$

Since $\xi \in \Lambda^-(\alpha)$, letting $s \to -\infty$, we have that

$$u(x) \leq u(q_\alpha) + b_{\alpha, q_\alpha}(x).$$

Since $q_\alpha \in \mathbb{A}(u)$ is arbitrary, we get that $u \leq v$. □

6 Examples

Example 6.1 A Lagrangian with $h = +\infty$.

Let $L : T\mathbb{R}^2 \to \mathbb{R}$ be $L(x, v) = \frac{1}{2}|v|^2 + \psi(x)$, where $|\cdot|$ is the euclidean metric on $\mathbb{R}^2$ and $\psi(x)$ is a smooth function with $\psi(x) = \frac{1}{|x|}$ for $|x| \geq 2$, $\psi \geq 0$ and $\psi(x) = 2$ for $0 \leq |x| \leq 1$.

Then

$$c(L) = -\inf \psi = 0,$$

because if γ_n is a smooth closed curve with length $\ell(\gamma_n) = 1$, $|\gamma_n(t)| \geq n$ and energy $E(\gamma_n) = \frac{1}{2}\dot{\gamma}_n^2 - \psi(\gamma_n) \equiv 0$, then

$$\begin{aligned} c(L) &\geq -\inf_{n>0} A_L(\gamma_n) = -\int_0^{T_n} \tfrac{1}{2}\dot{\gamma}_n^2 + \psi(\gamma_n) \\ &= -\int_0^{\frac{1}{|\dot{\gamma}_n|}} |\dot{\gamma}_n|^2 = -|\dot{\gamma}_n| \leq -\sqrt{\tfrac{2}{n}} \longrightarrow 0. \end{aligned}$$

On the other hand,

$$c(L) = -\inf \{ A_L(\gamma) \,|\, \gamma \text{ closed} \} \leq 0,$$

because $L \geq 0$.

Observe that since $L > 0$ and on compact subsets of $\mathbb{R}^2$, $L > a > 0$, then we have that

$$d_c(x, y) = \Phi_c(x, y) > 0 \text{ for all } x, y \in \mathbb{R}^2.$$

Hence $\hat{\Sigma}(L) = \varnothing$.

Suppose that $h(0,0) < +\infty$. Then $u(x) := h(x, 0)$ is in $\mathfrak{S}^+$. Let $\xi \in \Gamma^+(u) \cap T_0\mathbb{R}^2$ and write $x_\xi(t) = (r(t), \theta(t))$ in polar coordinates about the origin $0 \in \mathbb{R}^2$. Then $\limsup_{t \to +\infty} r(t) = +\infty$ because otherwise the orbit of ξ would lie on a compact subset of $E \equiv 0$ and then $\varnothing \neq \omega\text{-lim}(\xi) \subseteq \Sigma(L) = \varnothing$. Moreover,

$$|\dot{x}_\xi(t)| = \sqrt{\tfrac{2}{r(t)}}$$

and

$$L(\varphi_t\, \xi) = |\dot{x}_\xi(t)|^2 = \sqrt{\tfrac{2}{r(t)}}\; |\dot{x}_\xi(t)|.$$

Let $T_n \to +\infty$ be such that $r(T_n) \to +\infty$. Since $L + c = L \geq 0$, then

$$h(0,0) \geq \int_0^{+\infty} L(\varphi_t(\xi)) + c(L) = \int_0^{+\infty} \sqrt{\tfrac{2}{r(t)}}\, \left[\, |\dot r| + r\,|\dot\theta|\,\right]\, dt$$
$$\geq \limsup_{T_n} \int_0^{T_n} \sqrt{\tfrac{2}{r}}\, \dot r\, dt = \limsup_{n} \int_0^{r(T_n)} \sqrt{\tfrac{2}{r}}\, dr = +\infty.$$

□

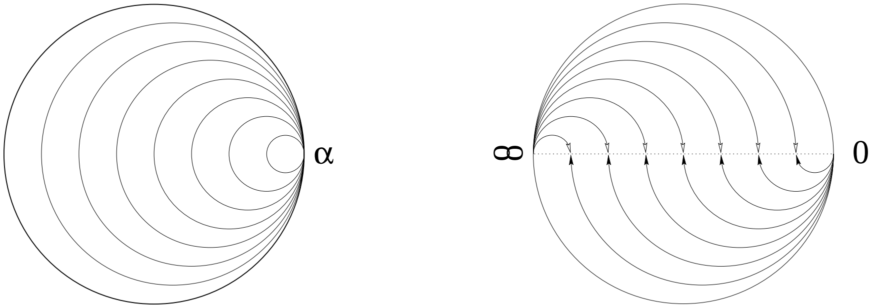

Fig. 2. Example 6.2

Example 6.2 $0 < h < +\infty$, $\widehat{\Sigma} = \varnothing$ and differentiable Busemann functions u with $\mathfrak{B}^-(u) = \mathfrak{B}^+(u) = \{\alpha\}$.

Let $\mathbb{H} := \mathbb{R}\times]0,+\infty[$ with the Poincaré metric $ds^2 = \frac{1}{y^2}\,(dx^2 + dy^2)$. Let $L : T\mathbb{H} \to \mathbb{R}$ be a Lagrangian of the form

$$L(x,v) = \tfrac{1}{2}\ \|v\|_x^2 + \eta_x(v),$$

where η_x is a 1-form on $\mathbb{H}$ such that $d\eta(v)$ is the area form an $|\cdot|_x$ is the Poincaré metric. The Euler-Lagrange equation is

(30) $$\tfrac{D}{dt}\,\dot x = Y_x(\dot x) = \dot x^{\perp},$$

where $Y_x : T\mathbb{H} \to T\mathbb{H}$ is a bundle map such that

$$d\eta_x(u,v) = \langle Y_x(u), v\rangle.$$

The energy function is $E(x,v) = \frac{1}{2}\ \|v\|_x^2$. On the energy levels $E < \frac{1}{2}$ the solutions of (30) are closed curves, and on $E = \frac{1}{2}$ the solutions are the horospheres parametrized by arc length.

Choose the form $\eta(x,y) = \frac{dx}{y}$, where $(x,y) \in \mathbb{H} = \mathbb{R}\times]0,+\infty[$. Then

$$L\big((x,y),(\dot x,\dot y)\big) = \frac{1}{2y^2}\left(\dot x^2 + \dot y^2\right) + \frac{\dot x}{y}\cdot$$

Observe that the form η is bounded in the Poincaré metric, so that the Lagrangian is superlinear and satisfies the boundedness condition.

It can be seen directly from the Euler-Lagrange equation that the curves $\dot{x} = -y$, $\dot{y} = 0$ are solutions with

$$L(\dot{x} = -y, \dot{y} = 0) + \tfrac{1}{2} \equiv 0. \tag{31}$$

The images of these curves are the stable horospheres associated to the geodesic $x = 0$, $\dot{y} = y$, parametrized by arc length.

We show that $c(L) = \frac{1}{2}$ and $h_c < +\infty$. Observe that if $v = (\dot{x}, \dot{y})$, $\dot{x} < 0$, then

$$L = \tfrac{1}{2}\ \|v\|^2 - \|(\dot{x}, 0)\| \geq \tfrac{1}{2}\ \|v\|^2 - \|v\| \geq -\tfrac{1}{2}. \tag{32}$$

Hence $L + \frac{1}{2} \geq 0$ and then $c(L) \leq \frac{1}{2}$.

Now fix $x \in \mathbb{H}$. For $r > 0$ let D_r be a geodesic disc of radius r such that $x \in \partial D_r$. Let γ_r be the curve whose image is the boundary of D_r oriented clockwise and with hyperbolic speed $\|\dot{\gamma}\| \equiv a$. Since $E(\gamma) = \frac{1}{2}\, a^2$, then

$$\begin{aligned} \int_{\gamma_r} L + \tfrac{1}{2}a^2 &= \int_{\gamma_r} v \cdot L_v = \int_{\gamma_r} \|v\|^2 + \int_{D_r} dA = a \cdot \text{length}(\gamma_r) - \text{area}(D_r), \\ &= a \cdot 2\pi\ \sinh(r) - 2\pi\ \cosh(r) = 2\pi\big[\tfrac{1}{2}\,(a-1)e^r - e^{-r}\big] + 2\pi. \end{aligned} \tag{33}$$

If $a < 1$, for $r > 0$ large, formula (33) is negative. Hence $c(L) \geq \frac{1}{2}$ and $c(L) = \frac{1}{2}$. Moreover,

$$h(x, x) \leq \liminf_{r \to +\infty} A_{L+\frac{1}{2}}(\gamma_r) = 2\pi < +\infty.$$

We prove that $\widehat{\Sigma} = \varnothing$. This implies that $h > 0$. First observe that if T is an isometry of $\mathbb{H}$, then $d(T_*\eta)$ is also the area form, so that $T_*\eta$ is cohomologous to η. This implies that given any two points $x, y \in \mathbb{D}$, there is a constant $b = b(x, y) \in \mathbb{R}$ such that for all $\gamma \in \mathcal{C}(x, y)$,

$$A_L(\gamma) = A_L(T \circ \gamma) + b(x, y).$$

In particular, the map dT leaves $\sigma(L)$ and $\widehat{\Sigma}(L)$ invariant. Since a horocycle h_1 can be sent by an isometry to another horocycle h_2 with $h_1 \cap h_2 \neq \varnothing$, then the horocycles can not be static because it would contradict the graph property.

The constant function $u : \mathbb{H} \to \{0\}$ satisfies $u \prec L + \frac{1}{2}$ because $L + \frac{1}{2} \geq 0$ and by (31) the vectors $v = (-y, 0) \in \Gamma^+(u) = \Gamma^-(u) \in \Sigma^-$ are semistatic. Its derivative $du = 0$ is sent by the inverse of the Legendre transform $v \mapsto L_v = \langle v, \cdot\rangle_x + \frac{dx}{y}$ to

$$\tfrac{1}{y^2}\ \langle v, \cdot\rangle_{\text{eucl}} = -\tfrac{dx}{y},$$

that is $v = (-y, 0)$. Also

$$H(du) = \tfrac{1}{2}\ \big\|du - \tfrac{dx}{y}\big\|^2 = \tfrac{1}{2}\ \big\|\tfrac{dx}{y}\big\|^2 = \tfrac{1}{2}.$$

Let $T : \mathbb{H} \hookleftarrow$ be an isometry of the hyperbolic metric. Write $\eta = \frac{dx}{y}$. Then $d\eta = A$ is the hyperbolic area 2-form. Since T is an isometry, then

$$d(T^*\eta) = T^*(d\eta) = d\eta.$$

Hence the form $T^*\eta - \eta$ is exact on $\mathbb{H}$ and there is a smooth function $v : \mathbb{H} \to \mathbb{R}$ such that

$$T^*\eta - \eta = -dv.$$

We show that v is a weak KAM solution. Observe that

$$\begin{aligned} L \circ dT(x,v) &= \tfrac{1}{2}\ \|v\|_x^2 + T^*\eta(x,v) \\ &= \tfrac{1}{2}\ \|v\|_x^2 + \eta(x,v) - dv(x,v) \\ &= L(x,v) - dv(x,v). \end{aligned} \tag{34}$$

Since by (32) $L \circ dT + \frac{1}{2} \geq 0$, then

$$dv \leq L + \tfrac{1}{2}. \tag{35}$$

Hence $v \prec L + \frac{1}{2}$. Moreover, the equality in (35) holds exactly when $L \circ dT(x,v) + \frac{1}{2} = 0$, i.e. when $dT(v) = (-y, 0) \in T_{(x,y)}\mathbb{H}$.

Since the isometries send horospheres to horospheres, they are self-conjugacies of the hamiltonian flow and hence the curves $\gamma(t) = T^{-1}(x - ty, y)$ realize v, i.e.

$$v(\gamma(t)) - v(\gamma(s)) = \oint_\gamma dv = \oint_\gamma L + \tfrac{1}{2}.$$

Here v is the Busemann weak KAM solution associated to the class $T(\infty) \in \partial\mathbb{H}$, on the sphere at infinity of $\mathbb{H}$.

We now show a picture of a non-Busemann weak KAM solution. We use the isometry $T : \mathbb{H} \hookleftarrow$, $T(z) = -\frac{1}{z}$, $z = x + iy \in \mathbb{C}$. The isometry $T = T^{-1}$ sends the line $t \mapsto -ty + iy$ to a horosphere with endpoint $0 \in \mathbb{C}$, oriented clockwise. Choose $v : \mathbb{H} \to \mathbb{R}$ such that $dv = \eta - T^*\eta$ and $v(0 + i) = 0$. Since T leaves the line $\mathrm{Re} z = 0$ invariant and $\eta = 0$ on vertical vectors, hence v is constant (equal to 0) on $\mathrm{Re} z = 0$.

Now we describe the weak KAM solution

$$w(z) := \min\{\, u(z), v(z) \,\} \in \mathfrak{S}^-.$$

Let $\gamma(t) = -ty + iy$. Then, using (34),

$$\begin{aligned} v(T^{-1}\gamma(t)) &= v(T^{-1}\gamma(0)) + \oint_{T^{-1}\circ\gamma} dv \\ &= 0 + \int_0^t \left[\, L \circ dT^{-1} \circ \dot\gamma + \tfrac{1}{2} \,\right] - \int_0^t \left[\, L \circ \dot\gamma + \tfrac{1}{2} \,\right] \\ &= 0 + \int_0^t \left[\, L \circ dT^{-1} \circ \dot\gamma + \tfrac{1}{2} \,\right] - 0. \end{aligned}$$

Since by (35) $L(x,v)+\frac{1}{2}>0$ when $v\neq -y+i0$, then $v(z)>0$ on $\mathrm{Re} z>0$ and $v(z)<0$ on $\mathrm{Re} z<0$. Thus

$$w(z)=\begin{cases}0=u(z) & \text{if } \mathrm{Re} z>0.\\ v(z) & \text{if } \mathrm{Re} z<0.\end{cases} \tag{36}$$

The cut locus of w is $\mathrm{Re} z=0$ and the basin of w is $\Gamma^-(w)=A\cup dT(A)$ where A is the set of vectors $(y,0)\in T_{x+iy}\mathbb{H}$.

□

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