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Regularity of topological and metric entropy of hyperbolic flows *

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Introduction

Let M be a compact smooth manifold and let $\mathscr{X}^r(M)$ be the Banach space of C^r vector fields on M endowed with the topology of uniform C^r convergence on compact subsets. For $X \in \mathscr{X}^r(M)$ let $h_{\text{top}}(X)$ be the topological entropy of X. Misiurewicz [18] proved that, for general C^r flows on $M^n, r<\infty$ and $n \leqq 3$, h_{top} need not be continuous. Yomdim and Newhouse [19, 23] proved that h_{top}: $\mathscr{X}^\infty(M^n) \to \mathbb{R}$ is upper-semicontinous. Katok proved that for 3-manifolds h_{top}: $\mathscr{X}^\infty(M^3) \to \mathbb{R}$ is lower semi-continuous. By combining these two results, one sees that h_{top}: $\mathscr{X}^\infty(M^3) \to \mathbb{R}$ is continuous. See [8] for a survey of regularity results for the topological entropy for general flows.

Let Λ be a hyperbolic basic set of $X \in \mathscr{X}^r(M)$, i.e. a compact hyperbolic transitive and isolated set. Isolated means that there exists a compact neighbourhood U of Λ such that Λ is the maximal X-invariant set of U, i.e., the set of points whose X-orbit is contained in U. By the standard theory of hyperbolic sets there exists a neighbourhood $\mathscr{U}$ of X in $\mathscr{X}^r(M)$ such that if $Y \in \mathscr{U}$ then the maximal Y-invariant set of U, that we shall denote Λ_Y, is a hyperbolic basic set and $X|_\Lambda$ and $Y|_{\Lambda_Y}$ are topologically equivalent.

Our objective is to study the regularity of the entropies of $Y|_{\Lambda_Y}$ arising from variational principles, as functions of $Y \in \mathscr{U}$. We begin by considering the topological entropy $h_{\text{top}}(Y|_{\Lambda_Y})$.

Theorem A. *The function* $\mathscr{U} \ni Y \to h_{\text{top}}(Y|_{\Lambda_Y}) \in \mathbb{R}$ *is* C^r.

For $r=1$ and $\Lambda = M$ this result was proved by Katok et al. [8]. In [9] Katok et al. proved the theorem above loosing one degree of differentiability. To obtain the full regulatity of $h_{\text{top}}(Y|_{\Lambda_Y})$ we use the techniques of Mañé in [15].

When Λ is an *attractor* of X (i.e. the ω-limit set of every x nearby Λ is contained in Λ) then Λ_Y is an attractor of Y for every $Y \in \mathscr{U}$. The set $W^s(\Lambda_Y)$ of points $x \in M$ whose ω-limit set under Y is contained in Λ_Y (called the *basin*

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of Λ_Y) is then open and there exists a unique Y-invariant probability μ_Y (called the Bowen-Ruelle-Sinai measure) on the Borel σ-algebra of Λ_Y such that for almost every $p \in W^s(\Lambda_Y)$ (with respect to the Lebesgue measure on M), if the forward Y-orbit of p is $x:[0, +\infty[\to M$, then

$$\int \varphi \, d\mu_Y = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \varphi(x(t))\, dt$$

for every continuous $\varphi: M \to \mathbb{R}$. Let $h_{\mu_Y}(Y|_{\Lambda_Y})$ be its metric entropy.

Theorem B. *The function* $\mathcal{U} \ni Y \to h_{\mu_Y}(Y|_{\Lambda_Y})$ *is* C^{r-2}.

It is interesting to apply this result to the geodesic flow of Riemannian manifolds with negative curvature. Given a compact boundaryless manifold M denote $R^r(M)$ the Banach manifolds of C^r Riemannian structures $g: TM \times TM \to \mathbb{R}$ and, if $g \in R^r(M)$ let $h(g)$ denote the entropy of the geodesic flow of g with respect to the Liouville measure. Let $A^r(M)$ be the (open) set of metrics $g \in R^r(M)$ whose geodesic flow is Anosov (in particular, metrics with negative curvature). Then the geodesic flow is generated by a C^{r-1} vector field for which the whole unit tangent bundle is a hyperbolic basic set and the Liouville measure is its Bowen-Ruelle-Sinai measure. Then Theorems A and B imply:

Corollary. *The entropy of the geodesic flow of* $g \in A^r(M)$, $r \leqq 3$ *with respect to the Liouville measure is a* C^{r-3} *function. The topological entropy of the geodesic flow of* $g \in A^r(M)$, $r \leqq 1$, *is a* C^{r-1} *function of* g.

For $r=4$ the first part of this corollary was proved by Knieper and Weiss [11]. Theorems A and B can be extended and unified to entropies with respect to equilibrium states. Take X, Λ, U and $\mathcal{U}$ as above. Denote $C^\alpha(M, \mathbb{R})$, $0<\alpha<1$, the Banach space of α-Hölder continuous real valued functions on M. Consider a function $\psi: \mathcal{U} \to C^\alpha(M, \mathbb{R})$ and, for $Y \in \mathcal{U}$, denote μ_Y the Y-invariant probability on the Borel σ-algebra of Λ_Y that is the equilibrium state of $\psi(Y)|_{\Lambda_Y}$. Let $h_{\mu_Y}(Y|_{\Lambda_Y})$ be its entropy and

$$P(Y) = P_Y(\psi(Y)) = h_{\mu_Y}(Y|_{\Lambda_Y}) + \int \psi(Y)\, d\mu_Y$$

the pressure of $(\psi(Y), \Lambda_Y, Y)$. The Ω-stability theorem gives for each Y in $\mathcal{U}$ a homeomorphism h_Y: $\Lambda \to \Lambda_Y$ such that it is Hölder continuous and sends orbits of (Λ, X) to orbits of (Λ_Y, Y). This topological equivalence h_Y is not unique because by composing with translations along orbits of Λ_Y we get other topological equivalences. However this is the only obstruction to the uniqueness of h_Y and on Proposition 1.2 we choose one such h_Y such that the map $\mathcal{U} \ni Y \mapsto h_Y \in C^\alpha(\Lambda, M)$ is C^{r-1}.

Theorem C. *If the map* $\mathcal{U} \ni Y \mapsto \psi(Y) \circ h_Y \in C^0(\Lambda, \mathbb{R})$ *is* C^s *and* $\mathcal{U} \ni Y \mapsto \psi(Y) \circ h_Y \in C^\alpha(\Lambda, \mathbb{R})$ *is* C^{s-1}, *then*

(a) $P: \mathcal{U} \to \mathbb{R}$ *is* C^s.
(b) $\mathcal{U} \ni Y \to h_{\mu_Y}(Y|_{\Lambda_Y})$ *is* C^{s-1}.
(c) *If* $0<\alpha<1$ *is small enough and denoting* $(C^\alpha(M, \mathbb{R}))^*$ *the dual Banach space of* $C^\alpha(M, \mathbb{R})$ *then* $\mathcal{U} \ni Y \to \mu_Y \in (C^\alpha(M, \mathbb{R}))^*$ *is* C^{s-1}.

In Sect. 1 we see that the map $\mathscr{U} \ni Y \to h_Y \in C^\alpha(\Lambda, M)$ is C^r for some appropriate α, so that Theorem C is valid when we consider for example a C^r map $\psi: \mathscr{U} \to C^r(U, \mathbb{R})$ and a neighbourhood U of Λ in M. Considering μ_Y as an element of the dual Banach space of $(C^0(M, \mathbb{R}))^*$, item (c) becomes false (see [15]).

Theorem A follows from item (a) of Theorem C and $\psi \equiv 0$. Theorem B follows from letting $\psi(Y)(p) = -\frac{d}{dt} \log|\det(D_p \varphi_Y(p,t)|_{E^u(\varphi_Y(p,t))})|_{t=0}$, where $\varphi_Y(p,t)$ is the flow of Y and $E^u(q)$ is the strong unstable subspace of Y at $q \in \Lambda_Y$. In Sect. 2 we prove that in this case the map $\mathscr{U} \ni Y \to \psi(Y) \circ h_Y \in C^\alpha(\Lambda, \mathbb{R})$ is C^{r-2}. It also follows from Theorem C that the map $\mathscr{U} \ni Y \to \int \psi(Y)\, d\mu_Y \in \mathbb{R}$ is C^{r-2}. Applying this for $\psi(Y)$ of Theorem B, we see that he sum of the positive Lyapunov exponents is a C^{r-2} function of the vector field Y and that the rate of escape of a neighbourhood U of Λ_Y, $\lim_{T \to \infty} \frac{1}{T} \log \operatorname{vol}\left(\bigcap_{t=0}^{T} \varphi_Y(U, -t)\right) = P_Y(\psi(Y))$ (see [5]) is a C^{r-1} function of the vector field $Y \in \mathscr{X}^r(M)$.

1 Preliminaries

Let M be a compact differentiable manifold and $\phi_t\colon M \to M$ a differentiable flow. A closed, ϕ_t-invariant set Λ without fixed points is *hyperbolic* if the tangent bundle restrited to Λ decomposes as the Witney sum of three $D\phi_t$-invariant subbundles $T_\Lambda M = \mathbf{E}^0 \oplus \mathbf{E}^s \oplus \mathbf{E}^u$ where $\mathbf{E}^0$ is the 1-dimensional subbundle tangent to the flow and there are constants $C, \lambda > 0$ satisfying:

(i) $|D\phi_t v| \leqq C e^{-\lambda t}|v|$ for $v \in \mathbf{E}^s$, $t > 0$.
(ii) $|D\phi_{-t} v| \leqq C e^{-\lambda t}|v|$ for $v \in \mathbf{E}^u$, $t > 0$.

A closed ϕ_t-invariant subset $\Lambda \subset M$ is a basic set for ϕ_t if

(a) Λ does not have fixed points and is hyperbolic.
(b) The periodic orbits in Λ are dense in Λ.
(c) $\phi_t|\Lambda$ is transitive (there is a dense orbit).
(d) There is an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{t \in \mathbb{R}} \phi_t(U)$.

From now on, fix an Axiom A basic set $\Lambda \subset M$ of the vector field $X \in \mathscr{X}^*(M)$ ($* = r$, $r+\alpha$, ∞ or some $\beta > 0$) and let $\phi\colon M \times \mathbb{R} \to M$, $\phi(p,t) = \phi_t(p)$ be its flow. We will always consider spaces of maps $\mathscr{X}^*(M) \subset C^*(M, TM)$, $C^*(\Lambda, M)$, $C^*(\Lambda, TM)$, etc. as Banach manifolds by composition with local charts. Let A, B be metric spaces. We say that a map $f: A \to B$ is *α-Hölder continuous* if

$$K(f) := |f|_\alpha := \sup_{d(x,y) < a} \frac{d(fx, fy)}{d(x,y)^\alpha} < \infty.$$

If B is a Banach space and $K \subset A$ is compact, we write

$$|f|_0 := \sup_{x \in K} |f(x)|$$

and $C^\alpha(K, B)$ for the Banach space of α-Hölder maps, with the *α-Hölder norm* $\|f\|:=|f|_0+|f|_\alpha$. Endow $C^\alpha(\Lambda, B)$ with the topology given by the Hölder norm on compact parts of Λ.

Consider the space

$$C^\alpha_\phi(\Lambda, M):=\left\{u\in X^\alpha(\Lambda, M)\,|\, D_\phi u(p):=\frac{d}{dt}u(\phi(p,t))|_{t=0} \text{ exists and is } \alpha\text{-Hölder}\right\},$$

with the topology of the α-Hölder norm $\|u\|+\|D_\phi u\|$.

1.1. Proposition [structural stability] (a) *There exist $0<\beta<1$ and a neighbourhood $\mathcal{U}\subset\mathcal{X}^r(M)$ of X and C^{r-1} maps $\mathcal{U}\to C^\beta_\phi(\Lambda, M)$: $Y\mapsto u_Y$ and $\mathcal{U}\to C^\beta(\Lambda,[\frac{1}{2}, +\infty[)$: $Y\mapsto \gamma_Y$ such that $Y\circ u_Y=\gamma_Y D_\phi u$.*
(b) *Moreover, the maps $\mathcal{U}\to C^0_\phi(\Lambda, M)$: $Y\mapsto \mathcal{U}_Y$ and $\mathcal{U}\to C^0(\Lambda, [\frac{1}{2}, +\infty[)$: $Y\mapsto\gamma_Y$ are C^r.*

Case (a) has been proved in [9]. Case (b) has been proved in [8] for $r=1$. That proof can be immediately generalized to an arbitrary positive integer r.

Note that for Y near X, the topological equivalence u_Y is uniquely determined by $u_Y(p)\in\exp(E^s_X(p)\oplus E^u_X(p))$, $Y\circ u_Y=\gamma D_\phi u_Y$ and u_Y near the identity.

1.2. Corollary. *For $Y\in\mathcal{U}$ consider the map τ_Y: $\Lambda\to\mathbb{R}^+$ defined by $\psi(u_Y(\phi_{-1}p), \tau_Y(p))=u_Y(p)$, where ψ is the flow of Y. Then*

$$\tau_Y(p)=\int_0^1 \gamma(\phi(p,s-1))\,ds$$

and the map $\mathcal{U}\to C^\beta(\Lambda,\mathbb{R}^+)$: $Y\mapsto\tau_Y$ is C^r.

2 Stability of the splitting

Let G be the Grassmann bundle of $u-$planes on TM, $u=\dim E^u_X$, i.e. the set of pairs (p, E) with $p\in M$ and $E\subset T_pM$ a u-dimensional subspace. G has a natural structure of compact differentiable manifold where a parametrization around a subspace $(p, E)\in G$ is $U\times L(E, E^\perp)\to G$, $(x, L)\mapsto D_x h$ (graph L), where $h: U\subset\mathbb{R}^n\to M$ is a parametrization of a neighbourhood of p and we identify $E\approx(D_x h)^{-1}E$. For $Y\in\mathcal{X}^r(M)$ near X, consider the topological equivalence u_Y of Proposition 1.1 and its splitting $TM|_{u_Y(\Lambda)}=E^0_Y\oplus E^s_Y\oplus E^u_Y$.

2.1. Proposition. *There exists a neighbourhood $\mathcal{U}$ of X in $\mathcal{X}^r(M)$ and $\beta>0$ such that the map $\mathcal{U}\to C^\beta(\Lambda, G)$: $Y\mapsto(p\mapsto E^u_Y\circ u_Y(p))$ is C^{r-2} and the map $\mathcal{U}\to C^0(\Lambda, G)$: $Y\mapsto E^u_Y\circ u_Y$ is C^{r-1}.*

Proof. Consider the product bundle π: $TM\oplus G\to M$ of triples (p, v, E) with $v\in T_pM$, $E\subset T_pM$. Then $TM\oplus G$ is naturally a compact $2n+u(n-u)$ manifold. For Y near X let $\tau_Y\in C^\beta(\Lambda,\mathbb{R}^+)$ be as in (1.2). Let $C^\beta_\phi(\Lambda, G)$ be the space of β-Hölder maps $\sigma:\Lambda\to G$, $\sigma(p)=(h(p), E(p))$ where $D_\phi h(p):=\left.\frac{d}{dt}\right|_{t=0} h(\phi(p,t))$ is β-Hölder, with the topology of the β-Hölder norm on h, $D_\phi h$ and E. Consider the map $H\colon \mathscr{X}^r(M)\times C^\delta_\phi(\Lambda,\mathbb{R}^+)\times C^\beta_\phi(\Lambda,G)\to C^\varepsilon(\Lambda,TM\oplus G)$, $H(Y,\gamma,\sigma)=(Y\circ h-\gamma D_\phi h, D_1\psi(h(\phi_{-1}p), \tau_Y(p))E(\phi_{-1}p))$ where $D_1\psi(q,s)=\frac{\partial\psi}{\partial q}(q,s)$, $q\in M$. Consider $\sigma\in C^b_\phi(\Lambda,G)$ satisfying the equation $H(Y,\gamma,\sigma)=(0,\sigma)$ and σ near $\sigma_0(p)=(p,E^u_X(p))$. Then it is easy to see that $h=u_Y$ and $F(q):=E\circ u_Y^{-1}(q)$ is an invariant u-subbundle for Y on $u_Y(\Lambda)$ near E^u_X. So that $q\mapsto F(q)$ should be the unstable subbundle for Y. We need to see the second component H_2 of H in its local form in order to apply the implicit function theorem.

We can extend the splitting $TM|_\Lambda=E^0_X\oplus E^s_X\oplus E^u_X$ to a C^γ-splitting $E^0\oplus E^s\oplus E^u$ (γ as in (1.1)) of a neighbourhood U of Λ. In local coordinates, a subspace $E\subset T_qM$, $E\cap(\mathbf{E}^0\oplus\mathbf{E}^s)=0$ is seen as a linear map $L\colon \mathbf{E}^u(q)\to\mathbf{E}^0(q)\oplus\mathbf{E}^s(q)$. The local expression for H_2 is

$$(1)\qquad H_2(Y,\sigma)(p)=[C(Y,\tau_Y p, h\phi_{-1}p)+D(Y,\tau_Y p, h\phi_{-1}p)L(\phi_{-1}p)]\cdot[A(Y,\tau_Y p, h\phi_{-1}p)+B(Y,\tau_Y p, h\phi_{-1}p)L(\phi_{-1}p)]^{-1}$$

where

$$D_1\psi_Y(q,\tau)=\frac{\partial\psi_Y}{\partial q}(q,\tau)=\begin{bmatrix}A(Y,\tau,q) & B(Y,\tau,q)\\ C(Y,\tau,q) & D(Y,\tau,q)\end{bmatrix}$$

in the splitting $\mathbf{E}^u\oplus(\mathbf{E}^0\oplus\mathbf{E}^s)$, for $Y\in\mathscr{X}^r(M)$ near X, $\tau\in\mathbb{R}$ and $q\in U$. We can replace $C^\beta_\phi(\Lambda,G)$ by $C^\beta_\phi(\Lambda,\mathscr{L})$ where $\mathscr{L}$ is the bundle of linear maps $L(\mathbf{E}^u,\mathbf{E}^0\oplus\mathbf{E}^s)$. Let $F(Y,\gamma,\sigma):=H(Y,\gamma,\sigma)-(0,\sigma)$. The formula (1) is analytic on L, C^{r-2} on h and C^{r-2} on Y by (1.2). So that for a suitable $\beta<\gamma$, F is C^{r-2}. We need to check that $D_{23}F(X,1,E^u_X)=D_{23}H_1\otimes(D_{23}H_2-\mathrm{Id})$ is invertible. From (1.1) we already know that $D_{23}H_1$ is invertible. Since we have chosen maps on the bundle $\mathscr{L}\to M$, the inverse $D_{23}F(X,1,E^u_X)$ should have the form $(D_{23}H_1)^{-1}\otimes(D_3H_2-\mathrm{Id})^{-1}$. It is a standart fact (see [21]) that $D_3H_2(X,1,E^u_X)$ is a contraction onto $C^0(\Lambda,\mathscr{L})$. We leave to the reader the proof that, since in (1) only h and L are not Lipschitz, we can choose $a>0$ in the definition of Hölder norm such that $D_3H_2(X,1,E^u_X)$ is a contraction onto $C^\beta(\Lambda,\mathscr{L})$. □

3 Symbolic dynamics

For $A=[A(i,j)]\in\{0,1\}^{n\times n}$ an $n\times n$ matrix of 0's and 1's we define

$$\Sigma=\Sigma_A:=\{\bar{x}=(x_i)_{i=-\infty}^{+\infty}\in\{1,\dots,n\}^{\mathbb{Z}}\,|\,A(x_i,x_{i+1})=1\ \ \forall i\in\mathbb{Z}\}$$
$$\Sigma^+=\Sigma_A^+:=\{\bar{x}=(x_i)_{i=0}^{+\infty}\in\{1,\dots,n\}^{\mathbb{Z}}\,|\,A(x_i,x_{i+1})=1\ \ \forall i\in\mathbb{N}\}$$

and $\sigma\colon\Sigma\hookleftarrow$ by $\sigma(\bar{x})=(y_i)_{i=-\infty}^{+\infty}$ where $y_i=x_{i+1}$. Similarly we define $\sigma_+\colon\Sigma^+\hookleftarrow$. Endow Σ (resp. Σ^+) with the metric $d(\tilde{x},\tilde{y})=b^N$ where $0<b<1$, $N=\max\{m\,|\,x_k=y_k, \forall|k|<m\}$. Then Σ (resp. Σ^+) is compact and σ becomes a homeomorphism which is called a *subshift of finite type* if $\sigma\colon\Sigma\hookleftarrow$ (resp. $\sigma_+\colon\Sigma^+\hookleftarrow$) is topologically transitive (i.e., for U, V non-empty open sets there is an $n>0$ with $f^nU\cap V\neq\phi$). For $\tau\colon\Sigma\to\mathbb{R}$ a positive continuous function consider the quotient space

$S(\Sigma,\tau):=\Sigma\times[0,+\infty[/\equiv$ where $\left(x, \sum_{k=0}^{n} \tau(\sigma^k x)\right)\equiv(\sigma^{n+1}x, 0)$, for any $n\in\mathbb{N}$. Then $S(\Sigma,\tau)$ is a compact metric space and the flow $f^t(x,s)=(x,s+t)$ on $\Sigma\times\mathbb{R}$ induces a flow f^t on $S(\Sigma,\tau)$ called the *suspension of σ at time τ.*

3.1 Lemma [3, 5] *There exists a topologically mixing subshift of finite type $\sigma\colon \Sigma\to\Sigma$, a positive $\tau\in C^{\alpha}(\Lambda,\mathbb{R}^+)$, for some $\alpha>0$, and a continuous surjection $\pi\colon S(\Sigma,\tau)\to\Lambda$ such that*

$$\begin{array}{ccc} S(\Sigma,\tau) & \xrightarrow{\ f^t\ } & S(\Sigma,\tau) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \Lambda & \xrightarrow{\ \phi_t\ } & \Lambda \end{array}$$

conmutes.

4 Entropy and equilibrium states

Let X be a compact metric space and $f^t\colon X\to X$ be a flow ($t\in\mathbb{R}$) or iterates of a homeomorphism $g=f^1$ ($t\in\mathbb{Z}$), and let $\varphi\colon X\to\mathbb{R}$ be a continuous function. For given $\varepsilon>0$ and $T>0$ (resp. $T\in\mathbb{N}$), a subset $E\subset X$ is called (n,ε)*-separated* if

$$x,y\in E,\, x\neq y \Rightarrow d(f^t x, f^t y)>\varepsilon \quad \text{for some } t\in[0,T].$$

One defines the *topological pressure* of φ, $P(F,\varphi)$, $F=(f^t)$ (resp. $P(g,\varphi)$) by

$$P(F,\varphi):=\lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{T}\log\Big(\sup\Big\{\sum_{x\in E}\exp S_T\varphi(x)\,\Big|\, E \text{ is } (T,\varepsilon)-\text{separated}\Big\}\Big)$$

where $S_T\varphi(x)=\int_0^T \varphi(f^t x)\,dt$ $\left(\text{resp. } S_T\varphi(x)=\sum_{n=0}^{T}\varphi(g^n x)\right)$. The *topological entropy* is defined to be the pressure of $\varphi\equiv 0$. Let $\mathcal{M}(g)$ be the set of g-invariant Borel probability measures, for flows $\mathcal{M}(F)=\bigcap_{t\in\mathbb{R}}\mathcal{M}(f^t)$, $F=(f^t)$. For a definition of *metric entropy* $h_\mu(g)$ and the variational principle

$$P(g,\varphi)=\sup_{\mu\in\mathcal{M}(g)}\left(h_\mu(g)+\int\varphi\,d\mu\right)$$

see [23]. An *equilibrium state* for φ with respect to g is a $\mu\in\mathcal{M}(g)$ which realizes this supremum. If one lets $\varphi^1(x):=\int_0^1\varphi(f^t x)\,dt$, then it is easy to see that $P(F,\varphi)=P(f^1,\varphi^1)$ and that for $\mu\in\mathcal{M}(F)$, $\int\varphi^1\,d\mu$. Since $\mathcal{M}(F)\subset\mathcal{M}(f^1)$ one has

$$h_\mu(f^1)+\int\varphi\,d\mu=h_\mu(f^1)+\int\varphi^1\,d\mu\leqq P(f^1,\varphi^1)=P(F,\varphi)$$

for $\mu \in \mathcal{M}(F)$. For $v \in \mathcal{M}(f^1)$ let μ be defined by $\int \psi \, d\mu = \int \psi^1 \, dv$ for ψ measurable, then $\mu \in \mathcal{M}(F)$ and ([6, p. 359–360]):

$$h_\mu(f^1) + \int \varphi^1 \, d\mu \geqq h_v(f^1) + \int \varphi^1 \, dv.$$

It follows that

$$P(F, \varphi) = \sup_{\mu \in \mathcal{M}(F)} \left(h_\mu(f^1) + \int \varphi \, d\mu\right).$$

An *equilibrium state* for φ (with respect to F) is a $\mu \in \mathcal{M}(F)$ which realizes this supremum. For $v \in \mathcal{M}(\sigma)$ and m the Lebesgue measure on $\mathbb{R}$, $v \times m$ gives measure 0 to the identifications $(x, \tau(x)) \equiv (\sigma(x), 0)$ on $Y = \{(x, t) \in \Sigma \times \mathbb{R} \mid 0 \leqq t \leqq \tau(x)\}$ so that $\mu_v = (v \times m(Y))^{-1} v \times m | Y$ induces a probability measure on $S(\Sigma, \tau)$. One can check that if $v \in \mathcal{M}(\sigma)$ then $\mu_v \in \mathcal{M}(F)$ for $F = (f^1)$ the suspension flow and that $\mathcal{M}(\sigma) \to \mathcal{M}(F)$: $v \to \mu_v$ is a bijection. It is known [4] that any $\varphi \in C^\alpha(\Sigma, \mathbb{R})$ has a unique equilibrium state w.r.t. σ for any $\alpha > 0$.

4.1 Lemma [Bowen, Franco-Sanchez] *Let* φ: $S(\Sigma, \tau) \to \mathbb{R}$ *be continuous,* $\phi(x) = \int_0^{\tau(x)} \varphi(x, t) \, dt$ *and* $p = P(F, \varphi)$. *Assume that* $\phi \in C^\alpha(\Sigma, \mathbb{R})$ *for some* $\alpha > 0$ *then*

(a) *There exists a unique equilibrium state* $\mu_\varphi \in \mathcal{M}(F)$ *for* φ w.r.t. F.
(b) $\mu_\varphi = \mu_{v_0}$ *where* v_0 *is the unique equilibrium state for* $\phi - p\tau$.
(c) $P(\sigma, \phi - p\tau) = 0$.

Proof. Let $\gamma = \phi - p\tau$. Since $\phi, \varphi \in C^\beta(\Sigma, \mathbb{R})$ for some $\beta > 0$, we have $\gamma \in C^\beta(\Sigma, \mathbb{R})$ so that γ has a unique equilibrium state v_0. By Fubini's theorem, for any $v \in \mathcal{M}(\tau)$, $(v \times m)(Y) = \int \tau \, dv$ and $\int \varphi \, d\mu_v = \dfrac{\int \phi \, dv}{\int \tau \, dv}$. A theorem of Abramov [1] states that

$$h_{\mu_v}(f^1) = \frac{h_v(\tau)}{\int \tau \, dv}.$$

Hence

$$\begin{aligned} p = P(F, \varphi) &= \sup_{\mu \in \mathcal{M}(F)} \left(h_\mu(f^1) + \int \varphi \, d_v\right) \\ &= \sup_{v \in \mathcal{M}(\tau)} \frac{\left(h_v(\tau) + \int \phi \, dv\right)}{\int \tau \, dv} \end{aligned}$$

because $v \mapsto \mu_v$ is a bijection. Thus $P(\sigma, \gamma) = \sup_v \left(h_v(\sigma) + \int (\phi - p\tau) \, dv\right) = 0$ with v attaining the supremum ($v = v_0$) precisely if μ_v is the unique equilibrium state for φ. □

By considering the conjugacy π of Lemma 3.1 we obtain equilibrium states for Axiom A basic sets $\phi_t : \Lambda \hookleftarrow$.

4.2. Corollary *For* $\phi_t : \Lambda \hookleftarrow$ *a basic set and* $\varphi : \Lambda \to \mathbb{R}$ *Hölder continuous,* φ *has a unique equilibrium state* w.r.t. ϕ_t *and* $\mu_\varphi(\Lambda) = \pi^* \omega(\Lambda) = \omega(\pi^{-1} \Lambda)$ *where* ω *is the equilibrium state for* $\varphi \circ \pi$ *on* $S(\Sigma, \tau)$ w.r.t. $F = (f^t)$.

For $\psi \in C^\alpha(\Sigma^+, \mathbb{R})$ we now introduce the Perron-Frobenius operator $\mathscr{L}_\psi : C^0(\Sigma^+, \mathbb{R}) \hookleftarrow$.

$$(\mathscr{L}_\psi \varphi)(x) := \sum_{\sigma_+(y) = x} \varphi(y) \exp(\psi(y)).$$

Say that $\psi, \varphi \in C^0(\Sigma, \mathbb{R})$ (resp. $C^0(\Sigma^+, \mathbb{R})$) are *homologous* if there exists $u \in C^0(\Sigma, \mathbb{R})$ such that $\psi = \varphi - u + u \circ \sigma$. It this case μ is an equilibrium state for ψ if and only if it is an equilibrium state for φ.

4.3. Theorem (Ruelle [20] see also [4]) *If $\psi \in C^\alpha(\Sigma^+, \mathbb{R})$, $0 < \alpha \leqq 1$, the spectrum of $\mathscr{L}_\psi : C^\alpha(\Sigma^+, \mathbb{R}) \hookleftarrow$ consists in a simple isolated eigenvalue $\lambda(\psi) > 0$ and a set contained in a* disc$\{z \in \mathbb{C} \,|\, |z| < \mu < \lambda(\psi)\}$. *The pressure is $P(\sigma_+, \psi) = \log \lambda(\psi)$. Moreover, there exists a strictly positive eigenfunction $h_\psi \in C^\alpha(\Sigma^+, \mathbb{R})$ uniquely determined up to a scalar multiple and a unique Borel probability ν_ψ on Σ^+ satisfying*

(a) $\mathscr{L}_\psi h_\psi = \lambda_\psi h_\psi$
(b) $\int h_\psi \, d\nu_\varphi = 1$
(c) $\mathscr{L}_\psi^* \nu_\psi = \lambda(\psi) \nu_\psi$
(d) $\mu_\psi := h_\psi \nu_\psi$ *is σ_+-invariant and is the unique equilibrium state for ψ* w.r.t. σ_+.
(e) $\mu_\psi = \mu_{\psi_1}$ *if and only if ψ is homologous to ψ_1.*
(f) *For all $\varphi \in C^0(\Sigma^+, \mathbb{R})$:* $\lim_{n \to +\infty} |\lambda(\psi)^{-n} \mathscr{L}_\psi^n \varphi - h_\varphi \int \varphi \, d\nu_\psi|_0 = 0$.

The following is just a reformulation of (f) for $\varphi = 1$.

4.4. Corollary. *If $\psi \in C^\alpha(\Sigma^+, \mathbb{R})$, $0 < \alpha \leqq 1$, then*

$$\lim_{n \to +\infty} \frac{1}{n} \log \sum_{\sigma_+^n(y) = x} \exp(S_n \psi)(y)$$

uniformly on $x \in \Sigma^+$, where $(S_n \psi)(y) := \sum_{k=0}^{n-1} \psi(\sigma_+^k y)$.

4.5. Corollary. *If $\psi \in C^0(\Sigma^+, \mathbb{R})$ the limit*

$$\lim_{n \to +\infty} \frac{1}{n} \log \sum_{\sigma_+^n(y) = x} \exp(S_n \psi)(y)$$

exists for all $x \in \Sigma^+$ and is independent of x.

Proof. Define $\Phi_n : C^0(\Sigma^+, \mathbb{R}) \hookleftarrow$ by $\Phi_n(\psi)(x) = \frac{1}{n} \log \sum_{\sigma_+^n y = x} \exp(S_n \psi)(y)$. Then

$$(\Phi_n'(\psi)\varphi)(x) = \sum_{\sigma_+^n y = x} \frac{\frac{1}{n}(S_n \varphi)(y) \exp(S_n \psi)(y)}{\sum_{\sigma_+^n y = x} \exp(S_n \psi)(y)}.$$ Hence $|(\Phi_n'(\psi)\varphi)(x)| \leqq |\varphi|_0$ for all $x \in \Sigma^+$. So that

$$|\Phi_n'(\psi)|_0 \leqq 1$$

for all n. Then the sequence of maps Φ_n: $C^0(\Sigma^+, \mathbb{R}) \hookleftarrow$ is uniformly Lipschitz and is pointwise convergent (by 4.4) in the dense subset $C^\alpha(\Sigma^+, \mathbb{R}) \subset C^0(\Sigma, \mathbb{R})$.

Therefore the sequence Φ_n converges uniformly on compact subsets of $C^0(\Sigma^+, \mathbb{R})$ to a continuous map Φ: $C^0(\Sigma^+, \mathbb{R}) \hookleftarrow$. Since $\Phi(\psi) \in C^0(\Sigma^+, \mathbb{R})$, $0 < \alpha \leqq 1$, then, by the density of $C^\alpha(\Sigma^+, \mathbb{R})$ in $C^0(\Sigma^+, \mathbb{R})$ and the continuity of Φ in $C^0(\Sigma^+, \mathbb{R})$, it follows that $\Phi(\psi)$ is a constant function for each $\psi \in C^0(\Sigma^+, \mathbb{R})$. □

Using (4.5), for $\psi \in C^0(\Sigma^+, \mathbb{R})$ define

$$P(\psi) = \lim_{n \to +\infty} \frac{1}{n} \log \sum_{\sigma_+^n y = x} \exp(S_n \psi)(y).$$

4.6. Corollary. *For all* $0 < \alpha \leqq 1$ *the maps*

$$\begin{aligned}
&P: C^\alpha(\Sigma^+, \mathbb{R}) \to \mathbb{R} \\
&C^\alpha(\Sigma^+, \mathbb{R}) \ni \psi \to \nu_\psi \in C^\alpha(\Sigma^+, \mathbb{R})^* \\
&C^\alpha(\Sigma^+, \mathbb{R}) \ni \psi \to h_\psi \in C^\alpha(\Sigma^+, \mathbb{R}) \\
&C^\alpha(\Sigma^+, \mathbb{R}) \ni \psi \to \mu_\psi = h_\psi \nu_\psi \in C^\alpha(\Sigma^+, \mathbb{R})^*
\end{aligned}$$

are real analytic.

Proof. For a given $\psi_0 \in C^\alpha(\Sigma^+, \mathbb{R})$ take disjoint circles γ_1 and γ_2 centered at $\lambda(\psi_0)$ and 0 such that their interiors are disjoint and contain $\mathrm{Sp}(\mathscr{L}_{\psi_0})$. Denote by S the space of continuous linear maps L: $C^\alpha(\Sigma^+, \mathbb{C}) \hookleftarrow$ endowed with the norm topology. Let V be a neighbourhood of $\mathscr{L}_{\psi_0}$ in S such that $\mathrm{Sp}(L) \subset \mathrm{int}(\gamma_1) \cup \mathrm{int}(\gamma_2)$ for all $L \in V$. Given $L \in V$ define the spectral projection $\pi_i(L)$: $C^\alpha(\Sigma^+, \mathbb{C}) \hookleftarrow i = 1, 2$ by

$$\pi_i(L) := \frac{1}{2\pi i} \int_{\gamma_i} (L - zI)^{-1} \, dz$$

and let $E_i(L) := \pi_i(L)(C^\alpha(\Sigma^+, \mathbb{C}))$. It is well known that [7, 3.3]

$$C^\alpha(\Sigma^+, \mathbb{C}) = E_1(L) \oplus E_2(L)$$
$$I = \pi_1(L) + \pi_2(L)$$
$$\dim E_1(L) = 1$$

$\pi_i(L)$ is a projection and a complex analytic function of L

$E_i(L)$ *is* L-invariant.

We can normalize h_{ψ_0} such that $h_{\psi_0} = \pi_1(\mathscr{L}_{\psi_0}) \cdot 1$. Take $v^* \in C^\alpha(\Sigma^+, \mathbb{C})^*$ such that $\langle v^*, h_{\psi_0} \rangle \neq 0$ and define for $L \in V$

$$h(L) := \pi_1(L) \cdot 1, \qquad \lambda(L) := \frac{\langle v^*, L h(L) \rangle}{\langle v^*, h(L) \rangle}.$$

The map h: $V \to C^\alpha(\Sigma^+, \mathbb{C})$ is analytic. The denominator of $\lambda(L)$ is different from zero if V is taken small enough. Then the map λ: $V \to \mathbb{C}$ is analytic. Define ν: $V \to C^\alpha(\Sigma^+, \mathbb{C})^*$ by

$$\langle \nu(L), \varphi \rangle = \frac{\pi_1(L)\varphi}{\pi_1(L) 1}.$$

The function $V \times C^\alpha(\Sigma^+, \mathbb{C}) \ni (L, \varphi) \mapsto \langle v(L), \varphi \rangle \in \mathbb{C}$ is analytic. Since $E_1(L)$ is 1-dimensional and L-invariant $Lh(L) = \lambda h(L)$ for some $\lambda \in \mathbb{C}$, using the definition of $\lambda(L)$ we see that

$$Lh(L) = \lambda(L)\, h(L).$$

For $L \in V$ consider the adjoint map L^*: $C^\alpha(\Sigma^+, \mathbb{C}) \hookleftarrow$

$$\langle L^* v(L), \varphi \rangle = \langle v(L), L\varphi \rangle = \frac{\pi_1(L) L\varphi}{\pi_1(L) 1} = \frac{L \pi_1(L)\varphi}{\pi_1(L) 1} = \lambda(L) \frac{\pi_1(L)\varphi}{\pi_1(L) 1} = \lambda(L) \langle v(L), \varphi \rangle$$

for any $\varphi \in C^\alpha(\Sigma^+, \mathbb{C})$. Hence

$$L^* v(L) = \lambda(L)\, v(L)$$

and $h(\mathscr{L}_\psi) = h_\psi$, $v(\mathscr{L}_\psi) = v_\psi$, $\log \lambda(\psi) = P(\psi)$ are real analytic functions of ψ for ψ real. □

Now we return frow Σ^+ to Σ. Say that $\psi \in C^\alpha(\Sigma^+, \mathbb{R})$ is "*homologous*" to $\varphi \in C^\alpha(\Sigma, \mathbb{R})$ if $\bar{\psi}$ is homologous to φ, where $\bar{\psi}((x_i)_{i=-\infty}^{+\infty}) = \psi((x_i)_{i=0}^{+\infty})$.

4.7. Lemma. *There exists a continuous linear map* A: $C^\alpha(\Sigma, \mathbb{R}) \to C^\beta(\Sigma^+, \mathbb{R})$, $\beta = \alpha/2$, *such that* $A(\psi)$ *is "homologous" to* ψ. *In particular* A^*: $C^\beta(\Sigma^+, \mathbb{R})^* \to C^\alpha(\Sigma, \mathbb{R})^*$, $(A^* \mu)(\varphi) = \mu(A\varphi)$ *is a continuous linear map.*

The proof appears in p. 11 of [4].

4.8. Corollary. *For any* $0 < \alpha \leqq 1$ *the maps*

$$\begin{aligned} &\textit{the pressure } P: C^\alpha(\Sigma, \mathbb{R}) \to \mathbb{R} \\ &C^\alpha(\Sigma, \mathbb{R}) \ni \psi \mapsto \mu_\psi \in C^\alpha(\Sigma, \mathbb{R})^* \\ &\textit{the entropy } C^\alpha(\Sigma, \mathbb{R}) \ni \psi \mapsto h(\mu_\psi) = P(\psi) - \int \psi\, d\mu_\psi \in \mathbb{R}^+ \end{aligned}$$

are analytic, where μ_ψ *is the equilibrium state of* ψ w.r.t. σ

Proof. Use the conmutative diagram

$$\begin{array}{ccc} C^\alpha(\Sigma, \mathbb{R}) & \xrightarrow{\text{equilibrium state}} & C^\alpha(\Sigma, \mathbb{R})^* \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A^*} \\ C^\beta(\Sigma^+, \mathbb{R}) & \xrightarrow{\text{equilibrium state}} & C^\beta(\Sigma^+, \mathbb{R})^* \end{array}$$

for $\beta = \alpha/2$. □

Here we reproduce the arguments of [15] in order to gain one more degree of differentiability by considering P: $C^0(\Sigma^+, \mathbb{R}) \to \mathbb{R}$, except for the use of Manning's curve [14] in order to realize $\frac{dP}{d\psi}$.

4.9. Corollary. *For all* $0 < \alpha \leqq 1$, v_ψ *is a weakly continuous function of* $\psi \in C^\alpha(\Sigma^+, \mathbb{R})$ *i.e.*

$$\lim_{n \to +\infty} \int \varphi\, dv_{\psi_n} = \int \varphi\, dv_\psi$$

for every convergent sequence $\psi_n \to \psi$ *in* $C^\alpha(\Sigma^+, \mathbb{R})$ *and all* $\varphi \in C^0(\Sigma^+, \mathbb{R})$.

Proof. Let $\psi_n \to \psi$ be a convergent sequence in $C^\alpha(\Sigma^+, \mathbb{R})$ and suppose that ν_{ψ_n} doesn't converge weakly to ν_ψ. Then we can assume that ν_{ψ_n} converges weakly to a probalitity $\nu \neq \nu_{\psi_n}$, and

$$\mathscr{L}_\psi^* \nu = \lim_{n \to +\infty} L_{\psi_n}^* \nu_{\psi_n} = \lim_{n \to +\infty} \lambda(\psi_n) \nu_{\psi_n} = \lambda(\psi) \nu.$$

Hence, $\nu \in C^0(\Sigma^+, \mathbb{R})^* \subset C^\alpha(\Sigma^+, \mathbb{R})^*$ is an eigenvector of $\mathscr{L}_\psi^*: C^\alpha(\Sigma^+, \mathbb{R})^* \hookleftarrow$ associated to the simple eigenvalue $\lambda(\psi)$ of $\mathscr{L}_\psi^*$. So that $\nu = \eta \nu_\psi$ for some $\eta \in \mathbb{R}$, but since ν and ν_ψ are probabilities $\nu = \nu_\psi$. □

4.10. Corollary. *If* $0 < \alpha \leqq 1$ *and* $\psi \in C^\alpha(\Sigma^+, \mathbb{R})$, *then the derivative* $P'(\psi)$: $C^\alpha(\Sigma^+, \mathbb{R}) \to \mathbb{R}$ *is given by*

$$P'(\psi)\varphi = \int \varphi \, d\mu_\psi.$$

Proof. For $s \in \mathbb{R}$ consider the inequality given by the variational principle

$$f(s) := P(\psi + s\varphi) \geqq h(\mu_\psi) + \int \psi \, d\mu_\psi + s \int \varphi \, d\mu_\psi =: g(s).$$

Then $f(s)$ is an analytic curve which lies above the straight line $g(s)$ and only touches it at $s = 0$, hence $P'(\psi)\varphi = f'(0) = g'(0) = \int \varphi \, d\mu_\psi$. □

4.11. Proposition [Mañé] *Let N be a Banach manifold and let* Φ: $N \to C^\alpha(\Sigma^+, \mathbb{R})$, $0 < \alpha \leqq 1$, *be a* C^k *map,* $k \geqq 1$, *such that* Φ: $N \to C^0(\Sigma^+, \mathbb{R})$ *is* C^{k+1}. *Then* $P \circ \Phi$: $N \to \mathbb{R}$ *is* C^{k+1}.

The proof of this requires two lemmas. For metric spaces K_1, K_2 we say that a map f: $K_1 \to K_2$ is *compact* if it maps bounded sets onto relatively compact sets. One can see that if a sequence of compact maps f_n: $K_1 \to K_2$ converges to a map f: $K_1 \to K_2$ uniformly on bounded sets, then f is compact.

4.12. Lemma. *Let* E_1, E_2 *be Banach spaces and* $U \subset E$, *an open set. If* f: $U \to E_2$ *is a* C^k *compact map, then for all* $x \in U$, *the derivatives* $f^{(i)}(x)$: $E_1 \times \overset{i}{\ldots} \times E_1 \to E_2$ *are compact for all* $1 \leqq i \leqq k$.

Proof. Given $x \in U$ let B be the unit ball centered at 0 and define mas f_n: $B \to E_2$ by

$$f_n(v) := n\left(f\left(x + \frac{1}{n} v\right) - f(x)\right).$$

Then the sequence f_n converges uniformly to $f'(x)|_B$. Since each map f_n is compact, it follows that $f'(x)$ is compact. Now suppose that we have proved that $f^{(i)}(x)$ is compact for $1 \leqq i < n$. Define maps f_n: $\to E_2$ by

$$f_n(v) = n^m\left(f\left(x + \frac{1}{n} v\right) - f(x) - \sum_{i=1}^{m-1} \frac{1}{i!} f^{(i)}(x)\left(\frac{1}{n} v, \overset{i}{\ldots} \frac{1}{n} v\right)\right)$$

Then the sequence f_n converges uniformly to the map $B \ni v \mapsto f^{(m)}(x)(v, \ldots, v) \in E_2$. Hence this map is compact. Since using the symmetry of the m-linear map $f^{(m)}(x)$ it is possible to write $f^{(m)}(x)(v_1, \ldots, v_m)$ as a linear combination of the vectors $f^{(m)}(x)(v_i, \ldots, v_i)$, $1 \leqq i \leqq m$, it follows that $f^{(m)}(x)$ is compact. □

4.13. Lemma. *Let E_0, E_1, E_2 be Banach spaces, $U \subset E_0$ an open set and suppose that $f: U \to E_1$, $L: E_1 \to E_2$ and $P: E_2 \to \mathbb{R}$ are maps satisfying*

(a) *L is linear and compact.*
(b) *f is C^k, $k \geqq 0$*
(c) *$L \circ f$ is C^{k+1}*
(d) *$P \circ L$ is C^{k+1}*
(e) *There exists a function T that to each $x \in E_1$ associates a continuous linear map $T(x): E_2 \to \mathbb{R}$ satisfying*

$$(P \circ L)'(x) = T(x)L \tag{1}$$

for all $x \in E_1$, and

$$\lim_{n \to +\infty} T(x_n)v = T(\lim x_n)v \tag{2}$$

for every convergent sequence $\{x_n\} \subset E_1$ and all $v \in E$.
Then $P \circ L \circ f$ is C^{k+1}.

Proof. Clearly $P \circ L \circ f$ *is* C^k because $P \circ L$ *is* C^k and f is C^k. Suppose that $k \geqq 1$. The derivative $(P \circ L \circ f)^{(k)}(x)$ can be written as the sum of

$$(P \circ L)'(f(x)) \cdot f^{(k)}(x) \tag{3}$$

and a linear combination of compositions of derivatives $(P \circ L)^{(i)}$ and $f^{(j)}$ with $1 < i \leqq k$ and $1 \leqq j < k$. Hence all these terms are C^1, because f is C^k and $P \circ L$ is C^{k+1}. This means that in order to prove that f is C^{k+1} we have only to prove that (3) is C^1. Observe that

$$\begin{aligned}(P \circ L)'(f(x+tw)) \cdot f^{(k)}(x+tw) - (P \circ L)'(fx) f^{(k)}(x) \\ = ((P \circ L)'(f(x+tw)) - (P \circ L)'(fx)) f^{(k)}(x+tw) \\ + (P \circ L)'(fx)(f^{(k)}(x+tw) - f^{(k)}(x)).\end{aligned}$$

Since $P \circ L$ is C^2 by (d) and the assumption $k \geqq 1$, it follows that

$$\lim_{t \to 0} \frac{1}{t} ((P \circ L)'(f(x+tw)) - (P \circ L)'(fx)) f^{(k)}(x+tw) = ((P \circ L)''(fx)(f'(x) \cdot w)) f^{(k)}(x).$$

Moreover, by (1):

$$(P \circ L)'(fx)(f^{(k)}(x+tw) - f^{(k)}(x)) = T(fx)((Lf)^{(k)}(x+tw) - (Lf)^{(k)}(x)).$$

Hence, since Lf is C^{k+1},

$$\lim_{t \to 0} \frac{1}{t} T(fx) \cdot ((Lf)^{(k)}(x+tw) - (Lf)^{(k)}(x)) = T(fx) \cdot ((Lf)^{(k+1)}(x) \cdot w).$$

Thus

$$\begin{aligned}\lim_{t \to 0} \frac{1}{t} ((P \circ L)'(f(x+tw)) f^{(k)}(x+tw) - (P \circ L)'(fx) f^{(k)}(x) \\ = (P \circ L)''(fx)(f'(x) \cdot w) f^{(k)}(x) + T(fx)(Lf)^{(k+1)}(x) \cdot w.\end{aligned}$$

Therefore, if we prove that this $(k+1)$-linear map is a continuous function of x, it will follow that (2) is C^1 and then that $P \circ L \circ f$ is C^{k+1}. Since $P \circ L$ is C^2 and f is C^k it follows that the first term of the sum depends continuously on x. To prove the continuity of $T(fx)(Lf)^{(k+1)}(x)$ first observe that if $x_n \to x$ and $S \subset E_2$ is a relatively compact set then, by (2), $T(x_n)|_S$ converges uniformly to $T(x)|_S$. Moreover, Lf is compact because L is compact and then, by the previous lemma, $(Lf)^{(k+1)}(y)$ is compact for all $y \in E_0$. Let B be the unit ball of $E_0 \times \overset{k+1}{\cdots} \times E_0$. Define

$$S = (Lf)^{(k+1)}(x) B \cup \Big(\bigcup_{n \geqq 1} (Lf)^{(k+1)}(x_n) B \Big).$$

This set is relatively compact because every sequence $\{u_n\} \subset S$ either has a subsequence contained in some $(Lf)^{(k)}(p) B$, $p \in \{x, x_1, \ldots\}$ (and then, since this set is relatively compact, has a convergent subsequence) or has a subsequence that can be written as

$$u_{n_j} = (Lf)^{(k+1)}(x_{m_j}) \ominus_{m_j}$$

with $\ominus_{m_j} \in B$ and $m_j \to +\infty$ when $j \to +\infty$. But now, by the compacity of $(Lf)^{(k+1)}(x)$, we can assume that the sequence $(Lf)^{(k+1)}(x) \ominus_{m_j}$, $j \geqq 1$, converges to a point $y \in E_2$ and then it is easy to prove that $(Lf)^{(k+1)}(x_{m_j}) \ominus_{m_j}$ converges to y. This concludes the proof of the relative compacity of S and then $T(x)|_S$ converges uniformly to $T(x)|_S$. Since $(Lf)^{(k+1)}(x) B \subset S$ and $(Lf)^{(k+1)}(x_n) B \subset S$ for all $n \geqq 1$, it follows that $T(x_n)(Lf)^{(k+1)}(x_n)|_B$ converges to $T(x)(Lf)^{(k+1)}(x)|_B$ uniformly. This completes the proof of the lemma when $k \geqq 1$. The case $k=0$ is handled by similar methods. □

Proof of (4.11) We apply Lemma 4.13 to an open set $U \subset N$, the Banach spaces $E_1 = C^\alpha(\Sigma^+, \mathbb{R})$ and $E_2 = C^0(\Sigma^+, \mathbb{R})$, the C^k map Φ: $U \to C^\alpha(\Sigma^+, \mathbb{R})$, the compact linear map i: $C^\alpha(\Sigma^+, \mathbb{R}) \to C^0(\Sigma^+, \mathbb{R})$ given by the inclusion and the function P: $C^0(\Sigma^+, \mathbb{R}) \to \mathbb{R}$. Hypothesis (a), (b) and (c) are obviously satisfied. Hypothesis (d) is satisfied because we proved (4.6) that P: $C^\alpha(\Sigma^+, \mathbb{R}) \to \mathbb{R}$ is real analytic. To check (2) associate, to each $\psi \in C^\alpha(\Sigma^+, \mathbb{R})$, the functional $T(\psi) \in C^0(\Sigma^+, \mathbb{R})^*$ given by $T(\psi)\varphi = \int \varphi \, d\mu_\psi = \int \varphi h_\psi \, d\nu_\psi$. Then, by (4.10), $(P \circ i)'(\psi)\varphi = T(\psi)\varphi$, thus proving property (1) of hypothesis (e). Property (2) follows from the fact that ν_ψ is, by (4.9), a weakly continuous function of $\psi \in C^\alpha(\Sigma^+, \mathbb{R})$ and h_ψ is a continuous (in fact real analytic) function of ψ by (4.6). Then (4.13) can be applied and proves that $P \circ i \circ \phi = P \circ \phi$ is C^{k+1}. □

5 Proofs of the theorems

5.1. Lemma. *There exists $\alpha > 0$, a subshift of finite type Σ, a neighbourhood $\mathscr{U} \subset \mathscr{X}^r(M)$ of X and C^{r-1} maps τ: $\mathscr{U} \to C^\alpha(\Sigma, \mathbb{R})$, k: $\mathscr{U} \to C^\alpha(\Sigma, M)$ such that τ: $\mathscr{U} \to C^0(\Sigma, \mathbb{R})$, k: $\mathscr{U} \to C^0(\Sigma, M)$ are C^r and for $Y \in \mathscr{U}$*

$$\begin{array}{ccc} S(\Sigma, \tau_Y) & \xrightarrow{\;f^t\;} & S(\Sigma, \tau_Y) \\ \Big\downarrow \overline{k_Y} & & \Big\downarrow \overline{k_Y} \\ \Lambda(Y) & \xrightarrow{\;\psi_Y\;} & \Lambda(Y) \end{array}$$

conmutes. Where ψ_Y *is the flow of* Y, $\overline{k_Y}(x,t)=\psi_Y(k_Y(x),t)$ *for* $x\in\Sigma$, $0\leq t<\tau_Y(x)$ *and* $\Lambda(Y)=\bigcap_{t\in\mathbb{R}}\psi_Y(U,t)$ *is the corresponding basic set for* $Y\in\mathcal{U}$ *where* U *is a small neighbourhood of* Λ.

Proof. Take Σ, π, τ as in Lemma 3.1 and u_Y, γ_Y as in Proposition 1.1. Let $k_Y(x):=u_Y\circ\pi(x)$ and

$$\tau_Y(x):=\int_0^{\tau(\pi x)}\gamma_Y(\phi(\pi x,s))^{-1}ds$$

where $\phi(x,t)$ is the flow of X. Then τ_Y is the solution of $\psi_Y(u_Y(\pi x),\tau_Y(\pi x)) =u_Y(\phi(\pi x,\tau_X(\pi x)))$, $x\in\Sigma$, with $\tau_X=\tau$. From this the lemma follows easily. □

5.2. *Proof of theorems.* Let $\mathcal{U}$ be as in Lemma 5.1. For $Y\in\mathcal{U}$, *let* $\varphi_t\in C^\alpha(\Sigma,\mathbb{R})$ be $\varphi_Y(x):=\int_0^{\tau_Y(x)}\psi(Y)\circ\overline{k_Y}(x,s)ds$. Then the map $\mathcal{U}\ni\mapsto\varphi_Y\in C^0(\Sigma,\mathbb{R})$ is C^s. The pressure $P_Y(\psi(Y))=:p_Y$ is given by $P(\varphi_Y-p_Y\tau_Y)=0$ where P is the pressure on (Σ,σ). By Proposition 4.11, the function G: $\mathcal{U}\times\mathbb{R}\to\mathbb{R}$, $G(Y,s):=P(\varphi_Y-p_Y\tau_Y)$ is C^s. By the same argument as in Corollary 4.10 we have $\frac{\partial G}{\partial s}(X,p_X)= -\int_X\tau_X d\nu_X<0$, where ν_Y is the equilibrium state of $-p_Y\tau_Y$ on (Σ,σ) for $Y\in\mathcal{U}$. So that the invertibility condition holds and the implicit function can be applied.

By Lemma 4.1(b) the equilibrium state μ_Y for $\psi(Y)$ on $(\Lambda(Y),Y)$ is given by

$$\mu_Y(\varphi)=\frac{\nu_Y(\phi)}{\int\tau_Y d\nu_Y},\qquad\text{for }\varphi\in C^0(M,\mathbb{R})$$

where $\phi(x)=\int_0^{\tau_Y(x)}\varphi(\overline{k_Y}(x,s))ds$, $x\in\Sigma$. Thus by Corollary 4.8, $Y\mapsto\mu_Y\in(C^\alpha(M,\mathbb{R}))^*$ is C^{s-1}. The differentiability of the metric entropy can be seen using the variational principle $h_\mu=P(\psi)-\int\psi\,d\mu$. This completes the proof of Theorem C.

In order to prove Theorem B, take $\mathcal{U}$ as in (2.1). For $Y\in\mathcal{U}$ consider the function φ_Y^u: $\Sigma\to\mathbb{R}$,

$$\varphi_Y^u(x):=-\log|\det D_1\phi_Y(u_Y\circ k_Y(x),\tau_Y(x))|_{E^u_{Y\circ u_Y\circ k_Y(x)}}|$$

where $\tau_Y:\Sigma\to\mathbb{R}$ is as in (5.1) and ϕ_Y is the flow of Y. Then by (2.1) and (4.1), the map $Y\mapsto\varphi_Y^u\in C^\alpha(\Sigma,\mathbb{R})$ is C^{r-2} for a suitable $\alpha>0$ and $Y\mapsto\varphi_Y^u\in C^\alpha(\Sigma,\mathbb{R})$ is C^{r-1}. An obviuous adaptation of the arguments above conclude the proof of Theorem B. □

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