Université de Bordeaux

Faculté de Mathématiques

ALGANT Master

Mémoire de Master 2

Algebraic Values of Transcendental Functions

José Ibrahim Villanueva Gutiérrez
Author

Dr. Yuri Bilu
Thesis Advisor

July 2014

Contents

Introduction

A José, Manuel, Angélica, Esperanza, Graciela,
que velan mis pasos,
y a los que siguen caminando conmigo,
mi Madre, mi Abuela, mi familia.

From the first examples of transcendental numbers, explicitly constructed by Liouville, and the nature of common numbers such as π, which is the key to solve the very old problem of *squaring the circle*, transcendental numbers have got the attention of mathematicians and wonderful results emerged from them. The idea of measuring how complicated a number is, have intrigued generations of mathematicians. Nowadays there is a solid machinery which allows to explore the jungle of numbers, nonetheless very little is known.

In this work we start giving the basic machinery, that is, the theory of heights, and we will give a proof of a particular case of the criterion of Schneider-Lang in the language of heights.

Then as is very common in mathematics, numbers not suffice, so we move to transcendental functions which are surprisingly common (as well as transcendental numbers) and assorted when we look at their possible images. We give the proof of a result which is outstanding (but non effective), which offers relatively small upper bounds for the number of algebraic values that a transcendental function take at algebraic numbers.

In the last part we hyper-specialize, that is, we take the Riemann zeta function, we take our magnifying glass between 2 and 3, and find that the number of rational values at these points is insignificant; of course we do not expect such points at all. This gives an insight of how fantastic the study of numerical functions can be.

I would like to thank the ALGANT consortium, who supported me to pursue this beautiful path by which a lot of great minds and persons have walked. Specially I am very grateful to my thesis advisor Yuri, who always had a cup of coffee and a hand for me. I would like to thank also my professors for sharing their invaluable knowledge with us.

Chapter 1

Transcendental numbers

1.1 Introduction

An *algebraic number* is a complex number which is the root of some nonzero polynomial with rational coefficients. Let $\overline{\mathbb{Q}}$ be the set of the algebraic numbers, then every $\alpha \in \overline{\mathbb{Q}}$ belongs to some number field K (a finite field extension of $\mathbb{Q}$), the degree d of α is the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

Definition 1.1.1. A complex number β is *transcendental* if $\beta \notin \overline{\mathbb{Q}}$.

Georg Cantor used his diagonalization argument to show that $\mathbb{C} \setminus \overline{\mathbb{Q}}$ is an uncountable set. Using approximation techniques to algebraic numbers, Liouville succeeded to give the first examples of transcendental numbers. Nevertheless, it was unknown whether e, π or $2^{\sqrt{3}}$ were transcendental numbers until the works of Hermite-Lindemann and of Gelfond-Schneider respectively.

In this chapter we introduce some of the machinery needed in the rest of this work and we show carefully the proof of a theorem from which the mentioned results follow.

1.2 Heights

For a number field K of degree $d = [K : \mathbb{Q}]$ let us denote M_K be its set of places, that is the set of non-trivial absolute values on K whose restrictions to $\mathbb{Q}$ are normalized as

$$\begin{aligned} |x|_\infty &= x, \text{ if } x \in \mathbb{Q}_{\geq 0}, \\ |p|_p &= \frac{1}{p}, \text{ if } p \text{ is a prime number.} \end{aligned}$$

For every $\alpha \neq 0$ in K, we have the *product formula*

$$\prod_{v \in M_K} |\alpha|_v^{d_v} = 1, \tag{1.1}$$

where d_v is the local degree $[K_v : \mathbb{Q}_v]$.

Let $P \in \mathbb{P}_K^n$ represented by a homogeneous non-zero vector $\mathbf{x}$ with homogeneous coordinates $\mathbf{x} = (x_0 : x_1 : \ldots : x_n)$.

Definition 1.2.1. We define the *projective height* of P as

$$H(\mathbf{x}) = \prod_{v \in M_K} \max_{0 \leq i \leq n} |x_i|_v^{d_v/d}. \tag{1.2}$$

Similarly we define the *logarithmic projective height* of α as

$$h(\alpha) = \frac{1}{d} \sum_{v \in M_K} d_v \max_{0 \leq i \leq n} \log |x_i|_v. \tag{1.3}$$

One shows easily that the projective height does not depend on the field K nor in the coordinates up to scalar multiplication. The natural embedding

$$\begin{aligned} \sigma : \mathbb{A}_K^n &\longrightarrow \mathbb{P}_K^n \\ (x_1, \ldots, x_n) &\mapsto (1 : x_1 : \ldots : x_n), \end{aligned}$$

provides a height function for affine points, defining the height of such a point as the projective height of its image.

Example 1.2.1. Let $\alpha \in K$. We define the *height* of α as

$$H(\alpha) = \prod_{v \in M_K} \max\{1, |\alpha|_v\}^{\frac{d_v}{d}}, \tag{1.4}$$

and the *logarithmic height* of α as

$$h(\alpha) = \frac{1}{d} \sum_{v \in M_K} d_v \log^+ |\alpha|_v, \tag{1.5}$$

where $\log^+ |\alpha|_v := \log \max\{1, |\alpha|_v\}$.

Notice that both definitions are equivalent up to taking logarithm, we shall call both of them *height*, also we will use the term height for affine points, considering the height of their image in a suitable projective space.

Proposition 1.2.1. *Let $\alpha = \alpha_1, \ldots, \alpha_n$ be algebraic numbers, then*

(i) $h(\alpha^r) = |r|h(\alpha)$, *for every* $r \in \mathbb{Q}$,

(ii) $h(\alpha_1 \cdots \alpha_n) \leq h(\alpha_1) + ... + h(\alpha_n)$,

(iii) $h(\alpha_1 + \ldots + \alpha_n) \leq \log(n) + h(\alpha_1) + \ldots + h(\alpha_n)$,

(iii) $h(\alpha) = h(\sigma(\alpha))$ *for all* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Proof. [Bo-Gu 06, I.1.5.14,15,17 & 18]. □

Definition 1.2.2. Let K be a number field. The height of a polynomial

$$f(T_1, \ldots, T_n) = \sum_{\mu_1,\ldots,\mu_n} \lambda_{\mu_1\ldots\mu_n} T_1^{\mu_1} \cdots T_n^{\mu_n} = \sum_{(\mu)} \lambda_{(\mu)} T_{(\mu)},$$

is the quantity

$$H(f) = \prod_{v \in M_K} \max_{\mu} \{1, |\lambda_{(\mu)}|_v\}^{d_v/d}. \tag{1.6}$$

Let $f_\alpha(x) = a_d(x - \alpha_1) \cdots (x - \alpha_d)$ be the minimal polynomial of α, where $\alpha = \alpha_1, ..., \alpha_n$ are its conjugates.

We define the *Mahler measure* of α (equivalently of f_α) as

$$M(\alpha) = M(f_\alpha) := |a_d| \prod_{i=1}^{d} \max\{1, |\alpha_d|\}, \tag{1.7}$$

this is equivalent to Jensen's formula

$$\log(M(f_\alpha)) = \log|a_d| + \sum_{i=1}^{d} \log^+ |\alpha_i|. \tag{1.8}$$

The Mahler measure is related to the height of α by the following formula

$$h(\alpha) = \frac{1}{d} \log M(\alpha). \tag{1.9}$$

The next theorem is a very important finiteness result.

Theorem 1.2.1 (Northcott)**.** *The number of algebraic numbers with bounded degree and bounded height is finite, i.e. for $D \in \mathbb{Z}_{\geq 1}$ and $N \in \mathbb{R}_{>0}$ the set*

$$E_{D,N} := \{\alpha \in \overline{\mathbb{Q}} \mid [\mathbb{Q}(\alpha) : \mathbb{Q}] \leq D \text{ and } h(\alpha) < N\} \tag{1.10}$$

is finite.

Proof. [Bo-Gu 06, I.1.6]. □

Moreover, the cardinality of $E_{D,N}$ which we will denote $\epsilon_{D,N}$ can be bounded effectively.

Proposition 1.2.2. *For every integer $D \geq 1$ and every real number $N \geq 0$, we have*

$$e^{D(D+1)(N-1)} < \epsilon_{D,N} \leq e^{D(D+1)(N+1)}. \tag{1.11}$$

Proof. [Su 06, Lemme 1.1]. □

There exist sharper results for the lower-bounding and for $\varepsilon_{d,N}$ the set of algebraic numbers with bounded height and exact degree $d \geq 1$.

Another important result about heights is the so called Siegel's lemma. For a system of linear equations over K a number field, it gives an upper bound (in terms of the height of the equations, the number of equations and dimension of the solutions) for the height of a nontrivial solution.

We denote $D_{K/\mathbb{Q}}$ the discriminant and $\mathcal{O}_K$ the ring of integers of a number field K.

Lemma 1.2.1 (Siegel's lemma)**.** *Let K be a number field of degree d, let r, n positive integers, assume that $r < n$ and let*

$$\begin{array}{ccccccc} a_{11}X_1 & + & \cdots & + & a_{1n}X_n & = & 0 \\ \vdots & & \ddots & & \vdots & & \vdots \\ a_{r1}X_1 & + & \cdots & + & a_{rn}X_n & = & 0 \end{array}$$

be a system of linear equations with $a_{ij} \in K$ (not all zero), denote $A = (a_{ij})$ and let $H(A)$ be the projective height of A as a point in $\mathbb{P}_K^{nr-1}$. There exists a nontrivial solution $\boldsymbol{x} = (x_1, ..., x_n) \in K^n$ such that

$$H(\boldsymbol{x}) \leq |D_{K/\mathbb{Q}}|^{1/2d}(\sqrt{n}H(A))^{\frac{r}{n-r}},$$

where $H(\boldsymbol{x})$ is the projective height considering $\boldsymbol{x}$ as a point in $\mathbb{P}_K^{n-1}$.

Proof. [Bo-Gu 06, 2.9.9]. □

The first version of Siegel's lemma was introduced by Siegel himself for integral linear systems and it is basically a consequence of the Pigeon-hole principle. There exist several variants of Siegel's lemma, the one we use here is proved using geometry of numbers.

Let $f \in \mathbb{C}[X_1, \ldots, X_t]$ be a polynomial in t variables over the complex numbers. We denote $L(f)$ the *length* of f, which is the sum of the modulus of its complex coefficients. The length behaves well under addition and multiplication

$$L(f+g) \;\leq\; L(f) + L(g) \;\text{ and }\; L(fg) \;\leq\; L(f)L(g). \tag{1.12}$$

An immediate result of the properties coming out from the height and the product formula is the following.

Lemma 1.2.2 (Liouville's Inequality)**.** *For an algebraic number $\alpha \neq 0$ in K and $v \in M_K$ we have*

$$|\alpha|_v \geq H(\alpha)^{-d/dv}. \tag{1.13}$$

Proof. By (i) in Proposition 1.2.1 we have $H(\alpha) = H(\alpha^{-1})$ and clearly $|\alpha|_v \leq H(\alpha)^d$. □

1.3 The transcendence of e and π

An *entire function* is a complex analytic function which is defined in the whole complex plane.

Let f be an entire function, we say that f has *order* $\leq \rho$ if there exists a number $C > 1$ such that for a large R we have

$$|f(z)| \leq C^{R^\rho},$$

whenever $|z| \leq R$. In general a meromorphic function is said to have order $\leq \rho$ if it is a quotient of entire functions of order $\leq \rho$.

Let f, g be real functions, we say that $f = O(g)$ if there exists $C > 0$ such that for x sufficiently big $|f(x)| \leq Cg(x)$.

Let

$$P(T_1, ..., T_N) = \sum \lambda_{(\mu)} T_{(\mu)} \in K[T_1, ..., T_N]$$

and let

$$Q(T_1, ..., T_N) = \sum \beta_{(\mu)} T_{(\mu)} \in \mathbb{R}_{\geq 0}[T_1, ..., T_N],$$

both of degree r. We say that Q *dominates* P at $v \in M_K$ if

$$|\lambda_{(\mu)}|_v \leq \beta_{(\mu)} \quad \forall \mu.$$

Theorem 1.3.1 (Schneider-Lang)**.** *Let K be a number field. Let $f_1, f_2, ..., f_N$ be meromorphic functions of order $\leq \rho$ such that $K(f_1, ..., f_N)$ has trascendental degree ≥ 2 over K. Suppose that the derivative $D = d/dz$ is an endomorphism of $K[f_1, ..., f_N]$. Let $w_1, ..., w_m \in \mathbb{C}$ none of them poles of the f_i's, and such that*

$$f_i(w_\nu) \in K$$

for $i = 1, ..., N$ and $\nu = 1, ..., m$. Then

$$m \leq 10\rho[K : \mathbb{Q}].$$

Proof. Let f and g be two functions among $f_1, ..., f_N$ which are algebraically independent over K. Let r be a multiple of $2m$.

Define

$$F = \sum_{i,j=1}^{r} b_{ij} f^i g^j \in K[f, g].$$

So F has degree $2r$. We have that $D^k F(z) \in K[f_1, ..., f_N]$ for every $k \geq 0$, then

$$D^k F(w_\nu) \in K \quad \forall \, \nu = 1, ..., m.$$

So let $n = \frac{r^2}{2m}$ and consider the system of linear equations in $(b_{11}, ..., b_{rr})$ over K

$$D^k F(w_\nu) = 0$$

for $0 \leq k < n$ and $\nu = 1, ..., m$. We have $r^2 = 2mn$ unknowns b_{ij} and mn equations, so by Siegel's lemma 1.2.1 we have that there exist a nontrivial solution $\mathbf{b} = (b_{ij}) \in \mathcal{O}_K^{2mn}$ such that

$$H(\mathbf{b}) \leq C_1 r H(A) \tag{1}$$

for C_1 depending only on K, and where A is the matrix of the system of linear forms over K.

Claim 1. *Let $w = w_i$ for some $i = 1, ..., m$ and let*

$$P(T_1, ..., T_N) = \sum \lambda_{(\mu)} T_{(\mu)} \in K[T_1, ..., T_N]$$

of degree $\leq r$. Set $f = P(f_1, ..., f_N)$, then there exists a real number C_2 such that for all positive integers k

$$H(D^k f(w)) \leq H(P) r^k k! C_2^{k+r}.$$

Proof of the claim. For every $v \in M_K$ define

$$b_v = \max_{\mu} \{|\lambda_{(\mu)}|_v\} \quad \text{and} \quad C_v = 1 + |f_1(w)|_v + ... + |f_N(w)|_v.$$

P is dominated at v by $b_v(1 + T_1 + ... + T_N)^r$, which implies $|f(w)|_v \leq b_v C_v^r$. It follows

$$\max\{1, |f(w)|_v\} \leq \max\{1, b_v\} \max\{1, C_v^r\},$$

hence taking the corresponding product over all the places we get

$$H(f(w)) \leq H(P) C_2^r,$$

from which the case $k = 0$ follows.

For the remaining cases we proceed by induction. There exist polynomials $P_i(T_1, ..., T_N)$ such that $Df_i = P_i(f_1, ..., f_N)$ and put $h = \max_{1 \leq i \leq N} \{\deg P_i\}$. There exists a unique derivation $\overline{D}$ on $K[T_1, ..., T_N]$ such that

$$\overline{D} T_i = P_i(T_1, ..., T_N).$$

For any $P \in K[T_1, ..., T_N]$ we can write

$$\overline{D}(P(T_1, ..., T_N)) = \sum_{i=1}^{N} \frac{\partial P}{\partial T_i}(T_1, ..., T_N) P_i(T_1, ..., T_N).$$

Writing

$$\frac{\partial P}{\partial T_i}(T_1, ..., T_N) = \sum \lambda_{(\mu)} \frac{\partial}{\partial T_i} T_{(\mu)}$$

we find that it is a polynomial (*over*) dominated at v by

$$b_v r (1 + T_1 + ... + T_N)^r,$$

moreover if we write

$$P_i(T_1, ..., T_N) = \sum \lambda_{(\mu_i)} T_{(\mu_i)},$$

and define $b_{i,v} = \max_{\mu_i}\{|\lambda_{(\mu_i)}|_v\}$, we have that P_i is a is dominated at v by

$$b_{i,v}(1 + T_1 + ... + T_N)^h,$$

therefore $\overline{D}P$ is dominated by

$$b_v C_3 r(1 + T_1 + ... + T_N)^{h+r}.$$

Now suppose that $\overline{D}^k P$ is dominated by

$$Q(T_1, ..., T_2) = b_v C_3 r^k k!(1 + T_1 + ... + T_N)^{hk+r},$$

one compute easily that $\overline{D}Q = b_v C_4 r^k k!(hk + r)(1 + T_1 + ... + T_N)^{h(k+1)+r}$ which dominates $\overline{D}^{k+1}P$, in fact since $hk + r \leq r(k + 1)$ we have that $\overline{D}^{k+1}P$ is dominated by

$$b_v C_4 r^{k+1}(k + 1)!(1 + T_1 + ... + T_N)^{h(k+1)+r}.$$

Using the last equation we obtain

$$|D^k f(w)|_v \leq b_v C_4 r^k k!(1 + |f_1(w)|_v + ... + |f_N(w)|_v)^{hk+r},$$

taking maximums and the corresponding products we get

$$H(D^k f(w)) \leq H(P) r^k k! C_2^{k+r},$$

this proves the claim. □

Since f, g are algebraically independent over K, our function F is not identically zero, so eventually the derivative would not vanish at all the points $w_1, ..., w_n$, let $s \geq n$ be the smallest integer for which that happens for a w, say w_1. We have then

$$\gamma := D^s F(w_1) \neq 0,$$

is in K.

Using Liouville's Inequality (1.13) we find a lower bound for the norm of γ and an upper bound obtained from the claim

$$1 \leq |\gamma| H(\gamma)^{[K:\mathbb{Q}]} \leq |\gamma| O(s^{5s})^{[K:\mathbb{Q}]} \tag{2}$$

when $s \to \infty$.

Now we will estimate the complex absolute value of γ by global arguments.

Let Θ be an entire function of order $\leq \rho$, such that Θf and Θg are entire and $\Theta(w_1) \neq 0$. Then $\Theta^{2r} F$ is entire and has zeros of exact order s at w_1 and $s_\nu \geq s$ at w_ν for $\nu = 2, ..., m$. We consider the entire function

$$G(z) = \frac{\Theta(z)^{2r} F(z)}{\prod_{v=1}^{m} (z - w_\nu)^s}.$$

By the maximum modulus principle, $|G(z)|$ is bounded by the maximum of G on the boundary of a disk of radius R, moreover if we take R be large, then $|z - w_\nu|$ approches R if z lies on the boundary of the disk, and consequently,

$$|G(z)| \leq \frac{s^{3s} C_5^{2rR^\rho}}{R^{ms}}.$$

Put $R = s^{\frac{1}{2\rho}}$, we get the estimate and the upper bound

$$|\gamma| \leq \frac{s^{4s} C_6^s}{s^{ms/2\rho}} \leq O(s^{5s}),$$

when r tend to infinity, so n and s tend to infinity. Combining this last inequality with (2), we obtain

$$s^{ms/2\rho} \leq O(s^{5s})^{[K:\mathbb{Q}]},$$

the desired result follows from taking logarithms. □

Corollary 1.3.1 (Hermite-Lindemann)**.** *If α is algebraic and nonzero, then e^α is transcendental. Hence e and π are transcendental.*

Proof. Suppose $\alpha \neq 0$ and e^α are algebraic, let $K = \mathbb{Q}(\alpha, e^\alpha)$. Consider the functions z and e^z, both of order ≤ 1. Also z and e^z are algebraically independent over K, since $e^z \in K[[z]] \backslash K[z]$, thus $K(z, e^z)$ has transcendental degree 2. The derivative d/dz is an endomorphism of $K[z, e^z]$. For every $m \geq 1$ our functions take algebraic values at $\alpha, 2\alpha, ..., m\alpha$. Then $m > 10\rho[K : \mathbb{Q}]$ for a sufficiently large m, contradiction. □

Corollary 1.3.2 (Gelfond-Schneider)**.** *If α is algebraic and $\neq 0, 1$ and if β is algebraic irrational, then $\alpha^\beta = e^{\beta \log \alpha}$ is transcendental.*

Proof. Suppose $\alpha \neq 0, 1$ and α^β are algebraic, with β quadratic irrational. Let $K = \mathbb{Q}(\alpha, \alpha^\beta)$. Consider the functions $e^{\beta t}$ and e^t, both of order ≤ 1.

Claim 2. *If $\omega_1, ..., \omega_n \in \mathbb{C}$ are pairwise distinct then $e^{\omega_1 z}, ..., e^{\omega_n z}$ are algebraically independent.*

Let $\omega_1, ..., \omega_n \in \mathbb{C}$ are pairwise distinct, consider $z_0 \in \mathbb{C}$ such that $(\omega_j - \omega_k)z_0$ is not a multiple of $2\pi i$ for every $j \neq k$. For that z_0, the numbers

$$\gamma_1 = e^{\omega_1 z_0}, ..., \gamma_n = e^{\omega_n z_0}$$

are pairwise distinct, consider the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_1^{n-1} & \gamma_2^{n-1} & \cdots & \gamma_n^{n-1} \end{pmatrix}$$

which determinant is non-zero, since it is given by $\prod_{1 \leq j,k \leq n} |\gamma_k - \gamma_j| \neq 0$, thus the columns and rows are linearly independent, hence $e^{\omega_1 z}, ..., e^{\omega_n z}$ are algebraically independent. □

From the claim we see that since $\beta \notin \mathbb{Q}$, $e^{\beta t}$ and e^t are algebraically independent over K. The derivative d/dz is an endomorphism of $K[e^{\beta t}, e^t]$. For every $m \geq 1$ our functions take algebraic values at $\log \alpha, 2 \log \alpha, ..., m \log \alpha$. Then $m > 10\rho[K : \mathbb{Q}]$ for a sufficiently large m, contradiction. □

Chapter 2

Algebraic values of transcendental functions

Definition 2.0.1. We say that f is an analytic *transcendental function* if the graph $(z, f(z))$ does not vanish identically for any non-trivial polynomial $P(X,Y) \in \mathbb{C}[X,Y]$, i.e. $z, f(z)$ are algebraically independent over $\mathbb{C}$.

A *transcendental entire function* is an entire function which is not a polynomial.

For a transcendental function f we denote S_f the set of algebraic values of f at algebraic points

$$S_f = \{\alpha \in \overline{\mathbb{Q}} \mid f(\alpha) \in \overline{\mathbb{Q}}\}. \tag{2.1}$$

Example 2.0.1. $S_{e^z} = \{0\}$ by Hermitte-Lindemman (Corollary 1.3.1). For the same reason for $P \in \overline{\mathbb{Q}}[x]$, $S_{e^{P(z)}} = \{$ zeros of $P\}$.

Example 2.0.2. If $f(z) = e^{\lambda z}$ where $\lambda \neq 0$ and e^z is algebraic, then by Gelfond-Schneider (Corollary 1.3.2) we have $S_f = \mathbb{Q}$.

2.1 The possible sets of values of a transcendental function

A natural question which arises is: Does there exist an entire function f for which its set of algebraic values is either empty or either all the algebraic integers? The answer is yes.

Theorem 2.1.1 (Stäckel)**.** *Let Σ be a countable subset of $\mathbb{C}$ and T a dense subset of $\mathbb{C}$, there is an entire transcendental function sending Σ into T.*

Hence if $\Sigma = \overline{\mathbb{Q}}$, as we already mentioned we can have $S_f = \emptyset$ when $T = \mathbb{C} \setminus \overline{\mathbb{Q}}$ and we can get $S_f = \overline{\mathbb{Q}}$ when $T = \overline{\mathbb{Q}}$.

F. Gramain showed that Stäckel's Theorem 2.1.1 apply if $\Sigma \subset \mathbb{R}$ and T is dense in $\mathbb{R}$. Then a real number field K could be mapped into $T = \mathbb{Z}[1/n]$, for $n \geq 2$.

We fix a transcendental function f. In order to study the set S_f we will study it locally.

Let us denote $D(0,R)$ the disk of radius R around the origin in the complex plane. If a function f is continuous in $\overline{D(0,R)}$, we denote

$$|f|_R = \max_{|z|\leq R} |f(z)|. \tag{2.2}$$

Schwarz' Lemma for complex analytic functions of one variable provides a sharp upper bound for the values of a function having a lot of zeroes.

Lemma 2.1.1 (Schwarz' Lemma)**.** *Let $R \geq r > 0$ be real numbers. Let f be an analytic function in an open neighbourhood of $\overline{D(0,R)}$ which vanishes at $w_1, \ldots, w_N \in \overline{D(0,r)}$ with multiplicity $\geq \kappa_i$ $(1 \leq i \leq N)$. Then*

$$|f|_r \leq |f|_R \prod_{i=1}^{N} \left(\frac{R^2 + r|w_i|}{R(r + |w_i|)} \right)^{-\kappa_i} \tag{2.3}$$

Proof. [Wa 00, Exercise 4.3] □

Liouville's Inequality for algebraic numbers can be generalized in the following way.

Lemma 2.1.2 (Polynomial Liouville's Inequality)**.** *Let K be a number field of degree d, let $v \in M_K^\infty$ and $\nu_1, \ldots, \nu_\ell$ be positive integers. For $1 \leq i \leq \ell$, let $\gamma_{i,1}, \gamma_{i,2}, \ldots, \gamma_{i,\nu_i}$ be elements of K. Let f be a polynomial in $\nu_1 + \cdots + \nu_\ell$ variables, with coefficients in $\mathbb{Z}$, which does not vanish at the point $\underline{\gamma} = (\gamma_{ij})_{1\leq j\leq \nu_i, 1\leq i\leq \ell}$. Assume f is of total degree at most N_i with respect to the ν_i variables corresponding to $\gamma_{i1}, \ldots, \gamma_{i\nu_i}$. Then*

$$\log |f(\underline{\gamma})|_v \geq -(d-1)\log L(f) - d\sum_{i=1}^{\ell} N_i h(1 : \gamma_{i1} : \cdots : \gamma_{i\nu_i}). \tag{2.4}$$

Proof. [Wa 00, Proposition 3.14]. □

Notice that if we let $\ell = 1$, $v_1 = 1$ and consider the polynomial $f(x) = x$ we recover (1.13).

Definition 2.1.1. Let $R > r > 0$ be real numbers. For every analytic function defined $D(0, R)$, for every integer $D \geq 1$ and every real number $N > 0$ we define

$$\Sigma_{D,N}(f, r) = \Sigma_{D,N} \tag{2.5}$$

as the set of $\alpha \in \overline{\mathbb{Q}} \cap \overline{D(0, r)}$ such that

$$f(\alpha) \in \overline{\mathbb{Q}},\ [\mathbb{Q}(\alpha, f(\alpha)) : \mathbb{Q}] \leq D,\ h(\alpha) \leq N,\ h(f(\alpha)) \leq N.$$

Then it follows that S_f is the union of all the $\Sigma_{D,N}$, more precisely

$$S_f \cap \overline{D(0, r)} = \bigcup_{D,N} \Sigma_{D,N}(f, r).$$

We will denote $\sigma_{D,N}$ the cardinality of $\Sigma_{D,N}$.

For a real number x, $[x] \in \mathbb{Z}$ denotes the integer part of x, it satisfies $0 \leq x - [x] < 1$.

2.2 An outstanding non effective theorem

The next theorem gives an upper bound for $\sigma_{D,N}(f, r)$ which depends uniquely on the domain of definition of the function, the degree and the height of the algebraic values.

Theorem 2.2.1 (A. Surroca)**.** *Let $R > r > 0$ be real numbers. Define*

$$c_0 = \log\left(\frac{R^2 + r^2}{2rR}\right),$$

and let δ be a real number such that $\delta > 2(6/c_0)^2$.

Let f be an analytic transcendental function on $D(0, R)$ and continuous on $\overline{D(0, R)}$.

(i) For every integer $D \geq 1$ there exists a sequence of real numbers $N \geq 0$ tending to infinity such that

$$\sigma_{D,N}(f, r) < \delta D^3 N^2 \tag{2.6}$$

(ii) For every real number $N > 0$, there is a sequence of integers $D \geq 2$ tending to infinity such that

$$\sigma_{D,N}(f, r) < \delta D^3 N^2 \tag{2.7}$$

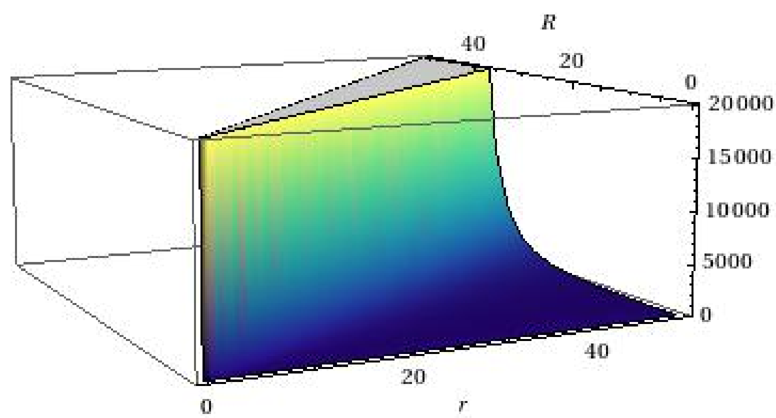

Figure 2.1: Graph of δ.

Notice that c_0 is small when $R - r \to 0$ and big when $R - r \to \infty$, and exactly the opposite happens for δ: when R and r are too close one to the other then its value is big and vice versa (see Figure 2.1).

If D and N are fixed then the set $\Sigma_{D,N}(f, r)$ is finite since it is a subset of $E_{D,N}$ (See Theorem 1.2.1).

Moreover, for D fixed and infinitely many integers N, the lower bound of $\epsilon_{D,N}$ given in Proposition 1.2.2 is bigger than $\sigma_{D,N}$. The same is true for fixed N and infinitely many integers D.

Proof of the Theorem.

First let us prove (i), we proceed by contradiction.

We give first an sketch of the proof.

(a) We suppose that there exists an integer $D \geq 1$ and an integer $N_0 \geq 1$ big enough such that for every $N \geq N_0$, we have

$$\sigma_{D,N} \geq \delta D^3 N^2.$$

For every $N \geq N_0$ we start by extracting a subset $S_{D,N}$ of $\Sigma_{D,N}$ for which its cardinality is well known.

(b) Using a Siegel's Lemma kind argument we find a non-zero polynomial P in two variables with integer coefficients which vanishes at every point in S_{D,N_0}. From which we define an analytic function F which is continuous in all $\overline{D(0,R)}$.

(c) Using induction, Liouville's Inequality and Schwarz lemma, we show that F vanishes at all points in $S_{D,N}$ for $N \geq N_0$.

(d) Last point implies that the function F will be the zero function, which is impossible since f is a transcendental over $\mathbb{C}(z)$.

(a)

Fix an integer $D \geq 1$ and a real $\delta > 2(6/c_0)^2$, suppose there exists a real number N_0 such that for every $N \geq N_0$

$$\sigma_{D,N} \geq \delta D^3 N^2.$$

Set

$$T = \left[\frac{c_0\delta}{6} D^2 N_0\right]. \tag{2.8}$$

Up to increasing N_0, the integer T satisfies the inequality

$$c_0\left[\delta D^3(N_0-1)^2\right] > D\log 2 + 4D\log T + T\left(4N_0 D + \log^+ R + \log^+ |f|_R\right).$$

Putting

$$u_1 = \log(2T^4 e^{2N_0 T} \max\{1, R^T\} \max\{1, |f|_R^T\}) \tag{2.9}$$

the last inequality becomes

$$c_0\left[\delta D^3(N_0-1)^2\right] > u_1 + (D-1)\log(2T^4) + 2TN_0(2D-1). \tag{2.10}$$

If $N \geq N_0$, we have $\sigma_{D,N} \geq \delta D^3 N^2$, then we can extract a subset $S_{D,N}$ of $\Sigma_{D,N}$ of exact cardinality $s_{D,N} = [\delta D^3 N^2]$.

Since D and N_0 are fixed, we denote $S_0 = S_{D,N_0}$ and $s_0 = s_{D,N_0}$, and we can write

$$S_0 = \{w_1, w_2, \ldots, w_{s_0}\}.$$

(b)

Using a Siegel's Lemma kind argument we construct a polynomial $P(X,Y) \in \mathbb{Z}[X,Y]$ which vanishes in every point of S_0.

Lemma 2.2.1. *There exists a polynomial $P \in \mathbb{Z}[X,Y]$ with the property that $P(1,Y)$ and $P(X,1)$ are of degree strictly less than T (see (2.8)) and such that*

$$P(w, f(w)) = 0, \quad \forall w \in S_0.$$

Moreover its coefficients are bounded in absolute value by $2T^2e^{2TN_0}$.

Proof. Let us write

$$P(X,Y) = \sum_{i=0}^{T-1}\sum_{j=0}^{T-1} c_{i,j}X^iY^j,$$

so consider the system of linear equations on the $c_{i,j}$'s variables given by

$$\begin{array}{rcl} P(w_1, f(w_1)) & = & 0 \\ \vdots & & \vdots \\ P(w_{s_0}, f(w_{s_0})) & = & 0, \end{array} \tag{2.11}$$

the system has a non-trivial solution $\mathbf{c} \in \mathbb{Z}^{T^2}$ provided $s_0 D < T^2$. This is in fact the case, since we have $\delta^2(c_0/6)^2 > 2\delta$, we find that (eventually increasing N_0)

$$s_0 D < 2\delta D^4 N_0^2 < T^2 \leq \left(\frac{c_0\delta}{6}\right)^2 D^4 N_0^2. \tag{2.12}$$

Proposition 2.2.1. *Let* $K_\ell = \mathbb{Q}(w_\ell, f(w_\ell))$ *and let* $d_\ell = [K_\ell : \mathbb{Q}]$*, similarly let* $d_{\ell,id} = [\mathbb{Q}(w_\ell) : \mathbb{Q}]$ *and* $d_{\ell,f} = [\mathbb{Q}(f(w_\ell)) : \mathbb{Q}]$*, with* $1 \le \ell \le s_0$*. Then there is a solution* $\boldsymbol{c} = (c_{i,j}) \in \mathbb{Z}^{T^2}$ *to the linear system (2.11) such that*

$$\max_{0\le i,j<T} |c_{i,j}| \le \left[\left(2^{s_0} \prod_{\ell=1}^{s_0} M_\ell \right)^{1/(T^2 - s_0 D)} \right], \tag{2.13}$$

where

$$M_\ell = T^{2d_\ell} M(w_\ell)^{(T-1)d_\ell/d_{\ell,id}} M(f(w_\ell))^{(T-1)d_\ell/d_{\ell,f}},$$

and the M *in the right denote the usual Mahler's measure (see 1.7).*

Proof. [Gr-Mi-Wa 86, Lemme 1.1]. □

The relation (1.9) between the Mahler's measure and the height yields

$$M_\ell = T^{2d_\ell} H(w_\ell)^{(T-1)d_\ell} H(f(w_\ell))^{(T-1)d_\ell},$$

furthermore since $d_\ell \le D$ and using the fact that $h(w_\ell)$ and $h(f(w_\ell))$ are bounded by N_0 we have

$$\max_{0\le i,j<T} |c_{i,j}| \le \left[\left(2^{s_0} T^{2s_0 D} e^{2s_0 N_0 D T} \right)^{1/(T^2 - s_0 D)} \right],$$

finally since $T^2 > 2s_0 D$, we have the bound

$$\max_{0\le i,j<T} |c_{i,j}| \le 2T^2 e^{2TN_0}.$$

□

Therefore we can define a function F continuous in all $\overline{D(0,R)}$ in the following way

$$F(z) = P(z, f(z)), \quad \forall\, z \in \overline{D(0,R)}. \tag{2.14}$$

(c)

We show now that F vanishes for all $w \in S_{D,N}$ for every $N \ge N_0$.

Lemma 2.2.2. *For every* $N \ge N_0$ *we have that if* $F(w) = 0$ *for every* $w \in S_{D,N}$ *then*

$$|F(w)| \le e^{u_1 - c_0 s_{D,N}}, \quad \forall\, w \in S_{D,N+1}, \tag{2.15}$$

where u_1 *is like in (2.9).*

Proof. Let $N \geq N_0$ and suppose $F(w) = 0$ for every $w \in S_{D,N}$. Considering $0 < r < R$ we can apply Schwar'z Lemma 2.1.1, from it we have

$$|F|_r \leq |F|_R \prod_{i=1}^{s_{D,N}} \left(\frac{R^2 + r|w_i|}{R(r + |w_i|)} \right)^{-1}, \tag{2.16}$$

further

$$\frac{R^2 + r|w|}{R(r + |w|)} \geq e^{c_0} > 0 \quad \forall\, w \in S_{D,N}, \tag{2.17}$$

which implies that

$$|F|_r \leq |F|_R e^{-c_0 s_{D,N}}. \tag{2.18}$$

Now we bound $|F|_R$ for $z \in \overline{D(0,R)}$,

$$\begin{aligned} |F|_R &\leq \left| \sum_{i=0}^{T-1} \sum_{j=0}^{T-1} c_{i,j} z^i f(z)^j \right| \\ &\leq T^2 \max_{i,j} |c_{i,j}| \max\{1, R^T\} \max\{1, |f|_R^T\} \\ &\leq 2T^4 e^{2N_0 T} \max\{1, R^T\} \max\{1, |f|_R^T\} \\ &= e^{u_1}, \end{aligned}$$

it follows that $|F(z)| \leq e^{u_1 - c_0 s_{D,N}}$, for all $z \in \overline{D(0,r)}$, in particular every $w \in S_{D,N+1}$ has complex absolute value $\leq r$, and the lemma follows. □

Lemma 2.2.3. *For every $N \geq N_0 + 1$ and for every $w \in S_{D,N}$, we have*

$$|F(w)| \leq e^{u_1 - c_0 s_{D,N-1}} \implies F(w) = 0.$$

Proof. Let $N \geq N_0 + 1$, $w \in S_{D,N}$ and suppose that $|F(w)| \leq e^{u_1 - c_0 s_{D,N-1}}$. We have $s_{D,N-1} = [\delta D^3 (N-1)^2]$ and since $N - 1 \geq N_0$, so N satisfies

$$c_0 s_{D,N-1} > u_1 + (D-1)\log(2T^4) + 2TN_0(D-1) - 2TND. \tag{2.19}$$

We can upper bound the length of f using Proposition 2.2.1 as follows

$$\begin{aligned} L(f) &\leq \sum_{i=0}^{T-1} \sum_{j=0}^{T-1} |c_{i,j}| \\ &\leq T^2 \max_{i,j} |c_{i,j}| \\ &\leq 2T^4 e^{2TN_0}, \end{aligned}$$

hence we have

$$\begin{aligned} e^{u_1 - c_0 s_{D,N-1}} &< e^{(1-D)\log(2T^4)+2TN_0(1-D)-2TND} \\ &\leq L(f)^{(1-D)}(e^{2N})^{-TD} \\ &\leq L(f)^{(1-D)}(H(w)H(f(w)))^{-TD}, \quad \forall w \in S_{D,N}. \end{aligned}$$

Thus by the hypothesis we get that $\forall w \in S_{D,N}$

$$|F(w)| < L(f)^{(1-D)}(H(w)H(f(w)))^{-TD},$$

so setting $|\cdot| = |\cdot|_v$, $K = \mathbb{Q}(w, f(w))$, $\ell = 2$, $N_1 = N_2 = T$, $\gamma = (w, f(w))$, $f = P$ and taking logarithm we get

$$\log|P(w)| < (1-D)\log L(f) - TD(h(w) + h(f(w))),$$

so by the Polynomial Liouville's Inequality (2.4), the polynomial vanishes at $(w, f(w))$, and this proves the lemma. □

Since $F(w) = 0$ for all $w \in S_0$, then for all $w \in S_{D,N_0+1}$ the inequality (2.15) of Lemma 2.2.2 holds, then Lemma 2.2.3 implies that F vanishes at S_{D,N_0+1}. Continuing the process we get the desired result.

(d)

We have shown that F vanishes for all $w \in S_{D,N}(f,r)$ for every $N \geq N_0$, that is

$$F(w) = 0, \quad \forall w \in \bigcup_{N \geq N_0} S_{D,N}(f,r). \tag{2.20}$$

The function F is continuous and non-zero in all $\overline{D(0,R)}$, furthermore is analytic in the compact $\overline{D(0,r)}$, this last contains every $S_{D,N}$, hence F vanishes in infinitely many points of $\overline{D(0,r)}$. This contradicts the fact that the zeros of the holomorphic function F are isolated.

This proves the part (i) of Theorem 2.2.1.

We sketch the proof of part (ii), it is quite similar to the proof of (i).

(a') We fix a real number N_0 and a real number $\delta > 2(6/c_0)^2$. Suppose that there exists an integer D such that for every $D' \geq D$ we have

$$\sigma_{D',N_0} \geq \delta D'^3 N_0^2.$$

We let T be as in (2.8). For every $D' \geq D$ we start by extracting a subset S_{D',N_0} of Σ_{D',N_0} for which its cardinality is well known.

This time we ask D to satisfy (up to increase it) the next inequality

$$c_0 \left[\delta(D-1)^3 N_0^2\right] > u_1 + D\log(2T^4) + 2TN_0(2D+1),$$

with u_1 as in (2.9).

(b') Using Siegel's Lemma kind argument we find a non-zero polynomial P in two variables which vanishes at every point in S_{D,N_0}. From which we define an analytic function F which is continuous in all $\overline{D(0,R)}$.

(c') Using induction, Liouville's Inequality and Schwarz lemma, we show that F vanishes at all points in $S_{D',N}$ for $D' \geq D$.

(d') Last point implies that the function F will be the zero function, which is a contradiction to the fact that f is a transcendental over $\mathbb{C}(z)$.

This completes the proof of the Theorem.

□

2.3 Unspecified sequences

Another natural question which arises is the following: Can we replace the statement *there exists a sequence of real numbers $N \geq 0$ tending to infinity* in Theorem 2.2.1 by *for N big enough*. The next theorem gives the answer: No.

Theorem 2.3.1 (A. Surroca)**.** *Let Φ be a positive function such that*

$$\lim_{x\to\infty} \frac{\Phi(x)}{x} = 0.$$

There exists a sequence of real numbers $(N_\nu)_{\nu\geq 1}$ strictly increasing and an entire transcendental function f such that for every algebraic number α and every positive integer k we have

$$f^{(k)}(\alpha) \in \mathbb{Q}(\alpha),$$

and such that for every integer $D \geq 1$ and every $k \geq D$,

$$\sigma_{D,N_\nu}(f,1) \geq \frac{1}{2} e^{D(D+1)\Phi(N_\nu)}. \tag{2.21}$$

Proof. [Su 06, Theorem 1.2]. □

Pila showed that if f is a real analytique function, defined in a closed interval and such that the image is not contained in any algebraic curve. For every positive real number ε there exists a constant $c(f,\varepsilon)$ such that the number of points $(x, f(x)) \in \mathbb{Q}^2$ of bounded height by N is less than

$$c(f,\varepsilon)e^{\varepsilon N}.$$

In particular Surroca's Theorem 2.3.1 shows that for N big enough both bounds get closer, so Pila's result is not far from been optimal.

Chapter 3

Rational values of the Riemann zeta function

The Riemann zeta function is a meromorphic function, obtained by analytic continuation of the series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{3.1}$$

to the whole complex plane with exception of a simple pole in $z = 1$.

The function $\zeta(z)$ satisfies the functional equation

$$\zeta(z) = \chi(z)\zeta(1 - z), \tag{3.2}$$

where $\chi(z) = 2^z \pi^{z-1} \sin(\frac{1}{2} z\pi)\Gamma(1 - z)$ and Γ is the Euler Γ-function. Seven classical methods to prove this equation can be find in [Ti 86, Theorem 2.1].

It is known that it has rational values at the non-positive integers and the irrationality of its values at the even positive integers. The French mathematician Roger Apéry showed in 1978 that $\zeta(3)$ is irrational, however it is unknown whether $\zeta(2k + 1)$ is irrational for $k \geq 2$.

n	$\cdots$	-4	-3	-2	-1	0	1	2	3	4	5	$\cdots$
$\zeta(n)$	$\cdots$	0	$\frac{1}{120}$	0	$-\frac{1}{12}$	$-\frac{1}{2}$	∞	$\frac{\pi^2}{6}$	$\notin \mathbb{Q}$	$\frac{\pi^4}{90}$	$?$	$\cdots$

In the beginning of the 21st century, the use of some linear forms allowed to show results like: *At least one of the numbers*

$$\zeta(5), \zeta(7), \zeta(9), \zeta(11), \zeta(19)$$

is irrational [V. V. Zudilin]. *At least one of the twenty two numbers*

$$\zeta(9), \zeta(11), \zeta(13), \ldots, \zeta(49), \zeta(51)$$

is irrational [V. V. Zudilin]. *The number of irrationals among*

$$\zeta(3), \zeta(5), \zeta(7), \ldots, \zeta(2D+1)$$

is at least $c \log D$, *for some* $c > 0$ [Ball & Rivoal].

This last result implies that there are infinitely many irrationals in the set $\{\zeta(2k+1) \mid k \geq 1\}$. However it is less known about the behaviour of ζ at non integral points.

Regarding the zeros of the ζ-function, apart from the trivial zeros at $\zeta(-2), \zeta(-4), ...$ from the functional equation (3.2) we can prove that the ζ function have an infinity of zeros in the strip $0 < \mathrm{Re}(z) < 0$.

3.1 Masser's effective theorem

The next Theorem, which restricts to an open interval in the real line (for definiteness) gives an insight of the number of rational values of the ζ-function.

The denominator d of a rational number x is the smallest positive integer d such that dx is integral.

Theorem 3.1.1 (Masser)**.** *There is a positive absolute constant* C *such that for any integer* $D \geq 3$ *the number of rational* z *with* $2 < z < 3$ *of denominator at most* D *such that* $\zeta(z)$ *is rational also of denominator at most* D *is at most* $C \left(\frac{\log D}{\log \log D} \right)^2$.

One expects that there are no such z at all. For an interval of length 1 the number of rational points of height less than H is about H^2, since $H(p/q) = \max\{|p|, |q|\}$.

If we apply Surroca's Theorem 2.2.1 to the function $f(z) = \zeta(z)$, setting $r = 1/2$, $R = 1 - \varepsilon$ for $\varepsilon > 0$ very small and fix $D = 1$, then there exists a sequence of positive real numbers tending to infinity such that

$$\sigma_{1,N}(\zeta, 1/2) \leq 1446(\log H)^2,$$

for H big enough.

Moreover if we put $f(z) = (z-1)\zeta(z)$ then f is an entire function so we can let R tend to infinity, in that way δ is an arbitrary positive small constant. However the sequence of real numbers is still unspecified.

The proof of the theorem splits in two main arguments. The first one resides in the analysis of the ζ-function, while the second one is an argument which follows from diophantine approximation.

In order to prove the Theorem we first prove two propositions.

3.2 On the zeros of $P(z,\zeta(z))$

For real $X \geq 0$ and $Y \geq 1$ denote $\mathcal{Z}(X,Y)$ the region of the complex plane defined by

$$\mathcal{Z}(X,Y) := \{z \in \mathbb{C} \mid -X \leq \mathrm{Re}(z) \leq X,\ 1 \leq \mathrm{Im}(z) \leq Y\}. \tag{3.3}$$

Proposition 3.2.1. *There is an effective absolute constant c such that for any integer $L \geq 1$, any real number $R \geq 2$ and any non-zero polynimial $P(z,w) \in \mathbb{C}[z,w]$ of degree at most L in each variable the function $P(z,\zeta(z))$ has at most*

$$cL(L + R\log R) \tag{3.4}$$

zeroes (counted with multiplicity) in $\overline{D(0,R)}$.

3.2.1 Five Analytic Lemmas

In order to prove the proposition we have to prove some lemmas first. The following lemma bounds the modulus of the derivative of an analytic function on a neighbourhood in which the function is not injective.

Lemma 3.2.1. *Let R, M be a positive real number and f a function analytic in an open subset containing $\overline{D(z_0,R)}$ and suppose that $|f'| \leq M$ in that disk. Suppose there is $\tilde{z}_0$ and δ with $f(\tilde{z}_0) = f(z_0)$ and $\tilde{z}_0 \in \overline{D(z_0,\delta)}$ with $\tilde{z}_0 \neq z_0$ where $\delta \leq \frac{R}{2}$. Then $|f'(z_0)| \leq 16\frac{\delta M}{R}$.*

Proof. This is a consequence of the Inverse Function Theorem. Suppose $f'(z_0) \neq 0$ otherwise we are done, consider the function

$$g(z) = \frac{f(z+z_0) - f(z_0)}{f'(z_0)},$$

notice that $0 = g(0) = g(\tilde{z}_0 - z_0)$ and $g(\tilde{z}_0 - z_0) = 1$. Applying the Lemma at [La 73, p. 124] with $r = \delta$, we have that there exist a unique $z \in \overline{D(0,\delta)}$ such that $g(z) = 0$ provided that

$$|g'(z_1) - g'(z_2)| \leq s,$$

for $0 < s < 1$ and for all $z_1, z_2 \in \overline{D(0,\delta)}$, in particular we can let $s = 1/2$. But we have two solutions 0 and $\tilde{z}_0 - z_0$, so we must have $1/2 < |g'(z_1) - g'(z_2)|$.

We estimate $|g'(z_1) - g'(z_2)|$ using Cauchy's Integral Formula. We have

$$g'(z_j) = \frac{1}{2\pi i} \int_{|z|=R} \frac{g'(z)}{(z - z_j)} dz,$$

for $j = 1, 2$. Then subtracting, changing variable $z \mapsto z - z_0$ and taking absolute value we have

$$|g'(z_1) - g'(z_2)| = \left| \frac{1}{2\pi i f'(z_0)} \int_{|z-z_0|=R} \frac{(z_1 - z_2) f'(z) dz}{(z - z_1 - z_0)(z - z_2 - z_0)} \right|,$$

now notice that

$$|z - z_j - z_0| \geq |z - z_0| - |z_j| \geq R - \delta \geq \frac{R}{2},$$

for $j = 1, 2$. From which the lemma follows. □

The proof of the next lemma follows from analysis of the Hadamard product of the derivative of the ζ-function.

Lemma 3.2.2. *For real $Y \geq 2$, $B \geq 0$ suppose $|\zeta'(z)| < \exp(-B)$ for $z \in \mathcal{Z}(3, Y)$. Then there exists a zero z_0 of ζ' with $|z - z_0| \leq cY \exp\left(-\frac{B}{cY \log Y}\right)$.*

Proof. The ζ-function has a simple pole of order 1 at $z = 1$, thus its derivative has only one pole of order 2 at $z = 1$. The function $(z-1)^2\zeta'(z)$ is entire of order 1, then Hadamard's factorization theorem [Ti 39, 8.24] implies that there exist complex numbers a and b such that

$$(z-1)^2\zeta'(z) = a \exp(bz) \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right),$$

where z_n are the zeros of ζ'.

We have the next decomposition for the product $\prod_{n=1}^{\infty} = \prod_{|z_n|<2|z|} \prod_{|z_n|\geq 2|z|}$, the product when $|z_n| \geq 2|z|$ is harmless since

$$\begin{aligned} \left| \prod_{|z_n|\geq 2|z|} \left(1 - \frac{z}{z_n}\right) \exp\left(\frac{z}{z_n}\right) \right| &\geq \prod_{|z_n|\geq 2|z|} \left| \mathrm{Re}\left(1 - \frac{z}{z_n}\right) \right| \left| \exp\left(\mathrm{Re}\left(\frac{z}{z_n}\right)\right) \right| \\ &\geq \prod_{|z_n|\geq 2|z|} \exp\left(-\left|\frac{z}{z_n}\right|^2\right), \end{aligned}$$

taking logarithms and considering the annuli $2^i|z| \leq |z_n| < 2^{i+1}|z|$ for $i \geq 1$ we have

$$-\log \left| \prod_{|z_n|\geq 2|z|} \right| \leq \sum_{|z_n|\geq 2|z|} \left|\frac{z}{z_n}\right|^2 \leq \sum_{i=1}^{\infty} 2^{-2i} \sum_{2^i|z|\leq |z_n| < 2^{i+1}|z|} 1.$$

Using Jensen's theorem [Ti 39, 3.61] for the function $(z-1)^2\zeta'(z)$ we can deduce that there are at most $cr\log r$ zeroes with $|z| \leq r$.

In the same fashion we get the same bound (up to multiplication by a constant) for $\left|\prod_{|z_n|<2|z|} \exp\left(\frac{z}{z_n}\right)\right|$, and also for $\left|\frac{a\exp(bz)}{(z-1)^2}\right|$. So using the hypothesis $|\zeta'(z)| < \exp(-B)$ we have

$$\prod_{z_n<2|z|} \left|1 - \frac{z}{z_n}\right| \leq \exp(-B + cY\log Y).$$

There exists n such that

$$\left|1 - \frac{z}{z_n}\right| \leq \exp\left(\frac{-B + cY\log Y}{N}\right),$$

where N is the number of zeroes in the disk $< 2|z|$. Then we have

$$\begin{aligned} |z_n - z| &\leq |z_n| \exp\left(\frac{-B}{N}\right) \exp\left(\frac{cY\log Y}{N}\right) \\ &\leq cY \exp\left(\frac{-B}{N}\right) \exp\left(\frac{cY\log Y}{N}\right) \\ &\leq cY \exp\left(\frac{-B}{cY\log Y}\right), \end{aligned}$$

since we can write $N = \hat{c}Y\log Y$, for some constant $\hat{c}$. □

The next lemma is an application of Lemma 3.2.1 in the special context of the ζ-function.

Lemma 3.2.3. *Let $Y \geq 2$, $\delta > 0$ be real numbers and suppose $w \in \overline{\mathbb{D}(0,1/2)}$ and $\zeta(\tilde{z}_0) = \zeta(z_0) = w$ for $\tilde{z}_0, z_0 \in \mathcal{Z}(3,Y)$ with $0 < |\tilde{z}_0 - z_0| \leq \delta \leq \frac{1}{4}$. Then $|\zeta'(z_0)| \leq c\delta Y^5$.*

Proof. Let $f = \zeta$ and $R = \frac{1}{2}$. For $z \in D(z_0, 3/4)$ we have that $\mathrm{Re}(z) < |\frac{15}{4}|$, we want to estimate the order of ζ in this disk. The order of ζ depends on the real and imaginary part of the variable z as follows:

$$\zeta(z) = \begin{cases} O(\mathrm{Im}(z)^{\frac{1}{2}-\mathrm{Re}(z)}) & \mathrm{Re}(z) \leq -\delta < 0, \\ O(\mathrm{Im}(z)^{\frac{3}{2}+\delta}) & \mathrm{Re}(z) \geq -\delta, \end{cases}$$

which follows from [Ti 86, 5.1]. Hence we have that for z as above $|\zeta(z)| \leq cY^5$. Now consider Cauchy's Integral Formula

$$\zeta'(z) = \frac{1}{2\pi i} \int_{|\xi - z| = 3/4} \frac{\zeta(\xi)}{(\xi - z)^2} d\xi,$$

taking modulus on both sides yields $|\zeta'(z)| \leq cY^5$. Therefore we make $M = cY^5$ and apply Lemma 3.2.1 to obtain $|\zeta'(z_0)| \leq c\delta Y^5$. □

3.2.2 On the distribution of the w-points of $\zeta(z)$

The next result is about the distribution of the w-points.

Lets denote $N(w,Y)$ the set of solutions of the equation

$$\zeta(z) = w,$$

for $0 \leq \mathrm{Re}(z) \leq 1$ and $1 \leq \mathrm{Im}(z) \leq Y$, with $Y > 1$. In the second part of [Bo-La-Li 13] it is proved that $N(w,Y)$ is finite for $Y > 1$ and they also showed the following result

$$N(w,Y) = \begin{cases} \frac{1}{2\pi} Y \log Y - \frac{1+\log 2\pi}{2\pi} Y + O(\log Y) & \text{pour } w \neq 1, \\ \frac{1}{2\pi} Y \log Y - \frac{1+\log 4\pi}{2\pi} Y + O(\log Y) & \text{pour } w = 1, \end{cases}$$

which is clearly not uniform near $w = 1$.

Lemma 3.2.4. *There is an absolute constant $r_0 > 0$ such that for any complex $w \in \overline{D(0,r_0)}$ and any real $Y \geq 2$ the number N of solutions (with multiplicity) of $\zeta(z) = w$ with $z \in \mathcal{Z}(3,Y)$ satisfies*

$$\left| N - \left(\frac{1}{2\pi} Y \log Y - \frac{1 + \log 2\pi}{2\pi} Y \right) \right| \leq c \log Y. \tag{3.5}$$

Proof. [Bo-La-Li 13, Chapitre II] □

The next result enable to recover the coefficients of a polynomial P from suitable values of $P(z,\zeta(z))$. This is done by using a sort of interpolation.

Lemma 3.2.5. *There are absolute constants $r_0 > 0$ and $c_0 > 0$ such that for any integer $L \geq 2$ there exist w_l ($l = 1,\dots,L$) and z_{kl} ($k,l = 0,\dots,L$) with*

$$\zeta(z_{kl}) = w_l \quad k,l = 0,\dots,L$$

and

$$|w_l| \leq r_0, \quad |w_l - w_i| \geq \frac{1}{cL^{1/2}} \quad i,l = 0,\dots,L;\ i \neq l,$$

$$|z_{kl} - 1|, \quad |z_{kl}| \leq \frac{c_0 L}{\log L}, \qquad \prod_{0\leq j\leq L, j\neq k} |z_{kl} - z_{jl}| \geq \exp\left(-cL^{3/2}\right) \ k,l = 0,\dots,L.$$

Proof. For any w take $|w| \leq r_0$ as in Lemma 3.2.4, so we have an inequality for the number N of solutions of $\zeta(z) = w$ in $Z(3,Y)$. Suppose that all the solutions are different, because if this fails then $\zeta'(z) = 0$ and such z are countable. We can also suppose that the z cannot be too close.

For each of the allowable w we have z_k with $\zeta(z_k) = w$ with say at least $\frac{1}{8}Y\log Y$ values of z_k. Thus we should take Y so as to make $0 \leq k \leq L$; that is Y of order $\frac{L}{\log L}$, and we can assume that Y is an integer. Further

$$|z_k - z_j| \geq \exp(-Y^{5/4})$$

for $j \neq k$. By Lemma 3.2.4 we find that the domains defined by $t \leq y < t+1$ with $t = 1,\dots,Y-1$ contains at most $c\log Y$ of the z_k. One of the families subsets defined by t odd or even contain at least the half of the z_k. If we estimate the product for a fixed j we find that at most $c\log Y$ of the z_j can satisfy $|z_k - z_j| \leq 1$. So the last inequality gives

$$\prod_{0\leq j\leq L, j\neq k} |z_k - z_j| \geq \exp(-Y^{11/8}) \geq \exp(-cL^{3/2})$$

as required. □

3.2.3 The proof

Proof of the Proposition 3.2.1. Let L be an integer ≥ 1 and R a real ≥ 2. Let $P \in \mathbb{C}[z, w]$ non-zero and such that $P(1, w)$ and $P(z, 1)$ have degree at most L. We can suppose up to multiplication by a complex number, that the coefficient norm of P is 1. Suppose that $F(z) = P(z, \zeta(z))$ has N zeroes (counted with multiplicity) in $\overline{D(0, R)}$, say $z_1, \ldots, z_N$. The function

$$\Phi(z) = \frac{(z-1)^L F(z)}{\prod_{i=1}^N (z - z_i)}$$

is entire. We set $\tilde{R} = \frac{c_0 L}{\log L} + R$, where c_0 is the constant found in Lemma 3.2.5. The maximum modulus principle yields

$$|\Phi|_{\tilde{R}} \leq |\Phi|_{5\tilde{R}}$$

for the supremum norms. The function $\hat{\zeta}(z) = (z-1)\zeta(z)$ has growth order 1 and it is classical that $|\hat{\zeta}|_r \leq r^{cr}$ for all $r \geq 2$. It follows that

$$|\Phi|_{5\tilde{R}} \leq \tilde{R}^{c\tilde{R}L}(4\tilde{R})^{-N}.$$

For any z with $|z - 1| \geq 1$ and $|z| \leq \tilde{R}$ we have

$$|F(z)| = |\Phi(z)||z-1|^{-L} \prod_{n=1}^N |z - z_n| \leq (2\tilde{R})^N |\Phi|_{\tilde{R}}.$$

Hence it follows that $|F(z)| \leq 2^{-N} \tilde{R}^{c\tilde{R}L}$. From Lemma 3.2.5 this holds for the $z = z_{kl}$ for $k, l = 0, \ldots, d$ as $\tilde{R} \geq \frac{c_0 L}{\log L}$. Thus

$$|P(z_{kl}, w_l)| \leq 2^{-N} \tilde{R}^{c\tilde{R}L} \text{ for } k, l = 0, \ldots, L.$$

We use now Lagrange interpolation formula twice:

$$P(z, w) = \sum_{l=0}^L \left(\prod_{0 \leq i \leq L, i \neq l} \frac{w - w_i}{w_l - w_i} \right) P(z, w_l)$$

and

$$P(z, w_l) = \sum_{k=0}^L \left(\prod_{0 \leq j \leq L, j \neq k} \frac{z - z_{jl}}{z_{kl} - z_{jl}} \right) P(z_{kl}, w_l).$$

The coefficient norm of P is at most $2^{-N}\tilde{R}^{c\tilde{R}L}$. As this norm is 1 and $\tilde{R}\log\tilde{R} \leq c(L + R\log R)$ we have that the proposition follows. This completes the proof of the proposition. $\square$

Notice that the result is best possible since in L for a fixed R, one can build a polynomial which vanishes at $(0, \zeta(0))$ with order at least $L^2 + 2L$.

On the other hand if we fix L, the result is also best possible in R, since if consider the polynomial $P(z, w) = w^L$, the ζ-function has around $R\log R$ zeroes within $\overline{D(0, R)}$ and they will appear with multiplicity at least L.

3.3 Interpolation

Definition 3.3.1. For a finite set S in $\mathbb{C}^2$ let $\omega(S)$ be the least degree of any curve passing through S, i.e. it corresponds to the total degree of a polynomial where the elements of S belong to its set of roots.

Proposition 3.3.1. *For any integers $d \geq 1$, $T \geq \sqrt{8d}$ and any real $A > 0$, $Z > 0$, $M > 0$, $H \geq 1$, let f_1, f_2 be functions analytic on an open neighbourhood of $\overline{D(0, 2Z)}$, with supremum norm $|f_i|_{2Z} \leq M$ for $i = 1, 2$ and set $\boldsymbol{f} = (f_1, f_2)$. Suppose that $\mathcal{Z}$ is a finite set of complex numbers such that*

(i) $z \in \overline{D(0, Z)}$ *for every* $z \in \mathcal{Z}$.

(ii) $|z' - z''| \leq \frac{1}{A}$ *for all* $z', z'' \in \mathcal{Z}$.

(iii) $[\mathbb{Q}(f_1(z), f_2(z)) : \mathbb{Q}] \leq d$ *for all* $z \in \mathcal{Z}$.

(iv) $H(f_1(z)) \leq H$ $H(f_2(z)) \leq H$ *for all* $z \in \mathcal{Z}$.

Then

$$\omega(f(\mathcal{Z})) \leq T \text{ provided } (AZ)^T > (4T)^{\frac{96d^2}{T}}(M + 1)^{16d}H^{48d^2}.$$

Proof. Take $S = \left[\frac{T^2}{8d}\right] \geq 1$ and pick $z_1, \ldots, z_S \in \mathcal{Z}$. If $f(\mathcal{Z})$ has less than S elements, then we will have $S - 1 < \frac{(T+1)(T+2)}{2}$, so there is a curve of degree at most T passing through $f(\mathcal{Z})$, i.e. $\omega(f(\mathcal{Z})) \leq T$. So suppose that this is not the case.

We will use Siegel's lemma to built a non-zero polynomial $P(w_1, w_2)$ of total degree at most T with

$$P(f_1(z_s), f_2(z_s)) = 0 \text{ for } s = 1, \ldots, S$$

and then we will use Scharwz lemma to prove that in fact it vanishes for every $z \in \mathcal{Z}$. Hence $f(\mathcal{Z}) \leq T$.[1]

Let $L = \left[\frac{T}{2}\right]$ and consider the polynomial

$$P(w_1, w_2) = \sum_{i=0}^{L}\sum_{j=0}^{L} \lambda_{ij} w_1^i w_2^j,$$

obviously $\deg(P(w_1, 1)), \deg(P(1, w_2)) \leq L$. Consider the system of equations

$$\begin{array}{rcl} P(f_1(z_1), f_2(z_1)) & = & 0 \\ \vdots & & \vdots \\ P(f_1(z_S), f_2(z_S)) & = & 0. \end{array} \tag{3.6}$$

The next lemma provides the conditions for a solution to exist.

Lemma 3.3.1. *Let $K_h = \mathbb{Q}(f_1(z_h), f_2(z_h))$ and let $d_h = [K_h : \mathbb{Q}]$, similarly let $d_{h,1} = [\mathbb{Q}(f_1(z_h)) : \mathbb{Q}]$ and $d_{h,2} = [\mathbb{Q}(f_2(z_h)) : \mathbb{Q}]$, with $1 \leq h \leq S$. Define $D = \sum_{1 \leq h \leq S} d_h$.*

For $(L+1)^2 > D$ there is a solution $\boldsymbol{c} = (c_{i,j}) \in \mathbb{Z}^{(L+1)^2}$ to the linear system (3.6) such that

$$\max_{0 \leq i,j < L+1} |c_{i,j}| \leq \left[\left(2^{\mu'} \prod_{h=1}^{S} M_h \right)^{1/((L+1)^2 - D)} \right], \tag{3.7}$$

where

$$M_h = (L+1)^{2d_h} M(f_1(z_h))^{Ld_h/d_{h,1}} M(f_2(z_h))^{Ld_h/d_{h,2}},$$

the M in the right denote the usual Mahler's measure (see 1.7) and . μ' $(\leq \mu)$ is the number of fields K_h which do not admit any real embedding.

[1] In fact we will use a similar technique we have already used while proving 2.2.1.

Proof. [Gr-Mi-Wa 86, Lemme 1.1]. □

So in order to apply the proposition to our setting we must first show that in fact $(L+1)^2 > D$, to do so let $d \geq d_h$ for every $1 \leq h \leq S$, then

$$D \leq dS < 2dS \leq 2d\left[\frac{T^2}{8d}\right] < \frac{T^2}{4} \leq (L+1)^2.$$

So using the relation (1.9) and by hypothesis we find that

$$\max_{0\leq i,j<L+1} |c_{i,j}| \leq \left(2^S \prod_{h=1}^{S} (L+1)^{2d_h} H(f_1(z_h))^{Ld_h} H(f_2(z_h))^{Ld_h}\right)^{1/((L+1)^2-D)} \quad (3.8)$$

$$\leq \left(2^S \prod_{h=1}^{S} (T+1)^{2d} H^{dT}\right)^{8/T^2} \quad (3.9)$$

$$\leq 2^{1/d}(T+1)^2 H^T. \quad (3.10)$$

Now we have to prove that the function given by the polynomial actually vanishes for every $z \in \mathcal{Z}$, that is $F(z) = P(f_1(z), f_2(z)) = 0$. So pick $z_0 \in \mathcal{Z}$ and consider the function

$$\Phi(z) = \frac{P(f_1(z), f_2(z))}{\prod_{s=1}^{S} z - z_s},$$

it is analytic on an open set containing $|z| \leq 2Z$. We have that $|\Phi(z_0)| \leq |\Phi|_{2Z}$. For z on the boundary we have $|z - z_s| \geq Z$ (by hypothesis (i)), and so $|\Phi|_{2Z} \leq |F|_{2Z} Z^{-S}$. We get

$$|F(z_0)| = |\Phi(z_0)| \prod_{s=1}^{S} |z_0 - z_s| \leq (AZ)^{-S} |F|_{2Z}.$$

Further we can bound the norm of F using the bound we found for the coefficients as follows $|F|_{2Z} \leq |P|(M+1)^T$. Moreover we also have $H(F(z_0)) \leq (T+1)^2|P|H^T$ (using standard domination argument), if $F(z_0) \neq 0$ Liouville's inequality (1.13) yields $H(F(z_0))^{-d} \leq |F(z_0)|$, which turns out to be a contradiction provided $(AZ)^S > (T+1)^{2d}(M+1)^T|P|^{2d}H^{dT}$, in this case $F(z_0) = 0$. Substituting the estimate of $|P|$ this will happen provided

$$(AZ)^S > 4(T+1)^{6d}(M+1)^T H^{3d},$$

since $AZ > 1$ and $S \geq \frac{1}{16d}T^2$ we get as expected

$$(AZ)^T > (4T)^{\frac{96d^2}{T}}(M+1)^{16d}H^{48d^2}.$$

□

Now we have all the ingredients to give the proof of the main theorem.

3.4 Proof of Masser's effective theorem

We prove a more general statement in terms of heights.

Theorem 3.4.1 (Masser)**.** *There is a positive absolute constant C such that there are at most $C\left(\frac{d^2\log 4H}{\log(d\log 4H)}\right)^2$ different complex numbers z with $|z-\frac{5}{2}| \leq \frac{1}{2}$ such that $[\mathbb{Q}(z,\zeta(z)):\mathbb{Q}] \leq d$, $H(z) \leq H$ and $H(\zeta(z)) < H$.*

This theorem implies the Theorem 3.1.1 since a bounded rational number of denominator at most D has height of order at most D.

Proof of the theorem. Let $T \geq \sqrt{8d}$ be an integer. Consider the entire functions $f_1(z) = z$ and $f_2(z) = (z-1)\zeta(z)$. By Proposition 3.3.1 we have that

$$\omega(\mathbf{f}(z)) \leq T$$

for any subset $\mathcal{Z}$ of points z with $|z-\frac{5}{2}| \leq \frac{1}{2}$ provided

(i) $Z \geq 3$ so $z \in \overline{D(0,3)}$ for every $z \in \mathcal{Z}$.

(ii) For $A = 1$ actually $|z' - z''| \leq \frac{1}{A}$ for all $z', z'' \in \mathcal{Z}$.

(iii) $[\mathbb{Q}(z),(z-1)\zeta(z)):\mathbb{Q}] \leq d$ for all $z \in \mathcal{Z}$, this is automatically satisfied since we want rational values at rational points.

(iv) $H(z) \leq \tilde{H}$ $H((z-1)\zeta(z)) \leq \tilde{H}$ for all $z \in \mathcal{Z}$.

(v) $(AZ)^T > (4T)^{\frac{96d^2}{T}}(M+1)^{16d}\tilde{H}^{48d^2}$.

So if z is to be counted we have $H(z) \leq H$ and by the height properties

$$H((z-1)\zeta(z)) \leq H((z-1))H(\zeta(z)) \leq 2H^2,$$

so we take $\tilde{H} = 2H^2$ and (iv) is satisfied.

Now we take

$$M \leq \max\{|z|_Z, |(z-1)\zeta(z)|_Z\} = Z^{cZ},$$

so (v) is true whenever $Z^T \geq c^{d^2} Z^{cdZ}(2H^2)^{48d^2}$ for positive absolute c. Taking $T \geq 2cdZ$ we get $Z^{cdZ} \geq c^{d^2}(2H^2)^{48d^2}$, which is valid if Z is a sufficiently large constant multiple of $\frac{d \log 4H}{\log(d \log 4H)}$, which is going to be in fact bigger than 3 as in (i) and $T > \sqrt{8d}$.

Applying Proposition 3.2.1 letting the degree be at most T and $R = 3$ we get $cT^2 = C\left(\frac{d^2 \log 4H}{\log(d \log 4H)}\right)^2$ possible points. □

3.5 Conclusions

All the constants calculated in these chapter are actually effective, they rely basically in the properties of the Riemann zeta function.

In the article of David Masser [Ma 11], he asked questions related to the study of the functions such as the Euler Gamma function for which very few things are known as well.

Bibliography

[Ahl 66] L. V. Ahlfors, *Complex Analysis.* McGraw-Hill, 1966.

[Bak 75] Alan Baker, *Transcendental Number Theory.* Cambridge University Press,1975.

[Bo-Gu 06] E. Bombieri and W. Gubler, *Heights in Diophantine Geometry.* Cambridge University Press, 2006.

[Bo-La-Li 13] H. Bohr, E. Landau, J.E. Littlewood, *Sur la fonction $\zeta(s)$ dans le voisinage de la droite $\sigma = \frac{1}{2}$.* Bull. Acad. Roy. Belg. Cl. Sci. 12; pp. 1144-1175, 1913.

[Bo-Jo 13] G. Boxall and G. Jones, *Algebraic values of certain analytic functions.* International Mathematics Research Notices, rnt239, 18 pages, 2013.

[Gr-Mi-Wa 86] F. Gramain, M. Mignotte and M. Waldschmidt, *Valeurs algébriques de fonctions analytiques.* Acta Arithmetica 47, no. 2, Pages 97-121, 1986.

[Ha-Wr 75] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers.* Oxford University Press, 1975.

[Hi-Si 00] M. Hindry and J. H. Silverman, *Diophantine Geometry: An Introduction.* Spinger-Verlag, Graduate texts in mathematics: 201, 2000.

[La 73] S. Lang, *Real Analysis.* Addison-Wesley Series in Mathematics, 1973.

[La 02] S. Lang, *Algebra.* Spinger-Verlag, Graduate texts in mathematics: 211, 2002.

[Ma 11] D. Masser, *Rational values of the Riemann zeta function*. Journal of Number Theory 131, Pages 2037-2046, 2011.

[Su 06] A. Surroca, *Valeurs algébriques de fonctions transcendantes*. International Mathematics Research Notices, Article ID 16834, Pages 1-31, 2006.

[Ti 39] E.C. Titchmarsh, *The Theory of Functions* Second Edition. Oxford Science Publications, 1939.

[Ti 86] E.C. Titchmarsh, *The Theory of the Riemann zeta-function* Second Edition. Oxford Science Publications, 1986.

[Wa 00] M. Waldschmidt, *Diophantine Approximation on Linear Algebraic Groups*. Spinger-Verlag, Grundlehren der mathematischen Wissenschaften: 326, 2000.