Introduction to Spectral Analysis

Hernando Ombao

Brown University
Biostatistics Program

February 18, 2011

Outline of Talk

- Time-Domain Analysis
- Spectral Analysis
- Coherence Analysis

Outline of Talk

Time-Domain Analysis

Spectral Analysis

Coherence Analysis

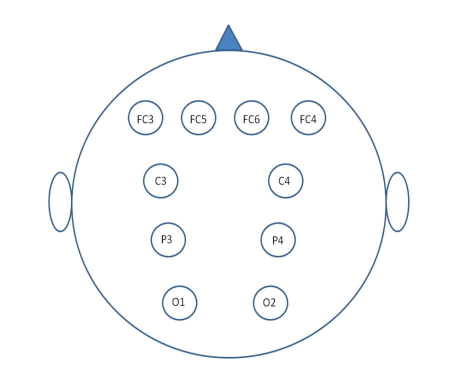
Outline of Talk

Time-Domain Analysis

Spectral Analysis

Coherence Analysis

Overview - Analysis of Brain Signals



- Data: multi-channel EEG, fMRI time series at several ROIs

Goals of our research

- Characterize and define dependence in a brain network
- Develop estimation and inference methods
- Develop classification methods that use connectivity as a biomarker
  - Predicting motor intent (Left vs. Right movement)
  - Differentiating patient groups (bipolar vs. healthy)

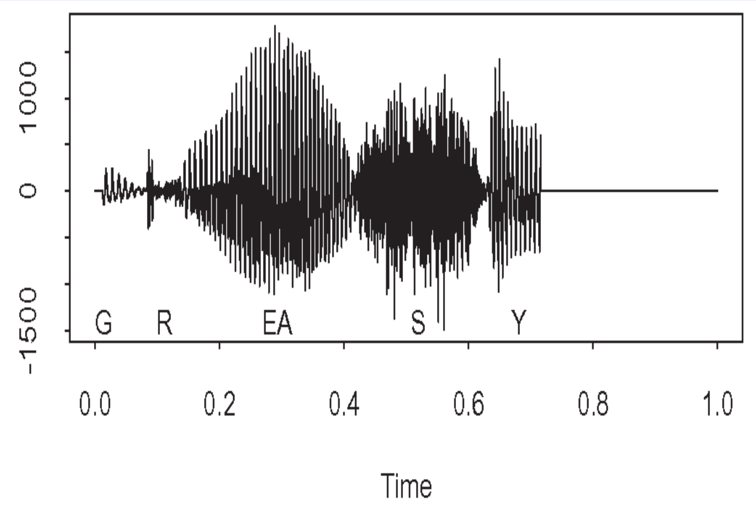
Overview - Analysis of Brain Signals

- Models and methods must incorporate information
  - Across trials, across subjects
- Models for estimating effect of a stimulus on brain network
- Model that use multi-modal data (EEG, fMRI, DTI)
- Dimension reduction: extract information from massive data that is most relevant for estimating dependence

Some References for Time Series Analysis

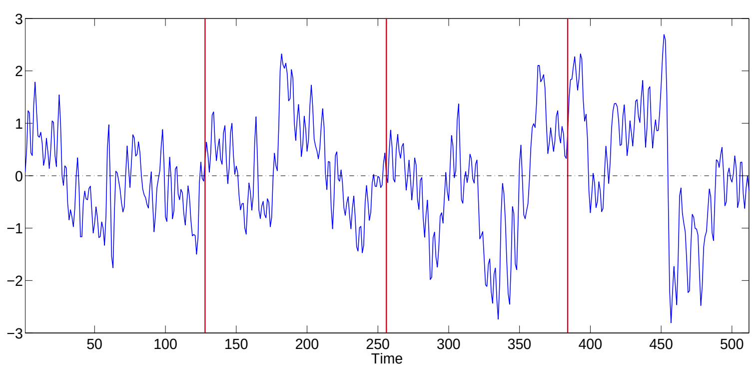
- Brillinger (1981) - Theory for Spectral Analysis.
- Brockwell and Davis (1991) - Theory book with emphasis on time domain analysis.
- Shumway and Stoffer (2007) - Combination of theory, methods, real-life examples.

Some Time Series Data



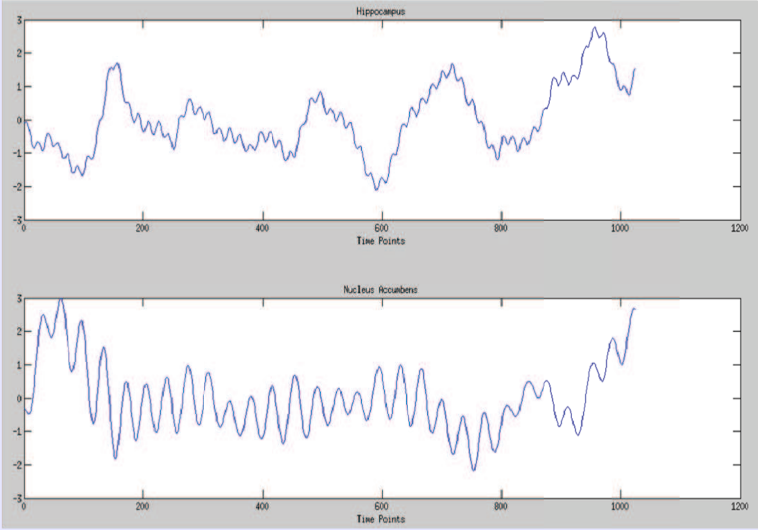
Speech Signal "GREASY"

SOME TIME SERIES DATA



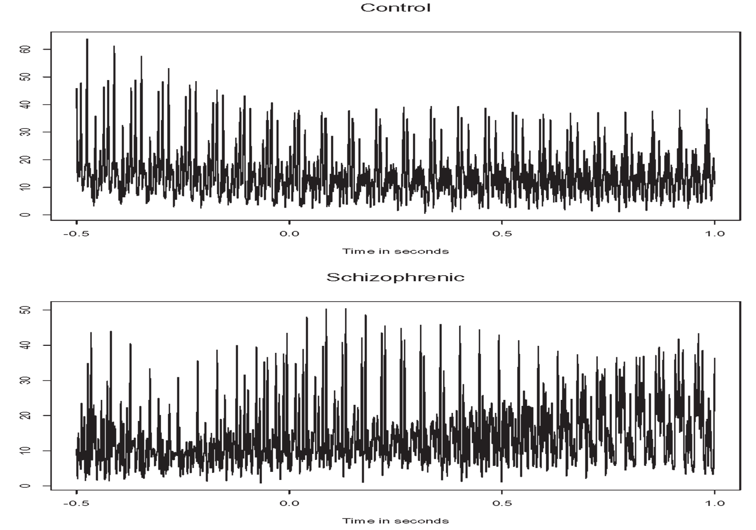
EEG Motor Experiment

Some Time Series Data



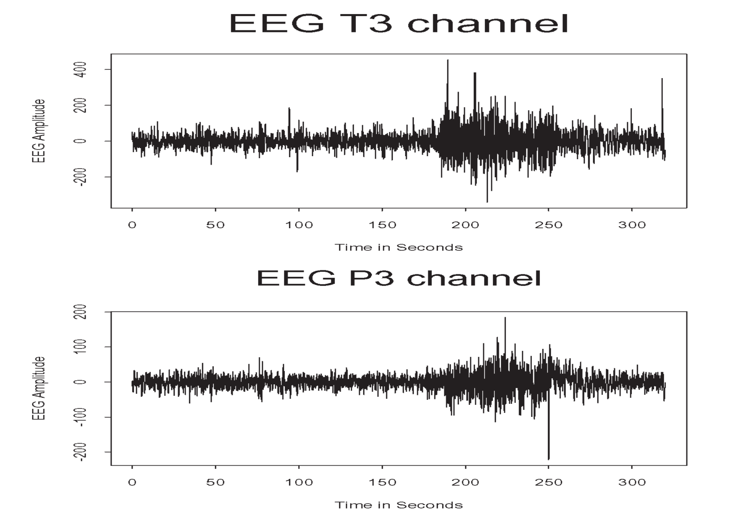
Local Field Potentials

Some Time Series Data



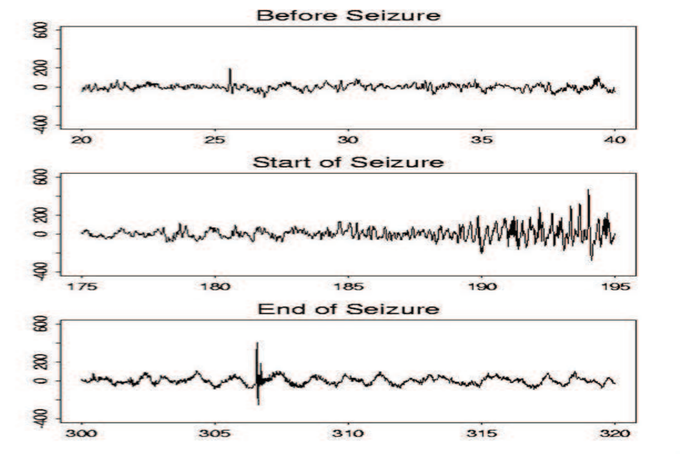
Magnetoencephalograms

Some Time Series Data



Epileptic Seizure EEGs

Some Time Series Data



Epileptic Seizure EEGs

Specific Goals in Analyzing Time Series Data

Time-domain Analysis

- Dependence. What is the correlation between $Y(t)$ and $Y(t+h)$?
- Prediction. Suppose that you have monthly sales data for 2000-2010, predict the monthly sales in January 2011 using
  - Past data for January 2000,2001, . . . (annual seasonality)
  - Immediate past months December, November, 2010 (lagged relationships)

Spectral-domain Analysis

- Signal decomposition. What oscillations are present in the time series?
- Coherence. What is the interactions between oscillations in different time series?

Basic Time Domain Analysis

- Time Series Data $[X(t), Y(t)]', t = 1, 2, \ldots$
- Mean $\mu(t) = [\mathbb{E}X(t), \mathbb{E}Y(t)]', t = 1, 2, \ldots$
- Variance

$$\gamma_{XX}(t,t) = \mathbb{C}\text{ov}[X(t), X(t)]$$
$$\gamma_{YY}(t,t) = \mathbb{C}\text{ov}[Y(t), Y(t)]$$

- Auto-covariance function

$$\gamma_{XX}(s,t) = \mathbb{C}\text{ov}[X(s), X(t)]$$
$$\gamma_{YY}(s,t) = \mathbb{C}\text{ov}[Y(s), Y(t)]$$

- Cross-covariance function

$$\gamma_{XY}(s,t) = \mathbb{C}\text{ov}[X(s), Y(t)]$$
$$\gamma_{XY}(t,s) = \mathbb{C}\text{ov}[X(t), Y(s)]$$

Basic Time Domain Analysis

$[X(t), Y(t)]'$ is a weakly stationary time series if

- Mean is constant in time

$$\mu(t) = [\mu_X, \mu_Y]' \text{ for all } t = 1, 2, \ldots$$

- Variance is constant in time

$$\gamma_{XX}(t,t) = \gamma_{XX}(0,0)$$
$$\gamma_{YY}(t,t) = \gamma_{YY}(0,0)$$

- Auto-covariance and cross-covariance depends only on the lag h

$$\gamma_{XX}(t+h,t) = \mathbb{C}\text{ov}[X(t+h), X(t)] = \gamma_{XX}(h,0) := \gamma_{XX}(h)$$
$$\gamma_{XY}(t+h,t) = \mathbb{C}\text{ov}[X(t+h), Y(t)] = \gamma_{XY}(h,0) := \gamma_{XY}(h)$$
$$\gamma_{XY}(t,t+h) = \mathbb{C}\text{ov}[X(t), Y(t+h)] = \gamma_{XY}(0,h) := \gamma_{XY}(-h)$$

Cross-correlation function

- Auto-correlation function

$$\rho_{XX}(h) = \mathbb{C}\mathrm{orr}[X(t+h), X(t)] = \frac{\gamma_{XX}(h)}{\gamma_{XX}(0)}$$

$$\rho_{YY}(h) = \mathbb{C}\mathrm{orr}[Y(t+h), Y(t)] = \frac{\gamma_{YY}(h)}{\gamma_{YY}(0)}$$

- Cross-correlation function

$$\rho_{XY}(h) = \mathbb{C}\mathrm{orr}[X(t+h), Y(t)] = \frac{\gamma_{XY}(h)}{\sqrt{\gamma_{XX}(0)\gamma_{YY}(0)}}$$

$$\rho_{XY}(-h) = \mathbb{C}\mathrm{orr}[X(t), Y(t+h)] = \frac{\gamma_{XY}(-h)}{\sqrt{\gamma_{XX}(0)\gamma_{YY}(0)}}$$

White Noise

- Time Series $X(t)$
- $\mathbb{E}X(t) = \mu$
- $\mathbb{V}\mathrm{ar}\, X(t) = \sigma_X^2$
- Auto-covariance function

$$\gamma_{XX}(h) = \begin{cases} \sigma_{XX}^2, & h = 0 \\ 0, & h \neq 0 \end{cases}$$

- Plot of the auto-covariance and auto-correlation functions.

Moving Average Model MA(q)

- $Z(t) \sim WN(0, \sigma_Z^2)$
- $X(t)$ is MA(q) if it has the representation

$$X(t) = Z(t) + \theta_1 Z(t-1) + \ldots + \theta_q Z(t-q)$$

- Intuition: applying a moving window of size $q+1$ on the white noise $\{Z(t)\}$
- Auto-covariance function

$$\gamma_{XX}(h) = \begin{cases} [1 + \theta_1^2 + \ldots + \theta_q^2]\sigma_Z^2, & h = 0 \\ \text{something}, & h = \pm 1 \\ \ldots, & \ldots \\ \text{something}, & h = \pm q \\ 0, & h = \pm(q+1), \ldots \end{cases}$$

Auto-regressive Model AR(p)

- $Z(t) \sim WN(0, \sigma_Z^2)$
- $X(t)$ is AR(p) if it has the representation

$$X(t) = \sum_{\ell=1}^{p} \phi_\ell X(t-\ell) + Z(t)$$

- Consider the simple case AR(1). When $|\phi_1| < 1$,

$$X(t) = \sum_{\ell=0}^{\infty} \phi_1^\ell Z(t-\ell)$$

- Causal: $X(t)$ depends only on the current and past noise values
- Auto-covariance function

$$\gamma_{XX}(h) = \begin{cases} \frac{\sigma_Z^2}{1-\phi_1^2}, & h = 0 \\ \phi_1^{|h|} \frac{\sigma_Z^2}{1-\phi_1^2}, & h = \pm 1, \pm 2, \ldots \end{cases}$$

Estimating the AR parameters

- $\phi = [\phi_1, \ldots, \phi_p]'$
- Conditional least squares criterion

$$S(\phi) = \sum_{t=p+1}^{T} [X(t) - (\phi_1 X(t-1) + \ldots + \phi_p X(t-p))]^2$$

- Conditional maximum likelihood
  - $X(t) = \phi_1 X(t-1) + \ldots + \phi_p X(t-p) + \epsilon(t);\ \ \epsilon(t) \sim N(0, \sigma^2)$
  - Define $m_p(t) = \phi_1 X(t-1) + \ldots + \phi_p X(t-p)$
  - Define $\mathcal{X}(t-1) = [X(t-1), X(t-2), \ldots]'$
  - $X(t)|\mathcal{X}(t-1)\ \sim\ N(m_p(t), \sigma^2)$
  - Conditional likelihood

$$\mathcal{L}_C(\phi) = f(X(p+1), \ldots X(T)\ |\mathcal{X}(p)) \quad (1)$$
$$= f(X(p+1)|\mathcal{X}(p)) \ldots f(X(T)|\mathcal{X}(T-1)) \quad (2)$$

Selecting the best order - acf and pacf plots

	ACF	PACF
MA	zero after q	tapers slowly
AR	tapers slowly	zero after p
ARMA	tapers slowly	tapers slowly

Selecting the best order - information criteria

- Data $X(t), t = 1, 2, \ldots, T$
- Set of candidate orders $p \in \{1, \ldots, P\}$
- Use data only for $t = P + 1, \ldots, T$
- For each p, estimate $\phi_1, \ldots, \phi_p$, compute the noise variance estimate

$$\widehat{\sigma}^2(p) = \frac{1}{T - P} \sum_{t=P+1}^{T} [X(t) - \widehat{m}_p(t)]^2$$

- Akaike information criterion (AIC)

$$AIC(p) = \log(\widehat{\sigma}^2) + (2p + 1)/(T - P)$$

- Bayesian information criterion (BIC)

$$BIC(p) = \log(\widehat{\sigma}^2) + (\log(p)p + 1)/(T - P)$$

- Choose p^* argmin of $AIC(p)$ or $BIC(p)$.

Some Examples in R

Time domain Models

`See file CorrelationsandModels`

Some Examples in R

Example - Time domain analysis of EEG

`See file CLASS-EEG`

Spectrum - gives variance decomposition

$X(t)$ STATIONARY TEMPORAL PROCESS

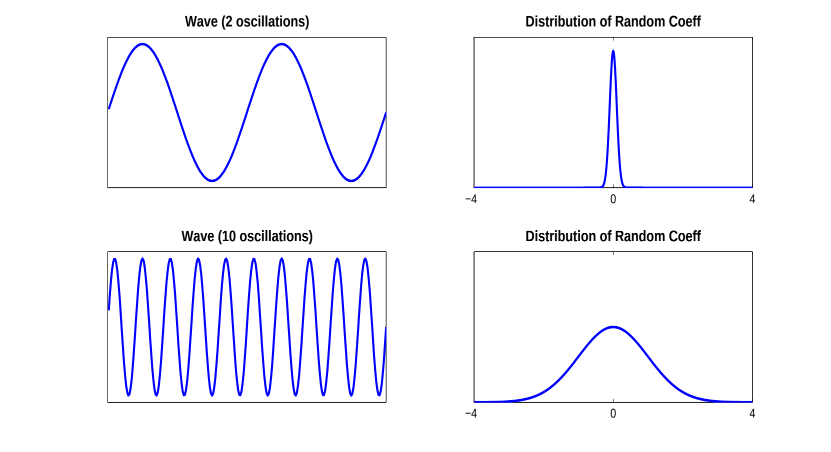
- Cramér Representation

  $X_t = \int \exp(i2\pi\omega t) dZ(\omega)$, $t = 0, \pm 1, \pm 2, \ldots$

- Basis Fourier waveforms $\exp(i2\pi\omega t), \omega \in (-\pi, \pi)$
- Random coefficients $dZ(\omega)$ – increment random process
  - $\mathbb{E} dZ(\omega) = 0$ and
  - $\mathbb{C}\mathrm{ov}[dZ(\omega), dZ(\lambda)] = \delta(\omega - \lambda) f(\omega) d\omega d\lambda$

Spectrum - gives variance decomposition

Mixing of oscillations



SPECTRUM - GIVES VARIANCE DECOMPOSITION

SPECTRUM – decomposition of variance

- $\mathbf{X} = [X(1), \ldots, X(T)]'$ - zero mean stationary time series
- Φ - columns are the orthogonal Fourier waveforms
- $\mathbf{d} = [d(\omega_0), \ldots, d(\omega_{T-1})]'$ - Fourier coefficients
- $\mathbf{X} = \Phi\mathbf{d}$
- $\mathbf{X}'\mathbf{X} = \mathbf{d}'\mathbf{d}$
- $\frac{1}{T}\mathbb{E}\mathbf{X}'\mathbf{X} = \frac{1}{T}\mathbb{E}\mathbf{d}'\mathbf{d}$
- $\mathbb{V}\mathrm{ar}\, X(t) \approx \int f(\omega) d\omega$

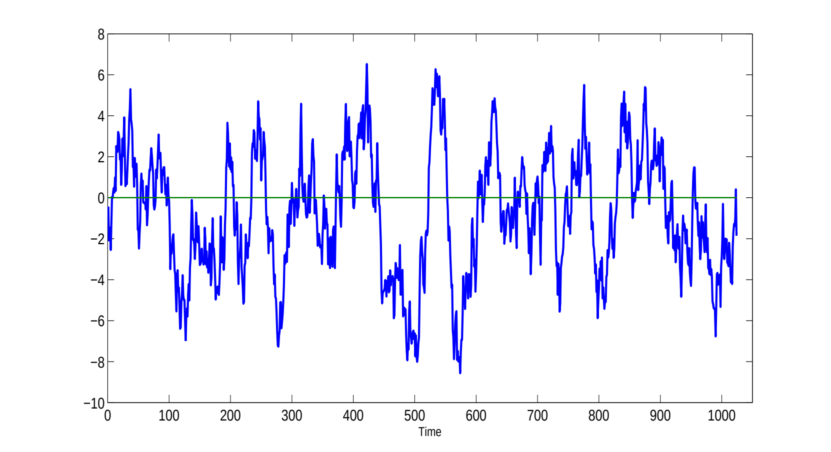
Spectrum - gives variance decomposition

A more formal derivation ...

- $X(t) = \int \exp(i2\pi\omega t)dZ(\omega)$
- $\gamma(h) = \mathbb{C}\mathrm{ov}[X(t+h), X(t)]$
- $f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) \exp(-i2\pi\omega h)$
- $\gamma(h) = \int_{-0.5}^{0.5} f(\omega) \exp(-i2\pi\omega h) d\omega$
- $\gamma(0) = \int f(\omega) d\omega$

SPECTRUM - GIVES VARIANCE DECOMPOSITION

$$\textbf{AR}(1)\textbf{:}\ X_t = 0.9X_{t-1} + \epsilon_t$$

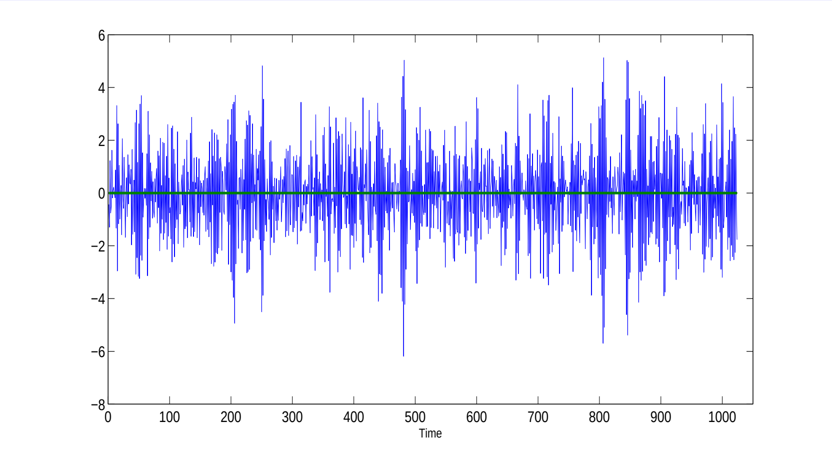


SPECTRUM - GIVES VARIANCE DECOMPOSITION

Spectrum of AR(1) with $\phi = 0.9$

Spectrum - gives variance decomposition

$$\textbf{AR}(1)\textbf{:}\ X_t = -0.9X_{t-1} + \epsilon_t$$

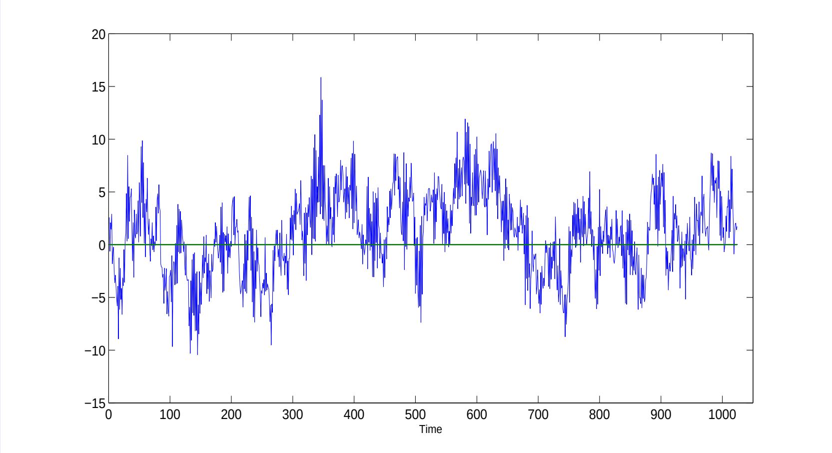


SPECTRUM - GIVES VARIANCE DECOMPOSITION

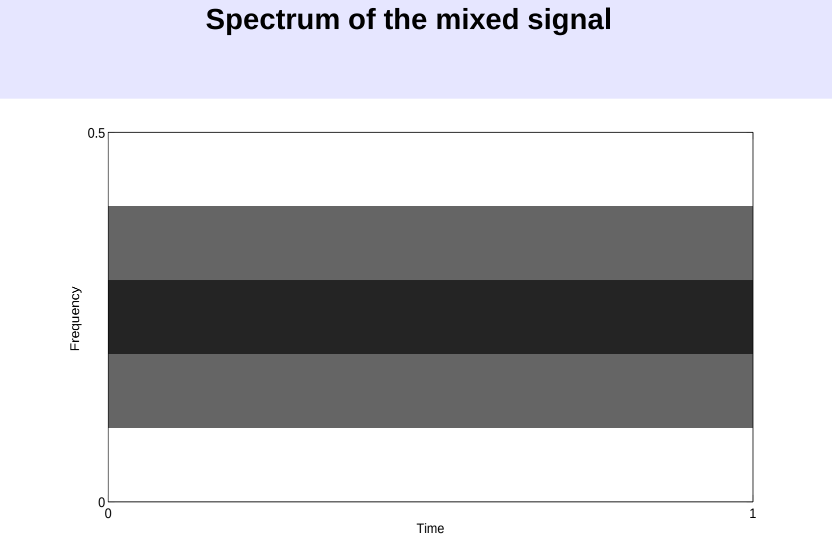
Spectrum of AR(1) with $\phi = -0.9$

Spectrum - gives variance decomposition

Mixture: Low + High Frequency Signal



SPECTRUM - GIVES VARIANCE DECOMPOSITION



Generating Mixtures of Oscillations

- Discrete Cramér representation

$$X(t) = \sum_{k=-(T/2-1)}^{T/2} A(\omega_k) \exp\left(\frac{2\pi kt}{T}\right)$$

- Fourier waveforms $\phi_k(t) = \exp\left(\frac{2\pi kt}{T}\right)$, $k = -(T/2-1), \ldots, T/2$
- Generate coefficients for $k = 0, T/2$: $A(\omega_k) \sim (0, f(\omega_k))$
- Generate coefficients for $k = 1, \ldots, (T/2-1)$:

$$A^R(\omega_k) \sim (0, \frac{f(\omega_k)}{2})$$

$$A^I(\omega_k) \sim (0, \frac{f(\omega_k)}{2})$$

$$A(\omega_k) = A^R(\omega_k) + iA^I(\omega_k)$$

- Generate coefficients for $k = -1, \ldots, -(T/2-1)$:

$$A(\omega_k) = A^R(-\omega_k) - iA^I(-\omega_k)$$

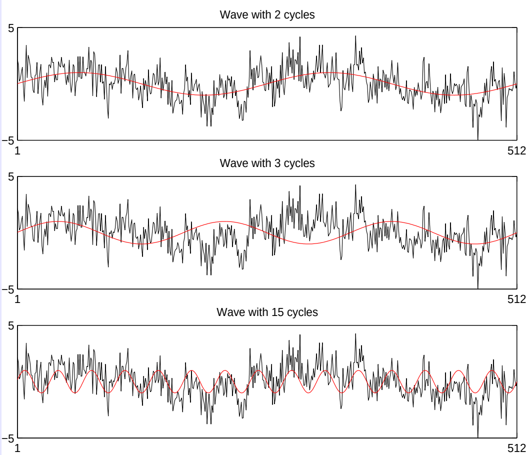
Generating Mixtures of Oscillations

`See file IntroSpectralModels`

Estimating the Spectrum

- $X_t, t = 1, \ldots, T$ (assume $\overline{X} = 0$)
- Estimate the spectrum
- Fourier coefficients $d(\omega_k) = \sum_t X_t \exp(-i2\pi\omega_k t)$

ESTIMATING THE SPECTRUM



- Fourier coefficients $d(\omega_k) = < \mathbf{X}, \phi_k >$
- Correlation between **X** and the waveform ϕ_k

Estimating the Spectrum

- Fourier periodograms $\mathcal{I}(\omega_k) = \frac{1}{T}|d(\omega_k)|^2$
- $\mathbb{E}\mathcal{I}(\omega) \approx f(\omega)$ but $\mathbb{V}\mathrm{ar}\,\mathcal{I}(\omega) = f^2(\omega)$
- $\widehat{f}(\omega) = \text{smooth}_{\lambda \in \mathcal{N}(\omega)}\, \mathcal{I}(\lambda)$
- Other approaches: wavelet denoising, parametric (ARMA)

Estimating the Spectrum

Examples in R

`See notes PeriodogramSmoothingNotes`

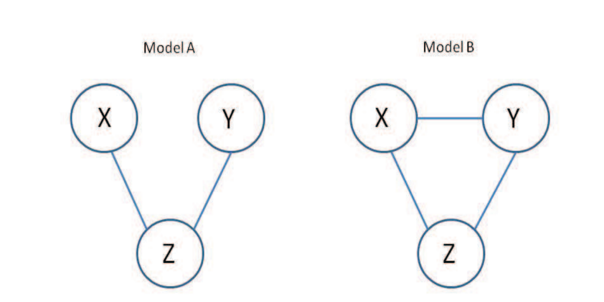
IMPORTANCE OF THE SPECTRUM

- SLEEP Studies
  - Depression study: among recoverers (IPT + flouxetine) [joint with Psychiatry, Univ Pittsburgh]
    - Alpha power *Post treatment* < *baseline*
    - Beta power *Post treatment* > *baseline*
- Cognitive Experiment working memory load
  - Gamma (32-50 hertz) power
- Heart Rate Variability
  - Feature of interest: high frequency power in inter-beat interval sequence in EKG (parasymphathetic modulation)
  - Across NREM periods in the entire night: increase among controls, near-constant among stress group.

Cross-coherence - a measure of dependence

- Three time series **X**, **Y**, **Z**
- Cross-dependence between **X** and **Y**
- Simple measures: cross-correlation and partial cross-correlation
- Indirect vs direct dependence

Cross-coherence - a measure of dependence

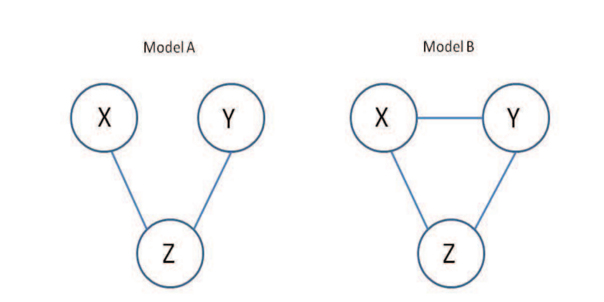


- Cross-correlation and Partial cross-correlation
- Indirect vs Direct

Cross-coherence - a measure of dependence

- Time series at 3 channels: $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$
- Cross-correlation $\rho(\mathbf{X}, \mathbf{Y}) = \frac{\mathbb{C}\mathrm{ov}(\mathbf{X},\mathbf{Y})}{\sqrt{\mathbb{V}\mathrm{ar}\,\mathbf{X}\,\mathbb{V}\mathrm{ar}\,\mathbf{Y}}}$
- Partial cross-correlation between **X** and **Y** given **Z**
  - Remove **Z** from **X**: $\epsilon_X = \mathbf{X} - \beta_X \mathbf{Z}$
  - Remove **Z** from **Y**: $\epsilon_Y = \mathbf{Y} - \beta_Y \mathbf{Z}$
  - $\rho(\mathbf{X}, \mathbf{Y} | \mathbf{Z}) = \frac{\mathbb{C}\mathrm{ov}(\epsilon_X, \epsilon_Y)}{\sqrt{\mathbb{V}\mathrm{ar}\,\epsilon_X\,\mathbb{V}\mathrm{ar}\,\epsilon_Y}}$

Cross-coherence - a measure of dependence



	Model A	Model B
Cross-Corr	Yes	Yes
Partial CC	NO	Yes

Cross-coherence - a measure of dependence

- When $\rho(\mathbf{X}, \mathbf{Y}|\mathbf{Z}) \neq 0$, we want to identify the frequency bands that drive the linear association.
- Notation

$$\boldsymbol{U}(t) = \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} \quad \boldsymbol{dZ}(\omega) = \begin{pmatrix} dZ_X(\omega) \\ dZ_Y(\omega) \\ dZ_Z(\omega) \end{pmatrix}$$

- Spectral representation of a stationary process

$$\boldsymbol{U}(t) = \int_{-0.5}^{0.5} \exp(i2\pi\omega t)\boldsymbol{dZ}(\omega).$$

- Formal definition of coherency (correlation of Fourier coefficients)

$$\rho_\omega(X, Y) = \mathbb{C}\mathrm{orr}[dZ_X(\omega), dZ_Y(\omega)]$$

Cross-coherence - a measure of dependence

- Filtered Signals

$$X_\omega(t) = \mathcal{F}_\omega X(t) \quad Y_\omega(t) = \mathcal{F}_\omega Y(t) \quad Z_\omega(t) = \mathcal{F}_\omega Z(t)$$

- Coherency at frequency band around ω

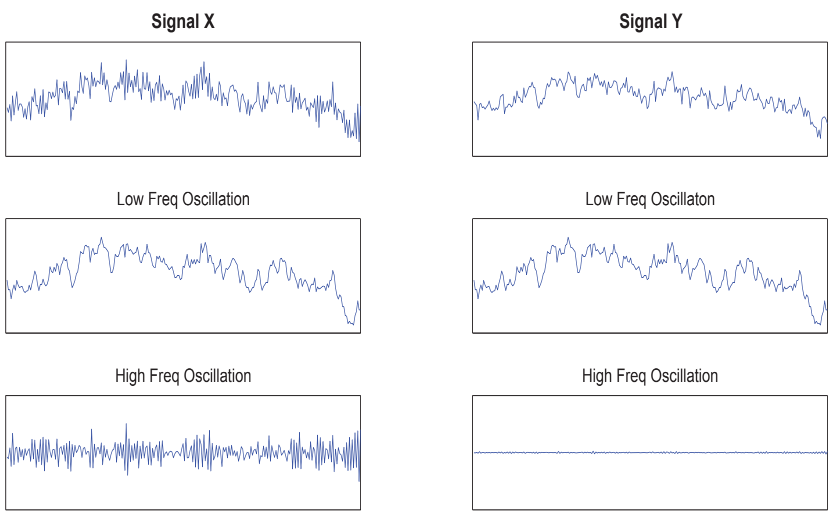
$$\rho_{X,Y}(\omega) = \mathbb{C}\mathrm{orr}[X_\omega(t), Y_\omega(t)]$$

- Partial coherence
  - Remove $Z_\omega(t)$ from $X_\omega(t)$: $\xi_\omega^X(t) = X_\omega(t) - \beta_X Z_\omega(t)$
  - Remove $Z_\omega(t)$ from $Y_\omega(t)$: $\xi_\omega^Y(t) = Y_\omega(t) - \beta_Y Z_\omega(t)$
  - $\rho_\omega^2(X, Y|Z) = \left| \frac{\mathbb{C}\mathrm{ov}(\xi_\omega^X(t), \xi_\omega^Y(t))}{\sqrt{\mathbb{V}\mathrm{ar}\, \xi_\omega^X(t)\, \mathbb{V}\mathrm{ar}\, \xi_\omega^Y(t)}} \right|^2$
- Relevant work:
  - Ombao and Van Bellegem (2008, IEEE Trans Signal Processing)
  - Fiecas and Ombao (2010, Annals of Applied Statistics)

An Illustration

- Latent Signals
  - $U_1(t)$ - low frequency
  - $U_2(t)$ - high frequency
- Observed Signals
  - $X(t) = U_1(t) + U_2(t) + Z_2(t)$
  - $Y(t) = U_1(t + \ell) + Z_1(t)$
- **X** and **Y** are linearly related through $\mathbf{U}_1$.

Cross-coherence - a measure of dependence



Cross-coherence - a measure of dependence

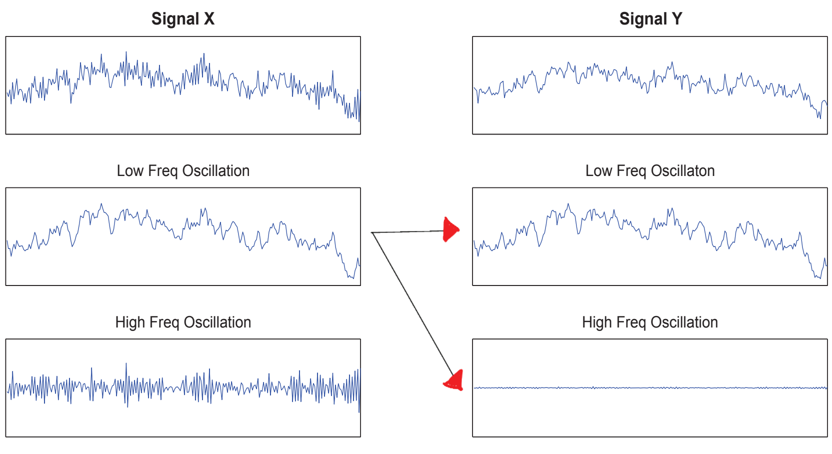
COHERENCE

- Identifies the oscillations that drive the linear association between **X** and **Y**.
- Identical Oscillations
  - Low freq oscillations in **X** vs Low freq oscillations in **Y**
  - High freq oscillations in **X** vs High freq oscillations in **Y**

Generalized Coherence

- Frequency bands
  - $(1, 4)$ Hertz - Delta
  - $(4, 8)$ Hertz - Theta
  - $(8, 12)$ Hertz - Alpha
  - $(16, 30)$ Hertz - Beta
  - $(30, 70)$ Hertz - Gamma
- Dependence between alpha oscillation activity in **X** and beta activity in **Y**

Generalized Coherence



Applications in the Neuroscience literature

- Working memory (hippocampus)
  - Gamma activity (amplitude) phase-locked to theta activity
  - Cross-frequency coupling predicts WM performance
- Reward processing (basal ganglia)
  - Bursts of high frequency activity (gamma) occurs preferentially during specific phases of lower frequency activity (alpha)

Generalized Coherence

Probability and Statistics

- Harmonizable Processes

$$X(t) = \int \exp(i2\pi\omega t)dZ(\omega)$$

  where $\{dZ(\omega)\}$ not necessarily uncorrelated
- References
  - Loéve 1955 (*Probability Theory*)
  - Martin 1982
  - Sharf (1990's onwards)
  - Hindberg and Hanssen 2007

Generalizations

- In the neuroscience literature
  - Ideas are present; several descriptive analysis
  - There is a need to introduce formal framework for testing
- In the signal processing literature
  - No framework for comparing across conditions and patient groups
  - No framework for replicated time series
- No models for studying how past alpha activity may predict future beta activity.

Generalized Coherence for Harmonizable Processes

Harmonizable process

$$X(t) = \int_{-0.5}^{0.5} \exp\{2\pi i \omega t\} dZ(\omega)$$

- $\{dZ(\omega)\}$ not necessarily uncorrelated.

Generalized Spectrum - Loeve Spectrum

$$\mathbb{C}\mathrm{ov}(dZ(\omega), dZ(\lambda)) = f(\omega, \lambda) d\omega d\lambda$$

- Allow correlation between different frequencies.

Generalized Coherence for Harmonizable Processes

Generalized Spectrum

- $\int \int |f(\omega, \lambda)| d\omega d\lambda < \infty$
- Relation with time varying covariance function

$$
\begin{aligned}
\gamma(s,t) &= \mathbb{E}[X(s)X^*(t)] \\
&= \int \int \exp\{2\pi i(s\omega - t\lambda)\} f(\omega, \lambda) d\omega d\lambda \\
\mathbb{V}\mathrm{ar}(X(t)) &= \int \int \exp\{2\pi i s(\omega - \lambda)\} f(\omega, \lambda) d\omega d\lambda
\end{aligned}
$$

- $f(\omega, \lambda) = \frac{1}{4\pi^2} \sum_s \sum_t \gamma(s,t) \exp\{-i(\omega s - \lambda t)\}$

Generalized Coherence (Dual frequency coherence)

$$X(t) \text{ at } \omega \longleftrightarrow X(t) \text{ at } \lambda \text{ ?}$$

Univariate $X(t)$

$$\rho^2_{XX}(\omega, \lambda) = \frac{|\mathbb{E}[dZ(\omega) dZ^*(\lambda)]|^2}{\mathbb{E}|dZ(\omega)|^2 \mathbb{E}|dZ(\lambda)|^2}$$

$$X(t) \text{ at } \omega \longleftrightarrow Y(t) \text{ at } \lambda \text{ ?}$$

Bivariate $X(t), Y(t)$

$$\rho^2_{XY}(\omega, \lambda) = \frac{\left|\mathbb{E}[dZ_X(\omega) dZ^*_Y(\lambda)]\right|^2}{\mathbb{E}|dZ_X(\omega)|^2 \mathbb{E}|dZ_Y(\lambda)|^2}$$

Generalized Coherence (Dual frequency coherence)

$$X(t) \text{ at } \omega \longleftrightarrow X(t) \text{ at } \lambda \text{ ?}$$

Univariate $X(t)$

$$\rho^2_{XX}(\omega, \lambda) = \frac{|\mathbb{E}[dZ(\omega)dZ^*(\lambda)]|^2}{\mathbb{E}|dZ(\omega)|^2\mathbb{E}|dZ(\lambda)|^2}$$

$$X(t) \text{ at } \omega \longleftrightarrow Y(t) \text{ at } \lambda \text{ ?}$$

Bivariate $X(t), Y(t)$

$$\rho^2_{XY}(\omega, \lambda) = \frac{\left|\mathbb{E}[dZ_X(\omega)dZ^*_Y(\lambda)]\right|^2}{\mathbb{E}|dZ_X(\omega)|^2\mathbb{E}|dZ_Y(\lambda)|^2}$$

Interpretation of dual frequency coherence

- Quantifies linear correlation between random oscillations at any pair of frequencies.
- When $\rho^2_{X,Y}(\omega, \lambda)$ close to 1 $\rightarrow$ $\Rightarrow$ linear relationship between $dZ_X(\omega)$ and $dZ_Y(\lambda)$
- The proportion of variance at ω in **X** that can be explained by the linear relationship between the
  - ω oscillation in **X**
  - λ oscillation in **Y**

INTERPRETATION OF DUAL FREQUENCY COHERENCE

- Quantifies linear correlation between random oscillations at any pair of frequencies.
- When $\rho^2_{X,Y}(\omega,\lambda)$ close to 1 $\rightarrow\ \Rightarrow$ linear relationship between $dZ_X(\omega)$ and $dZ_Y(\lambda)$
- The proportion of variance at ω in **X** that can be explained by the linear relationship between the
  - ω oscillation in **X**
  - λ oscillation in **Y**

Interpretation of dual frequency coherence

- Quantifies linear correlation between random oscillations at any pair of frequencies.
- When $\rho^2_{X,Y}(\omega, \lambda)$ close to 1 $\rightarrow$ $\Rightarrow$ linear relationship between $dZ_X(\omega)$ and $dZ_Y(\lambda)$
- The proportion of variance at ω in **X** that can be explained by the linear relationship between the
  - ω oscillation in **X**
  - λ oscillation in **Y**

Interpretation of dual frequency coherence

- Quantifies linear correlation between random oscillations at any pair of frequencies.
- When $\rho^2_{X,Y}(\omega, \lambda)$ close to 1 $\rightarrow \; \Rightarrow$ linear relationship between $dZ_X(\omega)$ and $dZ_Y(\lambda)$
- The proportion of variance at ω in **X** that can be explained by the linear relationship between the
  - ω oscillation in **X**
  - λ oscillation in **Y**

Interpretation of dual frequency coherence

- Quantifies linear correlation between random oscillations at any pair of frequencies.
- When $\rho^2_{X,Y}(\omega,\lambda)$ close to 1 $\rightarrow \;\Rightarrow$ linear relationship between $dZ_X(\omega)$ and $dZ_Y(\lambda)$
- The proportion of variance at ω in **X** that can be explained by the linear relationship between the
  - ω oscillation in **X**
  - λ oscillation in **Y**

Generalized Coherence

Harmonizable processes are generally non-stationary.

- A_1, B_1 iid $(0, \sigma_1^2)$;
- $A_2 = A_1 + Z_A$; $B_2 = B_1 + Z_B$;
- Consider the harmonizable sinusoidal process

$$X(t) = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)$$

- $\mathbb{V}\mathrm{ar}\, X(t) = \sigma_1^2 + \sigma_2^2 + \sigma_1^2 \cos[(\omega_1 - \omega_2)t]$

Generalized Coherence

Estimation

- $\mathbf{X}^r, \mathbf{Y}^r$ time series on trial $r = 1, \ldots, R$
- Fourier coefficient

$$d_X^r(\omega) = \sum_{t=1}^{T} X^r(t) \exp(-i2\pi\omega t)$$

- Generalized cross-periodgram for the r-th trial

$$\mathcal{I}_{X,Y}^r(\omega, \lambda) = d_X^r(\omega) d_Y^{r*}(\lambda)$$

- Estimate of the generalized spectrum

$$\widehat{f_{X,Y}}(\omega, \lambda) = \frac{1}{R} \sum_{r=1}^{R} \mathcal{I}_{X,Y}^r(\omega, \lambda)$$

GENERALIZED COHERENCE

Estimation

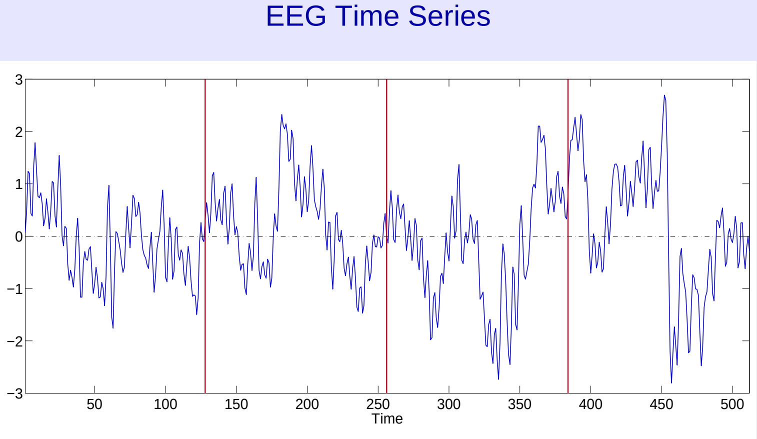
- Estimate of the generalized coherence

$$\widehat{\rho_{X,Y}}^2(\omega,\lambda) = \frac{|\widehat{f}_{X,Y}(\omega,\lambda)|^2}{\widehat{f}_{X,X}(\omega,\omega)\widehat{f}_{Y,Y}(\lambda,\lambda)}$$

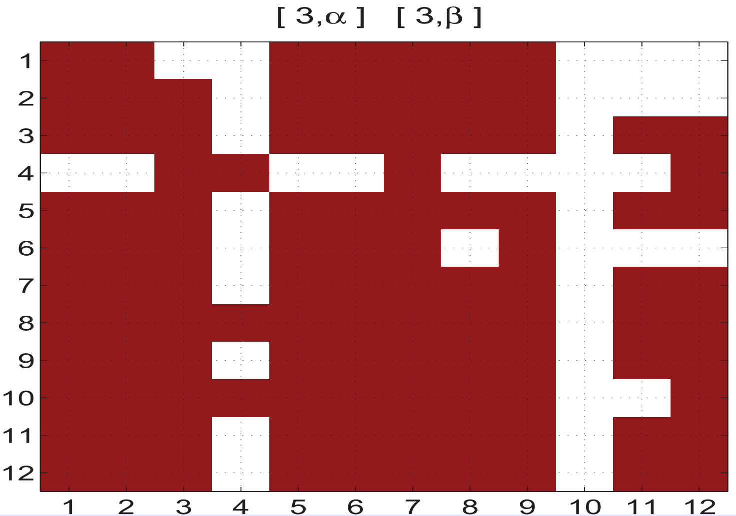
Preliminary EEG Data Analysis

- Visual Motor experiment (PI: J Sanes, Brown Neuroscience)
- Replicated trials ($r = 1, \ldots, R = 100+$)
- Time blocks $b = 1, \ldots, 4$
- Stimulus presented at block 3
- Each time block has $T = 128$ time points
- We computed Generalized Coherence
  - between different channels
  - between pairs of different frequencies
  - for both the same and successive time blocks

Preliminary EEG Data Analysis

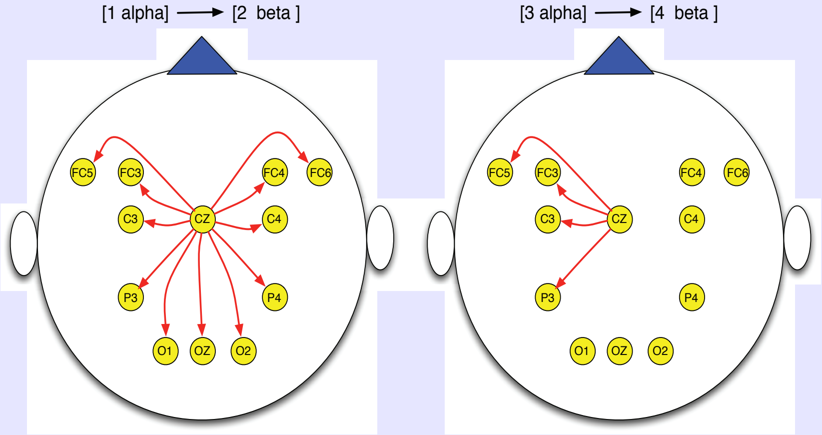


Alpha-Beta coherence at Block 3



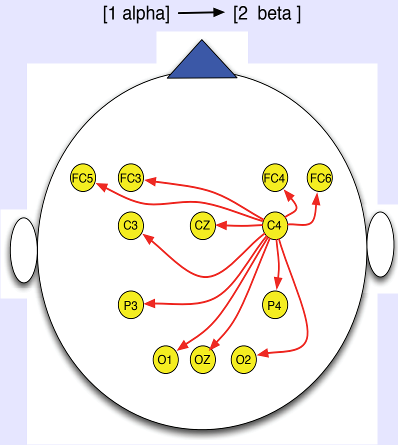
PRELIMINARY EEG DATA ANALYSIS

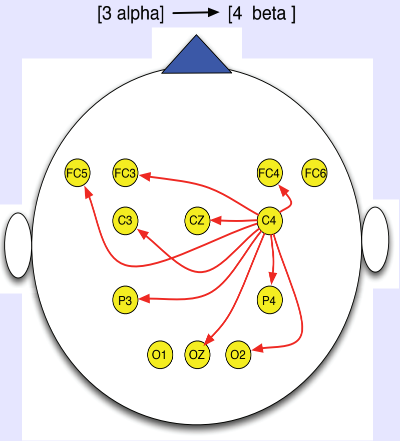
Connectivity – CZ seed channel



PRELIMINARY EEG DATA ANALYSIS

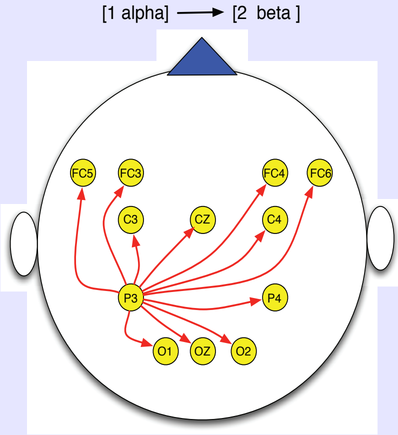
Connectivity – C4 seed channel

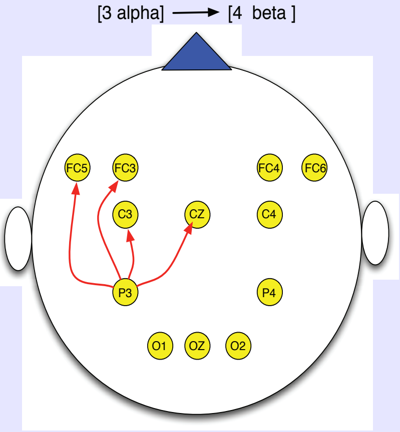




Preliminary EEG Data Analysis

Connectivity – P3 seed channel





Spectral Autoregressive Model

- $d_c(\alpha, b)$

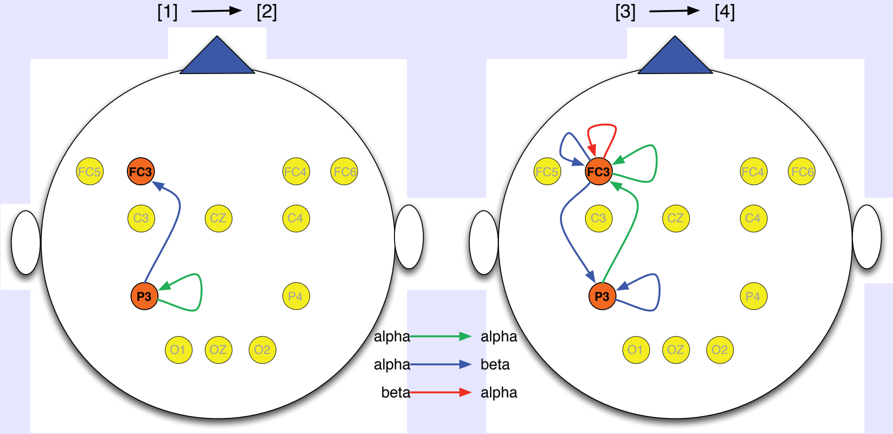
  alpha-band coefficient
  channel c
  time block b
- The Spectral-AR model

$$
\begin{aligned}
d_c(\alpha, b) &= d_c(\alpha, b-1) + d_c(\beta, b-1) + \\
&\quad d_{c'}(\alpha, b-1) + d_{c'}(\beta, b-1) + \epsilon_c(\alpha, b)
\end{aligned}
$$

Spectral Autoregressive Model

- Parietal-Frontal connectivity
- Complex-valued data
- Potential variations
  - Magnitute$(b-1) \ \rightarrow\ $ Magnitude(b)
  - Phase$(b-1) \ \rightarrow\ $ Phase(b)

Spectral Autoregressive Model



Current Work

- Establish conditions for mean-squared consistency of the estimator
- Testing for differences in generalized coherence - across conditions
- Penalized likelihood estimation method for the spectral-AR model

COLLABORATORS - BROWN NEURO-STATS

- **Graduate Students**

  Fiecas, Mark
  Gorrostieta, Cristina
  Joo, LiJin
  Kang, Hakmook

- **Undergraduate Student**

  Van Lunen, Daniel