Introduction to the Geometry of Stacks

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Abstract

Differential and algebraic stacks are generalizations of differential manifolds and algebraic varieties and schemes, and have found increasing use in modern differential and algebraic geometry. They have also found applications in mathematical physics.

These notes are designed to introduce a beginner to the basics of differential stacks. They are to be read along with the notes of my KIAS, Seoul lectures (2005) which give an introduction to algebraic stacks. The material will be presented in a series of lectures at CIMAT, Guanajuato in November 2008. The final version of the notes, which will contain some more material and more details, will be prepared after taking into account the suggestions/corrections received from the audience and the readers.

Contents

1 Functor of Points

Differential stacks (respectively, holomorphic stacks or algebraic stacks) are a generalization of manifolds (respectively, holomorphic manifolds or schemes). Manifolds or schemes are usually defined as certain topological spaces equipped with sheaves of rings. But following Grothendieck, these can also be thought of via their 'functors of points', which are certain set-valued functors. The basic new idea of Grothendieck in the invention of stacks (in early 1960's) is to go from set-valued functors to groupoid-valued functors, in order to take into account vital information about 'automorphisms' of the valued points. This section explains the approach to manifolds via their functors of points.

The simplest differential manifolds are $\mathbb{R}^n$ and its open subsets. Let $\mathcal{S}$ be the category formed by all open subsets U of $\mathbb{R}^n$ (for all $n \geq 0$) as the objects of the category, and with C^∞-maps $f : V \to U$ as the morphisms in the category. This is a full subcategory of the category of all C^∞ manifolds. Then corresponding to any arbitrary C^∞ manifold X, we can define a set-valued contravariant functor on $\mathcal{S}$, denoted by

$$h_X : \mathcal{S}^{op} \to Sets$$

as follows. For any object U in $\mathcal{S}$, we put

$$h_X(U) = Hom_{\mathcal{S}}(U, X)$$

which is the set of all C^∞-maps from U to X. Given any C^∞-map $f : V \to U$, we get a set-map

$$h_X(f) : h_X(U) \to h_X(V)$$

by composing with f. It can be seen that h_X is indeed a functor. It is called **the functor of points** of X. Elements of $h_X(U)$ are called **U-valued points** of X, and U is called the **level of definition** of these. We also say that any $f \in h_X(U)$ is a **point of X defined over U**.

Given any two manifolds X and Y and a C^∞ map $\varphi : X \to Y$, we get a natural transformation (that is, a morphism of functors)

$$h_\varphi : h_X \to h_Y$$

defined by composing with φ. Thus, we obtain a functor

$$h : C^\infty\text{-manifolds} \to Fun(\mathcal{S}^{op}, Sets)$$

under which $X \mapsto h_X$ and $\varphi \mapsto h_\varphi$, where $Fun(\mathcal{S}^{op}, Sets)$ is the category whose objects are all set-valued contravariant functors from $\mathcal{S}$, and whose morphisms are natural transformations between these.

Exercise Show that the functor $h : C^\infty\text{-manifolds} \to Fun(\mathcal{S}^{op}, Sets)$ is fully faithful. As a consequence, the category C^∞-manifolds is equivalent to a full subcategory of $Fun(\mathcal{S}^{op}, Sets)$.

Similarly, we can take $\mathcal{S}$ to be the category whose objects are all open subsets U of $\mathbb{C}^n$, for $n \geq 0$, and whose morphisms are all holomorphic maps $f : V \to U$ between such. This is a full subcategory of the category $\mathbb{C}$-manifolds of all holomorphic manifolds and holomorphic maps between them. Then as above, for each holomorphic manifold X we will get a functor of points h_X, and for each holomorphic map $\varphi : X \to Y$ we will get a natural transformation $h_\varphi : h_X \to h_Y$. Thus, again we will get a functor

$$h : \mathbb{C}\text{-manifolds} \to Fun(\mathcal{S}^{op}, Sets).$$

Once again, the reader may verify as an exercise that this functor is fully faithful.

Similarly, let $\mathcal{S} = Aff/S$ be the category of affine schemes over a base scheme S, which is a full subcategory of the category Sch/S of S-schemes. Once again, we get a fully faithful functor

$$h : Sch/S \to Fun(\mathcal{S}^{op}, Sets).$$

In the special case where $S = \operatorname{Spec} k$ for a commutative ring k, the opposite category of Aff/S is equivalent to the category Alg/k of commutative k-algebras. In this case, the functor h imbeds k-schemes as a full subcategory of the functor category $Fun(Alg/k, Sets)$.

The above constructions mean that an arbitrary C^∞-manifold (respectively, a holomorphic manifold or an S-scheme) can be regarded as a set-valued functor on the category $\mathcal{S}$ of particularly simple C^∞-manifolds (or holomorphic manifold or S-schemes).

Representability and strong representability

Let $\mathcal{S}$ be the base category of opens subsets of $\mathbb{R}^n$ with C^∞ maps. A set-valued contra-functor $\mathfrak{X}$ on $\mathcal{S}$ is said to be **representable by a manifold**, or **strongly representable**, if there exists a pair (X, η) consisting of a C^∞ manifold X together with a natural isomorphism of functors $\eta : h_X \to \mathfrak{X}$. It would be useful to call X as a **fine moduli space** for the functor $\mathfrak{X}$, and η a **universal family** (or a Poincaré family) on X.

We similarly define the notion of (strong) representability for the base category $\mathcal{S}$ of open subsets of $\mathbb{C}^n$ with holomorphic maps. In the differential or holomorphic category, the word 'strong' is often dropped for short.

In the algebraic set-up, where the base category $\mathcal{S}$ is Aff/S, the corresponding notion is expressed by saying that $\mathfrak{X}$ is **schematic** or **strongly representable**. A weaker condition, where the fine moduli space X is allowed to be an algebraic space over S (in the sense of Artin) which is not necessarily a scheme, is usually meant when one says that $\mathfrak{X}$ is **representable**.

Exercise 1.1 (Yoneda Lemma) Let U be any object of $\mathcal{S}$ and let $h_U : \mathcal{S}^{op} \to Sets$ be the corresponding strongly representable functor. Then for any functor $F : \mathcal{S}^{op} \to Sets$, the natural map

$$Hom(h_U, F) \to F(U)$$

is a bijection.

Exercise 1.2 If $\mathfrak{X} : \mathcal{S}^{op} \to Sets$ is strongly representable (or just representable when working in the algebraic category), show that the pair (X, η) which represents it is unique up to a unique isomorphism. Note that this statement is stronger than the Yoneda lemma, as X need not be in the base category $\mathcal{S}$.

The following descent property of Grothendieck is necessary (but not sufficient) for a functor $\mathfrak{X} : \mathcal{S}^{op} \to Sets$ to be representable.

Descent property: C^∞ or holomorphic case. Let U be any object of $\mathcal{S}$. Let $\mathfrak{X}|_U$ denote the pre-sheaf of sets defined on U which associates to any open subset $V \subset U$ the set $\mathfrak{X}(V)$, with restriction maps as defined by the functor $\mathfrak{X}$. If $\mathfrak{X} : \mathcal{S}^{op} \to Sets$ is strongly representable, then $\mathfrak{X}|_U$ is a sheaf of sets on U.

Descent property: algebraic case. When we are working in the algebraic category, the scheme U can be given the so called fpqc topology, which is finer than the Zariski topology. Then it is a basic theorem of Grothendieck's 'Theory of Descent' (see, for example the article of Vistoli in [FGA-E]) that if $\mathfrak{X}$ is schematic then $\mathfrak{X}|_U$ is a sheaf in the fpqc topology. Similarly, it can be shown (see Laumon and Moret-Bailly [L-MB]) that if $\mathfrak{X}$ is representable then $\mathfrak{X}|_U$ is a sheaf in the fpqc topology.

Some representable examples

1.3 For any U in $\mathcal{S}$, let $\mathcal{O}(U)$ be the set of regular functions on U. With pull-back of a regular function under any $\mathcal{S}$-morphism $f : V \to U$ defined as usual, this gives a functor $\mathcal{S}^{op} \to Sets$. This functor is represented by the pair $(\mathbf{A}^1, x)$ where $\mathbf{A}^1$ is the affine line (means the manifold $\mathbb{R}^1$ in C^∞ category, the manifold $\mathbb{C}^1$ in the holomorphic category, and $\mathrm{Spec}_{\mathcal{O}_S}\, \mathcal{O}_S[x]$ in S-schemes), and x is the standard coordinate function on it.

1.4 Let $n \geq 1$ be fixed. For any U in $\mathcal{S}$, let $G(U)$ be the set of all invertible $n \times n$-matrices in the ring $\mathcal{O}(U)$. With pull-back of a regular function under any $\mathcal{S}$-morphism $f : V \to U$ defined as usual, this gives a functor $\mathcal{S}^{op} \to Sets$. This functor is represented by the pair (GL_n, x) where x is the coordinate matrix $(x_{i,j})$. (Here, GL_n stands for the real Lie group $GL_n(\mathbb{R})$ or complex Lie group $GL_n(\mathbb{C})$ or the S-scheme $GL_{n,S} = \mathrm{Spec}_{\mathcal{O}_S}\, \mathcal{O}_S[x_{i,j}, y]/(\det(x_{i,j})y - 1)$ as the case may be.)

1.5 Let $n \geq 0$ be fixed. For any U in $\mathcal{S}$, consider surjective $\mathcal{O}_U$-linear homomorphisms $x : \mathcal{O}_U^{n+1} \to L$ where L is a rank 1 locally free sheaf of $\mathcal{O}_U$-modules. Two such homomorphisms $x : \mathcal{O}_U^{n+1} \to L$ and $x' : \mathcal{O}_U^{n+1} \to L'$ will be called equivalent if there exists an $\mathcal{O}_U$-linear isomorphism $\phi : L \to L'$ such that $x' = \phi \circ x$. Let $\mathfrak{X}(U)$ be the set of all equivalence classes. Given any $f : V \to U$, let $f^*(x) : \mathcal{O}_V^{n+1} \to f^*(L)$ be the pull-back of x. The association $x \mapsto f^*(x)$ preserves equivalences, so defines a map $\mathfrak{X}(f) : \mathfrak{X}(U) \to \mathfrak{X}(V)$. This defines a contra-functor $\mathfrak{X}$ from $\mathcal{S}$ to $Sets$. The most basic fact of projective geometry is that this functor is (strongly) represented by $(\mathbf{P}^n, q : \mathcal{O}_{\mathbf{P}^n}^{n+1} \to \mathcal{O}_{\mathbf{P}^n}(1))$ where q is the tautological quotient homomorphism. (Here, $\mathbf{P}^n$ stands for the real or complex projective space or the S-scheme $\mathbf{P}_S^n$ as the case may be.)

Some non-representable examples

1.6 For any U in $\mathcal{S}$, let $\Omega^n(U)$ be the set of all n-forms on U. These are the C^∞ or holomorphic differential n-forms, or regular kahler differentials $\Omega^1_{U/S}$, as the case may be. Pull-backs under any $\mathcal{S}$-morphism $f : V \to U$ are defined as usual.

We now that this functor is not representable if $n \geq 1$. If possible, let a manifold X_n together with an n-form $\alpha_n \in \Omega^n(X)$ represent this functor, so that given any manifold U in $\mathcal{S}$ and a n-form $\beta \in \Omega^n(U)$, there exists a unique map $f : U \to X$ such that $f^*(\alpha_n) = \beta$. Taking $\beta = 0$, as by assumption $n \geq 1$, we see that any constant f will do. So if f is unique, this forces X to be a point. But then $\alpha_n = 0$. On the other hand, a non-zero n-form β exists on some U (e.g. $U = \mathbb{R}^n$). Contradiction.

1.7 In fact, there is no finite dimensional versal family (X_n, α_n) for n-forms when $n \geq 1$. Such a family would have the property that given any manifold U in $\mathcal{S}$ and a n-form $\beta \in \Omega^n(U)$, and a point $P \in U$, there exists a neighbourhood V of P in U and a (possibly non-unique) map $f : V \to X$ such that $f^*(\alpha_n) = \beta|_V$. The case where $n \geq 2$ follows from linear algebra alone, while we can prove the case $n = 1$ using the fact that f^* commutes with exterior differentials, as follows. Consider the 1-form $\beta = \sum_{1 \leq i \leq n} x_{2i-1} dx_{2i}$ on $U = \mathbb{R}^{2m}$. If $\beta|_V = f^*(\alpha_1)$ in a non-empty neighbourhood V, then $d\beta|_V = f^*(d\alpha_1)$ in V. As the skew-symmetric bilinear form $d\beta$ has rank $2m$ (it is non-degenerate), it follows that $d\alpha_1$ must have rank $\geq 2m$ in the image of f, and so $\dim(X) \geq 2m$ for each m. This contradicts finite-dimensionality of X. (Of course, we could have a situation where X is a disjoint union of connected open manifolds whose dimensions are not bounded).

Exercise 1.8 For any U in $\mathcal{S}$, consider a 2-sheeted covering projection $p : \overline{U} \to U$ (note that p is surjective proper map which is a local diffeomorphism or a local holomorphic isomorphism or an étale morphism as the case may be). A covering $p : V \to U$ is defined to be equivalent to another covering $p' : \overline{U}' \to U$ if there exists an isomorphism $\varphi : \overline{U} \to \overline{U}'$ such that $p' \circ \varphi = p$. Under any $f : V \to U$, the usual pull-back of any 2-sheeted covering projection of U is a 2-sheeted covering projection of V, and this preserves equivalences, so we get a functor which attaches to U the set of equivalence classes of all 2-sheeted covering projections $p : \overline{U} \to U$. Show that this functor is not representable.

2 Preliminaries on fibered groupoids

Moduli problems are usually formulated as *groupoid-valued pseudo-functors* on the category of holomorphic manifolds (or on schemes, etc.), and these give rise to stacks. We begin with two examples of moduli problems to motivate the definition of a groupoid-valued pseudo-functor on a base category, working over the base category of holomorphic manifolds.

Example 2.1 Moduli problem for vector bundles Let X be a given compact Riemann surface. For any complex manifold U, let $Bun_X(U)$ be the category whose objects are all holomorphic vector bundles E_U on $X \times U$. We can regard such a vector bundle as a family of bundles on X parameterized by U. A morphism in $Bun_X(U)$ from E_U to F_U is a holomorphic isomorphism of vector bundles on $X \times U$. Given any holomorphic map $\varphi : V \to U$, from the family E_U we get a family denoted by $\varphi^*(E_U)$ over V, which is defined as the pullback of the bundle E_U on $X \times U$ under the map $\mathrm{id}_X \times \varphi : X \times U \to X \times V$. This defines a functor $\varphi^* : Bun_X(U) \to Bun_X(V)$. If $1_U : U \to U$ is the identity map, we have a natural isomorphism $\epsilon_U(E_U) : (1_U)^* E_U \to E_U$. If $W \xrightarrow{\psi} V \xrightarrow{\varphi} U$ are holomorphic maps, then we have a certain natural isomorphism $c_{\varphi,\psi} : \psi^* \circ \varphi^* \to (\varphi \circ \psi)^*$. The natural isomorphisms ϵ_U and $c_{\varphi,\psi}$ satisfy certain obvious properties (made explicit in Definition 2.3) which allows us to identify $(1_U)^* E_U$ with E_U and $\psi^* \circ \varphi^*(E_U)$ with $(\varphi \circ \psi)^*(E_U)$ for simplicity of notation, when there is no danger of confusion. (These identifications are justified by the Lemma 2.10.)

Example 2.2 Moduli problem for elliptic curves For any complex manifold U, let $\mathcal{M}_{1,1}(U)$ (or for simplicity of notation, $\mathcal{M}(U)$) denote the category whose objects are all families of elliptic curves E_U parameterized by U. Note that E_U is a holomorphic manifold with a proper holomorphic submersion $E_U \to U$ whose fibers are connected genus 1 curves, together with a given holomorphic section $U \to E_U$ called the zero section (which defines the zero element of the group structure on each fiber). A morphism in $\mathcal{M}(U)$ from E_U to F_U is an isomorphism of holomorphic manifolds which commutes with the projections to U and carries the zero section to the zero section. Given any holomorphic map $\varphi : V \to U$, again we can define the pull-back family $\varphi^*(E_U)$ over V as the fiber product

$$\varphi^*(E_U) = V \times_U E_U$$

and this defines a functor $\varphi^* : \mathcal{M}(U) \to \mathcal{M}(V)$. We again have a natural isomorphism $\epsilon_U(E_U) : (1_U)^* E_U \to E_U$. If $W \xrightarrow{\psi} V \xrightarrow{\varphi} U$ are holomorphic maps, then again we have a certain natural isomorphism $c_{\varphi,\psi} : (\varphi \circ \psi)^* \to \psi^* \circ \varphi^*$, which satisfies the good properties which allows us to identify $(1_U)^* E_U$ with E_U and $\psi^* \circ \varphi^*(E_U)$ with $(\varphi \circ \psi)^*(E_U)$ for simplicity of notation, when there is no danger of confusion.

The common features and properties we have describes in the above two examples are typical of moduli problems, and motivate the following definition. While reading the definition, one should keep in mind the above examples $\mathfrak{X} = Bun_X$ or $\mathfrak{X} = \mathcal{M}$. In both these examples, the base category $\mathcal{S}$ is the category of holomorphic manifolds.

Recall that a category in which each arrow is an isomorphism is called a **groupoid**.

Definition 2.3 A **groupoid-valued pseudo-functor on a base category** $\mathcal{S}$, also called a **groupoid-valued lax 2-functor on** $\mathcal{S}$, consists of the following.

(1) For each object U of $\mathcal{S}$, we are given a groupoid $\mathfrak{X}(U)$ (also denoted by $\mathfrak{X}_U$).

(2) For each arrow $\varphi : V \to U$ in $\mathcal{S}$, we are given a functor $\varphi^* : \mathfrak{X}(U) \to \mathfrak{X}(V)$.

(3) For each U in $\mathcal{S}$ we are given a natural isomorphism $\epsilon_U : (1_U)^* \to 1_{\mathfrak{X}(U)}$.

(4) For any pair of arrows $W \xrightarrow{\psi} V \xrightarrow{\varphi} U$ in $\mathcal{S}$, we are given a natural isomorphism $c_{\psi,\varphi} : \psi^* \circ \varphi^* \to (\varphi \circ \psi)^*$.

The above data is required to satisfy the following compatibility conditions, where $\star$ denotes the horizontal composition and $\bullet$ denotes the vertical composition of natural transformations.

(a) If $\varphi : V \to U$ is any arrow in $\mathcal{S}$, we have

$$c_{\varphi, 1_U} = \varphi^* \star \epsilon_U \text{ and } c_{1_V, \varphi} = \epsilon_V \star \varphi^*.$$

(b) If $X \xrightarrow{\eta} W \xrightarrow{\psi} V \xrightarrow{\varphi} U$ are arrows in $\mathcal{S}$, then

$$c_{\psi\eta, \varphi} \bullet (c_{\eta, \psi} \star 1_{\varphi^*}) = c_{\eta, \varphi\psi} \bullet (1_{\eta^*} \star c_{\psi, \varphi}).$$

The condition (a) says that for any object u of $\mathfrak{X}(U)$ we have $c_{\varphi, 1_U}(u) = \varphi^*(\epsilon_U(u)) : \varphi^*(1_U)^* u \to \varphi^* u$ and $c_{1_V, \varphi} = \epsilon_V(\varphi^* u) : (1_V)^* \varphi^* u \to \varphi^* u$. The condition (b) says that for any object u of $\mathfrak{X}(U)$ the following diagram commutes.

$$\begin{array}{ccc} \eta^*\psi^*\varphi^* u & \xrightarrow{\eta^*(c_{\psi,\varphi}(u))} & \eta^*(\varphi\psi)^* u \\ {\scriptstyle c_{\eta,\psi}(\varphi^* u)} \downarrow & & \downarrow {\scriptstyle c_{\eta,\varphi\psi}(u)} \\ (\psi\eta)^*\varphi^* u & \xrightarrow{c_{\psi\eta,\varphi}(u)} & (\varphi\psi\eta)^* u \end{array}$$

Definition 2.4 A **fibered groupoid** $(\mathfrak{X}, a)$ **over a base category** $\mathcal{S}$, also known as an $\mathcal{S}$**-groupoid**, consists of a category $\mathfrak{X}$ together with a functor $a : \mathfrak{X} \to \mathcal{S}$, such that the following conditions are satisfied.

(1) For each morphism $\varphi : V \to U$ in $\mathcal{S}$ and each object u in $\mathfrak{X}$ such that $a(u) = U$, there exists at least one object v in $\mathfrak{X}$ and a morphism $f : v \to u$ in $\mathfrak{X}$ such that $a(v) = V$ and $a(f) = \varphi$.

(2) Let $f : v \to u$ and $h : w \to u$ be a pair of morphisms in $\mathfrak{X}$, with common target u. Let $a(u) = U$, $a(v) = V$ and $a(w) = W$. Then for any $\psi : W \to V$ in $\mathcal{S}$ such that $a(f) \circ \psi = a(h)$, there exists a unique $g : w \to v$ in $\mathfrak{X}$ such that $a(g) = \psi$ and $h = f \circ g$.

The category $\mathfrak{X}$ is called the **total category**, the category $\mathcal{S}$ is called the **base category**, and the functor a is called the **structure functor**.

Definition 2.5 **The 2-category of all $\mathcal{S}$-groupoids** is defined as follows. The objects (0-cells) of this 2-category are all $\mathcal{S}$-groupoids $(\mathfrak{X}, a)$. A 1-morphism (1-cell) $F : (\mathfrak{X}, a) \to (\mathfrak{Y}, b)$ is a functor $F : \mathfrak{X} \to \mathfrak{Y}$ which commutes with the projections a and b, that is, $b \circ F = a$. A 2-isomorphism (2-cell) α from $F : (\mathfrak{X}, a) \to (\mathfrak{Y}, b)$ to $G : (\mathfrak{X}, a) \to (\mathfrak{Y}, b)$ is a natural isomorphism $\alpha : F \Rightarrow G$ of functors. The composition $\circ$ of 1-morphisms and the vertical composition $\bullet$ and the horizontal composition $\star$ of 2-morphisms is defined in the obvious way.

2.6 From a groupoid-valued pseudo-functor on $\mathcal{S}$ to an $\mathcal{S}$-groupoid. Let $((\mathfrak{X}(U)), (\epsilon_U), (c_{\psi,\varphi}))$ be a groupoid-valued pseudo-functor on $\mathcal{S}$. We associate to it an $\mathcal{S}$-groupoid $a : \mathfrak{X} \to \mathcal{S}$. The total category $\mathfrak{X}$ is defined as follows. An **object of** $\mathfrak{X}$ is a pair (U, u) where U is an object of $\mathcal{S}$ and u is an object of $\mathfrak{X}(U)$. Symbolically,

$$Ob(\mathfrak{X}) = \coprod_{U \in Ob(\mathcal{S})} Ob(\mathfrak{X}(U)).$$

A **morphism in** $\mathfrak{X}$ from (U, u) to (V, v) is a pair (φ, h) where $\varphi : U \to V$ is a morphism in $\mathcal{S}$, and $h : u \to \varphi^* v$ is an isomorphism in $\mathfrak{X}(U)$. The structure functor $a : \mathfrak{X} \to \mathcal{S}$ is defined by $(U, u) \mapsto U$ and $(\varphi, \alpha) \mapsto \varphi$.

2.7 From an $\mathcal{S}$-groupoid to a groupoid-valued pseudo-functor on $\mathcal{S}$. Let $(\mathfrak{X}, a)$ be an $\mathcal{S}$-groupoid. For each object U of $\mathcal{S}$, there is a (in general non-full) subcategory $\mathfrak{X}(U)$ of $\mathfrak{X}$ defined as follows. The objects of $\mathfrak{X}(U)$ are all objects u of $\mathfrak{X}$ for which $a(u) = U$, and the morphisms of $\mathfrak{X}(U)$ are all morphisms $f : u_1 \to u_2$ of $\mathfrak{X}$ for which $a(f) = \mathrm{id}_U$. As a consequence of the above conditions, each such $\mathfrak{X}(U)$ is a groupoid (all morphisms in $\mathfrak{X}(U)$ are isomorphisms).

Consider any morphism $\varphi : V \to U$ in $\mathcal{S}$ and an object u in $\mathfrak{X}$ such that $a(u) = U$. By condition (1), there exists a morphism $f : v \to u$ in $\mathfrak{X}$ such that $a(v) = V$ and $a(f) = \varphi$. Once and for all, we will make a choice of such an f for each φ (which will be possible subject to overcoming set-theoretic obstacles). Having made such a choice $f : v \to u$, we will denote v by $\varphi^*(u)$, and f by $\overline{\varphi}_u : \varphi^*(u) \to u$.

Given any pair of morphisms $f : v \to u$ and $h : w \to u$ with $a(f) = a(h) = \varphi$, by condition (2) applied by taking $W = V$ and $\psi = 1_V$, there exists a unique $g : w \to v$ with $h = f \circ g$. Therefore given any morphism $k : u_1 \to u_2$ in $\mathfrak{X}(U)$, by taking $u = u_2$, $f = \overline{\varphi}_2$ and $h = k \circ \overline{\varphi}_1$ in the above, it follows that there exists a unique morphism $g : \varphi^*(u_1) \to \varphi^*(u_2)$ in $\mathfrak{X}(U)$ such that

$$\overline{\varphi}_{u_2} \circ g = k \circ \overline{\varphi}_{u_1}$$

We denote g by $\phi^*(k)$. With the above definitions of φ^* on objects and morphisms in $\mathfrak{X}(U)$, it can be seen that for each $\varphi : V \to U$ in $\mathcal{S}$ we have defined a functor $\varphi^* : \mathfrak{X}(U) \to \mathfrak{X}(V)$.

For each u in $\mathfrak{X}(U)$, we define the morphism $\epsilon_U(u) : (1_U)^*(u) \to u$ to be equal to the chosen lift $(\overline{1_U})_u$ with target u of the identity $1_U : U \to U$. As u varies over $\mathfrak{X}(U)$, this can be seen to define a natural transformation

$$\epsilon_U : (1_U)^* \to 1_{\mathfrak{X}(U)}$$

for each object U in $\mathcal{S}$.

For any composable pair of arrows $W \xrightarrow{\psi} V \xrightarrow{\varphi} U$ in $\mathcal{S}$, and any object u in $\mathfrak{X}(U)$, reasoning as above we get a unique isomorphism $c_{\psi,\varphi}(u) : \psi^* \circ \varphi^*(u) \to (\varphi \circ \psi)^*(u)$ in $\mathfrak{X}(W)$ such that

$$\overline{\varphi}_u \circ \overline{\psi}_{\varphi^*(u)} = \overline{\varphi \circ \psi}_u \circ c_{\psi,\varphi}(u).$$

This can be seen to define a natural isomorphism

$$c_{\psi,\varphi} : \psi^* \circ \varphi^* \to (\varphi \circ \psi)^*.$$

It is clear from their constructions that the groupoids $\mathfrak{X}(U)$ with the functors φ^* and the natural isomorphisms ϵ_U and $c_{\psi,\varphi}$ form a groupoid-valued pseudo-functor on $\mathcal{S}$.

Definition 2.8 Given an $\mathcal{S}$-groupoid $(\mathfrak{X}, a)$, any data $((\varphi^*), (\epsilon_U), (c_{\psi,\varphi}))$ which makes $((\mathfrak{X}(U)), (\varphi^*), (\epsilon_U), (c_{\psi,\varphi}))$ a groupoid-valued pseudo-functor on $\mathcal{S}$ is called a **cleavage** of $(\mathfrak{X}, a)$. By the above (modulo set-theoretic difficulties), any $\mathcal{S}$-groupoid admits at least one cleavage. Suppose a particular cleavage is chosen for an $\mathcal{S}$-groupoid $(\mathfrak{X}, a)$. If for all U, we have $(1_U)^* = 1_{\mathfrak{X}(U)}$ with ϵ_U the identity natural transformation, and if moreover for each pair of composable morphisms we have $\psi^* \circ \varphi^* = (\varphi \circ \psi)^*$ with $c_{\psi,\varphi}$ the identity natural transformation, then we say that the chosen cleavage is a **splitting**, and $(\mathfrak{X}, a)$ – equipped with this cleavage – is a **split** groupoid.

Remark 2.9 We can choose the lifts $\overline{\varphi}_u : \varphi^*(u) \to u$ so that $(1_U)^* = 1_{\mathfrak{X}(U)}$ with ϵ_U the identity natural transformation (such a choice – though artificial – is always possible). One may ask whether moreover it is possible to choose the lifts $\overline{\varphi}_u : \varphi^*(u) \to u$ so that we will always have an equality $\psi^* \circ \varphi^* = (\varphi \circ \psi)^*$, with $c_{\psi,\varphi}$ the identity natural transformation. This is *not* always possible: for an elementary counter-example, the reader can see [Vistoli] Example 3.14. However, the following elementary result holds: see [Vistoli] Theorem 3.45.

Lemma 2.10 *If $a : \mathfrak{X} \to \mathcal{S}$ is any fibered groupoid, then there exists a fibered groupoid $b : \mathfrak{Y} \to \mathcal{S}$ such that*

(i) $\mathfrak{X}$ is equivalent to $\mathfrak{Y}$ as an $\mathcal{S}$-groupoid and

(ii) There exists a cleavage of $b : \mathfrak{Y} \to \mathcal{S}$ which is a splitting.

The groupoid $(\mathfrak{Y}, b)$ together with its splitting can be chosen to be functorial in $(\mathfrak{X}, a)$ in a suitable sense.

Remark 2.11 Because of the above lemma, we will be able to pretend for simplicity of notation that the various stacks we will deal with are equipped with cleavages that are splittings. In the rest of these notes, provided there is no danger of confusion, we will simply write $(1_U)^* = 1_{\mathfrak{X}(U)}$ and $\psi^* \circ \varphi^* = (\varphi \circ \psi)^*$, and moreover we will suppress any reference to ϵ_U and $c_{\psi,\varphi}$.

3 Stacks

The concept of a holomorphic stack (or an algebraic stack) may be regarded as an abstraction of some of the common features of different moduli problems. The two examples of moduli problems discussed above (namely, for vector bundles on X and elliptic curves) share certain important properties, known as the descent property and the effective descent property.

These properties make sense for any fibered groupoid $\mathfrak{X}$ over the base category $\mathcal{S}$ of holomorphic manifolds. For the sake of the following definition, we choose a cleavage of the stack, which we pretend to be a splitting for simplicity of notation. If u is an object of $\mathfrak{X}(U)$ and $V \subset U$ is an open subset, by the **restriction** $u|_V$ of u to V we will mean the object j^*u of $\mathfrak{X}(V)$, where $j : V \hookrightarrow U$ is the inclusion map. Given any morphism $f : u_1 \to u_2$ in $\mathfrak{X}(U)$, by the **restriction** $f|_V$ of f to V we will mean the morphism $j^*(f) : u_1|_V \to u_2|_V$. It can be verified that the definition below is independent of the choice of the cleavage.

Definition 3.1 A **stack** over the base category $\mathcal{S}$ of holomorphic manifolds is an $\mathcal{S}$-groupoid $(\mathfrak{X}, a)$ which satisfies the following descent property and effective descent property (with respect to some – hence, every – choice of a cleavage).

Descent property. Let u and u' be objects of $\mathfrak{X}(U)$, and let $f, g : u \to u'$ be isomorphisms. Let U_i be an open cover of U such that the restrictions $f_i, g_i : u|_{U_i} \to u'|_{U_i}$ to each U_i are equal to each other, that is, $f_i = g_i$ for each i. Then $f = g$. Moreover, if u and u' are objects of $\mathfrak{X}(U)$, if U_i is an open cover of U and if $f_i : u|_{U_i} \to u'|_{U_i}$ are isomorphisms such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for each pair of indices i, j, then there exists an isomorphism $f : u \to u'$ such that $f_i = f|_{U_i}$ for each i (such an f is unique by the previous statement).

Effective descent property. Let U_i be an open cover of U and let for each i there be given an object u_i of $\mathfrak{X}(U_i)$. For each pair of indices i, j let there be given an isomorphism $f_{i,j} : u_j|_{U_i \cap U_j} \to u_i|_{U_i \cap U_j}$ over $U_i \cap U_j$ such that when further restricted over $U_i \cap U_j \cap U_k$ we have the identity $f_{i,j} \circ f_{j,k} = f_{i,k}$. Then there exist an object u of $\mathfrak{X}(U)$ together with isomorphisms $h_i : u|_{U_i} \to u_i$ such that when restricted to $U_i \cap U_j$ we have $f_{i,j} = h_i \circ h_j^{-1}$.

Note. The descent property implies that the pair $(u, (h_i))$ is unique up to a unique isomorphism.

Remark 3.2 The role of a Grothendieck topology on the base category. Note that the above definition of a stack on the base category $\mathcal{S}$ of holomorphic manifolds made use of the notion of an open cover (U_i) of an object U of $\mathcal{S}$. The above definition will make sense for any base category $\mathcal{S}$ which is equipped with a Grothendieck topology, when (U_i) is replaced by an open cover $(U_i \to U)$ in the Grothendieck topology, and $U_i \cap U_j$ is replaced by the fibered product $U_i \times_U U_j$ with its projections to U_i and U_j.

3.3 For example, a **stack over the base category $\mathcal{S}$ of C^∞ manifolds** is defined exactly as in Definition 3.1 except that holomorphic manifolds and holomorphic maps are replaced by C^∞ manifolds and C^∞ maps, and open covers are taken in the usual **Euclidean topology**. If S is a given scheme, a **stack over the base**

category $\mathcal{S}$ of S-schemes is defined by taking the Grothendieck topology on $\mathcal{S}$ to be the **étale topology**, in which an open cover of an S-scheme U is a collection of étale morphisms $(U_i \to U)$ such that U is the union of the images of the U_i.

Example 3.4 The classifying stack BG. Let G be a Lie group. We can associate to it a stack BG over the base category of C^∞ manifolds, defined as follows. For any manifold U, let BG_U be the category whose objects are all C^∞ principal G bundles over U and whose arrows are all isomorphisms of principal bundles. Given any morphism $\varphi : V \to U$, a principal G-bundle P over U can be pulled back to define a principal G-bundle $\varphi^* P$ over V. This defines BG as a fibered category. The descent conditions are clearly satisfied.

Example 3.5 The classifying stack BG – holomorphic version. Let G be a complex Lie group. We can associate to it a stack BG over the base category of holomorphic manifolds, defined exactly as above, except that complex manifolds and holomorphic maps are replaced by C^∞ manifolds and C^∞ maps. The same notation BG is to be interpreted as C^∞ or holomorphic (or, for later use, algebraic) depending on the context.

The following material works equally well for all base categories $\mathcal{S}$ equipped with a Grothendieck topology (that is to say, all **sites** $\mathcal{S}$), such as C^∞-manifolds or holomorphic manifolds or the étale site of schemes.

3.6 Spaces as stacks. Any C^∞-manifold X can be regarded as a stack over the base category $\mathcal{S}$ of C^∞-manifold as follows. First note that to any set A we can associate a groupoid A' whose objects are elements of A and only morphisms are identities. Any set map $f : A \to B$ defines a unique morphism $f' : A' \to B'$ of groupoids, whose underlying map on objects is $f : A \to B$. This association defines fully faithful functor $\Phi : Sets \to Grpds$ from the category of sets to the category of groupoids. Now given any C^∞-manifold X, consider its 'functor of points' $h_X = Hom(-, X)$ which is a contravariant functor from the base category $\mathcal{S}$ of C^∞-manifold to $Sets$. Composing with Φ, we get a contra-functor $\Phi_X = \Phi \circ h_X : \mathcal{S} \to Grpds$. It can be verified easily (exercise) that Φ_X is a stack over $\mathcal{S}$.

A similar construction works in the holomorphic or algebraic categories, with the analogous properties.

Lemma 3.7 Yoneda Lemma for stacks *Let $\mathcal{S}$ be a site (a category equipped with a Grothendieck topology). Let $\mathfrak{X}$ be a stack over the base category $\mathcal{S}$. Let U be any object of $\mathcal{S}$, and let Φ_U be the corresponding stack over $\mathcal{S}$. Let $HOM(\Phi_U, \mathfrak{X})$ be the category whose objects are all 1-morphisms $F : \Phi_U \to \mathfrak{X}$ of $\mathcal{S}$-stacks, and whose morphisms $\alpha : F \to G$ (where $F, G : \Phi_U \to \mathfrak{X}$ are 1-morphisms of $\mathcal{S}$-groupoids) are 2-morphism $\alpha : F \Rightarrow G$ in the 2-category of $\mathcal{S}$-groupoids. We associate to $F : \Phi_U \to \mathfrak{X}$ the object $F(U)(\mathrm{id}_U)$ of $\mathfrak{X}(U)$ which is the image of the object id_U of $\Phi_U(U)$ under the functor $F(U) : \Phi_U(U) \to \mathfrak{X}(U)$. We associate to $\alpha : F \Rightarrow G$ the 1-morphism*

$\alpha(U)(\mathrm{id}_U) : F(U)(\mathrm{id}_U) \to G(U)(\mathrm{id}_U)$ in $\mathfrak{X}(U)$, where $\alpha(U) : F(U) \to G(U)$ is the natural transformation defined by α.

Then the above defines an equivalence of categories

$$HOM(\Phi_U, \mathfrak{X}) \to \mathfrak{X}(U).$$

This equivalence is functorial $\mathfrak{X}$ and contra-functorial in U.

3.8 In particular, if $\mathfrak{X} = \Phi_X$ where X is a manifold, then

$$HOM(\Phi_U, \Phi_X) = Hom_{\mathcal{S}}(U, X).$$

This is called the **weak Yoneda lemma**, and it shows that Φ is a fully faithful functor from $\mathcal{S}$ to $\mathcal{S}$-stacks. Hence we identify X with the stack Φ_X, and write the stack Φ_X simply as X.

Stackfication of an $\mathcal{S}$-groupoid

Given an $\mathcal{S}$-groupoid $\mathfrak{X}$, we can canonically associate to it a pair $(\overline{\mathfrak{X}}, i)$ consisting of a stack $\overline{\mathfrak{X}}$ over $\mathcal{S}$ and a 1-morphism of $\mathcal{S}$-groupoids $i : \mathfrak{X} \to \overline{\mathfrak{X}}$ with the following property. For any stack $\mathfrak{Y}$ over $\mathcal{S}$, the induced functor

$$i^* : HOM(\overline{\mathfrak{X}}, \mathfrak{Y}) \to HOM(\mathfrak{X}, \mathfrak{Y})$$

is an equivalence of categories.

The above can be done in two steps.

Step 1. To any $\mathcal{S}$-groupoid $\mathfrak{X}$, we canonically associate a pair $(\mathfrak{X}', i')$ consisting of a pre-stack $\mathfrak{X}'$ over $\mathcal{S}$ and a 1-morphism of $\mathcal{S}$-groupoids $i' : \mathfrak{X} \to \mathfrak{X}'$ with the following property. For any pre-stack $\mathfrak{Y}'$ over $\mathcal{S}$, the induced functor

$$(i')^* : HOM(\mathfrak{X}', \mathfrak{Y}') \to HOM(\mathfrak{X}, \mathfrak{Y}')$$

is an equivalence of categories. The pre-stack $\mathfrak{X}'$ is defined by taking $Ob(\mathfrak{X}'(U)) = Ob(\mathfrak{X}(U))$ for each object U in $\mathcal{S}$. Given any $x, y \in Ob(\mathfrak{X}'(U))$, we define the sheaf $\underline{Hom}_{\mathfrak{X}'(U)}(x, y)$ to be the sheafification of the pre-sheaf $\underline{Hom}_{\mathfrak{X}(U)}(x, y)$.

Step 2. To any pre-stack $\mathfrak{X}$ over $\mathcal{S}$, we canonically associate a pair $(\overline{\mathfrak{X}}, i)$ consisting of a stack $\overline{\mathfrak{X}}$ and a 1-morphism of $\mathcal{S}$-groupoids $i : \mathfrak{X} \to \overline{\mathfrak{X}}$ with the following property. For any stack $\mathfrak{Y}$ over $\mathcal{S}$, the induced functor

$$i^* : HOM(\overline{\mathfrak{X}}, \mathfrak{Y}) \to HOM(\mathfrak{X}, \mathfrak{Y})$$

is an equivalence of categories. Objects of $\overline{\mathfrak{X}}(U)$ are the triples $((U_i), (x_i), (g_{i,j}))$ consisting of an open cover (U_i) of U, a family of objects x_i in $\mathfrak{X}(U_i)$ and descent data $(g_{i,j})$ which consists of isomorphisms $g_{i,j} : x_j|_{U_i \cap U_j} \to x_i|_{U_i \cap U_j}$ which satisfy the co-cycle condition $g_{i,j} \circ g_{j,k} = g_{i,k}$ on $U_i \cap U_j \cap U_k$. Isomorphisms between objects of $\overline{\mathfrak{X}}(U)$ defined over two (possibly different) open covers are defined by passing to the common refinement given by intersecting the open covers. In the algebraic category (with étale topology), intersections are replaced by fiber products as usual.

3.9 Exercise Let $\mathcal{S}$ be the C^∞ base category, and let G be a Lie group. Let P be the $\mathcal{S}$-groupoid defined by taking P_U to be the groupoid which has a unique object with automorphism group $G(U)$ (the set of all C^∞ maps $U \to G$ together with point-wise multiplication). Show that the stackification of this is isomorphic to BG.

4 Lie groupoids and their quotient stacks

The material in this section equally applies to the C^∞ category and the holomorphic category. Depending on what category we choose, the term 'manifold' will mean one of these. The word 'morphism' between manifolds will mean a C^∞ map or a holomorphic map accordingly. A **submersion** is a morphism of manifolds $f : X \to Y$ such that at each $x \in X$ the tangent map $df_x : T_xX \to T_{f(x)}Y$ is surjective.

All this material also applies in the algebraic category after making suitable modifications. The main modification is to have **smooth morphisms** (see Hartshorne [AG]) in the sense of algebraic geometry (means flat of finite presentation, such that the sheaf of relative differentials is locally free of the correct rank) in place of submersions. Another important modification is to shift to étale topology whenever the implicit function theorem needs to be invoked.

4.1 If $X \to Y$ is a submersion of manifolds, then for any morphism $Z \to Y$ of manifolds, the fibered product $X \times_Y Z$ exists as a manifold, and the projection $p_2 : X \times_Y Z \to Z$ is again a submersion.

Definition 4.2 **A Lie groupoid** is a tuple $(X_0, X_1, e, s, t, m, i)$ where

(a) X_0 and X_1 are hausdorff manifolds.

(b) $e : X_0 \to X_1$, $s, t : X_1 \rightrightarrows X_0$ and $i : X_1 \to X_1$ are morphisms, such that s and t are submersions. (In particular, the fibered product $X_1 \times_{t,X_0,s} X_1$ exists as a manifold.)

(c) $m : X_1 \times_{t,X_0,s} X_1 \to X_1$ is a morphism.

We require this data to satisfies the condition that it defines a category C in which the points of X_0 are the objects of C, points of X_1 are the arrows of C, the map e attaches to each object its identity arrow, the maps s and t attach to each arrow its source object and its target object, the map m defines the composition of arrows in this category, and the map i associates to each arrow its inverse (in particular, each arrow is invertible).

Exercise 4.3 Express the above condition on the morphisms e, s, t, m, i in terms of commutativity of certain diagrams of manifolds and morphisms.

Example 4.4 Let G be a Lie group together with a right action $s : X \times G \to X$ on a manifold X. We put $X_0 = X$ and $X_1 = X \times G$. Let $t = p_1 : X \times G \to X$ be the projection. Let $e : X \to X \times G$ be the map $x \mapsto (x, e_G)$ where $e_G \in G$ is the identity element of G. Note that $X_1 \times_{t,X_0,s} X_1 = X \times G \times G$, and we define $m : X_1 \times_{t,X_0,s} X_1 \to X_1$ by $(x, g, h) \mapsto (x, gh)$. Let $i : X_1 \to X_1$ be the map $(x, g) \mapsto (xg, g^{-1})$. The above data defines a Lie groupoid.

4.5 For any $x \in X_0$, the inverse image $G_x \subset X_1$ of $(x, x) \in X_0 \times X_0$ under $(s, t) : X_1 \to X_0 \times X_0$ has the structure of a group under composition of arrows. This is called the **inertia** group of x. The **orbit** of x is the subset $O(x) \subset X_0$ which is the image of the map $t|_{s^{-1}(x)} : s^{-1}(x) \to X_0$. In categorical terms, this is the isomorphism class of x in X_0.

Example 4.6 To see how the above looks in a particularly bad case, take the Lie groupoid associated to the action of the additive Lie group $\mathbb{R}$ on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, given by $(x, y) \cdot t = (x + t, y + ct)$ where $c \in \mathbb{R}$ is irrational. The orbits are the 'winding lines' on the torus with slope c.

Proposition 4.7 *Let $X.$ be a Lie groupoid. Then for any $x \in X_0$, the subset G_x is is a closed submanifold of X_1, and this differential structure makes G_x a Lie group.*

Proof Note that as $x \in X_0$ is closed, G_x is a closed subset of X_1, and as the multiplication on G_x is the restriction to $G_x \times G_x \subset X_1 \times_{t,X_0,s} X_1$ of the map $m : X_1 \times_{t,X_0,s} X_1 \to X_1$, the multiplication is continuous. Similarly, the inverse operation on G is continuous. Hence G_x becomes a topological group.

As s is a submersion hence of maximal rank everywhere, for any $x \in X_0$ the subset $s^{-1}(x) \subset X_1$ is a closed submanifold. (Caution: This need not be connected or even equidimensional.) The topological group G_x acts on it by the continuous map $s^{-1}(x) \times G_x \to s^{-1}(x) : (a, g) \mapsto a \circ g$, which is the restriction of $m : X_1 \times_{t,X_0,s} X_1 \to X_1$ to $s^{-1}(x) \times G_x \subset X_1 \times_{t,X_0,s} X_1$. This action is C^∞ in the first factor $s^{-1}(x)$, that is, for any given $g \in G_x$ the induced map $a \mapsto a \circ g$ is a C^∞-automorphism of $s^{-1}(x)$. The map $t|_{s^{-1}(x)} : s^{-1}(x) \to X_0$ is set-theoretically the quotient map for the G_x-action on $s^{-1}(x)$, in particular, it is constant on G_x-orbits. Hence rank of $d(t|_{s^{-1}(x)})$ is constant on each fiber of $t|_{s^{-1}(x)} : s^{-1}(x) \to X_0$. Also note that the dimension of $s^{-1}(x)$ is constant along G_x-orbits.

Now let $a \in s^{-1}(x)$, such that rank of $d(t|_{s^{-1}(x)})$ is locally maximum at a. Let $y = t(a)$. Then by the above, the rank of $d(t|_{s^{-1}(x)})$ is locally maximum all along the fiber of $t|_{s^{-1}(x)}$ over y, Hence $Hom(x, y) = (s, t)^{-1}(x, y) \subset X_1$ is a closed submanifold of $s^{-1}(x)$ hence of X_1. As already noted, the dimension of $s^{-1}(x)$ is constant along each G_x-orbits. Hence $Hom(x, y)$ is a constant dimensional manifold, with a constant codimension in $s^{-1}(x)$.

If $b \in s^{-1}(x)$ with $z = t(b)$, then consider the injective map $\varphi : Hom(x, y) \to s^{-1}(x)$ defined by $f \mapsto b \circ a^{-1} \circ f$. This is a C^∞-map of manifolds, being induced by composition. Its image is the closed set $Hom(x,z)$, and the map gives a homeomorphism $\varphi' : Hom(x,y) \to Hom(x,z)$ (with inverse given by composition with $a \circ b^{-1}$). Moreover, both sides have right G_x-actions, and the above map is G_x-equivariant. As G_x acts transitively on $Hom(x,y)$, the rank of $d\varphi$ is constant. As φ is injective, $d\varphi$ is injective. Hence the image of φ, which equals $Hom(x,z)$, is a closed submanifold of $s^{-1}(x)$, diffeomorphic to $Hom(x,y)$.

This shows that each fiber $Hom(x,z)$ of $t|_{s^{-1}(x)} : s^{-1}(x) \to X_0$ is a closed submanifold of $s^{-1}(x)$. In particular, $Hom(x,x) = G_x$ is a closed submanifold of $s^{-1}(x)$ hence of X_1. As the group multiplication on G_x and the inverse on G are restrictions of certain C^∞ maps, this shows G_x with its submanifold structure and group structure is a Lie group as claimed.

Lemma 4.8 *Consider the following sequence of properties for a given morphism $f : X. \to Y$ of Lie groupoids.*

(a) The map $f_0 : X_0 \to Y_0$ is a submersion.

(b) The map $t \circ p_2 : X_0 \times_{f_0,Y_0,s} Y_1 \to Y_0$ is a submersion.

(c) For any morphism $g : Z. \to Y.$ of Lie groupoids, the fibered product

$$X_0 \times_{f_0,Y_0,s} Y_1 \times_{t,Y_0,g_0} Z_0$$

is a manifold.

(d) For any morphism $g : Z. \to Y.$ of Lie groupoids, the fibered product

$$X_1 \times_{f_0 s,Y_0,s} Y_1 \times_{t,Y_0,g_0 s} Z_1$$

is a manifold.

Then we have the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

Proof (a) $\Rightarrow$ (b) As pull-back of any submersion is a submersion, and by assumption f_0 is a submersion, its pull-back $p_2 : X_0 \times_{f_0,Y_0,s} Y_1 \to Y_1$ is a submersion. As composite of submersions is a submersion, it follows $t \circ p_2 : X_0 \times_{f_0,Y_0,s} Y_1 \to Y_0$ is a submersion.

(b) $\Rightarrow$ (c) As $t \circ p_2 : X_0 \times_{f_0,Y_0,s} Y_1 \to Y_0$ is a submersion,the fibered product

$$X_0 \times_{f_0,Y_0,s} Y_1 \times_{t,Y_0,g_0} Z_0 = (X_0 \times_{f_0,Y_0,s} Y_1) \times_{t \circ p_2,Y_0,g_0} Z_0$$

is again a manifold.

(c) $\Rightarrow$ (d) This follows from the equality

$$X_1 \times_{f_0 s,Y_0,s} Y_1 \times_{t,Y_0,g_0 s} Z_1 = (X_1 \times Z_1) \times_{s \times s, X_0 \times Z_0, p_{1,3}} (X_0 \times_{f_0,Y_0,s} Y_1 \times_{t,Y_0,g_0} Z_0)$$

The right-hand-side is a manifold as $s \times s : X_1 \times Z_1 \to X_0 \times Z_0$ is a submersion and $X_0 \times_{f_0,Y_0,s} Y_1 \times_{t,Y_0,g_0} Z_0$ is a manifold.

Definition 4.9 Let $f : X. \to Y.$ and $g : Z. \to Y.$ be morphisms of Lie groupoids. A **fibered product** of f and g is a 4-tuple $(P., F, G, \alpha)$ where $P.$ is a Lie groupoid, $F : P. \to Z.$ and $G : P. \to X.$ are 1-morphisms of Lie groupoids, and $\alpha : g \circ F \Rightarrow G \circ f$ is a 2-isomorphism, which is a universal attracting object in the 2-category formed by all such 4-tuples. This means given any other such 4-tuple $(P'., F', G', \alpha')$, there exists a 1-morphism $h : P'. \to P.$ and 2-isomorphisms $\beta : F \circ h \Rightarrow F'$ and $\gamma : G \circ h \Rightarrow G'$ such that

$$\alpha' = \gamma \bullet (\alpha \star \mathrm{id}_h) \bullet \beta^{-1}$$

and such that h is unique up to a 2-isomorphism. (Here, $\bullet$ denotes the vertical composition and $\star$ denotes the horizontal composition of 2-morphisms.)

We will denote the fibered product simply by $X. \times_{Y.} Z.$, and suppress the notations F, G, α, when there is no danger of confusion.

Remark 4.10 The above definition of fibered product in fact makes sense in any 2-category.

Proposition 4.11 *Let $f : X. \to Y.$ and $g : Z. \to Y.$ be morphisms of Lie groupoids. Suppose that the fibered product $X_0 \times_{f_0,Y_0,s} Y_1 \times_{t,Y_0,g_0} Z_0$ exists as a manifold (some sufficient conditions for this are given by Lemma 4.8 (a) and (b)). Then a fibered product Lie groupoid $X. \times_{Y.} Z.$ exists.*

Proof We define a fibered product $(P., F, G, \alpha)$ by putting

$$P_0 = X_0 \times_{f_0,Y_0,s} Y_1 \times_{t,Y_0,g_0} Z_0 \text{ and } P_1 = X_1 \times_{f_0s,Y_0,s} Y_1 \times_{t,Y_0,g_0s} Z_1.$$

By assumption, P_0 is a manifold, and by (c) $\Rightarrow$ (d) part of Lemma 4.8, this implies that P_1 is a manifold.

We define $e : P_0 \to P_1$ by $(x, b, z) \mapsto (e(x), b, e(z))$.

We define $s, t : P_1 \rightrightarrows P_0$ to be the maps

$$X_1 \times_{f_0s,Y_0,s} Y_1 \times_{t,Y_0,g_0s} Z_1 \rightrightarrows X_0 \times_{f_0,Y_0,s} Y_1 \times_{t,Y_0,g_0} Z_0$$

respectively defined by

$$(a, b, c) \mapsto (s(a), b, s(c)) \text{ and } (a, b, c) \mapsto (t(a), g_1(c) \circ b \circ f_1(a^{-1}), t(c)).$$

A point of $P_1 \times_{s,P_0,t} P_1$ therefore has the form $((a, b, c), (a', b', c'))$ where $a, a' \in X_1$, $b, b' \in Y_1$, and $c, c' \in Z_1$ with $f_0sa = sb$, $f_0sa' = sb'$, $tb = g_0sc$, $tb' = g_0sc'$ and $(s(a), b, s(c)) = (t(a'), g_1(c') \circ b' \circ f_1(a'^{-1}), t(c'))$.

We define the composition m on P by

$$(a, b, c) \circ (a', b', c') = (a \circ a', b', c \circ c').$$

We define $i : P_1 \to P_1$ by

$$(a, b, c)^{-1} = (a^{-1}, g_1(c) \circ b \circ f_1(a^{-1}), c^{-1}).$$

This data indeed defines a Lie groupoid P, as can be verified.

The morphisms $F_0 : P_0 \to Z_0$ and $G_0 : P_0 \to X_0$ are defined to be the respective projections from $P_0 = X_0 \times_{f_0,Y_0,s} Y_1 \times_{t,Y_0,g_0} Z_0$. The morphisms $G_1 : P_1 \to X_1$ and $F_1 : P_1 \to Z_1$ are defined to be the respective projections from $P_1 = X_1 \times_{f_0 s,Y_0,s} Y_1 \times_{t,Y_0,g_0 s} Z_1$ on its first and third factors.

Definition 4.12 Let $X.$ be a Lie groupoid. We associate to it a stack $[X.]$ defined as the stackification of the pre-stack P defined as follows. For any U in the base category $\mathcal{S}$, the objects of the groupoid P_U are all the morphisms $U \to X_0$, and the morphisms of P_U are all the morphisms $U \to X_1$.

4.13 Quotient manifolds. In the category of C^∞ or holomorphic manifolds, finite étale equivalence relations admit quotients. This is the assertion of the following important proposition. We have phrased it in the C^∞ case, and the analogous statement holds in the holomorphic case.

Proposition 4.14 *Let $X.$ be a Lie groupoid where X_0 and X_1 are hausdorff C^∞ manifolds, such that the following holds.*

(1) The map $(s,t) : X_1 \to X_0 \times X_0$ is a closed imbedding.

(2) The maps $s,t : X_1 \to X_0$ are proper local diffeomorphisms (means s and t are finite-sheeted covering projections).

Then the quotient stack $[X.]$ can be represented by a pair (Y,q) consisting of hausdorff C^∞ manifold Y, with the quotient map $q : X_0 \to Y$ a proper local diffeomorphism.

Proof We define the underlying topological space of Y as the quotient space of X_0 by the equivalence relation whose graph is the image of $(s,t) : X_1 \to X_0 \times X_0$, and define q to be the quotient map. We define the structure sheaf $\mathcal{O}_Y$ of C^∞ functions as follows. Given any $y \in Y$, choose any $x \in X_0$ such that $y = q(x)$. From the assumptions, x has a connected open neighbourhood $U \subset X_0$ such that

(i) U is evenly covered by s, that is, $s^{-1}(U)$ is a disjoint union $\cup_j V_j$ of open sets $V_j \subset X_1$ where each V_j maps isomorphically on to U under s,

(ii) the images $t(V_j)$ are pair-wise disjoint in X_0, and each V_j maps isomorphically on to $t(V_j)$ under t.

Note that such a U maps homeomorphically on to its image $q(U)$, which is open in Y. Moreover, such subsets $q(U)$ form a basis of open sets for Y. Now to define the sheaf $\mathcal{O}_Y$, we take a C^∞ function on $q(U)$ to be any $\phi : q(U) \to \mathbb{R}$ such that $\phi \circ q$ is a C^∞ function on U. We leave it to the reader to verify that this makes $(Y, \mathcal{O}_Y)$ a C^∞ manifold with q a C^∞ map, and the pair (Y,q) satisfies the conclusion of the theorem.

Remark 4.15 However, the corresponding proposition does not hold for schemes. The above proof fails because a finite étale map $s : X_1 \to X_0$ of schemes is not necessarily locally trivial in the Zariski topology. This is the motivation for the introduction of algebraic spaces by Artin. A famous counter-example to the above proposition in the algebraic category is due to Hironaka, where X_0 is a non-singular proper complex variety with a proper free action by $\mathbb{Z}/(2)$, and $X_1 = X_0 \times \mathbb{Z}/(2)$ (disjoint union of two copies of X_0) with s and t the projection and the action map.

Example 4.16 The classifying stack BG is isomorphic to the stack associated to the Lie groupoid $X.$ defined as follows. We take X_0 to be the one-point manifold, denoted by $*$. We take X_1 to be the underlying manifold of G. The map $e : X_0 \to X_1$ is the map $* \to G$ whose image is the identity element of G (again denoted by e). The maps $s, t : X_1 \rightrightarrows X_0$ are the constant maps, while m is the multiplication map $G \times G \to G$ and i is the inverse map $G \to G$. This groupoid is often briefly written as $G \rightrightarrows *$, and its quotient stack is written as $[*/G]$. We now define a 1-morphism $[*/G] \to BG$. For this, it is enough to define a 1-morphism of pre-stacks $F : P \to BG$ where P is the pre-stack associated to $G \rightrightarrows *$. By definition, for any U in $\mathcal{S}$, the groupoid P_U has a single object (which we denote by $*$) with automorphism group $G(U)$, the group of U-valued points of G. We associate to $*$ the trivial G-bundle $U \times G$ over U with right G-action by translation, and to any element of $G(U)$ we associate the G-equivariant automorphism (gauge transformation) of the bundle $U \times G \to U$ defined by left translation. This defines a functor $F_U : P_U \to (BG)_U$. These functors, as U varies over $\mathcal{S}$, define the 1-morphism of pre-stacks $F : P \to BG$. As BG is already a stack, it defines (uniquely up to a unique 2-isomorphism) a 1-morphism of stacks $F' : [*/G] \to BG$. In the reverse direction, for all principal G-bundles E on objects U of $\mathcal{S}$, choose once for all an open cover (U_i) of U and transition functions $(g_{i,j})$ for E. Define $BG \to [*/G]$ by sending E to be the object of $[*/G]_U$ which corresponds to the pair (f_0, f_1) consisting of the constant map $f_0 : \coprod U_i \to *$ and the map $f_1 = (g_{i,j}) : \coprod U_i \cap U_j \to G$.

Exercise 4.17 Show that the 1-morphism $F' : [*/G] \to BG$ is an isomorphism of stacks, with an inverse defined as above.

Proposition 4.18 *For any Morita equivalence $f : X. \to Y.$ the associated morphism of stacks $[f] : [X.] \to [Y.]$ is a 1-isomorphism.*

Proposition 4.19 *Let $X.$ and $Y.$ be Lie groupoids, and let $F : [X.] \to [Y.]$ be a 1-morphism between their quotient stacks. Then there exists a Morita equivalence $f : Z. \to X.$ and a morphism $g : Z. \to Y.$ such that the composite $[g] \circ [f]^{-1} : [X.] \to [Y.]$ is 2-equivalent to $F : [X.] \to [Y.]$.*

Example 4.20 Let S^1 be the unit circle in $\mathbb{C}$, with complex coordinate z. Let $\lambda \in S^1$ be a point of infinite order (that is, λ is not a root of unity – such a λ exists as S^1 is uncountable while points of order n form a subset of S^1 of cardinality n). Let $\mathbb{Z}$ act on S^1 by multiplication by λ. This defines a groupoid, with $X_0 = S^1$, $X_1 = S^1 \times \mathbb{Z}$, $s : X_1 \to X_0$ the map $(z, n) \mapsto z$, and $t : X_1 \to X_0$ the map

$(z,n) \mapsto z\lambda^n$. Note that both s and t are étale (in fact, they are covering projections), so we have an étale groupoid. As the automorphism group $(s,t)^{-1}(x,x)$ of any point $x \in X_0$ is trivial, the quotient stack $\mathfrak{X}$ is equivalent to a sheaf of sets on the base category $\mathcal{S}$. But $\mathfrak{X}$ cannot be represented by a differential manifold (exercise).

Example 4.21 The group $\mathbb{Z}$ acts by translation on the affine line $\operatorname{Spec}\mathbb{C}[x]$. The quotient stack of the corresponding Lie groupoid in the algebraic category of complex schemes cannot be represented by a complex scheme or algebraic space (exercise).

5 Differential stacks and analytic stacks

Let $\mathcal{S}$ be either the category of C^∞ manifolds or the category of holomorphic manifolds. The material in this section is valid in either case.

Any manifold X (that is, an object of $\mathcal{S}$) defines a stack over the base category $\mathcal{S}$, as follows. For any U in $\mathcal{S}$, we take the set of objects of $X(U)$ to be the set of all morphisms $Hom_{\mathcal{S}}(U,X)$. In algebro-geometric terms, objects of $X(U)$ are the U-valued points of X. We make this into a category, in which the only morphisms are the identities of the objects. Given any $\varphi : V \to U$, an object $x : U \to X$ of $X(U)$ pulls back to the object $x \circ \varphi : V \to X$ of $X(V)$. This defines X as a fibered category in groupoids over the base $\mathcal{S}$. The descent and effective descent conditions are clearly satisfied, so X is indeed a stack over $\mathcal{S}$.

Exercise 5.1 Show that the above defines a functor from $\mathcal{S}$ into the category of stacks over $\mathcal{S}$, and this functor is fully faithful. Moreover, for any stack $\mathfrak{X}$ and a manifold U, a 1-morphism of stacks $x : U \to \mathfrak{X}$ is the same as an object $x \in \mathfrak{X}_U$, and pull-backs correspond to composites.

Definition 5.2 A stack $\mathfrak{X}$ over $\mathcal{S}$ is called a **representable stack** if there exists a manifold X and an isomorphism of stacks $F : X \to \mathfrak{X}$. By the above exercise, such a pair (X,F), if it exists, is unique up to unique isomorphism.

Definition 5.3 Let $F : \mathfrak{X} \to \mathfrak{Y}$ be a 1-morphism of stacks over the base category $\mathcal{S}$. We say that F is a **representable submersion** (respectively, a **surjective representable submersion**) if the following condition is satisfied. For each manifold U and morphism $\phi : U \to \mathfrak{Y}$, the fibered product of stacks $\mathfrak{X} \times_{F,\mathfrak{Y},\phi} U$ is representable by a manifold and under any such a representation, the projection $\mathfrak{X} \times_{F,\mathfrak{Y},\phi} U \to U$ is a submersion of manifolds (respectively, a surjective submersion of manifolds).

Definition 5.4 Let $F : \mathfrak{X} \to \mathfrak{Y}$ be a 1-morphism of stacks over the base category $\mathcal{S}$. We say that F is a **representable 1-morphism** if the following condition is satisfied. For each manifold U and a representable submersion $\phi : U \to \mathfrak{Y}$ (as defined in Definition 5.3), the fibered product of stacks $\mathfrak{X} \times_{F,\mathfrak{Y},\phi} U$ is representable by a manifold.

5.5 Note in particular that any representable submersion $\phi : U \to \mathfrak{Y}$ as defined in Definition 5.3 is also a representable 1-morphism as defined in Definition 5.4.

5.6 Caution! The definition of a representable 1-morphism given above, where we work in the C^∞ or holomorphic categories, is weaker than the notion of a representable 1-morphism in the algebraic category. In the algebraic category, we call a 1-morphism $F : \mathfrak{X} \to \mathfrak{Y}$ if for an *arbitrary* 1-morphism $\phi : U \to \mathfrak{Y}$ where U is in $\mathcal{S} = Aff/S$, the fibered product of stacks $\mathfrak{X} \times_{F,\mathfrak{Y},\phi} U$ is representable by an algebraic space. The difference comes from the fact that arbitrary fibered products do not exist in the category of C^∞ or holomorphic manifolds.

5.7 In the C^∞ or holomorphic categories, any 1-morphism $\phi : U \to \mathfrak{Y}$, where U is in $\mathcal{S}$, is necessarily representable. The situation is quite different in the algebraic category.

Definition 5.8 Let P stand for any property of a morphism in $\mathcal{S}$ which is invariant under a submersive base-change. (The properties of being a surjection or being submersion or being proper are important such examples to keep in mind.) A representable 1-morphism $F : \mathfrak{X} \to \mathfrak{Y}$ of stacks over base $\mathcal{S}$ is said to have the property P if for each manifold U and representable submersion $\phi : U \to \mathfrak{Y}$, the projection morphism $p_2 : \mathfrak{X} \times_{F,\mathfrak{Y},\phi} U \to U$ (which is a morphism in $\mathcal{S}$ as by assumption F is a representable) has the property P.

Definition 5.9 A stack $\mathfrak{X}$ over the base category $\mathcal{S}$ of C^∞ manifolds (respectively, over the the base category $\mathcal{S}$ of holomorphic manifolds) is called a **differential stack** (respectively, a **holomorphic stack**) if there exists a manifold X in $\mathcal{S}$ and a surjective representable submersion $x : X \to \mathfrak{X}$ as defined in Definition 5.3. Any such surjective representable submersion $x : X \to \mathfrak{X}$ is called an **atlas** for $\mathfrak{X}$, or a **versal family**, or a **presentation** of $\mathfrak{X}$.

Lemma 5.10 *Let $F : \mathfrak{X} \to Y$ be a 1-morphism of stacks, where Y is a manifold. Suppose that $\phi : Z \to Y$ is a surjective submersion of manifolds, such that the fibered product stack $P = \mathfrak{X} \times_{F,Y,\phi} Z$ is representable by a manifold. Then the stack $\mathfrak{X}$ is a manifold.*

Proof As $\phi : Z \to Y$ is a surjective submersion, it admits local sections by the implicit function theorem. Hence there is a manifold W with a surjective étale map $\psi : W \to Y$ which factors as $W \to Z \to Y$. The further base change $P \times_Z W$ is representable. Now $\mathfrak{X}$ is obtained by effective descent under $W \to Y$.

As a consequence of this lemma, we have the following.

Proposition 5.11 *Let $\mathfrak{Y}$ be a differential stack with an atlas $y : Y \to \mathfrak{Y}$. Let $F : \mathfrak{X} \to \mathfrak{Y}$ be any 1-morphism. If the pull-back $\mathfrak{X} \times_{F,\mathfrak{Y},y} Y$ is a manifold, then F is a representable 1-morphism.*

5.12 The diagonal 1-morphism $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ of any differential stack $\mathfrak{X}$ is a representable 1-morphism. To see this, let $x : X \to \mathfrak{X}$ be an atlas for a stack $\mathfrak{X}$. We have an isomorphism of stacks

$$\mathfrak{X} \times_{\Delta,\mathfrak{X}\times\mathfrak{X},(x,x)} (X \times X) \simeq X \times_{x,\mathfrak{X},x} X,$$

while $X \times_{x,\mathfrak{X},x} X$ is a manifold. Now the desired conclusion follows as a corollary to Proposition 5.11.

5.13 In constant, the representability of the diagonal has to be made as an explicit assumption in the definition of an algebraic stack.

Proposition 5.14 *A stack over the base category $\mathcal{S}$ of C^∞ manifolds (respectively, over the the base category $\mathcal{S}$ of holomorphic manifolds) is a differential stack (respectively, a holomorphic stack) if and only if it is isomorphic to the quotient stack of a Lie groupoid in $\mathcal{S}$.*

Proof If $X.$ is a Lie groupoid, then the quotient map $X_0 \to [X.]$ is an atlas for the quotient stack. Conversely, if $x : X \to \mathfrak{X}$ is an atlas for a stack $\mathfrak{X}$, let $X_0 = X$, let $X_1 = X_0 \times_{\mathfrak{X}} X_0$, and let $s, t : X_1 \rightrightarrows X_0$ be the two projections. Let $e : X_0 \to X_1$ be the diagonal map, let $i : X_1 \to X_1$ interchange the two factors in $X_0 \times_{\mathfrak{X}} X_0$. We leave the definition of $\mu : X_1 \times_{s,X_0,t} X_1 \to X_1$ to the reader. Then $X. = (X_0, X_1, e, s, t, \mu, i)$ is a Lie groupoid, and $x : X_0 \to \mathfrak{X}$ is isomorphic to its quotient stack.

5.15 A differential stack $\mathfrak{X}$ is said to be **hausdorff** (or **separated**) if it satisfies any of the following three equivalent conditions.

(1) The diagonal 1-morphism $\Delta : \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is proper.

(2) There exists a versal family $X_0 \to \mathfrak{X}$ for which the map $(s, t) : X_1 \to X_0 \times X_0$ is proper (where $X_1 = X_0 \times_{\mathfrak{X}} X_0$, with projections s and t).

(3) For any versal family the map $(s, t) : X_1 \to X_0 \times X_0$ is proper.

The equivalence is left as as exercise, using the following cartesian diagram.

$$\begin{array}{ccc} \mathfrak{X} & \leftarrow & X_1 \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle(s,t)} \\ \mathfrak{X} \times \mathfrak{X} & \leftarrow & X_0 \times X_0 \end{array}$$

6 Sheaves and cohomology

General theory over arbitrary sites

6.1 A **pre-topology** T **on a category** $\mathcal{C}$ consists of a class T_U of indexed sets $\mathcal{U}$ of morphisms $\mathcal{U} = (u_i : U_i \to U)_{i\in I}$ for each object U of $\mathcal{C}$ such that the following conditions are satisfied.

(1) If U is any object and $f : V \to U$ is an isomorphism, then (f) is in T_U.

(2) If $\mathcal{U} = (u_i : U_i \to U)_{i\in I}$ is in T_U and if for each i, if $\mathcal{V}_i = (v_{i,j} : V_{i,j} \to U_i)_{J_i}$ is in T_{U_i}, then the indexed set of all composites $(u_i \circ v_{i,j} : V_{i,j} \to U)_K$, where $K = \coprod_I J_i$, is in T_U.

(3) If $\mathcal{U} = (u_i : U_i \to U)_{i\in I}$ is in T_U and if $f : V \to U$ is any morphism in $\mathcal{C}$, then the fibered products $V \times_{f,U,u_i} U_i$ exist, and the set of indexed set of all first projections $(p_i : V \times_{f,U,u_i} U_i \to V)_{i\in I}$ is in T_V.

A **site** consists of a category $\mathcal{C}$ together with a pre-topology T on it.

6.2 If $(\mathcal{C}, T)$ and $(\mathcal{C}', T')$ are sites, a **continuous functor** $f^{-1} : (\mathcal{C}, T) \to (\mathcal{C}', T')$ is a functor $f^{-1} : \mathcal{C} \to \mathcal{C}'$ such that for any T-cover $\mathcal{U} = (u_i : U_i \to U)_{i\in I}$ of an object U in $\mathcal{C}$, $f^{-1}\mathcal{U} = (f^{-1}u_i : f^{-1}U_i \to f^{-1}U)_{i\in I}$ is a T'-cover of $f^{-1}U$ in $\mathcal{C}'$.

Remark on notation: The notation $f^{-1} : (\mathcal{C}, T) \to (\mathcal{C}', T')$ for a continuous functor between sites involves an inverse for the following reason. In the simplest example where we have a continuous map $f : X' \to X$ between topological spaces, we get a functor $f^{-1} : Open(X) \to Open(X')$ – *in the opposite direction to that of* f – between their categories of open sets, sending any object $U \subset X$ of $Open(X)$ to the object $f^{-1}(U) \subset X'$ of $Open(X')$.

6.3 A composite of continuous functors is continuous. Thus, we get a category whose objects are sites and morphisms are continuous functors.

Sheaves on differential stacks

6.4 Let $\mathfrak{X}$ be an algebraic (or differential or ...) stack. We associate a site $\mathfrak{X}_{lis\text{-}et}$ with $\mathfrak{X}$ called the 'lisse-étale site' of $\mathfrak{X}$. The underlying category of the site $\mathfrak{X}_{lis\text{-}et}$ has as objects all pairs (U, u) where U is in the base category $\mathcal{S}$ and $u : U \to \mathfrak{X}$ is a smooth 1-morphisms (respectively, submersive 1-morphisms in the holomorphic or differential category). The morphisms $(U, u) \to (V, v)$ in $\mathfrak{X}_{lis\text{-}et}$ are pairs (f, α) consisting of a morphism $f : U \to V$ together with a 2-morphism $\alpha : u \Rightarrow v\circ f$. (Note that the morphisms $f : U \to V$ are *not* required to be smooth, only the morphisms $u : U \to \mathfrak{X}$ are required to be smooth.) Next we put a topology on this category. For any object $u : U \to \mathfrak{X}$ in $\mathfrak{X}_{lis\text{-}et}$, an open cover in the lisse-étale topology is a family of morphisms $(f_i, \alpha_i) : (U_i, u_i) \to (U, u)$ in $\mathfrak{X}_{lis\text{-}et}$, such that the family of the underlying morphisms $f_i : U_i \to U$ is an étale open cover of U (respectively, an open cover in the Euclidean topology in the holomorphic or differential category).

Remark 6.5 Note that we can regard $\mathfrak{X}$ as a category, which has a functor to the base category $\mathcal{S}$ of manifolds. Any morphism $u : U \to \mathfrak{X}$ is an object of this category (without the condition that u is lisse). In these terms, a morphism $(f, \alpha) : (U, u) \to (V, v)$ in $\mathfrak{X}_{lis\text{-}et}$ is exactly the same as a morphism in the category $\mathfrak{X}$. Thus, the underlying category of the site $\mathfrak{X}_{lis\text{-}et}$ is a strictly full subcategory of $\mathfrak{X}$.

6.6 Direct product in $\mathfrak{X}_{lis\text{-}et}$. Let (U, u), and (V, v) be objects of $\mathfrak{X}_{lis\text{-}et}$. We show that their direct product $(U, u) \times (V, v)$ exists in $\mathfrak{X}_{lis\text{-}et}$. Let

$$(U \times_{u,\mathfrak{X},v} V,\ \overline{v} : U \times_{u,\mathfrak{X},v} V \to U,\ \overline{u} : U \times_{u,\mathfrak{X},v} V \to V,\ \alpha : v \circ \overline{u} \Rightarrow u \circ \overline{v})$$

be the fibered product of $u : U \to \mathfrak{X}$ and $v : V \to \mathfrak{X}$ as stacks. As by assumption u is lisse, its base-change $\overline{u}$ is lisse. As by assumption v is lisse, the composite $v \circ \overline{u}$ is lisse, which shows that

$$(U \times_{u,\mathfrak{X},v} V, v \circ \overline{u}) \in Ob(\mathfrak{X}_{lis\text{-}et}).$$

We have the following morphisms in $\mathfrak{X}_{lis\text{-}et}$:

$$(\overline{v}, \alpha) : (U \times_{u,\mathfrak{X},v} V, v \circ \overline{u}) \to (U, u) \text{ and } (\overline{u}, 1_{v\circ\overline{u}}) : (U \times_{u,\mathfrak{X},v} V, v \circ \overline{u}) \to (V, v).$$

It is clear that the resulting triple

$$((U \times_{u,\mathfrak{X},v} V, v \circ \overline{u}),\ (\overline{v}, \alpha),\ (\overline{u}, 1_{v\circ\overline{u}}))$$

is the direct product of (U, u) and (V, v) in $\mathfrak{X}_{lis\text{-}et}$.

6.7 Fibered product in $\mathfrak{X}_{lis\text{-}et}$ when the first morphism is a submersion. Let (U, u), (V, v) and (W, w) be objects of $\mathfrak{X}_{lis\text{-}et}$, and let $(f, \alpha) : (V, v) \to (U, u)$ and $(g, \beta) : (W, w) \to (U, u)$ be morphisms in $\mathfrak{X}_{lis\text{-}et}$, such that $f : V \to U$ is a submersion. Then the fibered product $V \times_U W$ exists as a manifold, with projections $\overline{g} : V \times_U W \to V$ and $\overline{f} : V \times_U W \to W$. The morphism $\overline{f} : V \times_U W \to W$ is again a submersion, being base-change of f. As by definition of $\mathfrak{X}_{lis\text{-}et}$ the morphism $w : W \to \mathfrak{X}$ is a submersion, the composite $w \circ \overline{f} : V \times_U W \to \mathfrak{X}$ is again a submersion, which shows that

$$(V \times_U W, w \circ \overline{f}) \in Ob(\mathfrak{X}_{lis\text{-}et}).$$

We have 2-morphisms

$$\alpha * 1_{\overline{g}} \,:\, v \circ \overline{g} \,\Rightarrow\, u \circ f \circ \overline{g} \,=\, u \circ g \circ \overline{f}$$

and

$$\beta * 1_{\overline{f}} \,:\, w \circ \overline{f} \,\Rightarrow\, u \circ g \circ \overline{f}$$

where $*$ denotes the horizontal composition of 2-morphisms. As all 2-morphisms are invertible, this defines a 2-morphism

$$\gamma = (\beta * 1_{\overline{f}}) \cdot (\alpha^{-1} * 1_{\overline{g}}) \,:\, w \circ \overline{f} \,\Rightarrow\, v \circ \overline{g}$$

where $\cdot$ denotes the vertical composition of 2-morphisms. We thus have the following morphisms in $\mathfrak{X}_{lis\text{-}et}$:

$$(\overline{g}, \gamma) : (V \times_U W, w \circ \overline{f}) \to (V, v) \text{ and } (\overline{f}, 1_{w \circ \overline{f}}) : (V \times_U W, w \circ \overline{f}) \to (W, w).$$

It is clear that the resulting triple

$$((V \times_U W, w \circ \overline{f}), (\overline{g}, \gamma), (\overline{f}, 1_{w \circ \overline{f}}))$$

is the fibered product of $(f, \alpha) : (V, v) \to (U, u)$ and $(g, \beta) : (W, w) \to (U, u)$ in $\mathfrak{X}_{lis\text{-}et}$.

6.8 CAUTION: $\mathfrak{X}_{lis\text{-}et}$ lacks fibered products in general. The category of manifolds does not admit fiber products in general, so if $\mathfrak{X}$ is a differential stack, the category $\mathfrak{X}$ does not admit products or fiber products. As algebraic spaces admit fiber products, if $\mathfrak{X}$ is an algebraic stack, the category $\mathfrak{X}$ admits products as well as fiber products. The underlying category of $\mathfrak{X}_{lis\text{-}et}$ admits products in both differential and algebraic case, but **does not** admit fiber products in either of them.

Fortunately, this does not affect the definition of the site $\mathfrak{X}_{lis\text{-}et}$, for one of the conditions defining a site in SGA4 is that the pull-back of an open cover of an object should exist and be an open cover. In the case of the lisse-étale site, as the open covers are étale, the requisite fibered products do exist by 6.7. (The older Artin 1962 seminar notes required the underlying category to have arbitrary fibered products, which was excessive.)

The unfortunate consequence of the lack of arbitrary fibered products is that given a 1-morphism $F : \mathfrak{X} \to \mathfrak{Y}$ of algebraic or differential stacks, as $\mathfrak{Y}_{lis\text{-}et}$ does not have fibered products, the inverse image functor on sheaves (of sets or ...) $F^{-1} : Shv(\mathfrak{Y}_{lis\text{-}et}) \to Shv(\mathfrak{X}_{lis\text{-}et})$ is not in general exact (but is only right-exact) so we do not necessarily get a geometric morphism of topoi $(F^{-1}, F_*) : Shv(\mathfrak{X}_{lis\text{-}et}) \to Shv(\mathfrak{Y}_{lis\text{-}et})$.

So, in particular, we cannot conclude that on sheaves of abelian groups, the direct image functor functor F_* preserves injectives. This adversely affects the theory of derived functors RF_*.

Definition 6.9 Let $\mathfrak{X}$ be a stack. A **pre-sheaf** $\mathcal{P}$ on $\mathfrak{X}$ in the lisse-étale topology (that is, a pre-sheaf on the site $\mathfrak{X}_{lis\text{-}et}$) is a contra-functor from $\mathfrak{X}_{lis\text{-}et}$ to sets (or abelian groups, or ...). This means for each lisse $u : U \to \mathfrak{X}$ we are given an $\mathcal{P}(U, u)$, and for each $(f, \alpha) : (U, u) \to (V, v)$ we are given a restriction map $r_{(f,\alpha)} : \mathcal{P}(V, v) \to \mathcal{P}(U, u)$, which respects compositions. In particular, note that for any lisse $u : U \to \mathfrak{X}$, the group $Aut(u)$ (where u is regarded as an object of the groupoid $\mathfrak{X}_U$) acts on $\mathcal{P}(U, u)$. A pre-sheaf $\mathcal{F}$ is a **sheaf** if given any object (U, u) of $\mathfrak{X}_{lis\text{-}et}$ and its étale open cover $((f_i, \alpha_i) : (U_i, u_i) \to (U, u))_{i \in I}$, the following diagram is exact

$$\mathcal{F}(U, u) \to \prod_{i \in I} \mathcal{F}(U_i, u_i) \rightrightarrows \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \times_U U_j, u_j \circ \overline{f_i})$$

where the map $\mathcal{F}(U,u) \to \prod_{i\in I} \mathcal{F}(U_i,u_i)$ is induced by the restriction maps $r_{(f_i,\alpha_i)}$: $\mathcal{F}(U,u) \to \mathcal{F}(U_i,u_i)$, while the two maps $\prod_{i\in I} \mathcal{F}(U_i,u_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \times_U U_j, u_j \circ \overline{f_i})$ are induced by the restriction maps $r_{(\overline{f_j},\alpha_{i,j})} : \mathcal{F}(U_i,u_i) \to \mathcal{F}(U_i \times_U U_j, u_j \circ \overline{f_i})$ and $r_{(\overline{f_i},1)} : \mathcal{F}(U_j,u_j) \to \mathcal{F}(U_i \times_U U_j, u_j \circ \overline{f_i})$. In the above, $\alpha_{i,j} : u_j \circ \overline{f_i} \Rightarrow u_i \circ \overline{f_j}$ is part of the projection morphism $(\overline{f_j}, \alpha_{i,j}) : (U_i \times_U U_j, u_j \circ \overline{f_i}) \to (U_i,u_i)$ in $X_{lis\text{-}et}$, and is induced by α_i and α_j as described in detail above in Statement 6.7.

Definition 6.10 The 0 **th Cech cohomology set of a presheaf.** If $\mathcal{P}$ is a presheaf of sets (or groups or ...) on $\mathfrak{X}_{lis\text{-}et}$, (U,u) is an object of $\mathfrak{X}_{lis\text{-}et}$, and $\mathcal{U} = ((f_i,\alpha_i) : (U_i,u_i) \to (U,u))_{i\in I}$ is an étale open cover of (U,u), then the 0 th Cech cohomology set (or group or ...) of $\mathcal{P}$ on U w.r.t. $\mathcal{U}$ is a set $\check{H}^0(\mathcal{U},\mathcal{P})$ together with an injective map $\check{H}^0(\mathcal{U},\mathcal{P}) \to \mathcal{P}(U,u)$ which makes the following diagram exact.

$$\check{H}^0(\mathcal{U},\mathcal{P}) \to \prod_{i\in I} \mathcal{P}(U_i,u_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathcal{P}(U_i \times_U U_j, u_j \circ \overline{f_i})$$

6.11 Question If X is a manifold, then what is the relationship between the usual sheaves on X and sheaves on the site $X_{lis\text{-}et}$? How to recover the former from the latter? **Answer** A usual sheaf F on X give rise to sheaf $\mathcal{F}$ on $X_{lis\text{-}et}$ by putting $\mathcal{F}(U,u)$ to be $F(u(U))$ (note that $u(U)$ is open in X). The sheaves on $X_{lis\text{-}et}$ which arise this way are exactly the Cartesian sheaves.

Definition 6.12 Given a sheaf of sets $\mathcal{F}$ on $\mathfrak{X}_{lis\text{-}et}$, and a lisse $u : U \to \mathfrak{X}$, we get a sheaf on U in the étale topology, which we denote by $\mathcal{F}_{U,u}$. A sheaf $\mathcal{F}$ on $\mathfrak{X}_{lis\text{-}et}$ is called a **Cartesian sheaf** if for any morphism $(f,\alpha) : (U,u) \to (V,v)$ in $\mathfrak{X}_{lis\text{-}et}$, the restriction maps induce an isomorphism of sheaves $f^{-1}\mathcal{F}_{V,v} \to \mathcal{F}_{U,u}$ on U.

6.13 We have the sheaf of rings $\mathcal{O}_{\mathfrak{X}}$ defined by putting $\mathcal{O}_{\mathfrak{X}}(U,u) = \mathcal{O}(U)$ the ring of global regular functions on U. We can now define sheaves of $\mathcal{O}$-modules in the usual way. We will say that a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on $\mathfrak{X}_{lis\text{-}et}$ is a **Cartesian sheaf of $\mathcal{O}$-modules** if for any morphism $(f,\alpha) : (U,u) \to (V,v)$ in $\mathfrak{X}_{lis\text{-}et}$, the restriction maps induce an isomorphism of sheaves of $\mathcal{O}_U$-modules $f^*\mathcal{F}_{V,v} \to \mathcal{F}_{U,u}$ on U, where we use the notation $f^*\mathcal{F}_{V,v} = \mathcal{O}_U \otimes_{f^{-1}\mathcal{O}_V} f^{-1}\mathcal{F}_{V,v}$ as in algebraic geometry.

Example 6.14 The sheaf of $\mathcal{O}$-modules Ω^r on $\mathfrak{X}_{lis\text{-}et}$ is defined by $(U,u) \mapsto \Omega^r(U)$, the $\mathcal{O}(U)$-module of r-forms on U. If $r \geq 1$ then this is *not* a Cartesian sheaf of $\mathcal{O}$-modules. Only $\Omega^0 = \mathcal{O}$ is a Cartesian sheaf of $\mathcal{O}$-modules.

6.15 Cartesian sheaves via descent from an atlas We can ask whether any sheaf $\mathcal{F}$ on $\mathfrak{X}_{lis\text{-}et}$ admits a descent-theoretic description in terms of an atlas (X,x). The obvious candidate for such a description is a pair (F,φ) where F is a sheaf on X, and $\varphi : s^{-1}F \to t^{-1}F$ is an isomorphism over $X_1 = X \times_{s,\mathfrak{X},t} X$ which satisfies the cocycle condition over $X_2 = X \times_{s,\mathfrak{X},t} X \times_{s,\mathfrak{X},t} X$. This works for $\mathcal{F}$ if and only if $\mathcal{F}$ is a Cartesian sheaf on $\mathfrak{X}_{lis\text{-}et}$.

Lemma 6.16 *Let $\mathcal{F}$ be a sheaf on $\mathfrak{X}_{lis\text{-}et}$, and let $(f, \alpha) : (V, v) \to (U, u)$ be a morphism in $\mathfrak{X}_{lis\text{-}et}$ such that $f : V \to U$ is a surjective submersion (not necessarily étale). Then the following diagram of sets is exact.*

$$\mathcal{F}(U, u) \stackrel{r_{(f,\alpha)}}{\to} \mathcal{F}(V, v) \rightrightarrows \mathcal{F}(V \times_U V, v \circ p_2)$$

Proof. If $f : V \to U$ is a surjective étale map, then by definition of a sheaf in $\mathfrak{X}_{lis\text{-}et}$-topology, the above sequence is exact. More generally, if $f : V \to U$ is a surjective submersion, then there exists an étale cover $f' : V' \to U$ which factors via f. Let $v' = u \circ f'$, so that we have the 'commutative' diagram

$$\begin{array}{ccccc}
\mathcal{F}(U, u) & \to & \mathcal{F}(V, v) & \rightrightarrows & \mathcal{F}(V \times_U V, v \circ p_2) \\
 & & \downarrow & & \downarrow \\
\mathcal{F}(U, u) & \to & \mathcal{F}(V', v') & \rightrightarrows & \mathcal{F}(V' \times_U V', v' \circ p_2)
\end{array}$$

in which the bottom row is exact. Moreover, in the top row the two compositions coincide. It follows by diagram chasing that the top row is also exact. This proves the assertion.

Remark 6.17 The above shows that we will get the same sheaves on the lisse-lisse site $\mathfrak{X}_{lis\text{-}lis}$ (which is defined by taking open covers to be submersions, not necessarily étale) as on the lisse-étale site $\mathfrak{X}_{lis\text{-}et}$.

Global sections of sheaves

Definition 6.18 Global sections as inverse limit. If $\mathcal{F}$ is a (pre-)sheaf on a site $(\mathcal{C}, T)$, we get an inverse system of sets $\mathcal{F}(x)$ where x varies over objects of $\mathcal{C}$, and restriction maps $r_f : \mathcal{F}(y) \to \mathcal{F}(x)$ where $f : x \to y$ varies over morphisms of $\mathcal{C}$. The set of **global sections** of $\mathcal{F}$ is by definition the inverse limit

$$\Gamma(\mathcal{C}, \mathcal{F}) = \lim_{\leftarrow} (\mathcal{F}(x), (r(f)).$$

Being the inverse limit, note that $\Gamma(\mathcal{C}, \mathcal{F})$ comes equipped with restriction maps $\rho_{x,\mathcal{F}} : \Gamma(\mathcal{C}, \mathcal{F}) \to \mathcal{F}(x)$, which commute with the restriction maps $r_f : \mathcal{F}(y) \to \mathcal{F}(x)$. Note that the above definition depends just on the pre-sheaf $\mathcal{F}$ and is independent of the Grothendieck topology T.

6.19 If the site has a final object X, then the set of global sections of $\mathcal{F}$ is just $\mathcal{F}(X)$, with the usual restriction maps making it the inverse limit.

6.20 Let $(\mathcal{C}, T)$ be a site. If S is any set (or group, or ...) then the constant pre-sheaf $S_{\mathcal{C}}^{presh}$ on $\mathcal{C}$ is the constant contra-functor S on $\mathcal{C}$. The constant sheaf $S_{(\mathcal{C},T)}$ on $\mathcal{C}$ is its sheafification $a(S_{\mathcal{C}}^{presh})$.

Lemma 6.21 *Let $(\mathcal{C}, T)$ be a site, let $\mathbb{Z}_{\mathcal{C}}^{presh}$ and $\mathbb{Z}_{(\mathcal{C},T)}$ be the constant pre-sheaf and the constant sheaf $\mathbb{Z}$ on it. For pre-sheaf $\mathcal{P}$ of abelian groups on $\mathcal{C}$ and for any sheaf $\mathcal{F}$ of abelian groups on $(\mathcal{C}, T)$ and any object U in $\mathcal{C}$, let*

$$r_{\mathcal{P},U}^{presh} : Hom_{Presh(\mathcal{C})}(\mathbb{Z}_{\mathcal{C}}^{presh}, \mathcal{P}) \to \mathcal{P}(U)$$

and

$$r_{\mathcal{F},U} : Hom_{Sh(\mathcal{C},T)}(\mathbb{Z}_{\mathcal{C},T}, \mathcal{F}) \to \mathcal{F}(U)$$

be the maps which send a homomorphism to its evaluation on the section 1. *Then for any pre-sheaf $\mathcal{P}$ of abelian groups on $\mathcal{C}$ and for any sheaf $\mathcal{F}$ of abelian groups on $(\mathcal{C}, T)$, we have functorial isomorphisms*

$$\phi_{\mathcal{P}}^{presh} : Hom_{Presh(\mathcal{C})}(\mathbb{Z}_{\mathcal{C}}^{presh}, \mathcal{P}) \to \Gamma(\mathcal{C}, \mathcal{P})$$

and

$$\phi_{\mathcal{F}} : Hom_{Sh(\mathcal{C},T)}(\mathbb{Z}_{\mathcal{C},T}, \mathcal{F}) \to \Gamma(\mathcal{C}, \mathcal{F})$$

as inverse limits (that is, the maps r go to the corresponding maps ρ of statement 6.18).

6.22 Global sections over $\mathfrak{X}_{lis\text{-}et}$ as the 0 th Cech cohomology for an atlas. We next describe global sections of a sheaf $\mathcal{F}$ on $\mathfrak{X}_{lis\text{-}et}$ in terms of the 0 th Cech cohomology $\check{H}^0((x : X \to \mathfrak{X}), \mathcal{F})$ for an atlas $x : X \to \mathfrak{X}$ (here X is a manifold and x a surjective submersion). Note that this description works only for sheaves, not for pre-sheaves.

Let $X_1 = X \times_{\mathfrak{X}} X$ be the fibered product, which comes with projections $s, t : X_1 \rightrightarrows X_0$ and a 2-arrow $\alpha : x \circ s \Rightarrow x \circ t$. Note that (X, x) and $(X_1, x \circ s)$ are objects of the site $\mathfrak{X}_{lis\text{-}et}$. Consider the $\mathfrak{X}_{lis\text{-}et}$-morphisms

$$(s, \mathbf{1}_{x \circ s}) : (X_1, x \circ s) \to (X, x) \text{ and } (t, \alpha) : (X_1, x \circ s) \to (X, x)$$

where $\mathbf{1}_{x \circ s}$ denotes the identity 2-arrow of the 1-arrow $x \circ s$. Hence for any sheaf $\mathcal{F}$ on $\mathfrak{X}_{lis\text{-}et}$, we have the corresponding restriction maps

$$r_{(s, \mathbf{1}_{x \circ s})} : \mathcal{F}(X, x) \to \mathcal{F}(X_1, x \circ s) \text{ and } r_{(t,\alpha)} : \mathcal{F}(X, x) \to \mathcal{F}(X_1, x \circ s).$$

Let $\check{H}^0((x : X \to \mathfrak{X}), \mathcal{F})$, together with an injection $\rho_{(X,x)} : \check{H}^0((x : X \to \mathfrak{X}), \mathcal{F}) \to \mathcal{F}(X)$, be the equalizer for the above two restriction maps, so that we have an exact diagram of sets

$$\check{H}^0((x : X \to \mathfrak{X}), \mathcal{F}) \xrightarrow{\rho_{(X,x)}} \mathcal{F}(X, x) \rightrightarrows \mathcal{F}(X_1, x \circ s).$$

6.23 Refinement of atlas. With notation $\mathfrak{X}$, $\mathcal{F}$ and $x : X \to \mathfrak{X}$ as above, let $f : Y \to X$ be a surjective submersion of manifolds. Let $y = x \circ f : Y \to \mathfrak{X}$, and consider the corresponding exact diagram of sets

$$\check{H}^0((y : Y \to \mathfrak{X}), \mathcal{F}) \overset{\rho_{(Y,y)}}{\to} \mathcal{F}(Y, y) \rightrightarrows \mathcal{F}(Y_1, y \circ s)$$

made from (Y, y) in place of (X, x). Let $z = y \circ p_1 : Y \times_X Y \to \mathfrak{X}$. We have a map of manifolds $i : Y \times_X Y \to Y \times_{\mathfrak{X}} Y = Y_1$ under which a point $(P, Q) \in Y \times_X Y$ (note that $f(P) = f(Q) \in X$, so $y(P) = y(Q)$ as an object of $\mathfrak{X}(pt)$) is mapped to the triple $(P, 1_{y(P)}, Q)$ in $Y \times_{\mathfrak{X}} Y$. This gives a morphism $(i, \mathbf{1}_z) : (Y \times_X Y, z) \to (Y_1, y \circ s)$ in $\mathfrak{X}_{lis\text{-}et}$, and therefore a restriction map $r_{(i,\mathbf{1}_z)} : \mathcal{F}(Y_1, y \circ s) \to \mathcal{F}(Y \times_X Y, z)$.

With the above maps, we have the following 'commutative' diagram.

$$\begin{array}{ccccc}
\check{H}^0((x : X \to \mathfrak{X}), \mathcal{F}) & \to & \mathcal{F}(X, x) & \rightrightarrows & \mathcal{F}(X_1, x \circ s) \\
 & & \downarrow & & \downarrow \\
\check{H}^0((y : Y \to \mathfrak{X}), \mathcal{F}) & \to & \mathcal{F}(Y, y) & \rightrightarrows & \mathcal{F}(Y_1, y \circ s) \\
 & & \downarrow\downarrow & \swarrow & \\
 & & \mathcal{F}(Y \times_X Y, z) & &
\end{array}$$

The two rows and the middle column are exact diagrams of sets. The vertical map in the right column is injective as it is the restriction map for a morphism in $\mathfrak{X}_{lis\text{-}et}$ whose underlying map $Y_1 \to X_1$ is a submersive surjection. It follows by diagram chasing that we get an induced bijection

$$\check{H}^0((x : X \to \mathfrak{X}), \mathcal{F}) \overset{\sim}{\to} \check{H}^0((y : Y \to \mathfrak{X}), \mathcal{F})$$

which makes the left upper rectangle in the above diagram commute.

6.24 Generalizing the above, if (U, u) is any object of $\mathfrak{X}_{lis\text{-}et}$, we next define a restriction map $\rho_{(U,u)} : \check{H}^0((x : X \to \mathfrak{X}), \mathcal{F}) \to \mathcal{F}(U, u)$. Let

$$V = X \times_{x,\mathfrak{X},u} U$$

and let $v = u \circ p_U$, so that we have the 'commutative' diagram with exact rows

$$\begin{array}{ccccc}
\check{H}^0((x : X \to \mathfrak{X}), \mathcal{F}) & \to & \mathcal{F}(X, x) & \rightrightarrows & \mathcal{F}(X_1, x \circ s) \\
 & & \downarrow & & \downarrow \\
\mathcal{F}(U, u) & \to & \mathcal{F}(V, v) & \rightrightarrows & \mathcal{F}(V \times_U V, v \circ p_2)
\end{array}$$

Hence we get an induced map $\rho_{(U,u)} : \check{H}^0((x : X \to \mathfrak{X}), \mathcal{F}) \to \mathcal{F}(U, u)$. Recall that the sets $\mathcal{F}(U, u)$ and the restriction maps r for $\mathcal{F}$ define a filtered inverse system of sets indexed by the category $\mathfrak{X}_{lis\text{-}et}$, and the set of global sections is defined as the inverse limit. We leave it to the reader to verify that the set $\check{H}^0((x : X \to \mathfrak{X}), \mathcal{F})$ together with maps $\rho_{(U,u)}$ has the requisite universal property which makes it the inverse limit

$$(\check{H}^0((x : X \to \mathfrak{X}), \mathcal{F}), (\rho_{(U,u)})) = \varprojlim(\mathcal{F}(U, u), (r)).$$

The above shows that for any sheaf on the site $\mathfrak{X}_{lis\text{-}et}$, we have an alternative description of the global sections and their restrictions in terms of any chosen atlas (so in terms of any Lie groupoid whose quotient stack is the given stack).

Direct image and inverse image under $F : \mathfrak{X} \to \mathfrak{Y}$

6.25 Direct image for a representable 1-morphism. Let $F : \mathfrak{X} \to \mathfrak{Y}$ be a 1-morphism of differential stacks, such that F is representable. Let $\mathcal{F}$ be a sheaf on $\mathfrak{X}_{lis\text{-}et}$. For any object (V, v) of $\mathfrak{Y}_{lis\text{-}et}$, the fibered product stack $\mathfrak{X}_{F,\mathfrak{Y},v}V$ is then representable by a manifold U. Let $u : U \to \mathfrak{X}$ be the projection, which is smooth being the base change of $v : V \to \mathfrak{Y}$. Hence $\mathcal{F}(U, u)$ is defined, and we put $F_*(\mathcal{F})(V, v) = \mathcal{F}(U, u)$.

6.26 Direct image for a not-necessarily representable 1-morphism. Let $F : \mathfrak{X} \to \mathfrak{Y}$ be a 1-morphism of differential stacks, not necessarily representable. Let $\mathcal{F}$ be a sheaf on $\mathfrak{X}_{lis\text{-}et}$. We define the sheaf $F_*(\mathcal{F})$ on $\mathfrak{Y}_{lis\text{-}et}$ as follows. For any object (V, v) of $\mathfrak{Y}_{lis\text{-}et}$, consider all tuples $(U, u : U \to \mathfrak{X}, \phi : U \to V, \alpha)$ where (U, u) is in $\mathfrak{X}_{lis\text{-}et}$, $\phi : U \to V$ is any morphism, and α is a 2-commutator for the resulting square. Such tuples form a category, in which a morphism $(f, \beta, \gamma) : (U, u : U \to \mathfrak{X}, \phi : U \to V, \alpha) \to (U', u' : U' \to \mathfrak{X}, \phi' : U' \to V, \alpha')$ consisting of a morphism $f : U \to U'$ and 2-morphisms $\beta : u \Rightarrow u' \circ f$ and $\gamma : \phi \to \phi' \circ f$. Then we define $F_*(\mathcal{F})(V, v)$ to be the inverse limit of $\mathcal{F}(U, u)$ over all such tuples.

6.27 If we expand the category $\mathfrak{X}_{lis\text{-}et}$ to have as objects any smooth 1-morphisms of differential stacks $u : \mathcal{U} \to \mathfrak{X}$ (where $\mathcal{U}$ as well as u is not necessarily representable), then the above gets simplified: we can define $F_*(\mathcal{F})(V, v)$ to be the value of $\mathcal{F}$ on the pull-back of (V, v) under F. This would be equivalent to the above definition, as global sections over the base-change would be given exactly by the above inverse limit. Such an expansion of $\mathfrak{X}_{lis\text{-}et}$ is possible, and the categories of sheaves will be equivalent to the earlier notion.

Sheaf cohomology

6.28 (Grothendieck: Tohoku paper) An abelian category is said to satisfy (AB3) if direct sums parameterized by arbitrary sets exist (equivalently, direct limits parameterized by arbitrary small categories exist), and (AB5) if moreover direct limit over filtered sets is exact (means if I is a filtered set and each $A_i \to B_i \to C_i$ is exact then $colim_I A_i \to colim_I B_i \to colim_I C_i$ is exact. A generator for an abelian category is an object A such that if $f : B \to C$ is any morphism for which the composite $f \circ g : A \to C$ is zero for each $g : A \to B$, then $f = 0$. **Theorem** If an abelian category satisfies (AB5) and has a generating set, then it has enough injectives, and it admits direct products parameterized by arbitrary sets (equivalently, inverse limits parameterized by arbitrary small categories).

6.29 (See for example Artin: Grothendieck topologies) The categories P and S of presheaves and sheaves of abelian groups on any site are abelian categories. The inclusion functor $i : S \hookrightarrow P$ is left-exact and fully faithful, and it admits a left adjoint $a : P \to S$ (called the sheafification functor, made from applying a certain functor $+ : P \to P$ twice, so that $+ \circ + = i \circ a$). The functor a is exact. The category of presheaves P satisfies (AB5) (actually, even (AB4), (AB4*) and more) and has a generator. The category of sheaves S satisfies (AB5) and has a generating set. (These statements require some set-theoretic qualifications, which we will ignore here. In the context of differential manifolds and stacks where we need them, simple ad-hoc arguments can be given to take care of the set-theoretic requirements. In the general context, Grothendieck does it via universes.)

6.30 Let $\mathcal{C}$ be a category, an let U be any object. We define the presheaf $\mathbb{Z}_U^{presh}$ on $\mathcal{C}$ as follows. For any object V of $\mathcal{C}$, we put

$$\mathbb{Z}_U^{presh}(V) = \bigoplus_{Hom_{\mathcal{C}}(V,U)} \mathbb{Z}$$

If $\mathcal{C}$ is given a Grothendieck topology T, then the sheafification of $\mathbb{Z}_U^{presh}$ on the site $(\mathcal{C}, T)$ is denoted by $\mathbb{Z}_U$. This is the sheaf which represents the left-exact functor $\mathcal{F} \mapsto \mathcal{F}(U)$ from the category $S = Shv(\mathcal{C}, T)$ to the category of abelian groups. As U varies over an appropriate set of objects of $\mathcal{C}$ (which we may assume to exist), the $\mathbb{Z}_U$ form a set of generators for S, and hence the direct sum of these is a generator for S.

Definition 6.31 The functor $\Gamma(\mathfrak{X}, -)$ from the abelian category of sheaves of abelian groups on the site $\mathfrak{X}_{lis\text{-}et}$ to the category of abelian groups is left-exact. Its right derived functors $H^i(\mathfrak{X}, -)$ are the **cohomology group functors**.

Example 6.32 Let V be a finite-dimensional real vector space. Consider the stack BV. Then $H^1(BV, \mathcal{O}) = V^*$, if this can be computed by Cech cohomology for any atlas, in view of $\mathcal{O}$ being a fine sheaf. Consider the atlas $p : * \to BV$, and let $\mathcal{U} = \mathcal{U}(*, p)$ be the nerve of this cover. Then the Cech complex is

$$0 \to \mathcal{O}(*) \xrightarrow{\partial^0} \mathcal{O}(V) \xrightarrow{\partial^1} \mathcal{O}(V \times V) \to \dots$$

where $\partial^0 = 0$, and for any $f \in \mathcal{O}(V)$, the coboundary $\partial^1 f$ is defined by

$$(\partial^1 f)(x, y) = f(x) - f(x + y) + f(y)$$

Hence 1-cocycles f are just the C^∞ homomorphisms $f : V \to \mathbb{R}$, and they form the dual vector space V^*, while 1-coboundaries are zero. This shows $\check{H}^1(\mathcal{U}(V, p), \mathcal{O}) = V^*$.

6.33 Continuous map from the lisse-étale site to the étale site. Let X be a manifold, and consider the associated sites $X_{lis\text{-}et}$ and X_{et}. As any étale open $u : U \to X$ is a submersion, we have a continuous functor $\epsilon^{-1} : X_{et} \to X_{lis\text{-}et}$ under which any étale open $u : U \to X$ goes to $u : U \to X$ regarded as a lisse morphism. We have a pair of adjoint functors $(\epsilon^{-1}, \epsilon_*)$ where $\epsilon_* : Shv(X_{lis\text{-}et}) \to Shv(X_{et})$ is defined by $\epsilon_*(\mathcal{F})(U, u) = \mathcal{F}(U, u)$, and $\epsilon^{-1} : Shv(X_{et}) \to Shv(X_{lis\text{-}et})$ is defined as the sheafification of the presheaf $\mathcal{P}$ for which for any (V, v) in $X_{lis\text{-}et}$, we put $\mathcal{P}(V, v) = \mathcal{F}(v(V), j)$ where $j : v(V) \hookrightarrow X$ is the inclusion.

Proposition 6.34 Comparison of lisse-étale and étale cohomologies. *For any manifold X, the following properties hold.*

(1) The functor $\epsilon_* : Shv(X_{lis\text{-}et}) \to Shv(X_{et})$ is exact (not merely left-exact like all right-adjoints).

(2) The functor $\epsilon^{-1} : Shv(X_{et}) \to Shv(X_{lis\text{-}et})$ is exact (not merely right-exact like all left-adjoints).

(3) The functor $\epsilon_* : Shv(X_{lis\text{-}et}) \to Shv(X_{et})$ takes injective objects to injective objects.

(4) If $\mathcal{F}$ is a sheaf of abelian groups on $X_{lis\text{-}et}$, then $H^i(X_{lis\text{-}et}, \mathcal{F})$ is functorially isomorphic to $H^i(X_{et}, \epsilon_*\mathcal{F})$.

Proof. (1) Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be exact in $Shv(X_{lis\text{-}et})$. We know that $0 \to \epsilon_*\mathcal{F}' \to \epsilon_*\mathcal{F} \to \epsilon_*\mathcal{F}''$ is exact. It remains to show $\epsilon_*\mathcal{F} \to \epsilon_*\mathcal{F}''$ is epic, that is, given any $s'' \in (\epsilon_*\mathcal{F}'')(U, u)$ where (U, u) is an object of X_{et}, there exists an étale open cover $(V, v) \to (U, u)$ such that $s''|_{(V,v)}$ is in the image of $\epsilon_*\mathcal{F}(V, v)$. But note that for any $\mathcal{G}$ in $Shv(X_{lis\text{-}et})$ and any (U, u) in X_{et}, by definition $(\epsilon_*\mathcal{G})(U, u) = \mathcal{G}(U, u)$, and so the result follows.

(2) More generally, if a continuous functor $\mathcal{C} \to \mathcal{C}'$ on sites is such that $\mathcal{C}$ has finite limits, then the 'pull-back' functor from $Shv(\mathcal{C})$ to $Shv(\mathcal{C}')$ is given by sheafification of a filtered direct limit, so is exact. In the present case, $\epsilon^{-1} : X_{et} \to X_{lis\text{-}et}$ is a continuous functor and X_{et} has finite limits.

(3) As ϵ^{-1} is exact, its right adjoint ϵ_* preserves injectives.

(4) This is immediate from (1) and (3).

Lemma 6.35 *Let $x : X \to \mathfrak{X}$ be a smooth atlas of a differential stack, and let (X_n) be the corresponding simplicial object in $X_{lis\text{-}et}$. Then the complex of sheaves*

$$\mathbb{Z}_{X_0} \leftarrow \mathbb{Z}_{X_1} \leftarrow \mathbb{Z}_{X_2} \leftarrow \dots$$

is exact. Consequently, if I is an injective sheaf on $X_{lis\text{-}et}$ then the complex

$$I(X_0) \to I(X_1) \to \dots$$

is an exact complex of abelian groups.

Proof. As sheafification is exact, it is enough to show that the complex of presheaves

$$\mathbb{Z}_{X_0}^{presh} \leftarrow \mathbb{Z}_{X_1}^{presh} \leftarrow \mathbb{Z}_{X_2}^{presh} \leftarrow \dots$$

is exact. In other words, for any object (U, u) of $X_{lis\text{-}et}$, the sequence

$$\mathbb{Z}_{X_0}^{presh}(U, u) \leftarrow \mathbb{Z}_{X_1}^{presh}(U, u) \leftarrow \mathbb{Z}_{X_2}^{presh}(U, u) \leftarrow \dots$$

is exact. Let S be the set defined as

$$S = Hom_{X_{lis\text{-}et}}((U, u), (X, x)).$$

Then the above is the chain complex

$$\bigoplus_{S} \mathbb{Z} \leftarrow \bigoplus_{S^2} \mathbb{Z} \leftarrow \bigoplus_{S^3} \mathbb{Z} \leftarrow \dots$$

This is exact by a standard contracting homotopy (see e.g. Milne [M]), defined as follows. If S is empty, the exactness is clear as each term is zero. So assume S is non-zero, and choose an element $s_0 \in S$. For each $n \geq 1$, we define a map $S^n \to S^{n+1}$ by $(s_1, \dots, s_n) \mapsto (s_0, s_1, \dots, s_n)$. This induces a map $k_n : \bigoplus_{S^n} \mathbb{Z} \to \bigoplus_{S^{n+1}} \mathbb{Z}$, and (k_n) is the desired contracting homotopy. □

Theorem 6.36 *Let $\mathfrak{X}$ be a differential stack, and let $\mathcal{F}$ be any sheaf on $X_{lis\text{-}et}$ such that for every (U, u) in $X_{lis\text{-}et}$ and $p \geq 1$, we have $H^p(U_{et}, \mathcal{F}_{U,u}) = 0$. Then for any atlas $x : X \to \mathfrak{X}$, the natural map*

$$H^p(\mathfrak{X}_{lis\text{-}et}, \mathcal{F}) \to \check{H}^p((x : X \to \mathfrak{X}), \mathcal{F})$$

is an isomorphism for all $p \geq 0$.

Proof. Let $\mathcal{F} \to I^0 \xrightarrow{\delta} I^1 \to \dots$ be an injective resolution. Then we get a double complex

$$(C^{q,r} = I^q(X_r),\ \delta : C^{q,r} \to C^{q+1,r},\ \partial : C^{q,r} \to C^{q,r+1})$$

in which ∂ is the Cech differential. The complex

$$0 \to \Gamma(\mathfrak{X}, I^0) \xrightarrow{\delta} \Gamma(\mathfrak{X}, I^1) \to \dots$$

has an augmentation map to the double complex. Similarly, the Cech complex

$$0 \to \mathcal{F}(X_0) \xrightarrow{\partial} \mathcal{F}(X_1) \to \dots$$

has an augmentation map to the double complex. The r th row of this augmented double complex is the complex

$$0 \to \mathcal{F}(X_r) \to I^0(X_r) \to I^1(X_r) \to \dots$$

which is exact as by assumption $H^p((U_{et}, \mathcal{F}_{U,u}) = 0$ for $(U, u) = X_r$. The q th column of the augmented double complex is the complex

$$0 \to \Gamma(\mathfrak{X}, I^q) \to \Gamma(X_0, I^q) \to \dots$$

which is exact by Lemma 6.35. Hence the theorem follows. □

7 Differential forms and de Rham cohomology

7.1 Let $\mathfrak{X}$ be a differential stack. The **structure sheaf** $\mathcal{O}_{\mathfrak{X}}$ is a sheaf of rings on the site $\mathfrak{X}_{lis\text{-}et}$ defined by $\mathcal{O}_{\mathfrak{X}}(U,u) = \mathcal{O}(U)$ (the ring of global regular functions on U) for every (U,u) in $\mathfrak{X}_{lis\text{-}et}$. For any map $(f,\alpha) : (V,v) \to (U,u)$ in $\mathfrak{X}_{lis\text{-}et}$, the restriction map $r_{f,\alpha} : \mathcal{O}(U) \to \mathcal{O}(V)$ defined by $\phi \mapsto \phi \circ f$ for any $\phi : U \to \mathbb{R}$ in $\mathcal{O}(U)$. This is actually a sheaf of commutative $\mathbb{R}$-algebras.

7.2 Let $\mathfrak{X}$ be a differential stack. The **sheaf of differential p-forms** $\Omega^p_{\mathfrak{X}}$ is the sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules on the site $\mathfrak{X}_{lis\text{-}et}$ defined by $\Omega^p_{\mathfrak{X}}(U,u) = \Omega^p(U)$ (global differential p-forms on U) for every (U,u) in $\mathfrak{X}_{lis\text{-}et}$. For any map $(f,\alpha) : (V,v) \to (U,u)$ in $\mathfrak{X}_{lis\text{-}et}$, the restriction map $r_{f,\alpha} : \Omega^p(U) \to \Omega^p(V)$ is defined by pull-back of forms under $f : V \to U$.

7.3 An $\mathbb{R}$-linear homomorphism of sheaves $d^p_{\mathfrak{X}} : \Omega^p_{\mathfrak{X}} \to \Omega^{p+1}_{\mathfrak{X}}$ is defined by the exterior differentials $d : \Omega^p(U) \to \Omega^{p+1}(U)$ on all objects (U,u) in $\mathfrak{X}_{lis\text{-}et}$. By its definition, $d^{p+1}_{\mathfrak{X}} \circ d^p_{\mathfrak{X}} = 0$. The resulting complex $(\Omega_{\mathfrak{X}}, d_{\mathfrak{X}})$ is called the **de Rham complex** of $\mathfrak{X}$. Its hypercohomology $\mathbb{H}^p(\Omega_{\mathfrak{X}}, d_{\mathfrak{X}})$ is called the **de Rham cohomology** of $\mathfrak{X}$, and denoted by $H^p_{dR}(\mathfrak{X}_{lis\text{-}et})$.

7.4 De Rham's Theorem. For any differential stack $\mathfrak{X}$, the sequence

$$0 \to \mathbb{R}_{\mathfrak{X}} \hookrightarrow \Omega^0_{\mathfrak{X}} \xrightarrow{d} \Omega^1_{\mathfrak{X}} \xrightarrow{d} \dots$$

is an exact sequence of sheaves. Consequently, the map $\mathbb{R}_{\mathfrak{X}} \to (\Omega_{\mathfrak{X}}, d_{\mathfrak{X}})$ induces an isomorphism

$$H^p(\mathfrak{X}_{lis\text{-}et}, \mathbb{R}_{\mathfrak{X}}) \xrightarrow{\sim} \mathbb{H}^p(\Omega_{\mathfrak{X}}, d_{\mathfrak{X}}).$$

7.5 Hodge to de Rham spectral sequence. One of the two hypercohomology spectral sequences is of the form

$$E_2^{p,q} = H^p(\mathfrak{X}_{lis\text{-}et}, \Omega^q_{\mathfrak{X}}) \Rightarrow H^{p+q}_{dR}(\mathfrak{X}_{lis\text{-}et}).$$

The main result

Theorem 7.6 *Let $\mathfrak{X}$ be a differential stack, and let $x : X \to \mathfrak{X}$ be an atlas. Let (X_n) be the simplicial nerve of the atlas. Then the de Rham cohomology of $\mathfrak{X}$ is naturally isomorphic to the cohomology of the total complex of the resulting double complex $\Omega^p(X_r)$.*

Proof Let $(I^{p,q},\ d : I^{p,q} \to I^{p+1,q},\ \delta : I^{p,q} \to I^{p,q+1})$, where $p \geq -1$ and $q \geq 0$, be a Cartan-Eilenberg injective resolution of the augmented de Rham complex $0 \to \mathbb{R}_{\mathfrak{X}} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots$. In particular, we have an injective resolution

$$0 \to \mathbb{R}_{\mathfrak{X}} \to I^{-1,0} \xrightarrow{\delta} I^{-1,1} \xrightarrow{\delta} \dots$$

and an injective resolution

$$0 \to \Omega^p \to I^{p,0} \xrightarrow{\delta} I^{p,1} \xrightarrow{\delta} \dots$$

for each p. The maps $d : I^{p,q} \to I^{p+1,q}$ are induced by the inclusion $\mathbb{R}_{\mathfrak{X}} \to \Omega^0$ and by the de Rham differential $d : \Omega^p \to \Omega^{p+1}$. As the augmented de Rham complex is exact, the complexes

$$0 \to I^{-1,q} \xrightarrow{d} I^{0,q} \xrightarrow{d} I^{1,q} \xrightarrow{d} \dots$$

are exact.

Let $x : X \to \mathfrak{X}$ be a smooth atlas, and let (X_n) be the simplicial nerve of it. Then we get a triple complex of abelian groups

$$(T_r^{p,q},\ d : T_r^{p,q} \to T_r^{p+1,q},\ \delta : T_r^{p,q} \to T_r^{p,q+1},\ \partial : T_r^{p,q} \to T_{r+1}^{p,q})$$

defined as follows. We put

$$T_r^{p,q} = \begin{cases} I^{p,q}(X_r) & \text{for } p,q,r \geq 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The differential $\partial : T_r^{p,q} \to T_{r+1}^{p,q}$ is the Cech differential $\partial : I^{p,q}(X_r) \to I^{p,q}(X_{r+1})$, the differential $d : T_r^{p,q} \to T_r^{p+1,q}$ is induced by $d : I^{p,q} \to I^{p+1,q}$ and the differential $\delta : T_r^{p,q} \to T_r^{p,q+1}$ is induced by $\delta : I^{p,q} \to I^{p,q+1}$.

Now consider the double complex $(A_r^p, d : A_r^p \to A_r^{p+1}, \partial : A_r^p \to A_{r+1}^p)$ where

$$A_r^p = \begin{cases} \Omega^p(X_r) & \text{for } p,r \geq 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

with d induced by the differential $d : \Omega^p \to \Omega^{p+1}$ and ∂ induced by the Cech differential. We have an augmentation map

$$\alpha : A \to T$$

induced by the inclusions $\Omega^p \to I^{p,0}$. Note that Ω is an acyclic sheaf on X_r, so the complexes

$$0 \to \Omega^p(X_r) \to I^{p,0}(X_r) \to I^{p,1}(X_r) \to \dots$$

are exact. Hence the augmentation induces a quasi-isomorphism

$$Tot(\alpha) : Tot(A) \to Tot(T)$$

of the total complexes.

Similarly, consider the double complex $(B^{p,q}, d : B^{p,q} \to B^{p+1,q}, \delta : B^{p,q} \to B^{p,q+1})$ where

$$B^{p,q} = \begin{cases} \Gamma(\mathfrak{X}, I^{p,q}) & \text{for } p,q \geq 0, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

with d induced by the differential $d : \Omega^p \to \Omega^{p+1}$ and δ induced by $\delta : I^{p,q} \to I^{p,q+1}$). We have an augmentation map

$$\beta : B \to T$$

induced by the inclusions $\Gamma(\mathfrak{X}, I^{p,q}) \to I^{p,q}(X_0)$.

As each $I^{p,q}$ is an injective sheaf, each of the complexes

$$0 \to \Gamma(\mathfrak{X}, I^{p,q}) \to I^{p,q}(X_0) \xrightarrow{\partial} I^{p,q}(X_1) \xrightarrow{\partial} I^{p,q}(X_2) \xrightarrow{\partial} \dots$$

is exact by the description of global sections via an atlas together with the Lemma 6.35. Therefore the augmentation β induces a quasi-isomorphism

$$Tot(\beta) : Tot(B) \to Tot(T)$$

of the total complexes.

Hence we get an isomorphism in the derived category

$$Tot(\beta)^{-1} \circ Tot(\alpha) : Tot(A) \to Tot(B)$$

By definition, the cohomology of $Tot(B)$ is the de Rham cohomology of $\mathfrak{X}$. This proves the theorem.

A Appendix: 2-categories

Recall that a **groupoid** is a category in which all morphisms are isomorphisms.

Next we define what is a 2-category. While reading the definition, it will be useful to keep a basic example in mind: there is a 2-category C called 'the category of all categories', in which 0-cells are all categories, 1-cells are all functors, and 2-cells are natural isomorphisms between functors.

Definition A.1 A **2-category** C is given by the following data, which satisfies the following conditions.

We are given some **0-cells** or **'objects'**, which we denote by x, y, z etc. We are given some **1-cells** or **'arrows'**, denoted by f, g, h etc. We are given some **2-cells** or **'equivalences'**, denoted by α, β, γ etc.

Each 1-cell f has a given **source** and a **target**, which are 0-cells. The notation $f : x \to y$ will mean f has source x and target y. We also write $s(f) = x$ and $t(f) = y$. A **composite** 1-cell $g \circ f : x \to z$ is defined for any $f : x \to y$ and $g : y \to z$.

The 0-cells as objects, and 1-cells as morphisms with the given composition, are supposed to form a category $C^{(1)}$ called the 1-skeleton of C. Unlike usual categories in mathematics, this category $C^{(1)}$ may not be locally small, that is, all 1-cells $x \to y$ may not form a set. For any 0-cell x, we denote by 1_x the 1-cell which is the identity automorphism of x in $C^{(1)}$.

Each 2-cell α has a given **source** and a **target**, which are 1-cells. The notation $\alpha : f \Rightarrow g$ will mean α has source f and target g. We also write $s(\alpha) = f$ and $t(\alpha) = g$. It is assumed that if $\alpha : f \Rightarrow g$ is any 2-cell, then the 1-cells f and g have a common source 0-cell x and a common target 0-cell y (in other words, $s \circ s = s \circ t$ and $t \circ s = t \circ t$ as maps from the collection of all 2-cells to the collection of all 0-cells).

A composite of 2-cell $\beta \bullet \alpha : f \Rightarrow h$ is defined for any $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$. This is called as the **vertical composition** of 2-cells.

For any 0-cells x and y, we require that all 1-cells $f : x \to y$ as objects and all 2-cells α between such 1-cells as morphisms form a category. This category is denoted by $HOM(x, y)$.

It is required that the category $HOM(x, y)$ should be a groupoid. All the 2-cells are by assumption invertible, and α^{-1} will denote the inverse of α. The identity automorphism (which is a 2-cell of C) of any object f of $HOM(x, y)$ (which is a 1-cell of C) will be denoted by $\mathbf{1}_f$. For any 0-cell x, The identity automorphism $\mathbf{1}_{1_x}$ of the object 1_x of $HOM(x, x)$ will be denoted by $\mathbb{I}_x$. This will be called the **identity 2-cell of the 0-cell** x.

If x, y, z are 0-cells, $f, f' : x \to y$ and $g, g' : y \to z$ are 1-cells, and $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$ are 2-cells in C, then a 2-cell $\beta \star \alpha : g \circ f \Rightarrow g' \circ f'$ is defined, called the **horizontal composite** of the 2-cells α followed by β.

We require that horizontal composition be associative, and the identity 2-cells $\mathbb{I}_x$ of all the 0-cells x should be both left and right identities for horizontal composition.

Moreover, we require that vertical and horizontal compositions should commute. This means if x, y, z are 0-cells, $f, f', f'' : x \to y$ and $g, g', g'' : y \to z$ are 1-cells, and $\alpha : f \Rightarrow f'$, $\alpha' : f' \Rightarrow f''$ and $\beta : g \Rightarrow g'$, $\beta' : g' \Rightarrow g''$ are 2-cells in C, then we should have

$$(\beta' \star \alpha') \bullet (\beta \star \alpha) = (\beta' \bullet \beta) \star (\alpha' \bullet \alpha)$$

Remark A.2 In the 'small' case, all 0-cells, 1-cells and 2-cells in a 2-category C form sets which we denote by C_0, C_1, and C_2. The sources and targets are given by maps $s_1, t_1 : C_1 \to C_0$ and $s_2, t_2 : C_2 \to C_1$, with $s_1 s_2 = s_1 t_2$ and $t_1 s_2 = t_1 t_2$. Composite of 1-cells is a map

$$m_1 : C_1 \times_{s_1, C_0, t_1} C_1 \to C_1$$

There is a map $e_0 : C_0 \to C_1$ which attaches to any 0-cell its identity 1-cell, and a map $e_1 : C_1 \to C_2$ which attaches to any 1-cell its identity 2-cell. In these terms, the composite map $e_1 e_0 : C_0 \to C_2$ associates to any 0-cell its identity 2-cell.

Vertical composition of 2-cells is given by a map

$$m_2^{ver} : C_2 \times_{s_2, C_1, t_2} C_2 \to C_2$$

Horizontal composition is given by a map

$$m_2^{hor} : C_2 \times_{s_1 s_2, C_0, t_1 t_2} C_2 \to C_2$$

The inverse of a 2-cell is given by a map $i_2 : C_2 \to C_2$.

Thus, a small 2-category is given by data

$$(C_0, C_1, C_2, s_1, t_1, s_2, t_2, e_0, e_1, m_1, m_2^{ver}, m_2^{hor}, i_2)$$

The conditions which this data is required to satisfy can be expressed by the requirement of the commutativity of certain diagrams, which we leave to the reader.

Remark A.3 To any 2-category C we can associate a 1-category (means usual category) D (which we can call as the homotopy category of C in view of the next example) defined as follows. The objects of D are the same as the objects (0-cells) of C, while the morphisms $x \to y$ are equivalence classes of 1-cells $f : x \to y$ where two such f and g are regarded as equivalent if there exists a 2-cell $\alpha : f \Rightarrow g$ in C. Note however that D need not in general be locally small.

Example A.4 Let Top be the 2-category defined as follows. The 0-cells of Top are topological spaces denoted by X, Y, Z, etc. The 1-cells are continuous maps $f : X \to Y$, and their composition is defined as usual.

We now define 2-cells. Let $f, g : X \to Y$ be continuous maps, and let $H : X \times [0,1] \to Y$ and $H' : X \times [0,1] \to Y$ be homotopies from f to g. We will say that H' is a **re-parameterization** of H if there exists a continuous map $r : [0,1] \to [0,1]$ with $r(0) = 0$ and $r(1) = 1$ such that $H'(x,t) = H(x, r(t))$ for all $(x,t) \in X \times [0,1]$. Consider the equivalence relation generated by such re-parameterizations. A two cell $\alpha : f \Rightarrow g$ is an equivalence class $[H]$ of homotopies $H : X \times [0,1] \to Y$ from f to g under the above equivalence relation.

The vertical composition of 2-cells is defined by juxtaposing the time intervals of representing homotopies, which is well defined and associative at the level of the equivalence classes $[H]$. The horizontal composition of homotopies is defined in the obvious way, and it is well-defined on classes $[H]$. By going modulo 2-cells, we get from the 2-category Top the homotopy category Hot of topological spaces.

Remark A.5 Let X be any contractible space (for example, $X = \mathbb{R}^2$) regarded as an object in the 2-category Top. Note that a continuous map $f : Y \to X$ is unique up to homotopy, but need not be unique. Moreover, if $f, g : Y \to X$, then there could be more then one equivalence class of homotopies from f to g. This example motivates the definition of a terminal object in a 2-category.

Definition A.6 In any 2-category C, a **terminal object** is any 0-cell x with the properties that given any other 0-cell y, there exists a 1-cell $f : y \to x$, and for any two such 1-cells $f, g : y \to x$, there exists a 2-cell $\alpha : f \Rightarrow g$. (Such a 2-cell α need not be unique.)

Fibered products

Let C be a 2-category. Let $f : x \to y$ and $g : z \to y$ be 1-cells in C. We consider 4-tuples (w, F, G, α) in which w is a 0-cell, $F : w \to y$ and $G : w \to x$ are 1-cells, and $\alpha : f \circ G \Rightarrow g \circ F$ is a 2-cell. We now construct a category T in which the objects are all such 4-tuples. A morphisms in this category from (w', F', G', α') to (w, F, G, α) is a 3-tuple (h, β, γ) where $h : w' \to w$ is a 1-cell, and $\beta : G' \Rightarrow G \circ h$ and $\gamma : F' \Rightarrow F \circ h$ are 2-cells, such that the resulting polyhedron commutes, that is, we have

$$\alpha' = (\mathbf{1}_g \star \gamma^{-1}) \bullet \alpha \bullet (\mathbf{1}_f \star \beta)$$

where $\mathbf{1}_f$ and $\mathbf{1}_g$ respectively denote the identity 2-cells of the 1-cells f and g, $\star$ and $\bullet$ respectively denote the horizontal and the vertical composition of 2-cells in C, and γ^{-1} denotes the inverse 2-cell of γ. As horizontal composition of 2-cells is by assumption associative, the product notation is unambiguous.

If $(h', \beta', \gamma') : (w'', F'', G'', \alpha'') \to (w', F', G', \alpha')$ and $(h, \beta, \gamma) : (w', F', G', \alpha') \to (w, F, G, \alpha)$ are two such morphisms, we define their composite by putting

$$(h, \beta, \gamma)(h', \beta', \gamma') = (h \circ h', \beta \bullet \beta', \gamma \bullet \gamma')$$

This indeed makes sense as vertical and horizontal compositions of 2-cells commute. It can be verified that this composition law makes T into a category.

Let $(h_1, \beta_1, \gamma_1) : (w', F', G', \alpha') \to (w, F, G, \alpha)$ and $(h_2, \beta_2, \gamma_2) : (w', F', G', \alpha') \to (w, F, G, \alpha)$ be morphisms in T with common source (w', F', G', α') and common target (w, F, G, α). We now make T into a 2-category by defining a 2-cell $(h_1, \beta_1, \gamma_1) \Rightarrow (h_2, \beta_2, \gamma_2)$ to be a 2-cell $\delta : h_1 \Rightarrow h_2$ in C such that

$$(\mathbf{1}_G \star \delta) \bullet \beta_1 = \beta_2 \text{ and } (\mathbf{1}_F \star \delta) \bullet \gamma_1 = \gamma_2$$

We denote this 2-cell by $\delta_{(h_1,\beta_1,\gamma_1)\Rightarrow(h_2,\beta_2,\gamma_2)}$. We define the vertical composition $\bullet$ in T by putting

$$\delta'_{(h_2,\beta_2,\gamma_2)\Rightarrow(h_3,\beta_3,\gamma_3)} \bullet \delta_{(h_1,\beta_1,\gamma_1)\Rightarrow(h_2,\beta_2,\gamma_2)} = (\delta' \bullet \delta)_{(h_1,\beta_1,\gamma_1)\Rightarrow(h_3,\beta_3,\gamma_3)}$$

We now define the horizontal composition $\star$ in T. Let $(h_1', \beta_1', \gamma_1')$ and $(h_2', \beta_2', \gamma_2')$ be 1-cells in T from $(w'', F'', G'', \alpha'')$ to (w', F', G', α') and let (h_1, β_1, γ_1) and (h_2, β_2, γ_2) be 1-cells in T from (w', F', G', α') to (w, F, G, α). Let $\delta' : (h_1', \beta_1', \gamma_1') \Rightarrow (h_2', \beta_2', \gamma_2')$ and $\delta : (h_1, \beta_1, \gamma_1) \Rightarrow (h_2, \beta_2, \gamma_2)$ be 2-cells in T. We define their horizontal composition $\star$ in T by putting

$$\delta_{(h_1,\beta_1,\gamma_1)\Rightarrow(h_2,\beta_2,\gamma_2)} \star \delta'_{(h_1',\beta_1',\gamma_1')\Rightarrow(h_2',\beta_2',\gamma_2')} = (\delta \star \delta')_{(h_1\circ h_1',\beta_1\bullet\beta_1,\gamma_1\bullet\gamma_1')\Rightarrow(h_2\circ h_2',\beta_2\bullet\beta_2,\gamma_2\bullet\gamma_2')}$$

The reader can verify that the above data makes T a 2-category.

Definition A.7 Given any 1-cells $f : x \to y$ and $g : z \to y$ in a 2-category, consider the associated 2-category T of 4-tuples (w, F, G, α) which we described above. A terminal 0-cell (w, F, G, α) in T is called as the **fibered product** of f and g. A choice of such a tuple (if it exists) is denoted by $(x \times_{f,y,g} z, p_y, p_x, \alpha)$, or simply by $x \times_{f,y,g} z$ or even $x \times_y z$ if the other data is understood.

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Behrend: See various papers on arXiv.

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