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A SEMILINEAR MODEL FOR EXPONENTIAL DYNAMICS AND TOPOLOGY

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ABSTRACT. We present a model over the plane that recreates the same dynamics involved for the complex exponential family. This model is based on a one parameter family of continuous, semilinear maps. Under certain assumptions over the parameter value, we show the continuum obtained from the semilinear map resembles the one obtained for $E_\lambda(z)$.

1. INTRODUCTION

The complex exponential family $E_\lambda(z) = \lambda e^z$ exhibits both rich topology and interesting dynamics. It is known that if λ is real and $\lambda > 1/e$, then E_λ admits an invariant set in the strip $0 \leq \text{Im}z \leq \pi$ that is an indecomposable continuum [3]. This is a closed connected set which cannot be decomposed into two (not necessarily distinct) closed, connected sets. Such sets have a complicated topological structure [1], [5], [6], [7].

In contrast to this rich topology, the dynamics on this invariant set are quite tame. There is a unique repelling fixed point in the set. All other orbits either eventually land on the real line and then tend to ∞, or else they accumulate on both the orbit of 0 and a point at ∞.

Similar invariant indecomposable continua have been found in a variety of complex exponentials with λ complex [8]. There is also an uncountable collection of different, non-invariant indecomposable

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continua in the Julia sets of these maps [4]. Unfortunately, very little is known about the actual topology of these invariant sets. Also, how this topology depends on the parameter λ is an open question.

In an effort to simplify some of these questions, we propose in this paper a simpler family of maps that shares many of the topological and dynamical properties of the complex exponential family. Our map is a piecewise semilinear map of the strip $0 \leq \text{Im } z \leq 1$. This map mimics the behavior of E_λ, $\lambda > 1/e$ in that there is an invariant indecomposable continuum in this strip on which our map behaves dynamically like E_λ (see Figure 1). Because of the semilinearity, our map is significantly easier to work with in many respects. We illustrate this by computing a type of kneading invariant for this family of maps. This has allowed the second author to show that each member of this family is not topologically conjugate to any other member, despite the fact that their gross dynamical properties are the same (see [8]). Another advantage of our model is the fact that we can construct an uncountable number of curves homeomorphic to the real line that belong to the continuum. These curves are in fact composants. We conjecture that every point in the continuum that is not accessible from the exterior of the strip lies in such a curve.

2. Definition of the map

Let S denote the strip $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$. Let $L_1 \subset S$ be the open half strip

$$L_1 = \{(x, y) \in S \mid x > 0,\ 1/3 < y < 2/3\}.$$

Let $\lambda > 0$ (later we will further restrict λ). We define a family of homeomorphisms $h_\lambda : S - L_1 \to S - \{(\lambda, 0)\}$ as follows. We first decompose $S - L_1$ into three subregions:

$$\begin{aligned} H_1 &= \{(x,y) \mid x \geq 0,\ 0 \leq y \leq 1/3\} \\ H_2 &= \{(x,y) \mid x \geq 0,\ 2/3 \leq y \leq 1\} \\ H_3 &= \{(x,y) \mid x \leq 0\}. \end{aligned}$$

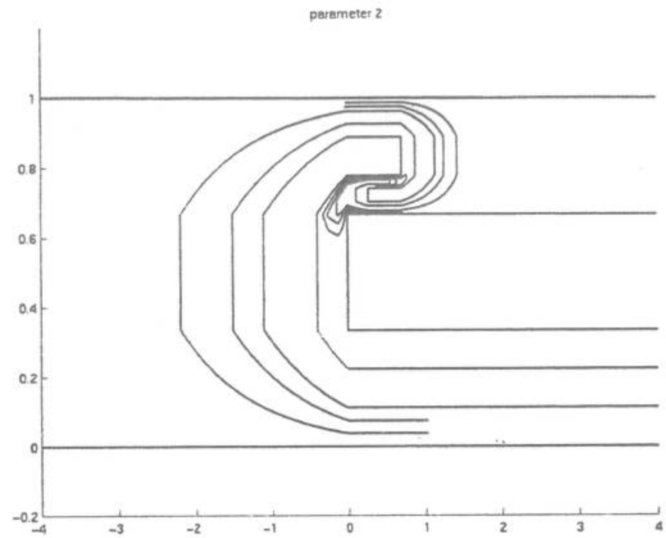

FIGURE 1. A partial picture of the continuum that corresponds to the semilinear family.

We further subdivide H_3 into 3 substrips of equal height:

$$\begin{aligned} A_1 &= \{(x,y)\,|\,x \le 0,\ 0 \le y \le 1/3\} \\ A_2 &= \{(x,y)\,|\,x \le 0,\ 1/3 \le y \le 2/3\} \\ A_3 &= \{(x,y)\,|\,x \le 0,\ 2/3 \le y \le 1\}. \end{aligned}$$

Finally, we define

$$h_\lambda(x,y) = \begin{cases} (\lambda + e^x, 3y) & \text{if } (x,y) \in H_1 \\ (\lambda - e^x, 3(1-y)) & \text{if } (x,y) \in H_2 \\ e^x(1,3y) + (\lambda, 0) & \text{if } (x,y) \in A_1 \\ e^x(3-6y,1) + (\lambda, 0) & \text{if } (x,y) \in A_2 \\ e^x(-1,3(1-y)) + (\lambda, 0) & \text{if } (x,y) \in A_3. \end{cases}$$

One checks easily the following facts:

(1) h_λ maps H_1 onto the half strip $\{(x,y) \in S\,|\,x \ge \lambda + 1\}$.
(2) h_λ maps H_2 onto the half strip $\{(x,y) \in S\,|\,x \le \lambda - 1\}$.
(3) h_λ is an expansion on $H_1 \cup H_2$ (strict expansion if $x \ge \delta > 0$).

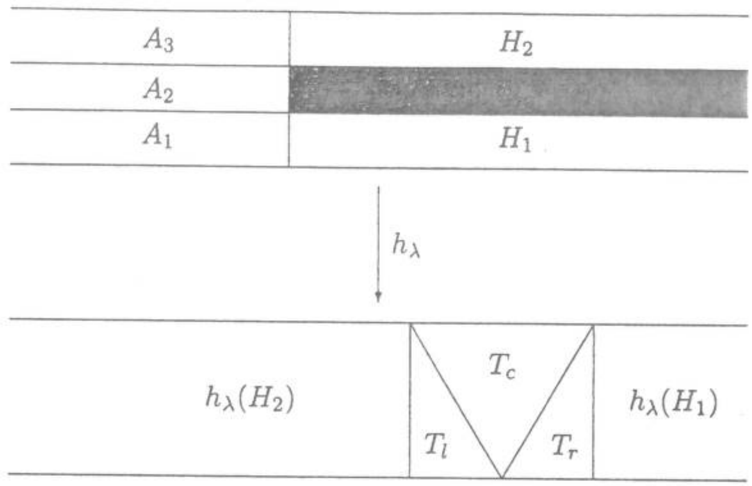

FIGURE 2. Domain and range for h_λ

(4) h_λ maps H_3 onto the rectangle $\{(x,y) \in S \mid \lambda - 1 \leq x \leq \lambda + 1\}$, missing only the point $(\lambda, 0)$.
(5) Note that h_λ is not a contraction on H_3. However, if x is sufficiently negative, then h_λ is a strong contraction. By a slight abuse of language we will call the image $h_\lambda(H_3) = \{(x,y) \in S \mid \lambda - 1 \leq x \leq \lambda + 1\}$ the *contracting rectangle*.
(6) h_λ maps $S - L_1$ homeomorphically onto $S - \{(\lambda, 0)\}$.
(7) On the real axis, h_λ is given by the function $x \to e^x + \lambda$. This function is conjugate to $x \to e^\lambda \cdot e^x$ via the conjugacy $x \to x - \lambda$.
(8) h_λ maps $y = 1$ to the half line $(-\infty, \lambda)$ on the real axis, and maps the boundary of L_1 onto the line $y = 1$.

In Figure 2, we display the domain and range of h_λ. Note that the image of H_3 consists of three triangular regions: on the right, T_r, the image of A_1; in the center, T_c, the image of A_2; and on the left, T_ℓ, the image of A_3. This is one aspect of the linearity of our map: E_λ would map this region to a semicircular region.

The map h_λ is similar to a complex exponential map of the form $E_\mu(z) = \mu e^z$, $\mu > 1/e$, on the strip $0 \leq y \leq \pi$. E_μ maps this strip onto the upper half plane (minus the origin) with a region analogous to our L_1 mapped above $y = \pi$. Thus, we think of L_1

as the set of points in S whose orbit leaves S after one iteration, although we do not define h_λ on L_1.

For later use, note that the image of a vertical line $x = \nu$ in H_3 is a *linear horseshoe curve.* This curve is the portion of the boundary of a rectangle formed by the three straight lines:

(1) $x = e^\nu + \lambda,\ \ 0 < y \le e^\nu$
(2) $y = e^\nu,\ \ \lambda - e^\nu \le x \le \lambda + e^\nu$
(3) $x = e^\nu - \lambda,\ 0 < y \le e^\nu$.

We call a region bounded by two such linear horseshoe curves a *linear horseshoe region.*

3. THE INDECOMPOSABLE CONTINUUM

For each $j \ge 2$, let $L_j = h_\lambda^{-1}(L_{j-1})$. Each L_j is an open simply connected subset of S that contains the half-strip $x > 0,\ 1/3^j < y < 2/3^j$, among (many) other points. Note also that the L_j are disjoint, since $h_\lambda(L_{j+1}) = L_j$. We think of L_j as the set of points that "escape from S" after j iterations of h_λ.

Let ∂L_j denote the boundary of L_j. If $(x, y) \in \partial L_j$, then $h_\lambda^j(x, y)$ lies on the line $y = 1$; $h_\lambda^{j+1}(x, y)$ lies on the real axis to the left of λ; and $h_\lambda^{j+2}(x, y)$ lies to the right of λ. Any point on ∂L_j thus has orbit that eventually tends to ∞ along the real axis.

Let Λ_λ denote the closure of $\cup_j \partial L_j$. Our main result in this paper is:

Theorem 3.1. *Λ_λ is an indecomposable continuum.*

To prove this, we will show that there exists a simply connected domain U whose boundary is Λ_λ and we will exhibit a prime end of U whose impression contains Λ_λ. Hence, by a result of Rutt [9], Λ_λ is either indecomposable or the union of two indecomposable continua [1]. The proof of Theorem 3.1 will rule out the latter case.

Denote by γ the union of the boundaries of the L_j. Now obviously, this choice of γ is neither continuous nor compact. However, we may compactify the set as in [3] by first compressing the strip S to a bounded horizontal strip and then by identifying points on the "backward" orbit of $(\lambda, 0)$. That is, we identify the point corresponding to $(-\infty, 0)$ with $(-\infty, 1), (\infty, 1)$ with $(\infty, 2/3)$, $(\infty, 1/3)$

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[1]We thank J. C. Mayer for showing us this argument using prime ends.

with $(\infty, 2/9)$, and so forth. Then we must show that the union of the preimages of $y = 0$ accumulates everywhere on itself.

Proposition 3.2. *The curve γ accumulates everywhere on itself.*

Proof: Note first that the boundaries of the L_j accumulate everywhere on $y = 0$. Indeed, the boundary of L_j contains a horizontal line segment of the form $x = t$, $y = 1/3^j$ with $t \geq 0$. Hence, the boundaries accumulate on $x \geq 0$. Consider any vertical line $x = \nu \leq 0$, $0 < y < \tau$, where $0 < \tau < 1/3$. Then h_λ maps this line to a vertical line of the form $(e^\nu + \lambda, 3ye^\nu)$ which crosses infinitely many of the horizontal lines above. Hence, the boundaries accumulate everywhere on $y = 0$. Now apply h_λ^{-k}. It follows that the boundaries of the L_j accumulate everywhere on each ∂L_k. In fact, we have shown more: Let $(x, y) \in \partial L_k$ and let $\mathcal{N}$ be a neighborhood of (x, y). Then there exists k_0 such that, if $k > k_0$, then ∂L_k meets $\mathcal{N}$. □

It follows that the closure of γ is Λ_λ. There is a natural embedding of the curve γ into the Riemann sphere that places its unique endpoint at infinity. The last Proposition implies that the points in γ are the only points accessible from the "exterior" of Λ_λ. Denote by U the exterior region of the curve. Clearly, U is an open simply connected region of the plane whose boundary is γ. Define by $\{A_k\}$ a chain of crosscuts such that A_k lies in the interior of U and the endpoints of each A_k are the point at infinity and the kth iterate of the backward orbit of $(\lambda, 0)$. It is easy to see that $\{A_k\}$ is a fundamental chain having N_k as the component of $U - A_k$ bounded by A_k and γ minus the arc $[h^{-k}(\lambda, 0), \infty]$.

Let η be the prime end associated to this fundamental chain. As γ accumulates everywhere on itself, the impression of η given by

$$I(\eta) = \bigcap_k \overline{N_k},$$

must contain γ. But $I(\eta)$ is by definition a compact set; hence, it contains the closure of γ.

By Rutt's result, $\overline{\gamma}$ is either indecomposable or the union of two indecomposable continua. Assume $\overline{\gamma} = A \cup B$, where A and B are indecomposable. Without loss of generality, assume that the point at infinity belongs to A and there is a point $p \in \gamma$ for which the arc $[p, \infty)$ of γ is also contained in A. If no such point exists, B must contain the arc $[p, \infty)$. But since B is a closed set in the topology inherited from the Riemann sphere, we must have $\infty \in B$. In this case, we choose B.

If $\gamma \cap B \neq \emptyset$, then for any $q \in \gamma \cap B$ there exists an open neighborhood $\mathcal{W}$ of q, such that $\mathcal{W} \cap A = \emptyset$. By Proposition 3.2, there exists $k_0 > 0$ such that, if $k > k_0$, $\overline{L_k}$ meets $\mathcal{W}$. Using the interior of L_k and $\mathcal{W}$, we can construct an open annulus that bounds the arc $[p, \infty]$ away from infinitely many points in γ. Since A is connected, then A is completely contained inside the open annulus.

But A is indecomposable and contains a ray of the form $[x, \infty]$, $x \in \mathbb{R}$. Then there is a point in $z \in A$ that lies above the real line and below ∂L_k (otherwise, A would be decomposable).

Then, there exists $n > k > k_0$ for which L_n lies in between z and $\mathbb{R}^+$, and moreover, L_n enters the neighborhood $\mathcal{W}$. We can construct a second annular region that will bound the point z away from the arc $[p, \infty]$. But this contradicts the fact that A is connected.

Hence, $\gamma \cap B = \emptyset$ and $\gamma \subset A$. Since A is closed, it follows that $B = \emptyset$ and $\overline{\gamma}$ is indecomposable. This ends the proof of Theorem 3.1. □

Proposition 3.3. *The curve γ is a composant of Λ_λ.*

Proof: Assume otherwise. Denote by κ the composant of the point at infinity. Let $z \in \kappa - \gamma$ and H be a proper subcontinuum that contains both z and ∞. Clearly, $\gamma \cap H \neq \emptyset$. Without loss of generality, assume there exists a point x on the real line such that the arc $[x, \infty]$ is the maximal arc that contains the point at infinity and belongs to $\gamma \cap H$. Let $A = H - [x, \infty]$. Since H is connected, A must accumulate on x. But as the regions L_k accumulate on the real line, there exists $N > 0$ for which A also accumulates on ∂L_k, for all $k > N$. Since H is closed, it follows that

$$\overline{\bigcup_{k>N} \partial L_k} \subset H.$$

By Proposition 3.2, $\overline{\bigcup_{k>N} \partial L_k} = \overline{\gamma}$, which contradicts the fact that H is a proper subcontinuum. □

4. Dynamics of h_λ

For the remainder of this paper, we will consider only λ-values larger than λ_0, where λ_0 is given by the following proposition.

Proposition 4.1. *There exists a unique $\lambda_0 > 1$ that solves the equation*

$$\lambda - \exp(\lambda - 1) = \ln(2/3).$$

The proof is straightforward calculus.

If $\lambda > \lambda_0 > 1$, it follows that the contracting rectangle lies to the right of $x = 0$ in S. For convenience, we introduce the real valued functions $f_\lambda(x) = \lambda + e^x$ and $g_\lambda(x) = \lambda - e^x$. Note that f_λ and g_λ are the real parts of h_λ on H_1 and H_2, respectively. In this section we will prove:

Theorem 4.2. *The map h_λ has a unique repelling fixed point p_λ in S. All other orbits have α-limit set given by $\{p_\lambda\}$. The ω-limit set is either*

(1) *the point at ∞ given by $x = \infty, y = 0$, in which case the orbit eventually lands on the real axis, or*
(2) *the orbit of $(\lambda, 0)$ together with points at ∞ to the left and right in S, in which case the orbit eventually cycles through H_1, H_2 and H_3, accumulating on the orbit of $(\lambda, 0)$.*

Before proving this theorem, consider the rectangle $R = \{(x, y) \in S \,|\, 0 \leq x \leq \lambda + 1\}$. Note that R contains the contracting rectangle since $\lambda > \lambda_0 > 1$. There are certain points in R whose orbit leaves S. For example, if $(x, y) \in R$ with $1/3 < y < 2/3$ and $x > 0$, then $(x, y) \in L_1$. Similarly, if $1/3^k < y < 2/3^k$ and $x \geq 0$, then $(x, y) \in L_k$. Let W denote R with all the open strips $1/3^k < y < 2/3^k$ removed. Then W is a union of closed rectangles R_k, $k = 0, 1, 2, \ldots$ where $R_k = \{(x, y) \in R \,|\, 2/3^{k+1} \leq y \leq 1/3^k\}$. Note that $R_0 \subset H_2$ but $R_j \subset H_1$ for $j \geq 1$.

We define the first return map $\phi_\lambda : W \to R$, by $\phi_\lambda(x, y) = h_\lambda^k(x, y)$ where $h_\lambda^k(x, y) \in W$, but $h_\lambda^i(x, y) \notin W$ for $i = 1, \ldots, k-1$. Note that the first iterate under h_λ of the left-hand boundary of R_k, $k \geq 1$, is the right-hand boundary of R_{k-1}. For technical reasons we will not consider these points as the first return points; rather, we will consider the next return as the first return points.

Proposition 4.3. *For any $k \geq 1$, $\phi_\lambda(R_k)$ is a linear horseshoe region that is located strictly below R_k in W.*

Proof: If $(x, y) \in R_k$, $k \geq 1$, then $h_\lambda^i(x, y) \in H_1$ for $i = 0, \ldots, k-1$, and $h_\lambda^k(x, y) \in H_2$. We claim that $h_\lambda^{k+1}(x, y) \in H_3$. Indeed, if $k = 1$, $h_\lambda^2(R_1)$ is the rectangle bounded on the right by $x = g_\lambda(f_\lambda(0)) = \lambda - \exp(\lambda + 1)$ which is negative for $\lambda > 0$. On the left, this rectangle is bounded by $x = g_\lambda(f_\lambda(\lambda+1))$. It follows similarly that $h_\lambda^{k+1}(R_k)$ is also a rectangle in H_3 whose right boundary is given by $x = g_\lambda \circ f_\lambda^k(0)$. It follows that the image of $h_\lambda^{k+1}(R_k)$ is a linear horseshoe region that lies in the contracting rectangle, hence in R. The maximal y-coordinate in $h_\lambda^{k+2}(R_k)$ is given by the y-coordinate of the image of the right hand boundary of $h_\lambda^{k+1}(R_k) \cap A_2$. This y-coordinate is $\exp(g_\lambda \circ f_\lambda^k(0))$. Thus, we need to show that

$$\exp(g_\lambda \circ f_\lambda^k(0)) < 2/3^{k+1}.$$

Notice first that the map $g_\lambda \circ f_\lambda^k(0) = \lambda - \exp(f_\lambda^k(0))$ is a decreasing function of λ when $\lambda > 0$. One checks easily that $g_\lambda \circ f_\lambda^k(0) < -\exp^{k+1}(0)$, which in turn implies that $\exp(g_\lambda \circ f_\lambda^k(0)) < 1/\exp^{k+2}(0)$. Since $3^{k+1} < 2\exp^{k+2}(0)$ holds for all k, the required inequality holds for $\lambda = 0$ and hence for all $\lambda > 0$. □

It follows from the above Proposition that any point in R_k, $k \geq 1$, whose forward orbit remains for all time in S, must repeatedly visit W, and each time this orbit returns to W it does so in an R_k with strictly larger k-value. This is not true in R_0, since the image of R_0 is the rectangle $\lambda - \exp(\lambda + 1) \leq x \leq \lambda + 1$ which contains R_0. Indeed we have:

Proposition 4.4. *There is a repelling fixed point p_λ in R_0.*

Proof: The real function $g_\lambda(x) = \lambda - e^x$ has a fixed point x_λ in the interval $0 \leq x \leq \lambda + 1$ since g_λ is decreasing. Indeed, $g_\lambda(0) = \lambda - 1 > 0$, and $g_\lambda(\lambda + 1) = \lambda - \exp(\lambda + 1) < 0$ when $\lambda > 1$. Similarly, $y \to 3(1 - y)$ has a fixed point at $y = 3/4$. Hence, $p_\lambda = (x_\lambda, 3/4)$ is fixed for h_λ and

$$Dh_\lambda(x_\lambda, 3/4) = \begin{pmatrix} -e^{x_\lambda} & 0 \\ 0 & -3 \end{pmatrix},$$

so p_λ is repelling. □

Proposition 4.5. *If $(x, y) \in R_0$ and $(x, y) \neq p_\lambda$, then the orbit of (x, y) must eventually leave R_0 and enter either L_1 or H_1.*

Proof: We divide R_0 into two rectangles:

$$\begin{aligned} U_0 &= \{(x, y) \in R_0 |\ x \geq \lambda - 1\} \\ U_1 &= \{(x, y) \in R_0 |\ x \leq \lambda - 1\}; \end{aligned}$$

i.e., U_0 lies in the contracting rectangle, U_1 does not.

The image of U_0 is the rectangle

$$\lambda - \exp(\lambda + 1) \leq x \leq \lambda - \exp(\lambda - 1) < 0$$

contained in H_3. We claim that, since $\lambda > \lambda_0$, the image of this rectangle, i.e., $h_\lambda^2(U_0)$, lies below $y = 2/3$ in the contracting rectangle. To see this, note that the vertical line $x = \lambda - \exp(\lambda - 1)$ is mapped to a linear horseshoe curve of height $\exp(\lambda - \exp(\lambda - 1))$. Hence, $h_\lambda^2(U_0)$ lies below $y = 2/3$ provided

$$\exp(\lambda - \exp(\lambda - 1)) < 2/3.$$

But this is precisely the condition that determines λ_0. Hence, $h_\lambda^2(U_0)$ meets only L_1 and H_1, not H_2, and so we are done in this case.

Now suppose $(x, y) \in U_1$. If $h_\lambda(x, y) \in L_1 \cup H_1$, then we are done. If $h_\lambda(x, y) \in H_3$, then $h_\lambda^2(x, y)$ lies in the contracting rectangle, and hence, in one of U_0, L_1, or H_1. In all three cases we are done. Finally, if $h_\lambda(x, y)$ remains in H_2, then there is a positive integer n such that $h_\lambda^n(x, y) \notin U_1$ (for otherwise (x, y) is the fixed point). Then $h_\lambda^n(x, y)$ lies in one of H_1, H_3, or U_0 and we are again done. □

5. Dynamics of ϕ_λ

In this section we describe the set of points in Λ_λ whose orbits under the return map ϕ_λ have the same "itinerary." The itinerary of (x, y) is a sequence of integers $(s_0 s_1 s_2 \ldots)$ with $s_j \geq 1$ such that $\phi_\lambda^k(x, y) \in R_{s_k}$ for each k. Note that we do not define the itinerary of points in R_0.

Recall that ϕ_λ maps each R_k for $k \geq 1$ to a linear horseshoe region inside the contracting rectangle that lies strictly below R_k. In particular, $\phi_\lambda(R_k)$ meets infinitely many R_j with $j > k$ in a pair of rectangles. $\phi_\lambda(R_k)$ may intersect other R_j with smaller indices, but this intersection will not be a pair of rectangles (see Figure 3).

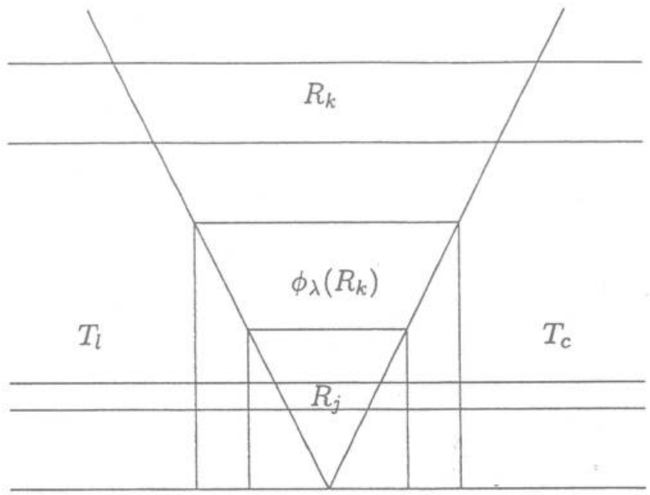

FIGURE 3. The image of $\phi_\lambda(R_k)$ meets infinitely many of the R_j in a pair of rectangles.

In this case, $\phi_\lambda(R_k)$ meets R_j in the two triangular regions T_ℓ and T_r lying below the diagonals in the contracting rectangle. We say that $\phi_\lambda(R_k)$ cuts completely across R_j in this case. Note that, if $\phi_\lambda(R_k)$ meets R_j in T_ℓ or T_r, then ϕ_λ maps vertical line segments in $\phi_\lambda^{-1}(R_j) \cap R_k$ to vertical line segments in R_j. Indeed, the only region where vertical lines are not preserved by ϕ_λ is A_2, and such an orbit does not visit A_2 before returning to W.

For any $k \geq 1$, let $\ell(k)$ denote the smallest integer such that $\phi_\lambda(R_k)$ cuts completely across $R_{\ell(k)}$. Clearly, $\ell(k) > k$, and, in fact, $\ell(k) >> k$ for k large.

Let ν be a continuous curve lying in some R_k. We say that ν is a *horizontal curve* if ν meets each vertical line $x = \nu$ with $0 \leq \nu \leq \lambda + 1$ exactly once. A *horizontal strip* is a region in R_k between two nonintersecting horizontal curves.

Now let $t = (t_1, t_2, t_3, \ldots)$ be a sequence of positive integers that satisfy $t_{k+1} \geq \ell(t_k)$ for $k \geq 1$. Such a sequence is called *admissible* for ϕ_λ. We have

Theorem 5.1. *Suppose t is an admissible sequence for ϕ_λ. Then*

$$\{(x,y) \in R_{t_1} \,|\, \text{the itinerary of } (x,y) \text{ is } (t_1, t_2, t_3, \ldots)\}$$

is a Cantor set of horizontal curves in Λ_λ.

Proof: Let $V_{t_1 \ldots t_k}$ denote the set of points whose itinerary begins with $t_1 \ldots t_k$. We have that $V_{t_1 t_2} = R_{t_1} \cap \phi_\lambda^{-1}(R_{t_2})$ is a pair of horizontal strips in R_{t_1}. Inductively, $V_{t_1 \ldots t_{k+1}}$ consists of 2^k horizontal strips in R_{t_1}, and $V_{t_1 \ldots t_{k+1}} \subset V_{t_1 \ldots t_k}$. It follows that the nested intersection of the $V_{t_1 \ldots t_k}$ is a Cantor set of horizontal components in R_{t_1}. Since ϕ_λ maps and contracts vertical vectors to vertical vectors, it follows that each of these components is actually a horizontal curve. Since each of these horizontal curves is an accumulation of horizontal curves that lie in ∂L_n for some n, it follows that these curves lie in Λ_λ. □

We now turn to the construction of a second curve μ that accumulates everywhere on itself and on γ. This curve is constructed by "connecting" various pieces of the Cantor set of curves that correspond to a given admissible sequence. The admissible sequence $(s_0 s_1 s_2 \ldots)$ involved will have the property that the digits s_j tend to ∞ extremely rapidly.

For concreteness, and to begin the construction, we choose $s_0 = 1$. So our sequence will be $(1 s_1 s_2 \ldots)$. We will specify the s_j inductively. Now $\phi_\lambda(R_1) = h_\lambda^3(R_1)$ is a linear horseshoe region that cuts completely across R_k for any $k \geq \ell(1)$. Choose any $s_1 \geq \ell(1)$. Then, for any choice of admissible sequence that begins $1 s_1$, there is a Cantor set of horizontal curves in R_1 corresponding to this itinerary. For any such choice, we may select one such horizontal curve, say τ_1. The curve $h_\lambda^3(\tau_1)$ is a horizontal segment lying in R_{s_1} inside one of the horizontal curves whose itinerary is $(s_1 s_2 s_3 \ldots)$. Call this horizontal curve τ_{s_1}. Now the preimage of τ_{s_1} under h_λ^3 is a curve that crosses R_1 in two horizontal curves, one of which is τ_1. Let $\tau_{1 s_1}$ denote the preimage of τ_{s_1} under h_λ^3. Note that $\tau_{1 s_1}$ extends beyond the boundaries of R_1 and is mapped in one-to-one fashion onto τ_{s_1} by h_λ^3 (see Figure 4).

We now continue this procedure inductively. For any $s_2 > \ell(s_1)$, the image of R_{s_1} under ϕ_λ is again a linear horsehoe region that cuts completely across R_{s_2}, and there is a Cantor set of horizontal curves in this strip that corresponds to any admissible itinerary $(s_2 s_3 s_4 \ldots)$. Note that $\phi_\lambda = h_\lambda^{s_1+2}$ in R_{s_1}. As above, $\phi_\lambda(\tau_{s_1})$ lies in one of these curves, say τ_{s_2}. Now pull back this horizontal curve by $h_\lambda^{-(s_1+2)}$. As above, we get a new curve $\tau_{s_1 s_2}$ that cuts completely

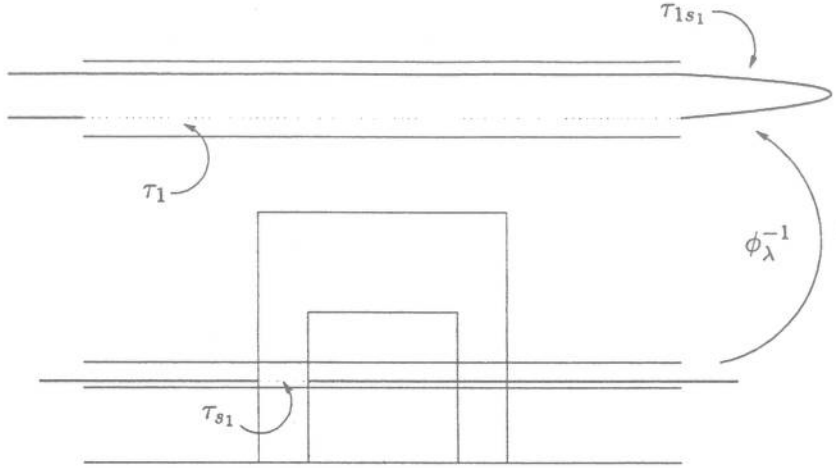

FIGURE 4. The curve τ_{1s_1} is the extension of τ_1 obtained by pulling back the curve τ_{s_1} under ϕ_λ^{-1}.

across R_{s_1} twice and extends τ_{s_1}. If we now pull $\tau_{s_1s_2}$ back by the original map, h_λ^{-3}, we obtain a new curve which we call $\tau_{1s_1s_2}$. This curve crosses the strip R_1 in at least four horizontal curves and extends τ_{1s_1}.

We now put additional restrictions on s_2 to control the behavior of $\tau_{1s_1s_2}$. Toward that end, we break up the right hand portion of the strip S into countably many rectangles $Q_j, j = 0, 1, \dots$ where

$$Q_j = \{(x, y) \in S \mid h_\lambda^j(0) \leq x \leq h_\lambda^{j+1}(0)\}.$$

Note that Q_0 contains all of the rectangles R_i, while Q_j contains the images of R_i under h_λ^j, provided that $i \geq j$. In particular, note that $h_\lambda^3(R_{s_1+3})$ is contained in Q_3 far to the right of R_{s_1}.

We now choose s_2 so that not only $\phi_\lambda(R_{s_1})$ cuts completely across R_{s_2}, but also $\phi_\lambda(R_{s_1+3})$ does as well. Equivalently, the four linear horseshoe regions $\phi_\lambda(R_{s_1+i})$ for $i = 0, 1, 2, 3$ cut completely across R_{s_2}. It follows that $\tau_{s_1s_2}$ is a curve that cuts twice completely across not only R_{s_1}, but also across the horizontal extensions of this rectangle in Q_j for $j = 1, 2, 3$. Now consider the pull back of this curve to $\tau_{1s_1s_2}$. This curve meets R_1 as above, but it must also cut completely across R_{s_1+3} (see Figure 5). That is, the extended curve $\tau_{1s_1s_2}$ meets both R_1 and the much lower R_{s_1+3}.

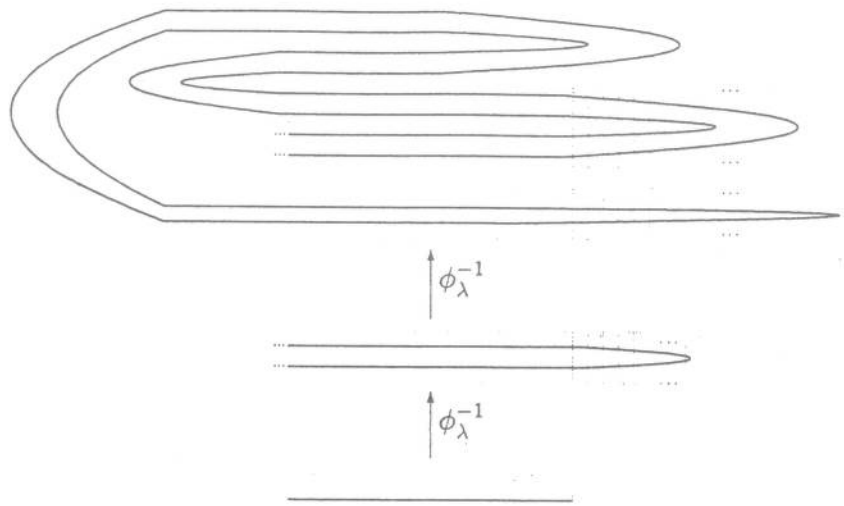

FIGURE 5. Two steps in the construction of μ.

By choosing s_2 larger so that many more linear horshoe regions of the form $\phi_\lambda(R_{s_1+k})$ cross R_{s_2}, we may guarantee that the extended curve crosses not only R_{s_1+3}, but also the extensions of this rectangle to the right into $Q_1, Q_2, \ldots Q_n$.

Continuing in such fashion, we may choose the s_j so that the curves $\tau_{1s_1s_2\ldots s_n}$ accumulate everywhere on the positive real axis as $n \to \infty$. If we choose μ to be the countable union of such extensions, we find a curve that accumulates everywhere on the positive reals. It is not hard to see that, in fact, μ accumulates on all of γ and, hence, on itself. Since μ is dense in Λ_λ, a similar argument as in Proposition 3.3 shows that μ is a composant of Λ_λ.

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