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EXISTENCE OF INDECOMPOSABLE CONTINUA FOR UNSTABLE EXPONENTIALS

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ABSTRACT. In the parameter plane for the complex exponential family $E_\lambda(z) = \lambda e^z$ there exist parameters for which the orbit of zero lies on dynamical curves which are invariant under a fixed power of E_λ. At the same time, the orbit of zero tends to infinity and in these cases, the Julia set for E_λ is the whole complex plane. We construct fundamental regions based on these dynamical curves. Inside each region, we show the existence of an invariant set that, once properly compactified, becomes an indecomposable continuum.

1. Introduction

In this paper, we work with complex parameters for which the orbit of λ under the exponential map lies on n dynamical curves and tends to infinity. For those parameters, it is known that the Julia set of E_λ is the whole complex plane (see [9] and [6]). Moreover, λ will lie on regions of the parameter plane for which a small perturbation can produce an attracting cycle. For these parameters, E_λ is called an unstable exponential (see [3]).

In Figure 1, we partially depict the composant of zero for the invariant set of the map $0.6\exp(z)$, previously described by R. L. Devaney. Our goal is to generalize the results presented in [4] for certain complex parameters. We do so by describing the construction of fundamental regions in the dynamical plane and showing the existence of invariant sets under E_λ^n inside each region. The

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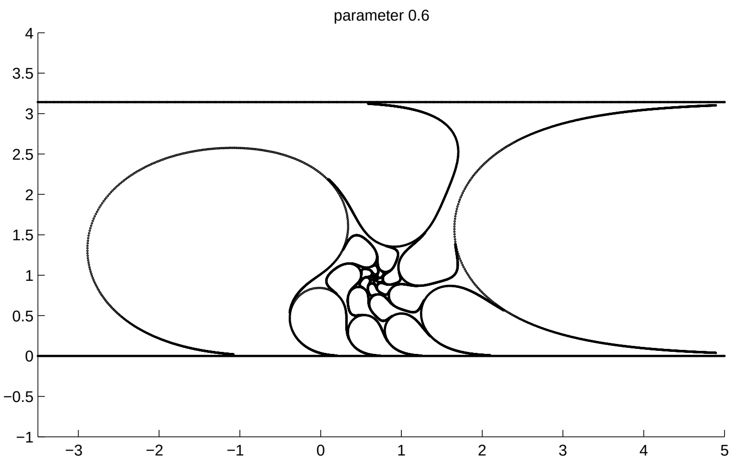

FIGURE 1. A partial picture of the invariant set Λ_λ for $\lambda = 0.6$. All curves extend toward infinity to the right without intersecting among themselves and wind around the repelling fixed point.

compactification of these invariant sets will result in continua with the same properties as the one described in [4].

We have structured this paper as follows: in Section 2 we review some results related to the dynamical curves for the complex exponential and describe the setup for our construction. The analysis of the fundamental regions, their topology and dynamics are given in Section 3, where we restrict our attention to the case $n = 2$ for clarity. The general case is presented in Section 4. A brief analysis of the dynamics restricted to each invariant set and some additional remarks are found in Secton 5.

2. Hairs in the Dynamical Plane

Let $\lambda \in \mathbb{C}$ and consider a partition of the complex plane minus the nonpositive real numbers (denoted by $\mathbb{C}^*$) by horizontal strips

$$R_k = \{z \in \mathbb{C} \mid (2k-1)\pi - \arg\lambda < \mathrm{Im} z < (2k+1)\pi - \arg\lambda\}$$

with arg λ taking values between $\pm\pi$ and $k \in \mathbb{Z}$. The strips are indexed so that k increases with increasing imaginary part. Define the itinerary of $z \in \mathbb{C}^*$ under E_λ in the usual way: $s = s_0 s_1 s_2 \ldots$

is the itinerary of z if and only if

$$s_j = k \quad \text{when} \quad E_\lambda^j(z) \in R_k.$$

When all s_j are nonzero integers, the itinerary is called *regular*; otherwise, it is called *irregular*. Let σ represent the *one-sided shift* map. An itinerary is *periodic* of period n if it is a fixed point for σ^n and $\sigma^j(s) \neq s$ for $j = 1, 2, \ldots, n-1$. We indicate the periodicity of an itinerary by writing $s = \overline{s_0 s_1 \ldots s_{n-1}}$.

Notice that E_λ maps the boundary of R_k onto the negative real axis, so $E_\lambda(R_k)$ is sent onto $\mathbb{C}^*$. No itinerary is defined for points whose orbit lands in the boundary of any R_k strip.

We recall some standard definitions found in [1]. A *hair* is a continuous curve extending from a particular point z_λ towards infinity in the right half plane. All points in a hair have the same itinerary. We call z_λ the *dynamical endpoint* of the hair. When restricted to periodic itineraries, it is the unique point in the hair with bounded orbit. Every other point in the hair has unbounded orbit (thus, each hair belongs to the Julia set). There exist similar objects in the parameter plane. Let H_s represent a continuous curve in the parameter plane for which, for every parameter λ in H_s, the orbit of λ under E_λ follows the given itinerary s in the dynamical plane.

We are solely interested in irregular periodic itineraries, that is, itineraries of the form $s = \overline{0 s_1 s_2 \ldots s_{n-1}}$ for which the singular value, $z = 0$, will follow. In general, C. Bodelón and his co-authors [1] have shown the existence of hairs for irregular itineraries in both the dynamical and parameter plane, but their definition is restricted to a region far to the right in the plane.

The general setup is as follows: given the itinerary $\overline{s_1 \ldots s_{n-1} 0}$ with $s_1 \neq 0$, let h_1 denote the dynamical curve on which λ lies. Then λ and any other point in h_1 have this itinerary. Then, there must exist a curve inside the strip R_0 that is the image of h_1 under E_λ^{n-1}. Pulling back by E_λ the piece of h_1 between λ and $E_\lambda^n(\lambda)$, we can uniquely extend the curve inside the strip R_0 to a curve on which the singular value lies. Denote this curve by h_0. Then all points in h_0 will follow the itinerary $s = \overline{0 s_1 s_2 \ldots s_{n-1}}$.

We require that for the parameter λ that lies on $H_{\overline{s_1 s_2 \ldots 0}}$,

$$E_\lambda^k(0) \longrightarrow \infty \quad \text{when} \quad k \longrightarrow \infty.$$

For $k = 1, \dots, n-1$, let $h_k = E_\lambda^k(h_0)$ denote the dynamical curve associated with the itinerary $\sigma^k(s)$. Notice that each h_k is forward invariant under E_λ^n, so each $E_\lambda^j(0)$ belongs to h_k if and only if $k \equiv j \pmod{n}$. We call each $E_\lambda^k(0)$ an *endpoint* of the curve h_k $(k = 0, \dots, n-1)$ in the strict topological sense.

As an example of this setup, notice that the results obtained in [4] apply to every λ in the parametrical curve $H_{\overline{0}}$ which is directly attached to the cusp of the cardioid in the parameter plane. $H_{\overline{0}}$ corresponds to the segment of the real line $\lambda \geq 1/e$. Clearly $E_\lambda^k(0)$ grows without bound. The itinerary of zero is $s = \overline{0}$ and the segment of the positive real line acts as the forward invariant curve h_0.

If λ belongs to the parametric hair H_s for our setup, as the orbit of zero tends to infinity, the Julia set for E_λ is the whole complex plane. By construction, each h_k and its endpoint $E_\lambda^k(0)$ are sent respectively *onto* the tail h_{k+1} and $E_\lambda^{k+1}(0)$ for $k = 0, \dots, n-2$. Since $E_\lambda^n(0) \in h_0$, the tail h_{n-1} is mapped *into* h_0.

3. Fundamental Regions

In this section, we describe the construction of the fundamental regions that will contain our indecomposable continua. For clarity, we restrict ourselves to the case $n = 2$, that is, for an itinerary of the form $s = \overline{0s_1}$. Thus, we assume the existence of two dynamical curves, h_0 and h_1 where zero and λ lie respectively on each curve. The general case will be discussed in Section 4.

To define our fundamental regions we will take sucessive preimages of a piece of h_0. Notice that since h_1 is sent into (but not onto) h_0, the preimage of the piece of h_0 from 0 to $E_\lambda^2(0)$ consists of infinitely many curves that extend without bound toward infinity in the left half plane. Each preimage has an endpoint at $E_\lambda(0) + i2k\pi$ with $k \in \mathbb{Z}$. Let α represent the unique component of the preimage directly attached to $E_\lambda(0)$. We call α an *extension* to the hair h_1 and $\hat{h}_1$ represents the hair with its extension.

Define by β_0 the curve among the preimage curves of α under E_λ such that β_0 is the extension to the hair h_0. This curve extends from 0 to infinity in the right half plane. In particular, there are two ways in which the extension β_0 may tend to infinity: far to the right, either $\operatorname{Im} \beta_0 > \operatorname{Im} h_0$ or $\operatorname{Im} \beta_0 < \operatorname{Im} h_0$.

Either of two orientations (the *upper* or *lower* orientation, respectively) may occur and they are completely determined by λ. To show this, we first need several definitions.

Definition 3.1. The extended curve $\hat{h}_1$ separates the complex plane into two open half planes. Let H^+ be the half plane above $\hat{h}_1$ and denote by H^- the half plane below $\hat{h}_1$.

Also $\hat{h}_0$ separates the complex plane into two open and simply connected regions. Denote by R_0^u the region relative to $\hat{h}_0$ with unbounded real part, and let R_0^b denote the region with bounded negative real part.

Proposition 3.2. Given $\hat{h}_0$ and $\hat{h}_1$ as described above, λ determines the orientation of β_0 as follows: if Im $\lambda > 0$ then β_0 has the upper orientation. Otherwise, β_0 has the lower orientation if Im $\lambda < 0$.

Proof: Consider the sign of $\operatorname{Im}\lambda$. If positive, this implies that $0 \in H^-$. Since $\hat{h}_0$ is the preimage of $\hat{h}_1$, and E_λ^{-1} sends any small neighborhood of zero far to the left half plane, the region R_0^u with unbounded real part must be the preimage of H^-. This implies that β_0 has the upper orientation. If $\operatorname{Im}\lambda < 0$, then $0 \in H^+$ and by a similar argument, it follows that β_0 has the lower orientation. $\square$

Let γ be a curve in the preimage of β_0 such that

(1) $\gamma \subset H^+$
(2) γ is the closest curve to $\hat{h}_1$ among all curves in the preimage of β_0.

Notice that γ and $\hat{h}_1$ extend towards infinity in the left and right half planes. In general, any preimage of a horizontal strip under the exponential is a horseshoe shaped region with its ends extending to the right half plane. Choose δ_0 to be the closest curve to $\hat{h}_0$ among all other preimage curves of γ (see Figure 2). Then, δ_0 and $\hat{h}_0$ will represent the boundary of a horseshoe shaped region. Therefore, we have

Definition 3.3. (Fundamental Regions) Let S_1 denote the closed strip that is bounded above by γ and below by $\hat{h}_1$. Also, let S_0 denote the closed horseshoe shaped region bounded by δ_0 and $\hat{h}_0$. Thus, S_0 represents the preimage of S_1.

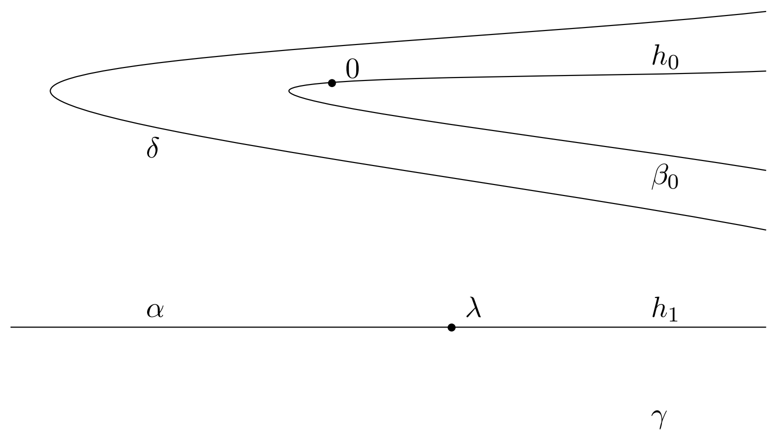

FIGURE 2. Tails and several preimages of α. Since $\operatorname{Im} \lambda < 0$, β_0 has the lower orientation.

By construction, if β_0 has the lower orientation, then δ_0 will be in the unbounded region R_0^u. If β_0 has the upper orientation, δ_0 will then lie on R_0^b. Since the following construction is independent of the orientation, we will assume through the rest of the paper that β_0 has the lower orientation.

By construction, a non-empty intersection of two fundamental regions can only happen if one region lies completely inside the other. The following proposition excludes the possibility of nested fundamental regions. The relevance of this result will become apparent in the next section.

Clearly S_1 cannot be in the interior of S_0 as the first region extends without bound toward the left and right half planes of the complex plane. For the case $n = 2$, there is only one case left to consider.

Proposition 3.4. Given λ, let S_0 and S_1 be the regions defined as above. Then S_0 cannot be contained in the interior of S_1.

Proof: We proceed by contradiction. Assume $S_0 \subset S_1$. Then E_λ^{-1} maps S_1 onto a horseshoe region, and so the preimage has bounded negative real part. Since S_0 contains the omitted value, the preimage of S_0 is a connected region with unbounded real part and must be contained inside the preimage of S_1. Thus, we have a contradiction. $\square$

3.1. Invariant Sets and Compactification

We first introduce a new partition of the complex plane in order to define itineraries for the orbits of points in which we are interested. Using these itineraries, we characterize the invariant sets under E_λ^2 inside S_0 and S_1.

We have previously selected a preimage of β_0, namely γ , lying in H^+. But there exist infinitely many preimages of β_0 that are no more than translations of γ^+ by $2\pi i$. We denote these curves by γ_j with $j \in \mathbb{Z}$. We index γ_j so that j increases with increasing imaginary part, so $\gamma_{j+1} = \gamma_j + 2\pi i$ for each j. In particular, we choose γ_0 so that the origin lies in the region bounded by γ_0 and γ_1. Also define $T_j, j \in \mathbb{Z}$, to be the open region bounded by γ_j and γ_{j+1}. Since γ_j is mapped onto β_0 by E_λ, then every T_j is mapped into $\mathbb{C} - 0$ injectively by E_λ. Thus, E_λ is an expansion inside each T_j and there must exist a repelling fixed point inside each T_j. This new partition allows us to define a new itinerary for almost any point in $\mathbb{C}$. We will use the same notation for these new itineraries.

Definition 3.5. Given $z \in \mathbb{C}$, we define its itinerary with respect to the partition T_j to be the sequence $s = s_0 s_1 s_2 \dots$ where

$$s_k = j \text{ if and only if } E_\lambda^k(z) \in T_j.$$

Although there is not a well defined itinerary for points that belong to any γ_j or any of its preimages, these points are eventually mapped into the hairs, and their dynamics are already known. Since neither the hairs nor any of its preimages intersect among themselves, this guarantees that S_0 is properly contained inside the region T_0.

Our goal is to study the set of points inside each S_k for $k = 0, 1$ and with an itinerary of the form $s = \overline{0 s_1}$, where $S_0 \subset T_0$ and $E_\lambda(S_0) \subset T_{s_1}$. Proposition 3.4 implies that $s_1 \neq 0$ and, for the general case, new itineraries associated with our construction will consist of non-repeating entries. This condition provides a better understanding of the parameters for which our construction is valid.

Since we have assumed β_0 has the lower orientation, $E_\lambda^2(S_0)$ is sent onto R_0^u. Therefore, there are points in S_0 that will leave this region under the action of E_λ^2. Let Λ_0 be the set of points in S_0 whose orbits under E_λ^2 never leave. Then any point in Λ_0 has itinerary $s = \overline{0 s_1}$ with $S_0 \subset T_0$ and $S_1 \subset T_{s_1}$.

Also, define

$$L_n = \{z \in S_0 | E_\lambda^{2k}(z) \in S_0, \text{ for } k = 1, 2, \ldots, n-1 \text{ but } E_\lambda^{2n}(z) \notin S_0\}$$

that is, L_n represents the set of points inside S_0 whose first $n-1$ iterations under E_λ^2 remain inside S_0 but its nth iteration leaves the region.

The next results follow from [4]:

(1) Each L_n is an open simply connected subset of S_0 that extends to infinity toward the right of S_0. Moreover, far to the right, each region L_n is bounded above by L_{n-1} and below by $\hat{h}_0$,
(2) $\Lambda_0 = S_0 - \cup_{n \in \mathbb{N}} L_n$,
(3) $\cup \partial L_n$ is dense in Λ_0,
(4) Λ_0 is a closed and connected subset of S_0.

In order to make Λ_0 a continuum, we first need to compactify the set in the plane by adding the backward orbit of zero under E_λ^2. To do so, we first compactify each curve $\hat{h}_0, \hat{h}_1$, the boundary curves of S_0, and all the curves ∂L_k by adding their endpoints at infinity. Then, we identify the endpoints at infinity with $E_\lambda^{-2}(0)$, $E_\lambda^{-4}(0)$, and so on.

Let Γ_0 represent the compactification of Λ_0. Γ_0 is then a curve obtained by joining the boundary of S_0 and all the boundaries of L_n by the endpoints at infinity. The density of $\cup \partial L_n$ allows us to show that Γ_0 will accumulate everywhere upon itself. By Montel's theorem, it follows that Γ_0 does not separate the plane. To show that Γ_0 is indecomposable, we make use of the next theorem due to S. B. Curry (see [2]).

Theorem 3.6. *Suppose X is a one-dimensional nonseparating continuum which is the closure of a ray that limits upon itself. Then X is indecomposable.*

Similar arguments show that S_1 yields an invariant set Λ_1 whose points have itinerary $\overline{s_1 0}$. Its compactification Γ_1 is obtained after adding the backward orbit of λ under E_λ^2.

4. General Case

After describing the construction of the case $n = 2$, the general case follows after certain remarks on the orientation of the β_k curves and nested regions are made.

Notice that h_{n-1} is the only curve sent into h_0, while all other curves h_k are sent onto h_{k+1} for $k = 0, \dots, n-2$. Then let α be the curve in the preimage of the piece of h_0 from 0 to $E_\lambda^n(0)$ that is directly attached to the point $E_\lambda^{n-1}(0)$.

Define β_k as the curve among the preimage curves of α under E_λ^{k+1-n} such that β_k is the extension to the curve h_k, $k = 0, 1, \dots, n-2$. Each curve extends from $E_\lambda^{k+1-n}(0)$ to infinity in the right half plane. As before, $\hat{h}_{n-1}$ separates the complex plane into H^+ and H^-.

Definition 4.1. For each $k = 0, 1, \dots, n-2$, $\hat{h}_k$ separates the complex plane into two open and simply connected regions. Denote by R_k^u the region relative to $\hat{h}_k$ with unbounded real part, and let R_k^b denote the region with bounded negative real part.

The orientation of each β_k is given by the next proposition.

Proposition 4.2. Given λ and $\hat{h}_0, \dots, \hat{h}_{n-1}$ as described above, λ determines the orientation of every β_k curve as follows:

(1) If Im $E_\lambda^{n-1}(0) > 0$, then β_{n-2} has the upper orientation. Otherwise, if Im $E_\lambda^{n-1}(0) < 0$, then β_{n-2} has the lower orientation.

(2) Recursively, the orientation of β_k is the same as β_{k+1} if $0 \in R_{k+1}^u$. Otherwise, β_k has the opposite orientation to β_{k+1} if $0 \in R_{k+1}^b$.

Proof: The first part follows from Proposition 3.2 applied to β_{n-2}.

To determine the orientation for the rest of the β_k curves, we need to consider the location of the singular value in terms of the extended curves. Assume we have determined the orientation of β_j for $j = n-2, n-3, \dots, k+1$. We wish to find the orientation for β_k. If $0 \in R_{k+1}^u$, then its preimage must be a region with unbounded real part. That is, E_λ maps

$$R_k^u \longrightarrow R_{k+1}^u \quad \text{and} \quad R_k^b \longrightarrow R_{k+1}^b,$$

implying that β_k must have the same orientation as β_{k+1}. If $0 \in R_{k+1}^b$, then we have

$$R_k^u \longrightarrow R_{k+1}^b \quad \text{and} \quad R_k^b \longrightarrow R_{k+1}^u$$

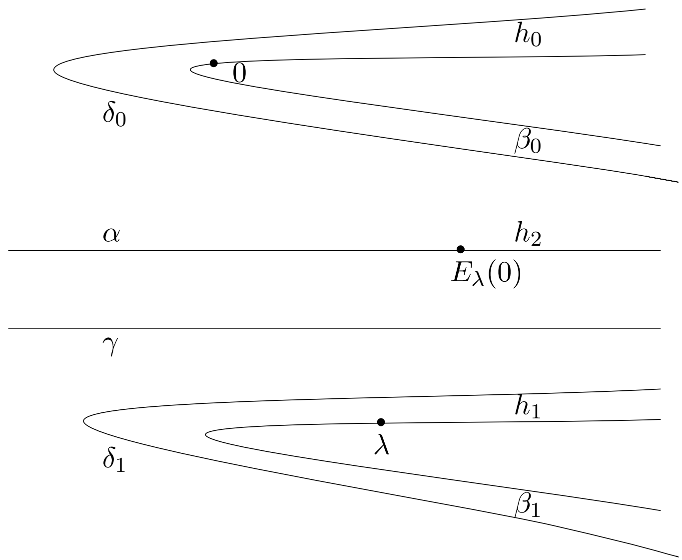

FIGURE 3. Tails and several preimages of α. Since Im $E_\lambda^2(0) < 0$, β_1 has the lower orientation. As $0 \in R_1^u$, β_0 has also the lower orientation.

and β_k must have an opposite orientation with respect to β_{k+1} as above. $\square$

Let γ be the suitable preimage of β_0 and denote by δ_k the suitable preimage of γ under E_λ^{k+1-n}. Then, γ and $\hat{h}_{n-1}$ bound the region S_{n-1}, which is homeomorphic to a horizontal strip. Similarly, δ_k and $\hat{h}_k$ bound the horseshoe shaped region S_k (see Figure 3).

As in Section 3, the orientation of β_k determines where δ_k lies with respect to R_k^u and R_k^b, but is not relevant in the construction. To exclude the possibility of nested fundamental regions we need the generalization of Proposition 3.4.

Proposition 4.3. Given λ, let $S_0, S_1, \ldots, S_{n-1}$ be the regions defined as above. Then no nested regions can occur, that is, for any $j, k = 0, 1, \ldots, n-1$, $S_k \nsubseteq S_j$, if $j \neq k$.

Proof: We proceed by contradiction. Let $0 \leq k < j \leq n-1$. First, assume $S_k \subset S_j$. Then, E_λ^{-k} maps the fundamental regions as

$$S_k \longrightarrow S_0 \quad \text{and} \quad S_j \longrightarrow S_{j-k}.$$

By assumption, S_{j-k} is a horseshoe region. To reach a contradiction, we need to take another preimage and apply Proposition 3.4.

If $S_j \subset S_k$, the contradiction is straightforward, as this case reduces to consider two horseshoe regions, the one with larger index nested in the second. By applying E_λ a finite number of times, we will obtain the strip S_{n-1} contained inside a horseshoe region. $\square$

Using the new set of fundamental regions and itineraries for the nth case, we can easily define the invariant sets Λ_k and their compactification. We summarize our results for the general case in the next theorem.

Theorem 4.4. *Let s denote an irregular periodic itinerary*

$$s = \overline{0s_1 \ldots s_{n-1}}$$

with $s_j \neq s_k$. Given n dynamical curves $h_0, h_1, \ldots, h_{n-1}$ as described before, there exists a parameter $\lambda \in \mathbb{C}$ such that, if Λ_s represents the set of points with itinerary s under E_λ, then Λ_s is invariant under E_λ^n. Once Λ_s is compactified in the plane, it contains an indecomposable continuum that does not separate the plane. In general, these properties hold for each $\Lambda_{\sigma^k(s)}$ with $k = 0, 1, \ldots, n-1$.

5. Further Remarks

We can extend the results in [4] to obtain a picture of the dynamics of points in each Λ_k. By construction, there is a unique fixed point p_k for the map E_λ^n when restricted to each invariant set. Thus, for any point $z \in \Lambda_k - \{p_k\}$, its α-limit set is p_k and its ω-limit set is either the point at infinity or the orbit of $E_\lambda^k(0)$ under the nth iterate of E_λ plus the point at infinity.

Little is known about the conjugacy classes of the invariants sets found in [4] or those presented here. Related results are addressed in [5] and [8] for a semilinear family that reproduces the dynamics of E_λ and its topology.

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