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Playing Catch-Up with Iterated Exponentials

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1. INTRODUCTION. Suppose that we have two animals that make the same number of strides per minute, but that one of them takes larger strides than the other. If the strides of the smaller animal (the prey) have length a and those of the larger animal (the predator) have length b, it is easy to see that a persistent predator will always be able to catch up with its prey. Let us assume that the prey starts one step ahead of the predator. After n steps the distance between the two is

$$nb - (n+1)a = n(b-a) - a$$

and consequently, if $n > a/(b-a)$, the predator will have overtaken its prey.

Let us now imagine a planet on which creatures move by jumps of increasing length. A creature on such a planet is at a distance a from where it started after one jump, a distance a^2 after two jumps, and a distance a^n after n jumps. Let us also assume that $a > 1$ so that creatures move away from their starting points. We can again ask whether a small creature that starts one step ahead of a predator can escape from it. We assume that the initial step of the predator is of size $b > a > 1$, so that if $b^n > a^{n+1}$, the smaller creature is in the maw (or the extraterrestrial equivalent) of its predator. A simple calculation shows that this happens if the predator is sufficiently persistent to make n jumps, where

$$n > (\log a)\left(\log \frac{b}{a}\right)^{-1}.$$

Of course, we can imagine an even stranger planet on which a creature makes an initial jump of size a, followed by a jump that moves it at distance a^a from its starting place, and another that brings it to a distance a^{a^a}, and so on. Thus the distance that such creatures travel is determined by *towers of a*.

Definition. Given any positive real number a, define the *zero tower* $T(a, 0)$ of a by $T(a, 0) = a$. Recursively, for $n \geq 1$ define the nth tower $T(a, n)$ of a by

$$T(a, n) = a^{T(a,n-1)}.$$

Ackermann has introduced a natural way of ordering operations on real numbers [1], so that addition is an operation of type 1, multiplication is of type 2, exponentiation is of type 3, the operation $T(a, n)$ is of type 4, and so on. We therefore live on a type 2 planet, since our movement in space is determined by operations of type 2. The two imaginary planets we have described are of types 3 and 4, respectively.

Therefore, creatures on planets of types 2 and 3 cannot escape their predators, even if they have a head start. Is the same true for creatures on a planet of type 4? Surprisingly, if their step grows to a sufficient size, they will be able to escape faster predators, no matter how persistent. More precisely, we will prove:

Theorem 1.1. *If $a \leq e^{1/e}$ and $b > a$, then there exists an index $n_0(a)$ such that*

$$T(b, n) > T(a, n + 1)$$

whenever $n > n_0(a)$. If $a > e^{1/e}$, then there exists a number $b_0(a)$ with $b_0(a) > 0$ such that

$$T(b, n) < T(a, n + 1)$$

for all n and all b in $(a, b_0(a)]$.

Thus creatures with step size greater than $e^{1/e}$ can escape predators whose initial step is smaller than $b_0(a)$, while smaller creatures always get caught.

2. PROOF OF THE THEOREM. Note that if we define $F_a(x) = a^x$ and denote the n-fold composition of F_a with itself by $F_a^n(x)$, then $F_a^n(a) = T(a, n + 1)$. The graph of $F_a(x)$ for $a < e^{1/e}$ intersects the line $y = x$ in two points $l(a)$ and $r(a)$, both of which are to the right of a. Under the iteration of $F_a(x)$, $l(a)$ is attracting, hence the sequence $F_a^n(a) = T(a, n + 1)$ approaches it from the left. In other words, a creature whose initial step size a satisfies $a < e^{1/e}$ tires quickly, takes progressively smaller steps, and never makes it past $l(a)$. It is easy to see that if $b > a$ then either the graph of $F_b(x)$ does not intersect the line $y = x$ or $l(b)$ is to the right of $l(a)$. In the first case the larger creature never tires, whereas in the second it moves toward the point $l(b)$. In either case it will overtake the smaller creature eventually (see Figure 1). The case $a = e^{1/e}$ can be treated similarly.

On the other hand, when $a > e^{1/e}$, the graph of $F_a(x)$ does not cross the diagonal, so F_a has no fixed points. In this case we show that the prey may escape to infinity and thereby elude its predator.

Note that if the initial steps are slightly larger than $e^{1/e}$ the creature will initially slow down until it makes it past the point $x = e$, after which it will catch a second wind, and make progressively larger leaps. Still, it is not clear whether the prey will be able to escape its predator.

To handle the case $a > e^{1/e}$ we convert the problem of comparing towers of powers of different bases to a problem of comparing iterates of an exponential map. We need the following lemma:

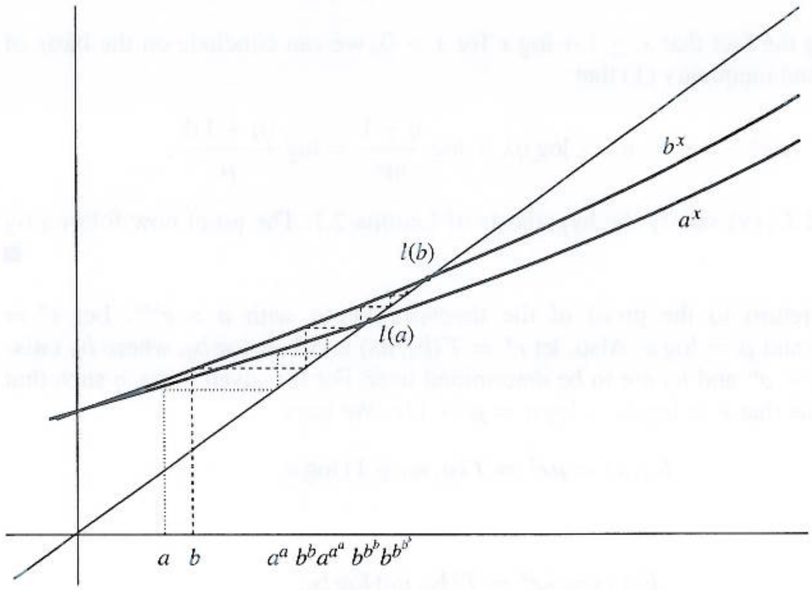

Figure 1. The towers of a and b converge to $l(a)$ and $l(b)$, respectively.

Lemma 2.1. *Fix λ and μ satisfying $\lambda > \mu > 0$, and let $\eta > 0$. If $x > 0$ and $y - x > \log((\eta+1)\lambda/\mu)$, then*

$$\mu e^y - \lambda e^x > \eta\lambda > 0.$$

Proof. We have

$$e^{y-x} > (\eta+1)\frac{\lambda}{\mu},$$

so

$$\mu e^{y-x} - \lambda > \eta\lambda,$$

and therefore

$$\mu e^y - \lambda e^x > e^x \eta\lambda > \eta\lambda. \qquad \blacksquare$$

Let $E_\lambda(x) = \lambda e^x$ and $E_\mu(y) = \mu e^y$ with $\lambda > \mu > 1/e$, and let x and y be as in Lemma 2.1. Choose η such that

$$\log\left(\frac{\eta+1}{\eta\mu}\right) < 1. \tag{1}$$

Analogous to the case of F_a, E_λ^j signifies the j-fold iteration of the exponential map E_λ.

Corollary 2.2. *Under the assumptions of Lemma 2.1, $E_\mu^j(y) > E_\lambda^j(x)$ holds for all $j \geq 1$.*

Proof. Using the fact that $x \geq 1 + \log x$ for $x > 0$, we can conclude on the basis of Lemma 2.1 and inequality (1) that

$$\mu e^y - \lambda e^x > \eta\lambda > \log \eta\lambda + \log \frac{\eta + 1}{\eta\mu} = \log \frac{(\eta + 1)\lambda}{\mu},$$

so $E_\lambda(x)$ and $E_\mu(y)$ satisfy the hypothesis of Lemma 2.1. The proof now follows by induction. ■

We now return to the proof of the theorem. Fix a with $a > e^{1/e}$. Let $e^y = T(a, n_0 + 1)$ and $\mu = \log a$. Also, let $e^x = T(b_0, n_0)$ and $\lambda = \log b_0$, where b_0 satisfying $a < b_0 < a^a$ and n_0 are to be determined later. For the given a, fix η such that (1) holds. Note that $\lambda = \log b_0 > \log a = \mu > 1/e$. We have

$$E_\mu(y) = \mu e^y = T(a, n_0 + 1) \log a$$

and

$$E_\lambda(x) = \lambda e^x = T(b_0, n_0) \log b_0.$$

We show that there exist n_0 and b_0 such that x and y satisfy the conditions of Lemma 2.1. Assume that we have done this. From Corollary 2.2 it follows that $E_\mu^j(y) > E_\lambda^j(x)$ for all j. In terms of towers, we have

$$T(a, n_0 + j) \log a > T(b_0, n_0 + j - 1) \log b_0$$

for all j. Using the fact that $\log b_0 > \log a$ and appealing to the monotonicity of the towers, we conclude that

$$T(a, n) > T(b, n - 1) \qquad (2)$$

for all b in (a, b_0) and for all n. It thus suffices to find n_0 and b_0 such that $x > 0$ and $y - x > (\eta + 1)\lambda$.

Because $a > 1/e$ the condition $x > 0$ follows automatically for any b_0 larger than a and any n_0. Since $T(a, n) - T(a, n - 1) \to \infty$ as $n \to \infty$, we can find n_0 such that

$$\big(T(a, n_0) - T(a, n_0 - 1)\big) \log a > (\eta + 3) \log a^a.$$

Therefore,

$$\big(T(a, n_0) - T(a, n_0 - 1)\big) \log a > (\eta + 3) \log b_0,$$

for any value of b_0 for which $a < b_0 < a^a$.

Let $b_1 (> a)$ be defined by

$$T(b_1, n_0 - 1) - T(a, n_0 - 1) = 1.$$

Similarly, we choose b_2 close enough to a so that

$$T(a, n_0 - 1)(\log b_2 - \log a) < \log b_2.$$

Clearly, $a < b_1, b_2 < a^a$. Let $b_0 = \min\{b_1, b_2\}$. From the definition of x and y it follows that

$$y - x = T(a, n_0) \log a - T(b_0, n_0 - 1) \log b_0$$

or

$$y - x = \big(T(a, n_0) - T(a, n_0 - 1)\big) \log a - T(a, n_0 - 1)(\log b_0 - \log a)$$
$$- \big(T(b_0, n_0 - 1) - T(a, n_0 - 1)\big) \log b_0.$$

That is, we have

$$y - x > (\eta + 3) \log b_0 - 2 \log b_0$$

or

$$y - x > (\eta + 1) \log b_0 = (\eta + 1)\lambda.$$

Hence, given any a we can produce b_0 and n_0 so that (2) holds.

3. REMARKS. The smallest initial step a predator needs to take to catch a prey with initial step of size a has a sharp lower bound $\gamma(a)$ given by

$$\gamma(a) = \sup\big\{b \ : \ T(a, n+1) > T(b, n) \text{ for all } n\big\}.$$

We call γ the *catch-up* function. The previous theorem implies that $\gamma(a) = a$ if $a \le e^{1/e}$ and $\gamma(a) > a$ if $a > e^{1/e}$. We can also define γ by letting $b_n(a)$ be the initial step size necessary to catch up in n steps, so that

$$T\big(b_n(a), n\big) = T(a, n+1).$$

Since n is the number of steps that a creature with initial step size $b_n(a)$ needs to take to catch the creature with initial step size a, it follows that $\gamma(a) = \lim_{n\to\infty} b_n(a)$.

The catch-up function has some interesting properties. Using estimates like those in the previous section, one can show that γ is an increasing function. Other properties of the function $\gamma(a)$ are more difficult to establish. We conjecture that the function is smooth. It cannot be analytic at the point $a = e^{1/e}$, and we conjecture that at the point $(e^{1/e}, \gamma(e^{1/e}))$ the graph of γ and the diagonal have a tangency of infinite order.

The catch-up problem has its origins in complex dynamics. The first and third authors have defined a piecewise semilinear family of continuous maps h_λ acting in the plane that has dynamical and topological properties similar to those exhibited by the complex exponential family λe^z (see [2] and [3]). This family acts exponentially in the x-coordinate (the action is conjugate to λe^x), but essentially linearly in the y-coordinate. In [6], the third author has shown that for any pair of parameters λ and μ the maps h_λ and h_μ are not topologically conjugate. The proof is based on the impossibility of catch-up as described earlier, now in the setting of h_λ.

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An Intuitive Derivation of Heron's Formula

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From elementary geometry we learn that two triangles are congruent if their edges have the same lengths, so it should come as no surprise that the edge lengths of a triangle determine the *area* of that triangle. On the other hand, the explicit formula for the area of a triangle in terms of its edge lengths, named for Heron of Alexandria (although attributed to Archimedes [**4**]), seems to be less commonly remembered (as compared with, say, the formulas for the volume of a sphere or the area of a rectangle).

One reason why Heron's formula is so easily forgotten may be that proofs are usually presented as unwieldy verifications of an already known formula, rather than as expositions that derive a formula from scratch in a constructive and intuitive manner. Perhaps the derivation that follows, while not truly elementary, will render Heron's formula more memorable for its symmetric and intuitive factorization.

The first step of this derivation is to recall that the square of the area of a triangle is a *polynomial* in the edge lengths. More generally, suppose that T is a simplex in $\mathbb{R}^n$ with vertices $x_0, x_1, \ldots, x_n$, where $x_0 = 0$, the origin. Let A denote the $n \times n$ matrix whose columns are given by the vectors $x_1, \ldots, x_n$, and suppose that the x_i are ordered so that A has positive determinant. The volume of T is then given by $\det(A) = n!\, V(T)$, implying that

$$V(T)^2 = \frac{1}{(n!)^2} \det(A^t A), \tag{1}$$

where A^t is the transpose of the matrix A. The entries of the matrix $A^t A$ are dot products of the form $x_i \cdot x_j$. From the identity

$$x_i \cdot x_j = \frac{1}{2}\left(|x_i|^2 + |x_j|^2 - |x_i - x_j|^2\right) \tag{2}$$

it then follows that the value of $V(T)^2$ is a *polynomial* in the *squares* of the edge lengths of T. Said differently, if T has edge lengths $a_{ij} = |x_i - x_j|$, then $V(T)^2$ is a polynomial in the variables $b_{ij} = a_{ij}^2$, as well as in the variables a_{ij} themselves. Since the determinant of an $n \times n$ matrix is a homogeneous polynomial of degree n in the matrix entries, the polynomial $f(a_{ij}) = V(T)^2$ is a *homogeneous polynomial of degree* $2n$. This polynomial is sometimes formulated in terms of linear algebraic