Sierpinski Carpets and Gaskets as Julia sets of Rational Maps

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0. Introduction. In recent years, it has been shown that the family of rational maps arising from *singular perturbations* of the simple polynomials $z \mapsto z^n$ have some interesting properties from a dynamical systems as well as a topological perspective. In this paper we survey some of these results. In addition, we provide proofs of these results in several special and illustrative cases. While the cases we describe here are by no means the most general, they do serve to illustrate the types of techniques that can be used in the general cases.

By a singular perturbation of z^n, we mean a map of the form $z \mapsto z^n + \lambda/z^m$ where λ is a complex parameter. Of primary interest is the Julia set of these maps. From an analytic viewpoint, the Julia set is the set of points at which the family of iterates of the map fails to be a normal family in the sense of Montel. From a dynamics point of view, the Julia set is the set of points on which the map is chaotic.

As is well known, the Julia set of z^n for $n \geq 2$ is just the unit circle. When we add the term λ/z^m for $\lambda \in \mathbb{C}$, $\lambda \neq 0$, several things happen. First of all, the degree of the rational map suddenly increases from n to $n+m$. Secondly, the superattracting fixed point at the origin becomes a pole, while ∞ remains a superattracting fixed point. As a consequence, an open set around the origin now lies in the basin of attraction of ∞. In between this neighborhood of 0 and the basin at ∞, the Julia set undergoes a significant transformation.

For example, if $1/n + 1/m < 1$, McMullen [13] has shown that, when $|\lambda|$ is small, the Julia set explodes from a single circle to a Cantor set of simple closed curves surrounding the origin. See Figure 1. When n and m do not satisfy the McMullen condition, the situation is quite different. In [2] it is shown that, in the cases $n = m = 2$ or $n = 2, m = 1$, there are infinitely many open sets of λ-values in any neighborhood of

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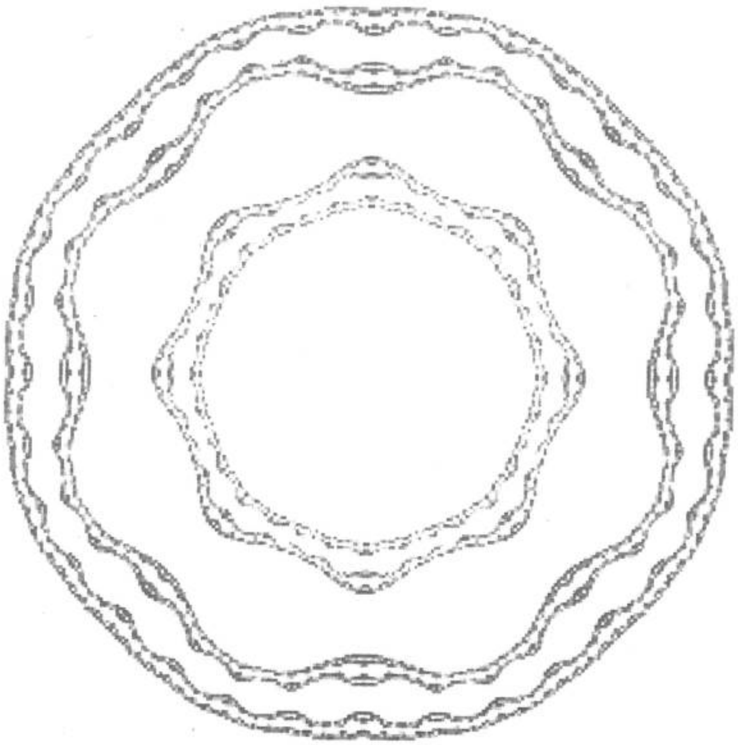

Figure 1: The Julia set for $z^4 + 0.04/z^4$ is a Cantor set of circles.

$\lambda = 0$ for which the Julia set is a Sierpinski curve. See Figure 2. A Sierpinski curve is an extremely rich topological space since this object is known to contain a homeomorphic copy of any one-dimensional, planar continuum, no matter how complicated this continuum is. It is also known that any two Sierpinski curves are homeomorphic [20]. However, from a dynamical systems point of view, it turns out that there are infinitely many dynamically distinct Sierpinski curve Julia sets in the sense that, if the parameters are drawn from disjoint open sets in the λ–plane, the corresponding maps are not topologically conjugate on their Julia sets.

When $n = 2$, $m = 2$ or $n = 2$, $m = 1$, there are many other interesting types of Julia sets in these families. For example, it is known [8] that there are infinitely many Julia sets in these families that have properties similar to a Sierpinski gasket. See Figure 3. These sets are topologically very different from the Sierpinski curves and it can be shown that, except for certain symmetric cases, these types of Julia sets are never homeomorphic.

In addition, in these two cases, there is a fundamental dichotomy for these rational maps that is similar in spirit to that for quadratic polynomials. This dichotomy states that if the critical points for these maps lie in the immediate basin of ∞, then the corresponding Julia set is a Cantor set, whereas if the critical points do not lie in this immediate basin, the Julia set is a connected set. The difference between the rational map case and the quadratic polynomial case is that the critical points for the rational maps may escape to ∞ without lying in the immediate basin of ∞, which is not possible for quadratic

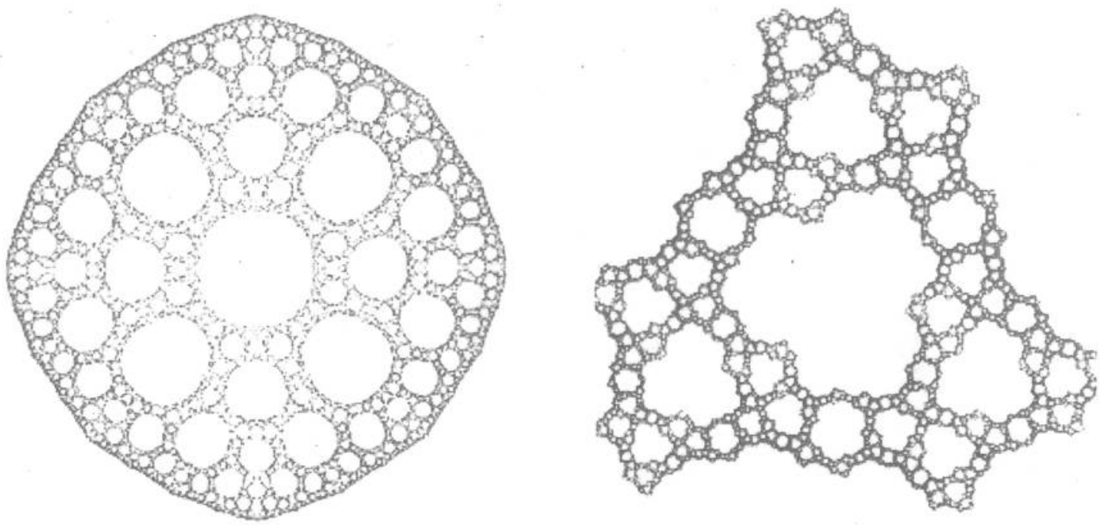

Figure 2: The Julia sets for (a) $z^2 - 0.06/z^2$, and (b) $z^2 + (-0.004 + 0.364i)/z$ are Sierpinski curves.

polynomials. As we show below, it is this situation that creates the Sierpinski curve Julia sets.

With this variety of different types of Julia sets in these families, it is little wonder that the parameter plane for these families is a rich topological object. Among other things, these parameter planes include infinitely many copies of "baby" Mandelbrot sets as well as other topologically interesting sets such as Cantor necklaces [3], [4]. See Figure 4.

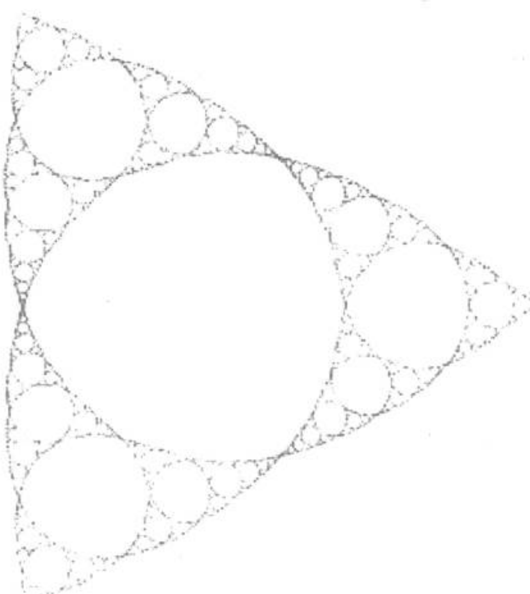

Figure 3: The Julia set for $z^2 + \lambda/z$ where $\lambda \approx -0.5925$ is a Sierpinski gasket.

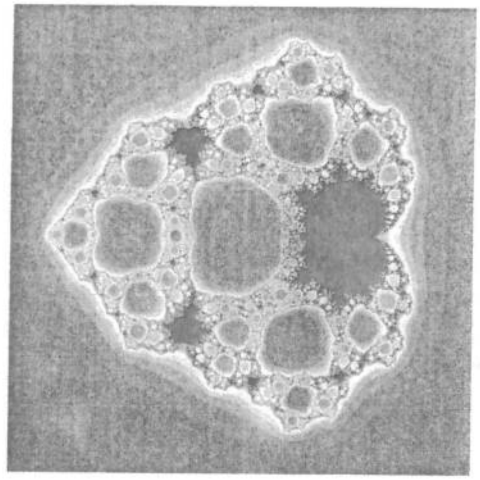

Figure 4: The parameter plane for $z^2 + \lambda/z^2$.

In this paper we restrict attention to the family of rational maps given by

$$F_\lambda(z) = z^2 + \frac{\lambda}{z^2},$$

although occasionally we will discuss the family

$$\tilde{F}_\lambda(z) = z^2 + \frac{\lambda}{z}.$$

Most of the results below hold for both families, though the proofs in the cases of F_λ and $\tilde{F}_\lambda$ are often quite different due to the presence of quite different symmetries in these two different families.

The authors wish to thank Pascale Roesch who made many fine suggestions concerning the original version of this paper.

Dedication. We are pleased to dedicate this paper to Bodil Branner, who is one of the finest mathematicians we have ever met. No, add to that: one of the finest **people** we have ever met.

1. Preliminaries. In this section we describe some of the basic properties of the family $F_\lambda(z) = z^2 + \lambda/z^2$. Observe that $F_\lambda(-z) = F_\lambda(z)$ and $F_\lambda(iz) = -F_\lambda(z)$ so that $F_\lambda^2(iz) = F_\lambda^2(z)$ for all $z \in \mathbb{C}$. Also note that 0 is the only pole for each function in this family. The points $(-\lambda)^{1/4}$ are prepoles for F_λ since they are mapped directly to 0. The four critical points for F_λ occur at $\lambda^{1/4}$. Note that $F_\lambda(\lambda^{1/4}) = \pm 2\lambda^{1/2}$ and $F_\lambda^2(\lambda^{1/4}) = 1/4 + 4\lambda$, so each of the four critical points lies on the same forward orbit after two iterations. We call the union of these orbits the *critical orbit* of F_λ.

Let $J = J(F_\lambda)$ denote the Julia set of F_λ. $J(F_\lambda)$ is the set of points at which the family of iterates of F_λ fails to be a normal family in the sense of Montel. Equivalently, $J(F_\lambda)$ is the closure of the set of repelling periodic points of F_λ (see [15]).

The point at ∞ is a superattracting fixed point for F_λ. Let B_λ be the immediate basin of attraction of ∞ and denote by β_λ the boundary of B_λ. The map F_λ has degree 2 at ∞ and so F_λ is conjugate to $z \mapsto z^2$ on B_λ if there is no critical point in B_λ. Otherwise, this conjugacy is defined only in a neighborhood of ∞. The basin B_λ is a (forward) invariant set for F_λ in the sense that, if $z \in B_\lambda$, then $F_\lambda^n(z) \in B_\lambda$ for all $n \geq 0$. The same is true for β_λ.

We denote by $K = K(F_\lambda)$ the set of points whose orbit under F_λ is bounded. In analogy with the situation for complex polynomials, we call K the filled Julia set of F_λ. K is given by $\mathbb{C} - \cup F_\lambda^{-n}(B_\lambda)$. Both J and K are completely invariant subsets in the sense that if $z \in J$ (resp. K), then $F_\lambda^n(z) \in J$ (resp. K) for all $n \in \mathbb{Z}$. The Julia set $J(F_\lambda)$ is the boundary of $K(F_\lambda)$; the proof is completely analogous to that for polynomials (see [15]).

Proposition (Four-fold Symmetry). *The sets B_λ, β_λ, $J(F_\lambda)$, and $K(F_\lambda)$ are all invariant under $z \mapsto iz$.*

Proof. We prove this for B_λ; the other cases are similar. Let $U = \{z \in B_\lambda \mid iz \in B_\lambda\}$. U is an open subset of B_λ. If $U \neq B_\lambda$, there exists $z_0 \in \partial U \cap B_\lambda$, where ∂U denotes the boundary of U. Hence $z_0 \in B_\lambda$ but $iz_0 \in \beta_\lambda$. It follows that $F_\lambda^n(iz_0) \in \beta_\lambda$ for all n. But since $F_\lambda^2(z_0) = F_\lambda^2(iz_0)$, it follows that $z_0 \notin B_\lambda$ as well. This contradiction establishes the result. $\square$

There is a second symmetry present for this family. Consider the map $H_\lambda(z) = \sqrt{\lambda}/z$. Note that we have two such maps depending upon which square root of λ we choose. H_λ is an involution and we have $F_\lambda(H_\lambda(z)) = F_\lambda(z)$. As a consequence, H_λ preserves both J and K. The involution H_λ also preserves the circle S_λ of radius $|\lambda|^{1/4}$ and interchanges the interior and exterior of this circle. We call S_λ the *critical circle*. Note that S_λ contains all four critical points as well as the four prepoles, and each of the involutions H_λ fixes a pair of the critical points of F_λ that are located symmetrically about the origin.

Write $\lambda = \rho \exp(i\psi)$ and $z = \rho^{1/4}\exp(i\theta) \in S_\lambda$. Then we compute

$$\begin{aligned} F_\lambda(z) &= \rho^{1/2}(\exp(2i\theta) + \exp(i(\psi - 2\theta))) \\ &= \rho^{1/2}((\cos(2\theta) + \cos(\psi - 2\theta)) + i(\sin(2\theta) + \sin(\psi - 2\theta))). \end{aligned}$$

If we set $x = \cos(2\theta) + \cos(\psi - 2\theta)$ and $y = \sin(2\theta) + \sin(\psi - 2\theta)$, then a computation shows that

$$\frac{d}{d\theta}\left(\frac{y}{x}\right) = 0.$$

Hence the image of the critical circle under F_λ is a line segment passing through the origin. F_λ maps S_λ onto this line in four-to-one fashion, except at the two endpoints, which are the critical values $\pm 2\sqrt{\lambda}$.

Note also that H_λ interchanges the circles centered at the origin and having radii $|\lambda|^{1/4}r$ and $|\lambda|^{1/4}/r$. Moreover, F_λ maps each of these two circles onto an ellipse that surrounds the image of the critical circle.

2. The Fundamental Dichotomy. We briefly recall the situation for the family of quadratic polynomials $Q_c(z) = z^2 + c$. Each map Q_c has a single critical point at 0 and so, like F_λ, Q_c has a single critical orbit. The fate of this orbit leads to the well known fundamental dichotomy for quadratic polynomials:

1. If $Q_c^n(0) \to \infty$, then $J(Q_c)$ is a Cantor set;
2. but if $Q_c^n(0) \not\to \infty$, then $J(Q_c)$ is a connected set.

The set of parameter values for which the quadratic Julia sets are connected is the well known Mandelbrot set. Our goal in this section is to prove a similar result in the case of F_λ. We remark that there is a more general form of this result called the escape trichotomy that holds in the more genral case of maps of the form $z^n + \lambda/z^m$. We refer to [6] for details.

Before stating this result, note that, unlike the quadratic case, there are two distinct ways that the critical orbit of F_λ may tend to ∞. One possibility is that one (and hence all) of the critical points lie in the immediate basin B_λ. The second possibility is that the critical points do not lie in B_λ but eventually map into B_λ. For quadratic polynomials this second possibility does not occur.

Theorem.

1. *If one and hence all of the critical points of F_λ lie in B_λ, then $J(F_\lambda)$ is a Cantor set.*
2. *If the finite critical points of F_λ do not lie in B_λ, then both $J(F_\lambda)$ and $K(F_\lambda)$ are compact and connected. In particular, if the finite critical points do not lie in B_λ but are mapped to B_λ by F_λ^n for some $n \geq 1$, then $J(F_\lambda)$ and $K(F_\lambda)$ are compact, connected, and locally connected sets.*

Proof. The proof that $J(F_\lambda)$ is a Cantor set when all critical points lie in B_λ is standard. See, for example, [15]. So suppose that no finite critical point lies in B_λ. Then we may extend the conjugacy between F_λ and z^2 to all of B_λ and so B_λ is a simply connected open set in $\overline{\mathbb{C}}$ and we have $F_\lambda \mid B_\lambda$ is two-to-one.

Since 0 is a pole of order two, there is an open, simply connected set T_λ containing 0 and having the property that F_λ maps T_λ in two-to-one fashion onto B_λ. This follows since each of the two involutions H_λ interchange B_λ and T_λ. One checks easily that T_λ possesses four-fold symmetry. Note that B_λ and T_λ are necessarily disjoint open sets. Note also that none of the critical points reside in T_λ. This follows since, if $\lambda^{1/4} \in T_\lambda$, then $-(\lambda^{1/4}) \in T_\lambda$ as well, and hence F_λ would be four-to-one on T_λ.

It is also true that none of the critical values lie in T_λ. We will assume this fact for now and provide a proof in the next section.

Now let $K_0 = \overline{\mathbb{C}} - B_\lambda$. K_0 is compact and connected with boundary β_λ. Let $K_1 = K_0 - F_\lambda^{-1}(B_\lambda) = K_0 - T_\lambda$. Since B_λ and T_λ are disjoint, K_1 is compact and connected. Now consider $F_\lambda^{-1}(T_\lambda)$. By our assumption above, none of the critical points of F_λ lies in $F_\lambda^{-1}(T_\lambda)$. Hence each component of $F_\lambda^{-1}(T_\lambda)$ is mapped in one-to-one fashion onto T_λ. Therefore, there are four disjoint components in this set, and each component is open, simply connected, and disjoint from both T_λ and B_λ.

We remark here that, if the critical points were to lie in $F_\lambda^{-1}(T_\lambda)$, then $F_\lambda^{-1}(T_\lambda)$ would be an annulus, not a collection of disks. This is the situation we will rule out later.

Thus we have that $K_2 = K_1 - F_\lambda^{-1}(T_\lambda)$ is a compact, connected set. Now we continue removing preimages of T_λ. Let $K_3 = K_2 - F_\lambda^{-2}(T_\lambda)$. If the orbit of the critical points of F_λ do not escape to ∞, then each component of $F_\lambda^{-2}(T_\lambda)$ is mapped one-to-one onto a component of $F_\lambda^{-1}(T_\lambda)$ and so $F_\lambda^{-2}(T_\lambda)$ consists of 16 simply connected open sets, each of which is disjoint from the previously removed open sets. Hence K_3 is compact and connected. Continuing in this fashion, assuming that the critical points do not escape, the components of $F_\lambda^{-n}(T_\lambda)$ $(n \geq 2)$ are mapped one-to-one onto components of $F_\lambda^{-n+1}(T_\lambda)$ and so $F_\lambda^{-n}(T_\lambda)$ consists of 4^{n-1} simply connected open sets, each of which is disjoint from the previously removed open sets. Hence, inductively, $K_n = K_{n-1} - F_\lambda^{-n+1}(T_\lambda)$ is compact and connected for all n. Therefore $K(F_\lambda) = \cap K_n$ is compact and connected. Since J is the boundary of K, J is also compact and connected.

If, on the other hand, one of the critical points lies in $F_\lambda^{-2}(T_\lambda)$, we claim that all of the preimages of T_λ under F_λ^2 are still open, simply connected, and disjoint, and that four of them are mapped two-to-one onto their images while the rest are mapped in one-to-one fashion.

To see this, suppose that one of the critical points, say c_λ, lies in a component V of $F_\lambda^{-2}(T_\lambda)$ that is mapped by F_λ onto a component of $F_\lambda^{-1}(T_\lambda)$. Call the image component W. Then a second critical point, $-c_\lambda$, is also mapped into W. Consequently, the set $-V$ containing $-c_\lambda$ is also mapped onto W. Now either V and $-V$ are disjoint, simply connected, and mapped two-to-one onto W, or else they form the same component of the preimage of W. In the latter case, there can be no other critical points in this component, for $\pm ic_\lambda$ are mapped to $-W$, which is disjoint from W. Hence F_λ is a degree four mapping onto a disk with exactly two critical points. This cannot happen by the Riemann-Hurwitz formula. Therefore the former case holds, and $\pm c_\lambda$ lie in disjoint components of $F_\lambda^{-2}(T_\lambda)$. Similarly, $\pm ic_\lambda$ lie in disjoint components of this set.

Thus, $F_\lambda^{-2}(T_\lambda)$ consists of a collection of non-intersecting, simply connected open sets lying in K_2. Hence K_3 is compact and connected. We now continue in the same fashion to show inductively that K_n is compact and connected. Therefore $K(F_\lambda) = \cap K_n$ is compact and connected, as is its boundary, $J(F_\lambda)$. This shows that J and K are compact and connected if the critical orbit escapes to ∞ but the critical points do not lie in B_λ. Also, since no critical points accumulate on J, the map is hyperbolic and so it is known [15] that J is locally connected.

We emphasize once again that the critical points for F_λ may not lie in B_λ yet the critical orbits may eventually enter B_λ. As shown in the above proof, this implies that the critical orbit passes through T_λ, the disjoint preimage of B_λ that contains the origin. We call T_λ the *trap door*, since any orbit that enters T_λ immediately "falls through" it and enters the basin at ∞. In this case we have a connected Julia set. In fact, we shall show in the next section that $J(F_\lambda)$ is a Sierpinski curve in the special case where this occurs and $|\lambda|$ is sufficiently small.

We denote the set of parameter values for which $J(F_\lambda)$ is connected by $\mathcal{M}$; $\mathcal{M}$ is called the *connectedness locus* for this family.

Proposition. *The connectedness locus lies on or inside the circle of radius* $3/16+\sqrt{2}/8 \approx 0.364$ *centered at* 0 *in the parameter plane.*

Proof. The critical values are given by $\pm 2\sqrt{\lambda}$. Consider the circle of radius $2|\sqrt{\lambda}|$ centered at 0. If z lies on this circle, we have

$$|F_\lambda(z)| \geq 4|\lambda| - \frac{1}{4}.$$

Note that

$$4|\lambda| - \frac{1}{4} > 2|\sqrt{\lambda}|$$

provided that

$$16|\lambda|^2 - 6|\lambda| + \frac{1}{16} > 0,$$

and this occurs if $|\lambda| > 3/16 + \sqrt{2}/8$. Hence F_λ maps the circle of radius $2|\sqrt{\lambda}|$ strictly outside itself for these λ-values.

Now the involution H_λ takes this circle to the circle of radius $1/2$ centered at the origin, and we have $2|\sqrt{\lambda}| > 1/2$ since $|\lambda| > 3/16 + \sqrt{2}/8$. It follows that F_λ maps the exterior of $|z| = 2|\sqrt{\lambda}|$ into itself in two-to-one fashion, so it follows that this entire region must lie in B_λ. Hence the critical values lie in B_λ in this case. From the proof of the fundamental dichotomy, the critical points cannot lie in the trap door and so they too must reside in B_λ. Therefore λ does not belong to $\mathcal{M}$. □

Note that these estimates for the size of $\mathcal{M}$ are the best possible, for if $\lambda = \lambda^* = -3/16 - \sqrt{2}/8$, then we have

$$\begin{aligned} F_\lambda^2(c_\lambda) &= -\frac{1+\sqrt{2}}{2} \\ F_\lambda^3(c_\lambda) &= \frac{1+\sqrt{2}}{2} \\ &= F_\lambda^4(c_\lambda). \end{aligned}$$

Hence the critical orbit lands on a fixed point for this particular λ-value. In $\mathcal{M}$, λ^* lies at the leftmost tip of the connectedness locus along the negative real axis. We shall deal with this particular λ-value and other related values in Section 4.

In analogy with the situation for quadratic functions, we also have the following *escape criterion*, though one can give significantly better estimates for this using the previous result.

Proposition. *Suppose $|\lambda| \leq 2$ and $|z| \geq 2$. Then $|F_\lambda^n(z)| > (1.5)^n |z|$, and therefore $z \in B_\lambda$.*

Proof. We have

$$|F_\lambda(z)| \geq |z|^2 - \frac{|\lambda|}{|z|^2} \geq |z|^2 - \frac{1}{2} \geq 1.5|z|.$$

Inductively, we have

$$|F_\lambda^n(z)| > (1.5)^n |z|$$

and the result follows.

3. Sierpinski Curve Julia Sets. In this section we describe the case where the critical points of F_λ have orbits that tend to ∞, but the critical points themselves do not lie in the immediate basin of ∞. The main result here is:

Theorem. *Suppose the critical orbit of F_λ tends to ∞, but the critical points do not lie in the immediate basin of ∞. Then $J(F_\lambda)$ is a Sierpinski curve.*

A *Sierpinski curve* is an interesting topological space that is, by definition, homeomorphic to the well known Sierpinski carpet fractal [12]. The Sierpinski carpet is a set that is obtained by starting with a square in the plane and dividing it into nine congruent subsquares, each of which has sides of length $1/3$ the size of the original square. Then the open middle square is removed, leaving eight subsquares in the original square. Then this process is repeated: remove the open middle third from each remaining square. This leaves 64 subsquares, each of which is $1/9$ the size of the original. Continuing, in the limit, the space that is obtained is the Sierpinski carpet. See [3].

It is straightforward to show that the Sierpinski carpet is a compact, connected, locally connected, nowhere dense subset of the plane. Moreover, each of the complementary domains (the removed open squares) is bounded by a simple closed curve that is disjoint from the boundary of every other complementary domain. It is known [20] that these properties characterize a Sierpinski curve: any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by pairwise disjoint simple closed curves is homeomorphic to the Sierpinski carpet. Hence any two Sierpinski curve Julia sets drawn from the family F_λ are homeomorphic. The interesting topology arises from the fact that a Sierpinski curve contains a homeomorphic copy of any one-dimensional plane continuum [17].

As an illustration of the proof of the theorem, we provide the details in the special case where $|\lambda| < 3^3/4^4 \approx 0.1$. For a proof for arbitrary λ, we refer to [7].

Proposition. *Suppose that $|\lambda| < 3^3/4^4$. Then the boundary of B_λ is a simple closed curve.*

Proof. Suppose z lies on the circle of radius $3/4$ centered at the origin. Then

$$|F_\lambda(z)| \leq |z|^2 + \frac{|\lambda|}{|z|^2} < \frac{9}{16} + \frac{3}{16} = 3/4$$

since $|\lambda| < 3^3/4^4$. Hence F_λ maps the circle of radius $3/4$ in two-to-one fashion onto an ellipse lying inside this circle. Also note that all critical points of F_λ lie inside this circle.

Let A_λ denote the annular region between the circle of radius $3/4$ and its preimage that lies outside this circle. Note that F_λ has degree two on A_λ as well as in the entire exterior region $r \geq 3/4$ since all critical points lie in $r < 3/4$. Let U_λ denote the disk in the complement of A_λ that contains the origin.

We now use quasiconformal surgery to modify F_λ to a new map E_λ which agrees with F_λ in the region outside A_λ but which is conjugate to $z \mapsto z^2$ in the interior of U_λ with a fixed point at the origin. To obtain E_λ, we first replace F_λ in U_λ with the map $z \mapsto z^2$ on $|z| < 3/4$. Then we extend E_λ to A_λ so that the new map is continuous and

1. maps A_λ two-to-one onto $U_\lambda - E_\lambda(U_\lambda)$;
2. agrees with E_λ on the inner boundary of A_λ;
3. and agrees with F_λ on the outer boundary of A_λ.

The map E_λ is continuous and has degree 2 with two superattracting fixed points, one at 0 and one at ∞. We define a new complex structure on $\mathbb{C}$ that is preserved by E_λ in the usual manner. Hence E_λ is quasiconformally conjugate to z^2 on all of $\mathbb{C}$. Therefore the boundary of the basin of attraction of ∞ for E_λ is a simple closed curve. Since E_λ agrees with F_λ in the exterior portion of A_λ containing ∞, the same is true for F_λ. This proves that β_λ is a simple closed curve when $|\lambda| < 3^3/4^4$. □

In particular, when $|\lambda| < 3^3/4^4$, since all of the critical points lie inside the circle of radius $3/4$ centered at the origin, the only way the critical orbits can escape to ∞ in this case is by passing through the trap door. Therefore we have:

Corollary. *The region $|\lambda| < 3^3/4^4$ lies in the interior of the connectedness locus.*

Before moving on, we use the above technique to fill in the hole we left in the previous section:

Proposition. *If the finite critical points are not in B_λ (so $B_\lambda \neq T_\lambda$), then the critical values of F_λ do not lie in T_λ.*

Proof. Let $\pm v_\lambda$ denote the critical values of F_λ. Recall that $F_\lambda(v_\lambda) = F_\lambda(-v_\lambda)$. Suppose for the sake of contradiction that the critical values of F_λ lie in the trap door. Let γ be a simple closed curve in B_λ that separates both ∞ and $F_\lambda(v_\lambda)$ from the boundary of B_λ. Let Γ be the closed disk in the Riemann sphere that is bounded by γ and contains both ∞ and $F_\lambda(v_\lambda)$. Let Λ denote the preimage of Γ in T_λ. Λ contains 0 and $\pm v_\lambda$ in its interior.

Consider $F_\lambda^{-1}(\Lambda)$. We claim that $F_\lambda^{-1}(\Lambda)$ is an annulus that is disjoint from T_λ and also surrounds T_λ. We first observe that $F_\lambda^{-1}(\Lambda)$ must be a connected set. If this were not the case, then this set would consist of at most two components, since each preimage of Λ necessarily contains at least two of the four critical points. So suppose $F_\lambda^{-1}(\Lambda)$ consists of two disjoint components, C_+ and C_-. If the critical point c_λ belongs to C_+, then $-c_\lambda$ belongs to C_- since both of these points are mapped to the same critical value. Then the critical point ic_λ belongs to one of these sets, say C_+, so $-ic_\lambda \in C_-$.

Now apply the involution H_λ to C_+. Recall that there are two such involutions, and each fixes a pair of critical points. We choose the one that fixes c_λ and $-c_\lambda$. Since $F_\lambda(H_\lambda(z)) = F_\lambda(z)$, we have $H_\lambda(C_+) = C_+$ and $H_\lambda(C_-) = C_-$. But $H_\lambda(ic_\lambda) = -ic_\lambda$ and so we cannot have $H_\lambda(C_+) = C_+$. This contradiction shows that $F_\lambda^{-1}(\Lambda)$ cannot consist of two disjoint components.

So let $C = F_\lambda^{-1}(\Lambda)$. Since C contains 4 critical points and is mapped with degree 4 onto a simply connected region, the Riemann-Hurwitz formula implies that C must be an annulus. As in the previous Proposition, we may replace F_λ by a new map that agrees with F_λ outside C and is globally conjugate to $z \mapsto z^2$. As before, this proves that the boundary of B_λ is a simple closed curve. So too is its preimage, the boundary of T_λ.

Now the region between B_λ and T_λ is an annulus A that is bounded by these two simple closed curves. Let Q denote the preimage of T_λ lying in A. As above, Q is an annulus. A is then the union of three subannuli, A_{in}, Q, and A_{out}, where A_{in} is the inner annulus between T_λ and Q, and A_{out} is the outer annulus between Q and B_λ. F_λ maps both A_{in} and A_{out} two-to-one onto A. Therefore the modulus of A_{in} and A_{out} is one-half the modulus of A. But the modulus of the third annulus Q is positive, and the modulus of A is the sum of the moduli of A_{in}, A_{out}, and Q. This yields a contradiction. Hence the critical values cannot lie in T_λ as claimed. □

We now use this result to prove:

Theorem. *Suppose $|\lambda| < 3^3/4^4$ and that the critical points of F_λ tend to ∞ but do not lie in the the immediate basin B_λ of ∞. Then $J(F_\lambda) = K(F_\lambda)$ is a Sierpinski curve.*

Proof. It suffices to show that $J(F_\lambda)$ is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by simple closed curves that are disjoint. The fact that both J and K are compact, connected, and locally connected was shown in the previous section. Since all of the critical orbits tend to ∞, it follows that $J=K$ and hence, using standard properties of the Julia set, J is nowhere dense.

It therefore suffices to show that all of the complementary domains are bounded by disjoint simple closed curves. By the earlier Proposition, B_λ is bounded by a simple closed curve β_λ lying strictly outside the circle of radius 3/4. Using the involution H_λ, the boundary of the trap door is given by $H_\lambda(\beta_\lambda)$, and so this region is bounded by a simple closed curve disjoint from β_λ.

As in the previous section, the preimage of T_λ consists of four simply connected open sets whose boundaries are simple closed curves that are mapped onto the boundary of T_λ, which we denote by τ_λ. The boundaries of these components are disjoint from β_λ, since this curve is invariant under F_λ. They are disjoint from τ_λ since the boundary of the trap door is mapped to β_λ whereas the boundary of the components are mapped to τ_λ, and we know that $\tau_\lambda \cap \beta_\lambda = \emptyset$. Finally, the boundary of each component is disjoint from any other such boundary for a point in the intersection would necessarily be a critical point. If this were the case, then the critical orbit would eventually map to β_λ, contradicting our assumption that the critical orbit tends to ∞. Hence the first preimages of T_λ are all bounded by simple closed curves that are disjoint from each other as well as the boundaries of B_λ and T_λ. Continuing in this fashion, we see that the preimages $F_\lambda^{-n}(T_\lambda)$ are similarly bounded by simple closed curves that are disjoint from all earlier preimages of β_λ. This gives the result. □

While these Sierpinski curve Julia sets are all homeomorphic, it is known that there are infinitely many open sets of parameter values $\mathcal{O}_j$ having the property that, if λ_1 and λ_2 belong to distinct $\mathcal{O}_j$'s, then F_{λ_1} and F_{λ_2} are not topologically conjugate on their respective Julia sets. See [2]. The basic reason for this lack of topological conjugacy is the fact that, in different $\mathcal{O}_j$'s, the number of iterations for the critical orbit to enter B_λ is different.

4. Sierpinski Gasket Julia sets. In this section we turn our attention to a different type of Julia set that occurs for certain members of the family F_λ. We assume in this section that the critical points of F_λ all lie on the boundary of the immediate basin of ∞ and that the critical orbit is preperiodic. We call such maps *Misiurewicz-Sierpinski* maps, or MS maps, for short.

Since all of the critical points are preperiodic, the Julia set of an MS map is the complement of the union of all preimages of B_λ, just as in the Sierpinski curve case. Hence we may construct this set inductively as in the proof of the fundamental dichotomy. Let K_0 denote $\mathbb{C} - B_\lambda$. It is known that the boundary of K_0 is a simple closed curve (for a proof, see [6]). Let $K_1 = K_0 - T_\lambda$ and for $k \geq 1$ set

$$K_{k+1} = K_k - F_\lambda^{-k}(T_\lambda).$$

Then

$$J(F_\lambda) = \bigcap_{k=0}^{\infty} K_k.$$

This construction yields a very different type of Julia set in the case of MS maps. To see this, note first that, using the involution H_λ, the critical points lie in the boundary τ_λ of the trap door as well as in β_λ. It can be shown [6] that, in fact, the critical points are the only points lying in the intersection of β_λ and τ_λ. Thus, when we remove the trap door from K_0 to form K_1, we are essentially removing an open *generalized square*, a region bounded by a simple closed curve with four *corners* that are the four critical points. The four corners divide the boundary of the square into four curves that we call *edges*. In particular, if we remove the four critical points from K_1, then the resulting set consists of four disjoint sets $I'_0, \ldots, I'_3$. We assume that I'_0 contains the fixed point p_λ that lies in β_λ and that the other I'_j are indexed in the counterclockwise direction. Let I_j denote the closure of I'_j, so that I_j is just I'_j with two critical points added. Then, by four-fold symmetry, F_λ maps I_j in one-to-one fashion onto K_0.

Since there are no critical points in any of the preimages of the trap door, K_{k+1} is obtained by removing 4^k generalized squares from K_k. Each of these removed squares is mapped homeomorphically onto the trap door by F_λ^k and hence each has exactly four corners lying in the boundary of K_k. By definition, these corners are the preimages of the critical points.

This process is reminiscent of the deterministic process used to construct the Sierpinski gasket (sometimes called the Sierpinski triangle). To construct this set, we start with a triangle and remove a middle triangle so that three congruent triangles remain, each of which meets the other two triangles at a unique point. We then continue this process, removing 3^k triangles at the k^{th} stage. In the limit we obtain the Sierpinski gasket. In analogy with this construction, and despite the fact that the removed sets are generalized squares rather than triangles we call the Julia set of an MS map a *generalized Sierpinski gasket*.

If we consider the degree three family

$$\tilde{F}_\lambda(z) = z^2 + \frac{\lambda}{z},$$

then there are analogous MS parameters for which $J(\tilde{F}_\lambda)$ is a generalized Sierpinski gasket where "triangles" are removed instead of squares. For example, when $\lambda \approx -0.5925$, the Julia set of $\tilde{F}_\lambda$ is actually homeomorphic to the Sierpinski triangle. See Figure 3.

We have the following result. See [8] for the complete proof.

Theorem. *Suppose F_λ and F_μ are two MS maps with $\lambda \neq \mu$ and the imaginary parts of both λ and μ are positive. Then $J(F_\lambda)$ is not homeomorphic to $J(F_\lambda)$.*

We make the assumption in this theorem that the imaginary parts of the parameters are positive because the Julia sets of F_λ and $F_{\bar{\lambda}}$ are easily seen to be homeomorphic. The Julia sets of two MS maps in the family F_λ are displayed in Figure 5. In the first example, the parameter value $\nu = -3/16 - \sqrt{2}/8 \approx -0.36428$ lies at the leftmost tip of the connectedness locus. The critical points can be clearly identified as the four corners of the trap door and are mapped after three iterations onto the repelling fixed point p_ν that lies in β_ν. The second example corresponds to $\mu \approx -0.01965 + 0.2754i$

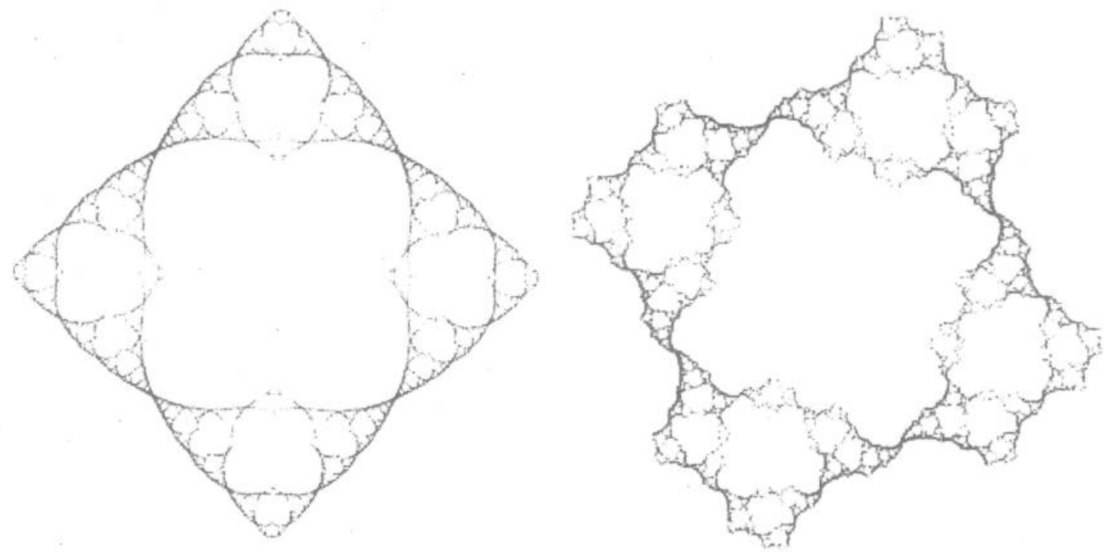

Figure 5: $J(F_\lambda)$ for $\lambda \approx -0.36428$ and $\lambda \approx -0.01965 + 0.2754i$.

for which the critical points are mapped to p_μ after four iterations. Rather than present the full details of the proof of the above theorem, we will illustrate the principal ideas using these two examples.

Note first that, in both of these images, every preimage of the boundary of the trap door seems to have two corners lying in the boundary of a previous preimage. This configuration holds true for every MS map as we show next.

Proposition. Let τ_λ^k be the union of all of the components of $F_\lambda^{-k}(\tau_\lambda)$ and let A be a particular component in τ_λ^k with $k \geq 1$. Then exactly two of the corner points of A lie in a particular edge of a single component of τ_λ^{k-1}.

Proof. The case $k = 1$ is seen as follows. Recall that $J(F_\lambda)$ is contained in the union of four closed sets $I_0, \ldots, I_3$ that meet only at the critical points and that are mapped by F_λ in one-to-one fashion onto $\mathbb{C} - B_\lambda$. Hence F_λ maps each $I_j \cap J(F_\lambda)$ for $j = 0, \ldots, 3$ in one-to-one fashion onto all of $J(F_\lambda)$, with $F_\lambda(I_j \cap \beta_\lambda)$ mapped onto one of the two halves of β_λ lying between two critical values (which, by assumption, are not equal to any of the critical points). Hence $F_\lambda(I_j \cap \beta_\lambda)$ contains exactly two critical points. Similarly, $F_\lambda(I_j \cap \tau_\lambda)$ maps onto the other half of β_λ and so also meets two critical points. The preimages of these four critical points are precisely the corners of the component of τ_λ^1 that lies in I_j. Thus we see that each component of τ_λ^1 meets the boundary of one of the I_j's in two points lying in β_λ and two points lying in τ_λ. In particular, two of the corners lie in the edge of τ_λ that meets I_j.

Now consider a component in τ_λ^k with $k > 1$. F_λ^k maps each component in τ_λ^k onto τ_λ and therefore F_λ^{k-1} maps the components in τ_λ^k onto one of the four components of τ_λ^1. Since each of these four components meets a particular edge of τ_λ in exactly two

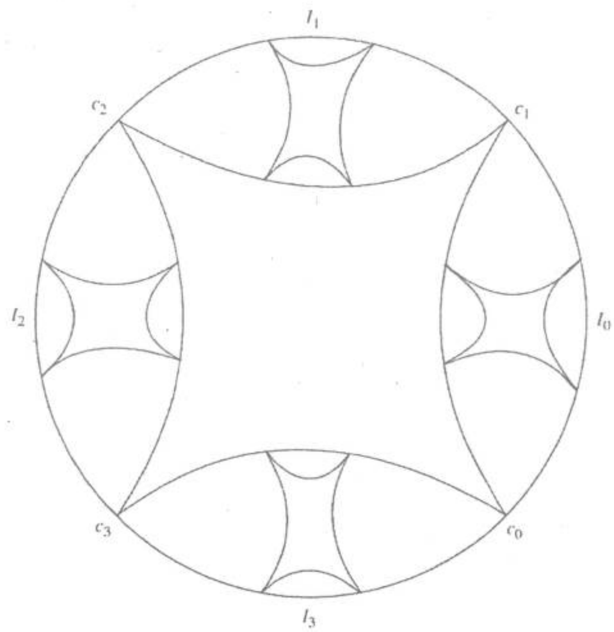

Figure 6: A topological representation of the boundaries β_λ, τ_λ, the four components of τ_λ^1 and the critical points. These curves satisfy the above configuration for every MS map.

corner points, it follows that each component of τ_λ^k meets an edge of one of the components of τ_λ^{k-1} in exactly two corner points as claimed.

Figure 6 provides a caricature of β_λ, τ_λ and τ_λ^1 which is valid for any MS λ-value. We seek a topological criterion that allows us to conclude that the Julia sets of two MS maps are not homeomorphic. The following result provides a topological characterization of the critical points that is helpful in this regard (see [8] for the proof).

Proposition. *Suppose F_λ is an MS map. The four corners of the trap door is the only set of four points in the Julia set whose removal disconnects $J(F_\lambda)$ into exactly four components. Any other set of four points removed from $J(F_\lambda)$ will yield at most three components.*

Suppose now that λ and μ are both MS parameters. If there exists a homeomorphism $h : J(F_\lambda) \longrightarrow J(F_\mu)$, then it follows from the Proposition that:

1. h maps the corners of τ_λ to the corners of τ_μ, and
2. the corners of each component of $F_\lambda^{-k}(\tau_\lambda)$ are mapped to the corners of a unique component of $F_\mu^{-k}(\tau_\mu)$.

As we will show below, the configuration of the corners with respect to the curve β_λ provides enough information to determine when two Julia sets are homeomorphic. This configuration, on the other hand, is completely determined by the orbit of the critical points.

To specify such an orbit, we define the itinerary $S(z)$ of a point $z \in J(F_\lambda)$ in the natural way by recording how its orbit meanders through the regions $I_0, \ldots, I_3$. That is, $S(z) = (s_0 s_1 s_2 \ldots)$ where each s_k is an integer j between 0 and 3 that specifies which I_j the point $F_\lambda^k(z)$ lies in. So that itineraries are unique, we modify the I_j slightly by removing one of the critical points from each I_j (so this set is no longer closed). In particular, let c_0 be the critical point of F_λ that lies in the fourth quadrant and in I_0. Then let $c_1 = ic_0$, $c_2 = -c_0$, and $c_3 = -ic_0$. We specify that only the critical point c_j now lies in I_j. Using these I_j's, the fixed point p_λ lies in I_0 so its itinerary is given by $S(p_\lambda) = \overline{0}$. Its preimage $q_\lambda \in \beta_\lambda$ lies in I_2; hence $S(q_\lambda) = 2\overline{0}$. Since the critical point c_j lies in I_j only, we have a well determined itinerary for each c_j. For example if $\lambda = \nu \approx -0.36428$, then one computes easily that the itinerary of the critical point c_1 is given by $S(c_1) = 112\overline{0}$. Using the symmetries of the map F_λ, it is easy to see that $S(c_0) = 032\overline{0}$, $S(c_2) = 232\overline{0}$ and $S(c_3) = 312\overline{0}$. When $\lambda = \mu \approx -0.01965 + 0.2754i$, $S(c_1) = 1112\overline{0}$ and thus $S(c_2) = 2312\overline{0}$, $S(c_3) = 3112\overline{0}$ and $S(c_0) = 0312\overline{0}$.

For each MS λ–value we define the *k–skeleton* of the Julia set, denoted by $\mathbb{J}(F_\lambda, k)$, as the union of β_λ, τ_λ and τ_λ^j for $j = 1, \ldots, k$. The k–skeleton not only provides the configuration of the first k preimages of τ_λ along β_λ, but if we define $\mathbb{J} = \lim_{k \to \infty} \mathbb{J}(F_\lambda, k)$, then the closure of $\mathbb{J}$ is equal to the Julia set, $J(F_\lambda)$.

We may construct a homeomorphism $\varphi = \varphi_\lambda$ that maps β_λ to the unit circle S^1 and "straightens" any other curve in $\mathbb{J}(F_\lambda, k)$ to a smooth curve, except at the images of the corners. Let

$$M(F_\lambda, k) = S^1 \cup \varphi(\tau_\lambda) \cup \varphi(\tau_\lambda^1) \cup \ldots \cup \varphi(\tau_\lambda^k).$$

The set $M(F_\lambda, k)$ represents a topological model in the plane of the k–skeleton of the Julia set. Since F_λ acts like $z \to z^2$ when restricted to β_λ, the model inherits the dynamics of the angle doubling map $D(\theta) = 2\theta \bmod 2\pi$ in S^1. Thus, to any point $z \in \beta_\lambda$ we can naturally associate an angle $\theta(z) \in [0, 2\pi]$ given by the angle of $\varphi_\lambda(z)$ in S^1. We may assume that $M(F_\lambda, k)$ satisfies the same symmetry relations as $J(F_\lambda)$ and that the four half-open regions I_j are mapped to corresponding regions in $M(F_\lambda, k)$.

To illustrate the construction of the model, consider our first example $\nu \approx -0.36428$. In this case, the critical point c_1 has itinerary $(112\overline{0})$. Hence $\theta(c_1) = \pi/4$ and, by symmetry, $\theta(c_2) = 3\pi/4$, $\theta(c_3) = 5\pi/4$ and $\theta(c_0) = 7\pi/4$. Since every model inherits the configuration of the Julia set for MS maps, each component of $\varphi(\tau_\nu^1)$ has two corners lying on an edge of $\varphi(\tau_\nu)$ and the two remaining corners must lie on $S^1 = \varphi(\beta_\nu)$. We let $x_0 = \varphi(F_\nu^{-1}(c_0))$ and $x_1 = \varphi(F_\nu^{-1}(c_1))$, so that x_0 and x_1 are the two corners that lie in $I_0 \cap S^1$. Similarly, we let x_2 and x_3 be the corresponding corners in $I_0 \cap \varphi(\tau_\nu)$. So we have $\theta(x_0) = D^{-1}(7\pi/4) = 7\pi/8$ and $\theta(x_1) = D^{-1}(\pi/4) = \pi/8$. A rotation by a multiple of $\pi/2$ provides the angles of the corners lying on $I_j \cap S^1$. For example, if w_0 and w_1 are the corresponding corners of $\varphi(\tau_\nu^1)$ lying in $I_1 \cap S^1$, then $\theta(w_0) = 3\pi/8$ and $\theta(w_1) = 5\pi/8$. Figure A shows in fact the angles along S^1 in $M(F_\nu, 1)$.

To construct $M(F_\nu, 2)$, we will determine first the configuration of the corners of a single component in $\varphi(\tau_\nu^2) \subset I_0$. Let U be the component of τ_ν^2 that is contained in

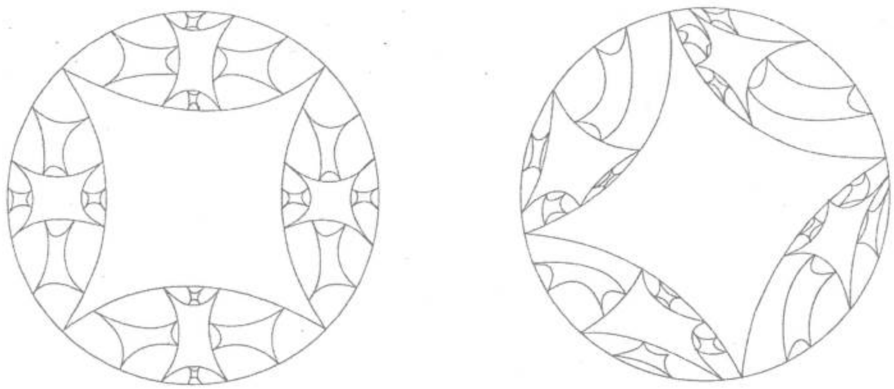

Figure 7: The model $M(F_\nu, 2)$ for $\nu \approx -0.36428$ is displayed to the left and $M(F_\mu, 2)$ with $\mu \approx -0.01965 + 0.2754i$ is displayed to the right. Note that $\theta(c_1(\nu)) = \pi/4$ and $\theta(c_1(\mu)) = \pi/16$.

the "triangle" T_ν defined by x_1, $\varphi(c_1)$ and x_2. Label the corners of U by y_0, y_1, y_2 and y_3 and assume y_2 and y_3 lie in the edge $[x_2, x_1] \subset \tau_\nu^1$. To compute the location of the remaining two corners, we first note the arc $[x_1, \varphi(c_1)] \subset S^1$ is mapped under D to an arc γ in $I_1 \cap S^1$. Clearly $\theta(\gamma) = [\pi/4, \pi/2]$ and thus, the corner w_0 lies in γ. Pulling back γ into I_0 by D we obtain the corner $y_0 \in [x_1, \varphi(c_1)]$.

A similar argument can be applied to the arc $\alpha = [\varphi(c_1), x_2]$ to obtain the location of y_1. Since D is not defined in this arc, consider first $\varphi^{-1}(\alpha) = [c_1, F_\nu^{-1}(c_2)]$ in $I_0 \cap J(F_\nu)$. This arc is mapped by F_ν to $[F_\nu(c_1), c_2] \subset I_1 \cap \beta_\nu$. Then, the homeomorphism φ sends $F_\nu(\varphi^{-1}(\alpha))$ onto $[D(\varphi(c_1)), \varphi(c_2)] \subset I_1 \cap S^1$. This arc has angles $[\pi/2, 3\pi/4]$ and thus, the corner w_1 lies in it. Pulling back this point by the proper composition of maps yields the point $y_1 \in \alpha$.

A similar process can be carried out to obtain $M(F_\mu, 2)$ for our second example where $\mu \approx -0.01965 + 0.2754i$. See [8] for the details. Figure 7 shows the models for the two MS maps discussed above. Both models display the configuration of the corner points with respect to the angles of corners along S^1.

To show the Julia sets of F_ν and F_μ are not homeomorphic, we proceed by contradiction. Assume there exists a homeomorphism $h : J(F_\nu) \to J(F_\mu)$. Recall that a homeomorphism must send critical points to critical points and corners of components in τ_ν^k to corners of components in τ_μ^k. Without loss of generality, assume that h maps $c_1(\nu)$ to $c_1(\mu)$ and $I_0(\nu)$ onto $I_0(\mu)$.

Restricting h to the 2-skeletons of the Julia sets, we can define a new homeomorphism $\widetilde{h}$ defined by the following diagram

$$\begin{array}{ccc} \mathbb{J}(F_\nu, 2) & \xrightarrow{\varphi_\nu} & M(F_\nu, 2) \\ h\big\downarrow & & \big\downarrow\widetilde{h} \\ \mathbb{J}(F_\mu, 2) & \xrightarrow{\varphi_\nu} & M(F_\mu, 2). \end{array}$$

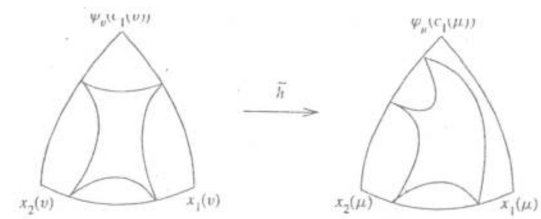

Figure 8: The map $\widetilde{h}$ must preserve the configurations of the corners along the edges of T_ν and T_μ.

Let T_ν denote the triangle in $M(\nu, 2)$ with vertices $x_1(\nu)$, $\varphi_\nu(c_1(\nu))$ and $x_2(\nu)$. This triangle contains a unique component $U(\nu) \subset \tau_\nu^1$ that defines a configuration of its corners along the edges of T_ν. Define T_μ analogously (see Figure 8). Since there exists an edge in T_μ that contains no corners of $U(\mu)$, $\widetilde{h}$ cannot possibly send the configuration given by $U(\nu)$ to the configuration of $U(\mu)$, and we have reached a contradiction.

The procedure to prove that any two MS maps have topologically distinct Julia sets is similar in spirit to this construction, although in the general case one needs to proceed to the k-skeleton to show this where k may be large.

5. The Connectedness Locus. Recall that the connectedness locus $\mathcal{M}$ for the family F_λ is the set of all λ-values for which $J(F_\lambda)$ is connected. In this section, we mimic the Douady-Hubbard proof [11] to show that $\overline{\mathbb{C}} - \mathcal{M}$ is conformally equivalent to the open unit disk $\mathbb{D}$. See also [18].

As we have already noted, the point at infinity is a superattracting fixed point. Consequently, the map F_λ is locally, analytically conjugate to the map $z \mapsto z^2$ in a neighborhood of ∞. In our case, there exists an analytic homeomorphism ϕ_λ defined in a neighborhood of ∞ such that

1. $\phi_\lambda(\infty) = \infty$
2. $\phi_\lambda'(\infty) = 1$
3. $\phi_\lambda \circ F_\lambda(z) = (\phi_\lambda(z))^2$.

This homeomorphism is often called the Böttcher coordinate of F_λ. For $|z| \geq \max\{|\lambda|, 2\}$, one can use the triangle inequality to show that $|F_\lambda(z)| > \frac{3}{2}|z|$, and in this case ϕ_λ is given by the infinite product representation

$$\phi_\lambda(z) = z \prod_{k=0}^{\infty} \left(1 + \frac{\lambda}{{z_k}^4}\right)^{1/2^{k+1}}$$

where $z_k = F_\lambda^k(z)$.

Let $O^{-(0)}$ denote the backwards orbit of the pole. That is,

$$O^-(0) = \bigcup_{k=0}^{\infty} F_\lambda^{-k}(0).$$

Associated to ϕ_λ is the nonnegative rate-of-escape function

$$G_\lambda : \mathbb{C} - O^-(0) \to \mathbb{R}$$

given by

$$G_\lambda(z) = \lim_{k\to\infty} \frac{1}{2^k} \log_+ |z_k|,$$

where $\log_+$ is the maximum of log and 0. This function has the following properties:

1. Let A_λ be the entire basin of ∞. Then $G_\lambda^{-1}(0) = \mathbb{C} - A_\lambda$.
2. $G_\lambda(z_{k+1}) = 2G_\lambda(z_k)$.
3. $G_\lambda(iz) = G_\lambda(z)$.
4. $G_\lambda(H_\lambda(z)) = G_\lambda(z)$.
5. G_λ is harmonic on $A_\lambda - O^{-(0)}$.
6. $G_\lambda(z) = \log|z| +$ bounded terms as $|z| \to \infty$.
7. $G_\lambda(z) = \log|\phi_\lambda(z)|$ if $\phi_\lambda(z)$ is defined.

The nonzero level sets of G_λ are called the *equipotential curves* for A_λ.

We know that the immediate basin B_λ of ∞ assumes one of two topological types:

1. If the critical points are not in B_λ, then B_λ is simply connected, and the domain of ϕ_λ can be extended to all of B_λ. Therefore, $F_\lambda : B_\lambda \to B_\lambda$ is a two-to-one branched cover with the branch point at ∞.
2. If the finite critical points are in B_λ, then the Julia set of F_λ is a Cantor set, and B_λ is the Fatou set.

In case 1, the involution H_λ determines the remaining inverse image T_λ of B_λ. That is, if $T_\lambda = H_\lambda(B_\lambda)$, then $F_\lambda : T_\lambda \to B_\lambda$ is also a two-to-one branched cover of B_λ with branch point at 0, and

$$F_\lambda^{-1}(B_\lambda) = B_\lambda \cup T_\lambda.$$

The distinction above results in a nice division of parameter space for the family $F_\lambda(z)$ into two disjoint subsets. We now focus on those values of λ for which case 2 holds, i.e., those λ for which the Julia set is topologically conjugate to the one-sided shift on four symbols. Consequently, we call this subset $\mathbb{C} - \mathcal{M}$ of parameter space the shift locus. Given $\lambda \in \mathbb{C} - \mathcal{M}$, let L_λ denote the component of $\{z \mid G_\lambda(z) > G_\lambda(\lambda^{1/4})\}$ that contains ∞. Then ϕ_λ extends naturally to all of L_λ. We mimic the Douady-Hubbard uniformization of the complement of the Mandelbrot set by defining the map $\Phi : \mathbb{C} - \mathcal{M} \to \mathbb{C} - \overline{\mathbb{D}}$ as

$$\Phi(\lambda) = \phi_\lambda\left(\frac{1}{4} + 4\lambda\right).$$

There are three things that need to be verified to show that Φ determines a conformal equivalence between $\mathbb{C} - \mathcal{M}$ and $\mathbb{C} - \overline{\mathbb{D}}$:

1. The map Φ is holomorphic.
2. It extends to a holomorphic map from $\mathbb{C} - \mathcal{M} \cup \{\infty\}$ to $\overline{\mathbb{C}} - \overline{\mathbb{D}}$.
3. The extension is a proper map of degree one.

First, Φ is holomorphic because $\frac{1}{4} + 4\lambda$ lies in L_λ for all $\lambda \in \mathbb{C} - \mathcal{M}$ and $\phi_\lambda(z)$ varies analytically in both z and λ.

For step 2, we use the infinite product representation of ϕ_λ. For $|\lambda| > \frac{9}{16}$, we have

$$\phi_\lambda\left(\frac{1}{4} + 4\lambda\right) = \left(\frac{1}{4} + 4\lambda\right) \prod_{k=0}^{\infty} \left(1 + \frac{\lambda}{\lambda_k^4}\right)^{1/2^{k+1}}$$

where $\lambda_k = F_\lambda^k(\frac{1}{4} + 4\lambda)$. From this expression, we see that

$$\frac{\Phi(\lambda)}{\lambda} \to 4$$

as $\lambda \to \infty$, and we can extend Φ to a holomorphic map from $\mathbb{C} - \mathcal{M} \cup \{\infty\}$ to $\overline{\mathbb{C}} - \overline{\mathbb{D}}$ by setting $\Phi(\infty) = \infty$.

The map Φ is proper if $|\Phi(\lambda)| \to 1$ as $\lambda \to \partial\mathcal{M}$. To show this, we compute $G_\lambda(\frac{1}{4} + 4\lambda)$ as $\lambda \to \partial\mathcal{M}$ using two lemmas.

Lemma. *The boundary $\partial\mathcal{M}$ of the connectedness locus is contained in the annulus*

$$\frac{3^3}{4^4} \le |\lambda| \le \frac{3+2\sqrt{2}}{16}.$$

The first inequality was proved in Section 3 and the second in Section 2.

Lemma. *Let $\lambda_k = F_\lambda^k(\frac{1}{4} + 4\lambda)$ for $k = 0, 1, 2, \ldots$. Then $\partial\mathcal{M}$ is a subset of*

$$\{\lambda \,|\, |\lambda_k| \le 2 \text{ for all } k = 0, 1, 2, \ldots\}.$$

Proof. From the results of Section 2 we know that if $|\lambda| > 2$, then $\lambda \in \mathbb{C} - \mathcal{M}$, so we may assume that $|\lambda| \le 2$.

Suppose $|\lambda_k| > 2$ for some k. Then the escape criterion from Section 2 guarantees that $\lambda_k \in B_\lambda$. Either $\lambda \in \mathbb{C} - \mathcal{M}$, or $\lambda \in \mathcal{M}$ with $\lambda_j \in T_\lambda$. In the first case we have $\lambda \notin \partial\mathcal{M}$. In the latter case, an open neighborhood of λ also has $\lambda_j \in T_\lambda$ and therefore λ is in the interior of $\mathcal{M}$. $\square$

For each $\lambda \in \mathbb{C} - \mathcal{M}$, let m_λ correspond to the last iterate such that $|\lambda_{m_\lambda}| \le 2$. Note that the previous lemma implies that $m_\lambda \to \infty$ as $\lambda \to \partial\mathcal{M}$. For a fixed $\lambda \in \mathbb{C} - \mathcal{M}$, we drop the subscript on m_λ, and we estimate $G_\lambda(\frac{1}{4} + 4\lambda)$ by considering

$$\sqrt[2^n]{|\lambda_m + n|}$$

for $n = 1, 2, \ldots$. Note that

$$\sqrt[2^n]{|\lambda_{m+n}|} = \sqrt[2^n]{\frac{|\lambda_{m+n}|}{|\lambda_{m+n-1}|^2}} \sqrt[2^n-1]{\frac{|\lambda_{m+n-1}|}{|\lambda_{m+n-2}|^2}} \cdots \sqrt{\frac{|\lambda_{m+1}|}{|\lambda_m|^2}} |\lambda_m|.$$

Assuming $|\lambda| < 1$, we estimate all but the last two factors by

$$\sqrt[2^k]{\frac{|\lambda_{m+k}|}{|\lambda_{m+k-1}|^2}} \leq \sqrt[2^k]{1+\frac{1}{|\lambda_{m+k-1}|^4}}.$$

If $k > 1$, we know that $|\lambda_{m+k-1}| > 2$, and we apply Bernoulli's inequality $\sqrt[k]{1+x} \leq 1 + x/k$ to obtain

$$\sqrt[2^k]{\frac{|\lambda_{m+k}|}{|\lambda_{m+k-1}|^2}} \left(1+\frac{1}{16 \cdot 2^k}\right).$$

The term

$$\sqrt{\frac{|\lambda_{m+1}|}{|\lambda_m|^2}}$$

requires special attention because $|\lambda_m| \leq 2$. For this part of the argument, it is convenient to work within the disk $|\lambda + \frac{1}{16}| \leq \frac{1}{2}$. Fix a λ within this disk. Let A_0 be the annulus bounded by the circle of radius 2 centered at the origin and the ellipse that is its image under F_λ.

Since the critical values of F_λ cannot be in A_0, A_0 has two preimages. One preimage A_1 has the circle of radius 2 centered at the origin as one boundary component, and the other preimage of A_0 is $H(A_1)$. Note that A_1 lies outside the unit disk.

Similarly, the critical values of F_λ cannot be in A_1, and therefore A_1 has two preimages. One preimage A_2 has a boundary in common with A_1 and the other preimage is $H(A_2)$.

Proceeding in this manner, we can produce annuli A_n such that

1. F_λ maps A_n onto A_{n-1} in a two-to-one fashion, and
2. one boundary component of A_n is also a boundary component of A_{n-1} as long as the critical values of F_λ do not lie in A_{n-1}.

If the critical values do not lie in A_n for some n, then the Julia set of F_λ is connected and

$$\{z||z| \geq 2\} \cup \bigcup_{n=1}^{\infty} A_n$$

exhaust the immediate basin of infinity. Consequently, $\lambda_m \in A_1$ and $\lambda_{m+1} \in A_0$ for $\lambda \in \mathbb{C} - \mathcal{M}$. Therefore, we have

$$\sqrt{\frac{|\lambda_{m+1}|}{|\lambda_{m2}|}} \leq \frac{\sqrt{265}}{8}.$$

Combining these two estimates with the fact that $|\lambda_m| < 2$, we obtain

$$\sqrt[2^n]{|\lambda_{m+n}|} \leq \frac{\sqrt{265}}{4} \prod_{k=2}^{n}\left(1+\frac{1}{16 \cdot 2^k}\right) < \frac{\sqrt{265}}{4} \prod_{k=2}^{\infty}\left(1+\frac{1}{16 \cdot 2^k}\right).$$

Since the right-hand side of this inequality converges to some number C (independent of those λ within the annulus under consideration), we have

$$\lim_{n \to \infty} \sqrt[2^n]{|\lambda_{m+n}|} \leq C.$$

This inequality implies that $G_\lambda(\lambda_m) \leq \log C$. Consequently,

$$2^m G_\lambda\left(\frac{1}{4} + 4\lambda\right) \leq \log C.$$

Since $m \to \infty$ as $\lambda \to \partial\mathcal{M}$, we conclude that

$$G_\lambda\left(\frac{1}{4} + 4\lambda\right) \to 0$$

as $\lambda \to \partial\mathcal{M}$, and Φ is a proper map.

We see that Φ has degree one by noting that $\Phi^{-1}(\infty) = \{\infty\}$. We have proved:

Theorem. *The complement $\overline{\mathbb{C}} - \mathcal{M}$ of the connectedness locus is conformally equivalent to a disk in the Riemann sphere.*

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