Families of Baker domains for meromorphic functions with countable many essential singularities

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ABSTRACT
Given a transcendental meromorphic function that satisfies an asymptotic behaviour near infinity, P. Rippon and G. Stallard showed in [15] that such function has a family of invariant Baker domains associated to its essential singularity at infinity. In this work, we extend this result to meromorphic functions with a countable number of essential singularities. Furthermore, for any $p \in \mathbb{N}$ we provide a closed form description of a large class of functions with countable many essential singularities that exhibit p families of invariant Baker domains. This closed form allow us to provide explicit examples of the results.

KEYWORDS
Meromorphic functions; Fatou set; Baker domains; asymptotic representations.

AMS CLASSIFICATION
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1. Introduction and statement of results

For a given meromorphic function $f : \Omega \to \widehat{\mathbb{C}}$, where $\Omega \in \{\mathbb{C}, \widehat{\mathbb{C}}\}$, we define the iterates of f by $f^0 = \text{Id}$ and $f^n = f(f^{n-1})$ for $n \geq 1$. The *Fatou set* of f, denoted here by $\mathcal{F}(f)$, is the set of all points $z \in \mathbb{C}$ for which the family of iterates $\{f^n\}_{n\geq 0}$ is defined, meromorphic and forms a normal family in a neighbourhood of the point z. Its complement, $\mathcal{J}(f) = \widehat{\mathbb{C}} \setminus \mathcal{F}(f)$, is known as the *Julia set* of f. Both sets are completely invariant under f. In particular, if $U \subset \mathcal{F}(f)$ is a maximal component of normality for f, then given any integer $n \geq 0$, there exists a component $U_n \subset \mathcal{F}(f)$ (also maximal) so that $f^n(U) \subset U_n$. If $U_n = U$ and $n \geq 1$ is the minimal integer that satisfies this, U is called a *periodic component of period* n. See [2] for a full classification of periodic Fatou components and further properties of Fatou and Julia sets of meromorphic functions. The kind of Fatou components that one shall be concerned with in this work are defined as follows.

Definition 1.1. Let $U \subset \mathcal{F}(f)$ be a maximal domain of normality so that $f : U \to U$ is analytic. If there exists $z_0 \in \partial U$ such that $f^n(z) \to z_0$ as $n \to \infty$ for every $z \in U$

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and $f(z_0)$ is not defined, then U is called an invariant *Baker domain* of f and z_0 is called the *Baker point* of U.

It follows directly from Definition 1.1 that rational maps do not have Baker domains, while transcendental meromorphic functions may exhibit Baker domains associated to their essential singularity at infinity. The first known example of a transcendental entire function with an invariant Baker domain was given by P. Fatou, [8]. Other examples of transcendental entire and meromorphic functions are described in the survey by P. Rippon, [14]. Recently, D. Martí-Pete provided examples of analytic self-maps of the punctured plane with Baker domains associated to the ommited values, see [11].

Rational, entire transcendental functions and analytic self-maps of the punctured plane are closed under composition. In contrast, if f is a transcendental meromorphic function with at least one pole that is not an ommited value in $\mathbb{C}$, then that pole becomes a finite essential singularity of f^2, thus making f^2 to lie outside the class of f. Indeed, let $z_0 \in \mathbb{C}$ be a pole of f of order $p \geq 1$ and assume there exists at least one value $w_0 \in \mathbb{C}$ so that $f(w_0) = z_0$. One can express f as $f(z) = (z - z_0)^{-p} g(z)$, where g is a transcendental meromorphic function that is analytic in a neigbourhood of z_0. The second iterate of f can be written as

$$f^2(z) = \left((z - z_0)^{-p} g(z) - z_0\right)^{-p} g\left((z - z_0)^{-p} g(z)\right).$$

From the above expression one easily sees that w_0 is now a pole of f^2 and z_0 becomes a finite essential singularity for f^2. To overcome this problem, one may consider a larger class of meromorphic functions previously investigated by A. Bolsch in [4]. The following definition is motivated by Definition 1.1 also in [4].

Definition 1.2. A function f is said to belong to *class* $\mathcal{K}$ if there exists a countable compact set $A(f) \subset \widehat{\mathbb{C}}$ such that f is a non-constant analytic function in $\widehat{\mathbb{C}} \backslash A(f)$ but in no other superset.

Thus, a function in class $\mathcal{K}$ has countably many essential singularities. Moreover, it follows from [4, Theorem 1.2 (iii)] that if $f \in \mathcal{K}$, then $f^2 \in \mathcal{K}$ with $A(f^2) = A(f) \cup f^{-1}(A(f))$, In consequence, $\mathcal{K}$ is closed under composition.

The extension of the Julia-Fatou theory of iteration in class $\mathcal{K}$ can be found in the dissertations of Bolsch [4] and M. Herring [9]. Among many results, it was shown in Theorem 4.1.1 in [9] that the classification of invariant Fatou components for transcendental meromorphic functions holds for functions in class $\mathcal{K}$. Our main purpose is to provide sufficient conditions over a large class of functions with countable many essential singularities that exhibit an infinite collection of invariant Baker domains.

Families of Baker domains

Consider the case of the entire transcendental function $f(z) = z + e^{-z}$. It was proved by I. N. Baker and P. Domínguez in [1, Theorem 5.1] the existence of an invariant Baker domain $U \subset \mathbb{C}$ where $\Re(f^n(z)) \to \infty$ as $n \to \infty$ and where ∂U is tangent to the lines $y = \pm\pi$. Since $f(z + 2\pi i) = f(z) + 2\pi i$, then for each $k \in \mathbb{Z}$, $U_k := U + 2\pi i k$ is again an invariant Baker domain with boundary tangent to the lines $y = (2k \pm 1)\pi$.

Following the terminology introduced in [15], one says that $f(z) = z + e^{-z}$ exhibits a *family of Baker domains*: that is, $\mathcal{F}(f)$ contains an infinite and countable collection of invariant Baker domains associated to the essential singularity at infinity and each

Baker domain lies inside a strip $|\Im(z)-2\pi k| < \pi$, for each $k \in \mathbb{Z}$. Also in [15], P. Rippon and G. Stallard studied a class of transcendental meromorphic functions that share an asymptotic behaviour near infinity, namely, functions that satisfy

$$\sup\{|\mathrm{Arg}((f(z)-z)e^z)| : z \in R(t,s)\} \to 0, \text{ as } t \to \infty \tag{AB}$$

over a half strip $R(t,s) = \{z : t \leq \Re(z), |\Im(z)| \leq s\}$, with $s > 0$ and where $\mathrm{Arg}(z)$ denotes the principal argument. For further reference, we provide the full statement.

Theorem 1.3 (Theorem 1 in [15]). *If f is a meromorphic function that satisfies condition (AB), then:*

(1) for each $k \in \mathbb{Z}$ there is an invariant Baker domain U_k of f such that, for each $0 < \theta < \pi$, U_k contains a set of the form

$$V_k(\theta) = \{z : v_k(\theta) < \Re(z), |\Im(z) - 2k\pi| < \theta\};$$

(2) the U_k are distinct Baker domains;
(3) if $z \in U_k$, then $|\Im(f^n(z)) - 2k\pi| \to 0$ and $\Re(f^n(z)) \to \infty$ as $n \to \infty$;
(4) each U_k contains a singularity of f^{-1}.

Based on this theorem, the authors constructed examples of transcendental entire functions of the form $g(z) = z(1 + e^{z^p})$ with $p \in \mathbb{N}$, so that on each sector $|\mathrm{Arg}(z) - 2\pi ij/p| < \pi/p$, $j = 0, \ldots, p-1$, there exists a family of invariant Baker domains. One says that g exhibits *p-families of Baker domains* associated to its unique essential singularity at infinity.

Since the required asymptotic behavior in (AB) is a local condition near infinity, it is not difficult to conclude that Theorem 1.3 can be extended to functions with more than one isolated essential singularity. This observation is at the core of the proof of our results.

Statement of results

Our first result presents a generalization of Theorem 1.3 in which condition (AB) is replaced by a new asymptotic behaviour (AB′) where a displacement in the argument is now allowed. This new asymptotic condition was inspired by the Remark after Theorem 6.1 in [14].

Theorem A. *Let f be a non-constant, complex-valued function that is analytic on the right half plane $H_0 = \{z \in \mathbb{C} : 0 < x_0 < \Re(z)\}$ and suppose that, for each $s > 0$ and some $\alpha \in \mathbb{R}$,*

$$\sup\{|\mathrm{Arg}((f(z)-z)e^z e^{i\alpha})| : z \in R(t,s)\} \to 0, \text{ as } t \to \infty. \tag{AB$'$}$$

If f has an isolated essential singularity at ∞, then f satisfies the following conditions:

(1′) *for each $k \in \mathbb{Z}$ there is an invariant Baker domain U_k of f such that, for each $0 < \theta < \pi$, U_k contains a set of the form*

$$V_k'(\theta) = \{z : v_k(\theta) < \Re(z), |\Im(z) - 2k\pi - \alpha| < \theta\};$$

(2′) *the U_k are distinct Baker domains;*

($3'$) *if* $z \in U_k$*, then* $|\Im(f^n(z)) - 2k\pi - \alpha| \to 0$*,* $(f^{n+1}(z) - f^n(z)) \to 0$ *and* $\Re(f^n(z)) \to \infty$ *as* $n \to \infty$*;*

($4'$) *each* U_k *contains a singularity of* f^{-1}*.*

Proof. The proof is almost verbatim to the proof of Theorem 1.3 in [15] with the following variations. First, if f is a function that satisfies condition (AB$'$), then it is clear that the definition of $V_k'(\theta)$ and the estimate on $\Im(f^n(z))$ in ($3'$) must take into account the displacement by α.

The condition $(f^{n+1}(z) - f^n(z)) \to 0$ that appears in ($3'$) can be implicitly derived from the proof of Theorem 1.3. Indeed, Lemma 2.1 in [15] states that if $s > 0$, then

$$\sup\{|f(z) - z| : z \in R(t, s)\} \to 0, \quad \text{as} \quad t \to \infty.$$

The justification behind the above condition requires that $\operatorname{Arg}((f(z) - z)e^z)$ be sufficiently small when $z \in R(t, s+1)$ which is always possible due to (AB). Similarly, the principal argument of $(f(z) - z)e^z e^{i\alpha}$ can be made sufficiently small if f now satisfies condition (AB$'$), and thus the rest of the proof of Lemma 2.1 follows through. The lemma, combined with $\Re(f^n(z)) \to \infty$ as $n \to \infty$ gives the desired result.

□

The main result in this article is the following.

Theorem B. *Let* $\Omega \in \{\mathbb{C}, \widehat{\mathbb{C}}\}$ *and* $g : \Omega \to \widehat{\mathbb{C}}$ *be either a transcendental meromorphic function or a rational function. Consider a function in class* $\mathcal{K}$ *given by*

$$f(z) = z + exp(g(z)),$$

with $A(f) = \overline{g^{-1}(\infty)}$*. If* $z_0 \in A(f)$ *is a pole of* g *of order* $p \geq 1$*, then* $f(z)$ *has p-families of Baker domains with* z_0 *as its Baker point. Furthermore, each family lies in a sector of opening* $2\pi/p$ *in a small neighbourhood of* z_0*.*

From the classification of Baker domains described by Cowen and König (see Section 3 for further details) we are able to conclude the following.

Corollary. *Under the assumptions of Theorem B, each Baker domain* U_k *of* f *with Baker point at* z_0 *is parabolic of type I.*

We provide three examples of functions in class $\mathcal{K}$ of the form $f(z) = z + \exp(g(z))$ that satisfy conditions of Theorem B. The distinction among these examples can be resumed as follows.

- Lemma 3.2 applied to the case when $g(z) = z(z^2 - c^2)^{-1}$, with $c \in \mathbb{C}^*$, concludes that the point at infinity is a parabolic fixed point of f.
- When $g(z) = z^{-3} + \exp(z)$, then the point at infinity becomes a Baker point of f associated to infinitely many families of Baker domains parabolic of type I (Proposition 4.3).
- When $g(z) = (\sin(z))^{-1}$, the point at infinity is a Baker point of f associated to two invariant Baker domains parabolic of type II (Proposition 4.5).

Outline

The proof of Theorem B is given in Section 2. A brief introduction to the classification of Baker domains and the proof of the Corollary is found in Section 3. In Section 4 we provide several examples of functions in class $\mathcal{K}$ of the form $f(z) = z + \exp(g(z))$, when g is either a rational or a transcendental meromorphic function.

2. Proof of Theorem B

Let $z_0 \in A(f)$ be a pole of g of order $p \geq 1$ and set $\omega = \exp(2\pi i/p)$. For each $j = 1, 2, \ldots, n-1$, define the A_j sector by $A_j = \omega^j A_0$, where

$$A_0 := \left\{ z \in \mathbb{C}^* : \frac{-\pi}{p} < \arg(z) < \frac{\pi}{p} \right\}$$

and arg has been chosen so that the positive real axis is the angle bisector of A_0. Select $t, s > 0$ so that $\tan(s/t) = \pi/p$ and denote by $R_0 = R_0(t,s)$ the half strip contained in $\overline{A_0}$ and defined by

$$R_0(t,s) := \{ z : t \leq \Re(z),\ |\Im(z)| \leq s \}.$$

Similarly, let $R_j = R_j(t,s) = \omega^j R_0$ for each j.

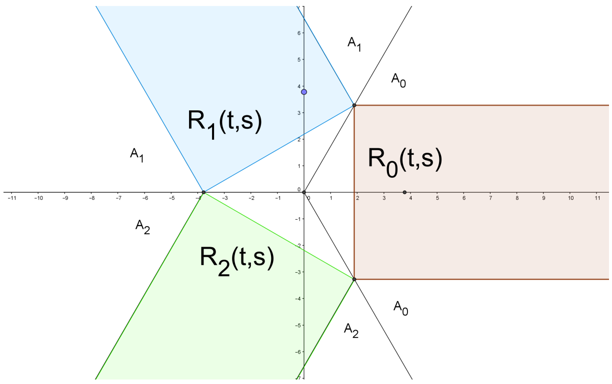

Figure 1. Sectors and half strips for the case $p = 3$.

In order to understand the asymptotic behavior of $g(z)$ as $z \to z_0$ along a prescribed direction, consider a punctured disk D^* centred at z_0 and radius sufficiently small so that, for each $z \in D^*$, the Laurent series expansion of g at z_0 is given by

$$g(z) = \frac{a_{-p}}{(z-z_0)^p} + \frac{a_{-p+1}}{(z-z_0)^{p-1}} + \ldots + a_0 + a_1(z - z_0) + \ldots, \tag{1}$$

with $a_{-p} \neq 0$. Selecting the same branch of $\arg(z)$ as before, there exists $b \in \mathbb{C}$ that satisfies $(-b)^p = -a_{-p}$ (the minus sign in front of b will become clear later on) so the Möbius map $M(w) = z_0 - b/w$ sends D^* into a punctured neighbourhood D^*_∞ of infinity and where for each $w \in D^*_\infty$,

$$G(w) = g(M(w)) = -w^p + \alpha_{p-1}w^{p-1} + \ldots + \alpha_0 + \frac{\alpha_{-1}}{w} + \ldots. \tag{2}$$

Proposition 2.1. *Let $D_\infty \subset \widehat{\mathbb{C}}$ denote a disk centred at infinity and consider a function G analytic in the punctured neighbourhood D_∞^* with power series expansion given by*

$$G(w) = -w^p + \alpha_{p-1}w^{p-1} + ... + \alpha_1 w + \alpha_0 + \dots,$$

for some $p \geq 1$. Then, for each $j = 0, \dots, p-1$,

$$\lim_{\substack{w\to\infty \\ w\in R_j(t,s)}} \Re(G(w)) = -\infty.$$

Proof. First, observe that $\lim_{w\to\infty} -G(w)/w^p = 1$ in any direction that w approaches infinity. Denote this asymptotic equivalence as $G(w) \sim -w^p$ when $w \to \infty$. Fix j and consider $w \in R_j(t,s)$. Since the height of each half strip is bounded, then $\arg(w) \to 2\pi j/p$ as $w \to \infty$ over R_j (that is, $\arg(w)$ converges to the angle bisector of A_j as $w \to \infty$ along R_j). Then $\arg(w^p) = p \arg(w) \to 2\pi j$. This, together with the fact $G(w) \sim -w^p$ as $w \to \infty$ gives the assertion on the limit and the proposition is proved. $\square$

Consider the asymptotic expression of Equations (1) and (2) given by

$$g(z) = a_{-p}(z - z_0)^{-p}(1 + o(1)), \qquad z \to z_0, \tag{3}$$
$$G(w) = -w^p(1 + o(1)), \qquad w \to \infty. \tag{4}$$

Using the change of coordinates defined by the Möbius map M, we now consider the conjugate function $F(w) = M^{-1} \circ f \circ M(w)$, which, after some computations, its expression can be reduced to

$$F(w) = \frac{w}{1 - \frac{w}{b}\exp(G(w))}.$$

The negative sign in the definition of b now becomes apparent in the denominator of F, and it will allow one to express $F(w)$ using the geometric series. To do so, recall that from Proposition 2.1,

$$\lim_{R_j(t,s)\ni w\to\infty} \Re(G(w)) = -\infty$$

over each sector A_j, independently of $s, t > 0$. Hence, given $0 < r < 1$, for each $s > 0$ there exists $t_s := t(s) > 0$ such that

$$\left|\frac{w}{b}\exp(G(w))\right| < r, \quad w \in R_j(t_s, s).$$

Consider the set $V_j := \bigcup_{s>0} R_j(t_s, s) \subset A_j$ and observe that

$$V_j \subset \left\{ w \in A_j : \left|\frac{w}{b}\exp(G(w))\right| < r < 1 \right\}.$$

Clealry, V_j is a nonempty subset of A_j with unbounded real part. Since the geometric series for $\frac{w}{b}\exp(G(w))$ is uniformly convergent inside V_j, we can express $F(w)$ as

$$\begin{array}{rcl} F(w) & = & w\left(1 + (\frac{w}{b}\exp(G(w))) + (\frac{w}{b}\exp(G(w)))^2 + ...\right), \\ & = & w + \frac{w^2}{b}\exp(G(w))(1 + (\frac{w}{b}\exp(G(w))) + (\frac{w}{b}\exp(G(w)))^2 + ...), \end{array}$$

for $w \in V_j$. Without loss of generality, we restrict our analysis to the case $j = 0$ and drop the subscripts from the regions A_0, R_0 and V_0, as previously defined. The asymptotic form of F can be expressed as

$$F(w) = w + \frac{w^2}{b}\exp(G(w))(1 + o(1)), \qquad \Re(w) \to \infty,\ w \in R(t,s). \tag{5}$$

Let $\phi^{-1}(z) = z^{1/p}$ be the unbranched inverse of $w^p = \phi(w)$ defined in the sector A. Clearly, $\phi|_V$ is well defined and single valued. Denote by $\mathbf{V} = \phi(V)$ and consider the conjugate function $\mathbf{F} := \phi \circ F \circ \phi^{-1}$ where $\mathbf{F} : \mathbf{V} \to \mathbb{C} \backslash \mathbb{R}_-$. For $t, s > 0$ sufficiently large, the half strip $\mathbf{R}(t,s)$ is simply defined by $\mathbf{R}(t,s) = \{z \in \mathbf{V} : t \leq \Re(z),\ |\Im(z)| \leq s\}$. Since $\mathbf{R}(t,s) \subset \mathbf{V}$, then $\phi^{-1}(\mathbf{R}(t,s)) \subset V$, so, for $z \in \mathbf{R}(t,s)$

$$F(\phi^{-1}(z)) = z^{1/p} + z^{1/p}H(z)(1 + H(z) + (H(z))^2 + ...),$$

where $H(z) = z^{1/p}b^{-1}\exp(G(z^{1/p}))$. From Equation (2), we obtain

$$G(z^{1/p}) = -z(1 - \beta_1 b z^{-1/p} + \beta_2 b z^{-2/p} \mp ...)$$

and consequently

$$G(z^{1/p}) = -z(1 + o(1)), \qquad \text{as } \Re(z) \to \infty,\ z \in \mathbf{R}(t,s). \tag{6}$$

Combining the conclusion of Proposition 2.1, Equations (5) and (6) we obtain for $\Re(z) \to \infty$ inside $\mathbf{R}(t,s)$,

$$F(\phi^{-1}(z)) = z^{1/p} + \frac{z^{2/p}}{b}\exp(-z(1 + o(1))(1 + o(1)).$$

Then,

$$\begin{array}{rcl} \mathbf{F}(z) & = & \left(z^{1/p} + \frac{z^{2/p}}{b}\exp(-z(1 + o(1))(1 + o(1))\right)^p \\ & = & z + \frac{z^2}{b^p}\exp(-pz(1 + o(1))(1 + o(1))^{p+1} \\ & = & z + \frac{z^2}{b^p}\exp(-pz(1 + o(1))(1 + o(1)), \quad \Re(z) \to \infty. \end{array}$$

Since $p \geq 1$ is an integer, the map $z \mapsto pz$ does not affect the angle bisector of $\mathbf{R}(t,s)$, so one can consider the conjugation $z \mapsto p\mathbf{F}(\frac{z}{p})$ which, for simplicity, it is denoted again by $\mathbf{F}$. One obtains

$$\mathbf{F}(z) = z + \frac{z^2}{pb^p}\exp(-z(1 + o(1))(1 + o(1)), \tag{7}$$

as $\Re(z) \to \infty$, $z \in \mathbf{R}(t,s)$. It follows that

$$\arg\left((\mathbf{F}(z)-z)e^{z}\right) = \arg\left(\frac{z^2}{pb^p}\exp(zo(1))(1+o(1))\right).$$

Setting $\alpha = -\arg b^{-p}$, the previous expression becomes

$$\begin{aligned} \arg((\mathbf{F}(z)-z)e^{z}e^{i\alpha}) &= \arg\left(z^2\exp(zo(1))(1+o(1))\right), \\ &= (2\arg(z) + \Im(z)o(1))(1+o(1)) \end{aligned} \tag{8}$$

which converges to zero as $\Re(z) \to \infty$, $z \in \mathbf{R}(t,s)$ with bounded imaginary part. Taking supremum, we conclude that $\mathbf{F}$ satisfies condition (AB$'$) of Theorem A. Then $\mathbf{F}(z)$ possesses a family of Baker domains, $\mathbf{U}_k, k \in \mathbb{Z}$ with ∞ as its Baker point. As the same analysis can be performed on each of the sectors A_j under the appropiate change of variable, then F has in fact p-families of Baker domains, each one inside a sector A_j near the point at infinity.

Finally, since each change of coordinates was conformal in its respective domains, we conclude that f has p-families of Baker domains associated to z_0. Each family is contained in a sector of opening $2\pi/p$ of D^*, which corresponds to a sector A_j under the map $M^{-1}(z)$. □

3. Classification of Baker domains

The classical result of Denjoy-Wolff establishes that if f is a holomorphic, fixpoint-free self-map of a hyperbolic domain U, then its sequence of iterates converge uniformly to a boundary point ω of U. In [5], C.C. Cowen studies the case when U concides with the right-half plane $\mathbb{H}_+ = \{z : \Re(z) > 0\}$ and $\omega = \infty$, showing the existence of a semi-conjugacy between (f,U) and (T,Ω) via an analytic function $\varphi : \mathbb{H}_+ \to \Omega$. The local dynamics of the pair (T,Ω) provides a linear model for the dynamics of (f,U) classified as

- *parabolic of type I* if $\Omega = \mathbb{C}$ and $T(w) = w+1$,
- *parabolic of type II* if $\Omega = \mathbb{H}_+$ and $T(w) = w \pm i$,
- *hyperbolic type* if $\Omega = \mathbb{H}_+$ and $T(w) = aw$, $a > 1$.

Although one may feel tempted to directly apply Cowen's classification to the case of a transcendental meromorphic function f acting on an invariant Baker domain U, it is not necessarily true that U may be simply connected (see for example [12] where the author provides examples of 2-periodic Baker domains of infinite connectivity). A generalization of the existence of semi-conjugacies for a multiply connected hyperbolic domain was presented by H. Königs in [10, Theorem 3]. For our purposes, it will be sufficient to consider the work by K. Barański et al. in [3], adjusting their terminology to Cowen's classification.

First, recall the following definition given in [3]: consider a hyperbolic domain U in $\mathbb{C}$ and ζ be an isolated point of the boundary of U in $\widehat{\mathbb{C}}$. There exists a neighbourhood $V \subset \widehat{\mathbb{C}}$ of ζ such that $V \setminus \{\zeta\} \subset U$. If $f(U) \subset U$, Picard's Theorem implies that f extends holomorphically to V. If $f(\zeta) = \zeta$ one says that ζ is an *isolated boundary fixed point* of f.

Theorem 3.1 (Theorem A in [3]). *Let U be a hyperbolic domain in $\mathbb{C}$ and let $f : U \to U$ be a holomorphic map without fixed points and without isolated boundary fixed points. Then the following statements are equivalent:*

(a) U is parabolic of type I.
(b) $\rho_U(f^{n+1}(z), f^n(z)) \to 0$ as $n \to \infty$ for some $z \in U$.
(c) $\rho_U(f^{n+1}(z), f^n(z)) \to 0$ as $n \to \infty$ almost uniformly on U.
(d) $|f^{n+1}(z) - f^n(z)|/\mathrm{dist}(f^n(z), \partial U) \to 0$ as $n \to \infty$ for some $z \in U$.
(e) $|f^{n+1}(z) - f^n(z)|/\mathrm{dist}(f^n(z), \partial U) \to 0$ as $n \to \infty$ almost uniformly on U.

ρ_U denotes the hyperbolic metric in (b) and (c), while the euclidean metric is used in (d) and (e).

Lemma 3.2. *Let $f(z) = z + \exp(g(z))$ be a function in class $\mathcal{K}$, where g is either a transcendental meromorphic function or a rational function. Then f has no isolated boundary fixed points.*

Proof. It is clear that f has no fixed points, except when $g(\infty) \in \mathbb{C}$ and thus g is rational. In this case one obtains $\lim_{z\to\infty} f(z) = \infty$, so the point at infinity is a fixed point of f. Let $g(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials without common factors and degrees $p = \deg(P)$ and $q = \deg(Q)$. Since $g(\infty) \in \mathbb{C}$, then $0 \le p \le q$. If one writes $g'(z) = R(z)/(Q(z))^2$ it follows that $\deg(R) \le 2q - 1 < \deg(Q^2)$, and thus $g'(\infty) = 0$. This implies that

$$\lim_{z\to\infty} \frac{1}{f'(z)} = \lim_{z\to\infty} \frac{1}{1 + g'(z)\exp(g(z))} = 1,$$

that is, ∞ is a parabolic fixed point for f, and hence, it cannot be an isolated boundary point of its Leau domain. □

Proof of Corollary. Consider again the expression of the function $\mathbf{F}$ given in Equation (7) and its asymptotic behavior given in Equation (8). Since the supremum of the argument tends to zero as $\Re(z) \to \infty$ along the half strip $\mathbf{R}(t, s)$, then $\mathbf{F}$ satisfies the hypothesis of Theorem A. In particular, from condition $(1')$ each Baker domain $\mathbf{U}_k$ of $\mathbf{F}$ contains a set of the form

$$\mathbf{V}'_k(\theta) = \{z : 0 < v_k(\theta) < \Re(z), |\Im(z) - 2k\pi - \alpha| < \theta\}$$

which implies that $\mathrm{dist}(\mathbf{F}^n(z), \partial\mathbf{U}_k)$ is bounded away from 0. From condition $(3')$ one has $(\mathbf{F}^{n+1}(z) - \mathbf{F}^n(z)) \to 0$ as $n \to \infty$. Then

$$\frac{|\mathbf{F}^{n+1}(z) - \mathbf{F}^n(z)|}{\mathrm{dist}(\mathbf{F}^n(z), \partial\mathbf{U}_k)} \to 0 \text{ as } n \to \infty.$$

It follows directly from Theorem 3.1 that each $\mathbf{U}_k$ is parabolic of type I. The conformal conjugacies that relate $\mathbf{F}$ to f preserve the classification of each Baker domain U_k. □

4. Examples

We now present three examples of functions in class $\mathcal{K}$ of the form $f(z) = z + \exp(g(z))$ in terms of the choice of the function g.

Example 4.1 (*One-parameter family in $\mathcal{K}$*)**.** Consider first an analytic family of rational functions given by

$$g_c(z) = \frac{z}{z^2 - c^2}, \quad \text{with} \;\; c \in \mathbb{C}. \tag{9}$$

Thus when $c \neq 0$, g_c represents a singular perturbation of the Möbius map $z \mapsto 1/z$, where the simple pole at the origin splits into two simple poles at $\pm c$. Then one obtains an analytic family of functions in class $\mathcal{K}$

$$f_c(z) = z + \exp\left(\frac{z}{z^2 - c^2}\right), \tag{10}$$

with $A(f_c) = \{\pm c\}$. The point at infinity becomes a parabolic fixed point for each f_c as a consequence of Lemma 3.2. Theorem B guarantees the existence of a family of Baker domains parabolic of type I at each Baker point $\pm c$ with a displacement in the bisector of each U_k given by $\alpha = -\Im(c)/(4|c|^2)$.

Remark 1. In [7] the author provides a global study of the dynamical properties of f_c, showing that for certain values of c, each Baker domain U_k is simply connected.

Since the expression of g_c is sufficiently simple, one can verify directly the conditions of Theorem A. Indeed, for $c \neq 0$, consider the Möbius map $M(z) = -\frac{1}{2(z-c)}$ which sends the ordered triplet $(-c, c, \infty)$ into the ordered triplet $(\frac{1}{4c}, \infty, 0)$. Setting $w = u + iv$ and $w = M(z)$, one obtains a new function dependent on $c \neq 0$,

$$\begin{aligned} F_c(w) &= M \circ f \circ M^{-1}(w), \\ &= M\left(M^{-1}(w) + \exp\left(g_c \circ M^{-1}(w)\right)\right), \end{aligned}$$

which reduces to

$$F_c(w) = \frac{w}{1 - 2w\exp\left(\frac{2w(2cw-1)}{1-4cw}\right)}. \tag{11}$$

F_c now has one of its essential singularities at the point at infinity. For convenience, let $G_c(w) = \frac{2w(2cw-1)}{1-4cw}$. The power series expansion in a neighbourhood at infinity can be easily computed to obtain

$$G_c(w) = -w - \frac{w^2}{4c} - \frac{w^3}{8c^2} - \frac{w^4}{16c^3} + O(w^5).$$

From Proposition 2.1, $\lim_{\Re(w)\to\infty} \Re(G_c(w)) = -\infty$ and a straightforward computation shows that

$$\lim_{\Re(w)\to\infty} \Im(G_c(w)) = -\left(\frac{\Im(c)}{4|c|^2} - \Im(w)\right). \tag{12}$$

Hence, there exists $u_v := u(v) > 0$, such that

$$|2w\exp(G_c(w))| = 2|w|\exp(\Re(G_c(w))) < r < 1,$$

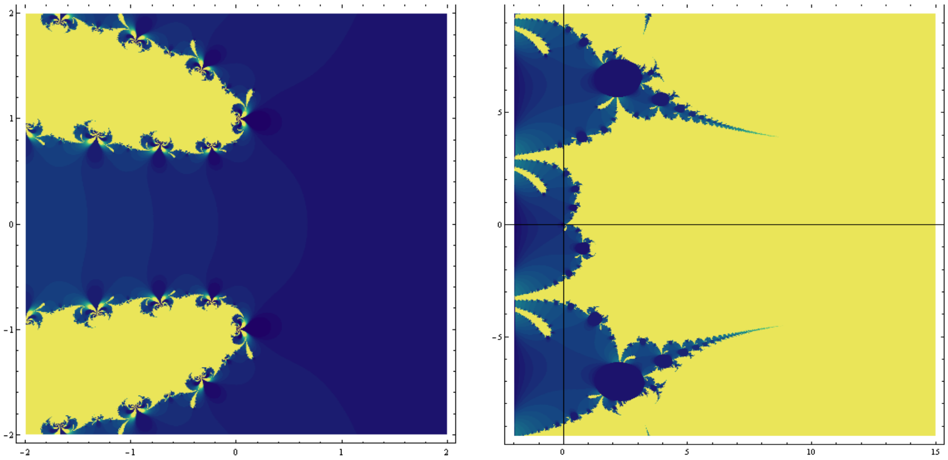

Figure 2. Left: Family of Baker domains for f_c and $c = i$ are displayed in yellow. In shades of blue is the parabolic basin associated to infinity. Right: Dynamical plane for the conjugated function F_c under M. In shades of blue is the parabolic basin and in yellow the Baker domains. Observe the small displacement between $\mathbb{R}^+$ and the bisector of U_0.

for w in a half strip $R(u_v, v)$. Setting $V_r = \bigcup_{v>0} R(u_v, v)$, $F_c(w)$ can be expressed as the series

$$F_c(w) = w\left(1 + 2we^{G_c(w)} + \left(2we^{G_c(w)}\right)^2 + ...\right),$$

which is uniformly convergent in V_r. Then, as $\mathfrak{R}(w) \to \infty$ along V_r, $F_c(w) = w + 2w^2e^{G_c(w)}(1 + o(1))$. Mutiplying by e^we^{-w}, one has $F_c(w) = w + 2w^2e^{G_c(w)+w}e^{-w}(1 + o(1))$ and thus

$$\begin{aligned} \arg((F_c(w) - w)e^w) &= \arg\left((2w^2e^{G_c(w)+w})(1 + o(1))\right), \\ &= (2\arg(w) + \mathfrak{I}(G_c(w) + w))(1 + o(1)). \end{aligned}$$

Since $\arg(w) \to 0$ and (12) holds if $\mathfrak{R}(w) \to \infty$, then it is clear that

$$\arg\left((F_c(w) - w)e^we^{i\alpha}\right) \to 0$$

with $\alpha = -\frac{\mathfrak{I}(c)}{4|c|^2}$. Thus $F_c(w)$ meets the hypothesis of Theorem A. An analogous approach can be performed for the essential singularity at $-c$. Figure 2 displays the dynamical planes of f_c and F_c when $c = i$. The image on the right shows the displacement of the bisector of U_0 from the line $y = 0$ by a factor of $\alpha = -\frac{1}{4}$.

Example 4.2 (*Multiple pole at the origin*)**.** Consider the case when $g(z) = z^{-3} + \exp(z)$. Then $A(f) = \{0, \infty\}$. Theorem B implies the existence of 3-families of Baker domains of parabolic type I at $z_0 = 0$. Since g is a trancendental meromorphic function, the point at infinity is no longer a fixed point of f, and in particular, f has no isolated boundary fixed points. One can say more about the dynamics near this essential singularity.

Proposition 4.3. *The function $f(z) = z + \exp(z^{-3} + e^z)$ has at $z = \infty$ a Baker point for infinitely many families of Baker domains parabolic of type I.*

Proof. First observe that

$$\begin{aligned} f(z+2\pi ik) &= z+\exp\left(\frac{1}{(z+2\pi ik)^3}+e^z\right)+2\pi ik, \\ &= z+\exp\left(\frac{1}{z^3}-O\left(\frac{1}{z^4}\right)+e^z\right)+2\pi ik, \\ &= z+\exp\left(\frac{1}{z^3}+e^z\right)\frac{1}{\exp(O(z^{-4}))}+2\pi ik, \end{aligned}$$

when $|z| >> 1$. Then $f(z+2\pi ik)\sim f(z)+2\pi ik$ for all $k\in\mathbb{Z}$. So it is sufficient to restrict the analysis of f inside the horizontal strip $0<\Im(z)<2\pi$. To do so, consider first the conjugated function

$$F(w)=\exp\circ f\circ\log(w)=w\exp\left(\frac{e^w}{w^3}\right),$$

where log denotes the branch of logarithm with arg taking values in $(0,2\pi)$. In order to verify the asymptotic behavior as $\Re(w)\to\infty$, one must consider a second conjugacy under the map $w\mapsto -w$. One obtains

$$G(w)=w\exp\left(\frac{-e^{-w}}{w^3}\right)=w\left(1-\frac{e^{-w}}{w^3}+O\left(\frac{e^{-2w}}{w^6}\right)\right).$$

Then $\mathrm{Arg}\,((G(w)-w))e^w)=\mathrm{Arg}\left(-\frac{1}{w^3}+O(\frac{e^{-w}}{w^6})\right)\to 0$ as $\Re(w)\to\infty$. From Theorem 1.3 one concludes that G has a family of invariant Baker domains associated to infinity in the far right plane. Similarly, f has a family of invariant Baker domains on each horizontal strip $|\Im(z)-k\pi|<\pi$, $k\in\mathbb{Z}$. Since f has no fixed points (and thus no isolated boundary fixed points), we can apply Theorem 3.1 to conclude that each Baker domain associated to infinity is parabolic of type I. □

Example 4.4 (*A transcendental meromorphic function*)**.** One can consider the case when $g(z)=1/\sin(z)$, so by Theorem B, each point $z_k=k\pi, k\in\mathbb{Z}$ is a Baker point of f associated to a family of Baker domains. Observe that ∞ is also an essential singularity which is the accumulation point of z_k. One can say more.

Proposition 4.5. *The point at infinity is a Baker point of f with two invariant Baker domains parabolic of type II.*

Proof. It is enough to restrict our analysis to the upper half plane. First consider $H_R=\{z:\Im(z)>R>e^2\}$ and $z\in H_R$. As $\sin(z)=(e^{iz}-e^{-iz})/2i$, the asymptotic expression of $f(z)$ inside H_R can be written as

$$f(z)=1+z+O(e^{-iz})\quad\text{when}\quad \Re(z)>>1, z\in H_R. \tag{13}$$

Thus, $\Re(f^n(z))\to\infty$ as $n\to\infty$ inside H_R. Also, inside H_R and $n>>1$,

$$\Im(f^n(z))=\Im(z)+\Im(O(e^{-if^n(z)}))\leq\Im(z)+|O(e^{-if^n(z)})|, \tag{14}$$

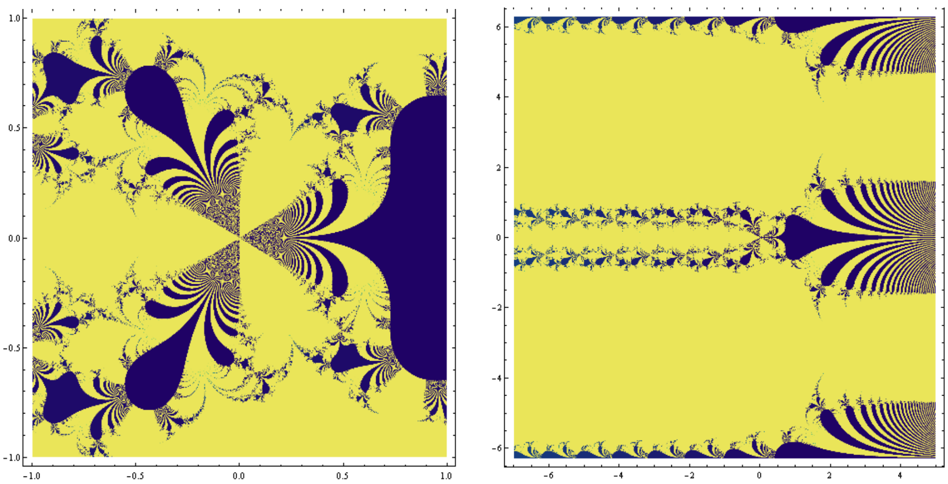

Figure 3. Dynamical plane for f when $g(z) = z^{-3} + e^z$. Baker domains are displayed in yellow, the Julia set in blue. Left: Nearby the origin there exists 3-families of Baker domains, each one inside a sector of opening $2\pi/3$. Right: Zooming away from the origin, Baker domains associated to infinity are contained in horizontal strips of height 2π.

and $|O(e^{-if^n(z)})| \leq \exp(e^{-\Im(f^n(z))} - 1) < 1$. From these estimates it follows that $f(H_R) \subset H_R$. Thus, there exists a Baker domain U contained in the upper half plane, where $H_R \subset U$ and $\Re(f^n(z)) \to \infty$ as $n \to \infty$ for $z \in U$. To show that U is parabolic of type II in terms of König's classification (see [10, Theorem 3]), we first restrict the analysis to the set H_R. Then, one must show that for $z_0 \in H_R$ and

$$c_n(z_0) = \frac{|f^{n+1}(z_0) - f^n(z_0)|}{\text{dist}(f^n(z_0), \partial H_R)},$$

then $\liminf_{n\to\infty} c_n(z_0) > 0$ and $\inf_{z_0 \in U} \limsup_{n\to\infty} c_n(z_0) = 0$. If n is sufficiently large, combining (13) and (14), one obtains

$$\text{dist}(f^n(z_0), H_R) = \Im(f^n(z_0)) - R \leq \Im(z_0) + 1 - R.$$

Thus, when $n >> 1$,

$$\begin{aligned} c_n(z_0) &= \frac{|1 + f^n(z_0) + O(e^{-if^n(z_0)}) - f^n(z_0)|}{\Im(f^n(z_0)) - R}, \\ &\geq \frac{1 - |O(e^{-if^n(z_0)})|}{\Im(z_0) + 1 - R} > 0, \end{aligned}$$

and one obtains $\liminf_{n\to\infty} c_n(z_0) > 0$ for $z_0 \in H_R$. On the other hand,

$$\limsup_{n\to\infty} c_n(z_0) = \limsup_{n\to\infty} \frac{|1 + O(e^{-if^n(z_0)})|}{\text{dist}(f^n(z_0), \partial H_R)} < \frac{2}{\text{dist}(f^n(z_0), \partial H_R)}$$

and since $\Im(f^n(z_0)) \geq \Im(z_0) - \exp(e^{-\Im(f^n(z_0))} - 1) > \Im(z_0) - 1$, it follows that

$$\inf_{z_0 \in H_R} \limsup_{n\to\infty} c_n(z_0) = \inf_{z_0 \in H_R} \frac{2}{\Im(z_0) - (R+1)}.$$

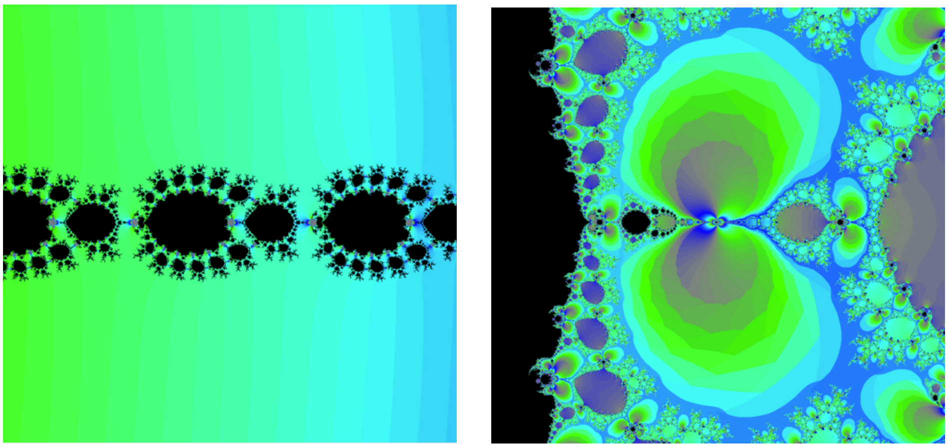

Figure 4. Left: Dynamical plane for $f(z) = z + \exp(1/\sin(z))$ centred at the origin. The black regions represent Baker domains parabolic of type I (and their preimages) associated to points z_k. The invariant Baker domains parabolic of type II associated to infinity are in colour. Right: Dynamical plane of $1/(f(1/z))$, centred at the origin. The upper and lower half planes from the left image are displayed as two petals at the origin.

So as $\Im(z_0) \to \infty$, the above inequality converges to zero. As $H_R \subset U$, one concludes that U must be parabolic of type II. The analysis for the domain in the lower half plane is similar. $\square$

Remark 2. This last example has been mentioned in [6], where the authors employ a different approach from the one provided in this article.

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