HERMAN RINGS OF ELLIPTIC FUNCTIONS

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Abstract. It has been shown by Hawkins and Koss that over any given lattice, the Weierstrass $\wp$ function does not exhibit cycles of Herman rings. We show that, regardless of the lattice, any elliptic function of order two cannot have cycles of Herman rings. Through quasiconformal surgery, we obtain the existence of elliptic functions of order at least three with an invariant Herman ring. Finally, if an elliptic function has order $o \geq 2$, then we show there can be at most $o - 2$ invariant Herman rings.

1. Introduction

Given a transcendental meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ with a unique essential singularity at infinity, we consider the dynamical system determined by its iterates. The *Fatou set* of f, denoted by $\mathcal{F}(f)$, is defined as the set of points $z \in \mathbb{C}$ that have an open neighborhood $U = U(z)$ where the family of iterates $\{f^n|U\}$ is well-defined and form a normal family. The *Julia set*, $\mathcal{J}(f)$ equals the complement of $\mathcal{F}(f)$ with respect to $\widehat{\mathbb{C}}$. In particular, $\mathcal{J}(f) = \overline{\bigcup_{n\geq 0} f^{-n}(\infty)}$.

If f is an elliptic function with respect to a lattice Λ (that is, a meromorphic function which is Λ-periodic), its order, denoted by o_f, is defined as the number of poles of f in a given period parallelogram, counting multiplicity. Furthermore, f takes each value $z \in \widehat{\mathbb{C}}$ exactly o_f times. Hence f has no omitted or asymptotic values, and its singular set consists of a finite number of critical values. One can conclude that f neither exhibits wandering nor Baker domains, and the classification of periodic Fatou components of any elliptic function reduces to (super)attracting, parabolic and rotation domains (Siegel disks or Herman rings). In general, it is a difficult problem to determine the existence of Herman rings since, in contrast to other Fatou components, a Herman ring is not associated to a periodic orbit. On the other hand, each cycle of Herman rings is closely related to the post-critical set and poles of the function.

In the context of elliptic functions, Hawkins and Koss showed in [HK04] that over any given lattice Λ, the Weierstrass $\wp$-function has no cycle of Herman rings. The proof relies on two facts: the set of poles for $\wp_\Lambda$ coincides with the set of periods and $\wp_\Lambda$ is an even function. In the particular case when H is a positively invariant Herman ring under $\wp_\Lambda$, one can assume that the origin lies inside its bounded complementary component. By showing that H is symmetric with respect to 0 (that is, $z \in H$ if and only if $-z \in H$) one obtains $\wp_\Lambda(-z) = \wp_\Lambda(z)$, a contradiction with the injectivity of $\wp_\Lambda|H$.

Non-existence results have been obtained for other elliptic functions (mostly of even order) over some particular lattices, see [Ko09], [CK15], [MP16] and [HM18]. It has been conjectured that even elliptic functions of any order do not exhibit Herman rings.

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Our first result (analogous to Shishikura's result for rational functions of degree two) is based on the explicit expression of an elliptic function of order 2 and the symmetries it exhibits with respect to its poles.

Theorem 1.1. *An elliptic function of order 2 has no cycles of Herman rings.*

Although similar ideas as for $\wp_\Lambda$ apply to the case of a double pole, a new geometrical approach is needed when the elliptic function has two simple poles.

We are also concerned with the existence of Herman rings for elliptic functions. In [Sh87], Shishikura describes a quasiconformal surgery technique between two rational functions, each one with an n-cycle of Siegel disks, to construct a new rational function with an n-cycle of Herman rings. The extension of this method to transcendental meromorphic functions is found in [DF07] and more recently in [BF14]. We adapt these techniques to perform surgery between an elliptic function f_Λ of order $o \geq 2$ with an invariant Siegel disk, and a rational function W of degree $d \geq 2$ with an invariant Siegel disk, to obtain a quasiregular function $g : \mathbb{C} \to \widehat{\mathbb{C}}$ outside a discrete set of poles with an invariant annular domain, where it is conjugated to a rigid irrational rotation (Theorem 4.2). If $\psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ denotes the quasiconformal map that conjugates g to a transcendental meromorphic function $\mathscr{G}$ and $\tilde{\Lambda} = \psi(\Lambda)$, we show that $\tilde{\Lambda}$ is a lattice (Lemma 4.7) and furthermore,

Theorem 1.2. *The function $\mathscr{G} : \mathbb{C} \to \widehat{\mathbb{C}}$ is a doubly periodic meromorphic function with respect to the lattice $\tilde{\Lambda}$, it has order $o + d - 1 \geq 3$ and possesses an invariant Herman ring.*

Since an elliptic function is a meromorphic function of finite type, then it must have a finite number of Herman rings, [Zh00, Theorem 1]. In addition, it is known that a meromorphic function with n poles has at most n invariant Herman rings (see [DF07], [FP12]). Taking into account that any elliptic function has at least two poles (counting multiplicity) on each period parallelogram, and that each invariant Herman ring can be turn into a Siegel disk through quasiconformal surgery, we obtain the following upper bound.

Theorem 1.3. *Let f be an elliptic function of order $o_f \geq 2$. Then f can have at most $o_f - 2$ invariant Herman rings.*

If f has $H_1, \dots, H_m$ invariant Herman rings, let B_i denote the compact complementary component of $\mathbb{C} \setminus H_i$ for each $i = 1, \dots, m$. From Theorem 1.3, one can easily derive the following result that, in turn, it can be implemented to refine the upper bound of invariant Herman rings according to the number of poles and their multiplicities.

Corollary 1.4. *If f has $m \geq 1$ invariant Herman rings, then the complement of $\bigcup_{i=1}^m (H_i \cup B_i) + \Lambda$ contains the residue classes of (at least) two simple poles or (at least) the residue class of a multiple pole.*

The paper is organized as follows. In Section 2 we review some properties of elliptic functions, their dynamics and introduce some notation. The proof of Theorem 1.1 is found in Section 3. In Section 4 we describe the quasiconformal surgery construction and provide the proof of Theorem 1.2. Finally, the proof of Theorem 1.3 and the implementation of Corollary 1.4 to refine the upper bound are discussed in Section 5.

2. Preliminaries

In this section we gather some standard results of elliptic functions and their dynamics. We refer the reader to the classical exposition of Du Val [Du73] for a comprehensive account on elliptic functions and their properties. The dynamics of meromorphic functions can be consulted in [Be93]. The works by Hawkins and Koss in [HK04] and [HK05] have established most of the fundamental results regarding the dynamics of elliptic functions and, in particular, the dynamics of the Weierstrass $\wp$ function.

A *lattice* $\Lambda \subset \mathbb{C}$ is a collection of complex numbers that form a discrete group with respect to addition. Λ is said to be *trivial, simple* or *double* if $\Lambda = \{0\}$, $\Lambda \cong \mathbb{Z}$ or $\Lambda \cong \mathbb{Z}\times\mathbb{Z}$, respectively. Given any nontrivial lattice Λ, its group representation can be naturally defined by the group homomorphism

$$T : \Lambda \to \mathrm{GL}(2, \mathbb{C}),\ T(\lambda) = T_\lambda : z \mapsto z + \lambda.$$

Clearly $T(\lambda_1 + \lambda_2) = T(\lambda_1) \circ T(\lambda_2)$ for any $\lambda_1, \lambda_2 \in \Lambda$ and the representation is faithful. We say that Λ acts via translations over the complex plane. A *residue class* of the action of Λ in $\mathbb{C}$ is defined as

$$[z] = \{w \in \mathbb{C} \mid \exists \lambda \in \Lambda,\ T_\lambda(w) = z\}.$$

From now on, a lattice Λ will always refer to a double lattice $\Lambda \cong \mathbb{Z} \times \mathbb{Z}$.

If $\lambda_1, \lambda_2 \in \mathbb{C}$ are $\mathbb{R}$-linearly independent and satisfy $\mathrm{Im}(\lambda_2/\lambda_1) > 0$, then the collection of all entire linear combinations of λ_1 and λ_2 determines a lattice. In this case we write $\Lambda = \langle\lambda_1, \lambda_2\rangle = \{m\lambda_1 + n\lambda_2 \mid m, n \in \mathbb{Z}\}$. An *elliptic function* $f : \mathbb{C} \to \widehat{\mathbb{C}}$, is a transcendental meromorphic function that is doubly periodic with respect to Λ. We write $\mathcal{E}(\Lambda)$ to denote the field of all elliptic functions with respect to Λ. Let Q be a fundamental domain for Λ and assume no poles lie on its boundary. The *order* of an elliptic function $f \in \mathcal{E}(\Lambda)$, denoted o_f, is the number of poles of f in the interior of Q, counting multiplicity. A consequence of Liouville's theorem is that $o_f \geq 2$ for any non-constant elliptic function.

Denote by $\wp_\Lambda$ the Weierstrass $\wp$ function which is doubly periodic with respect to Λ. Its analytic expression is given by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda^*}\left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right), \tag{2.1}$$

where $\Lambda^* = \Lambda - \{0\}$. The above expression shows that $\wp_\Lambda$ is an even meromorphic function with double poles at the residue class $[0] = \Lambda$ and at no other points. Since (2.1) converges uniformly on compact sets not containing lattice points, term by term differentiation gives $\wp'_\Lambda(z) = -2\sum_{\lambda\in\Lambda}(z-\lambda)^{-3}$, which is an odd elliptic function over Λ. Another useful analytic expression is the power series development of $\wp_\Lambda$ at the origin, namely

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)E_{2k+2}(\Lambda)z^{2k},$$

where $E_{2k}(\Lambda) = \sum_{\lambda\in\Lambda^*}\lambda^{-2k}$ is the Eisenstein series of order $2k$. This expression, combined with the power series development of $\wp'_\Lambda$ can be used to show the relation

$$(\wp'_\Lambda)^2 = 4(\wp_\Lambda)^3 - g_2\wp_\Lambda - g_3, \tag{2.2}$$

where $g_2 = g_2(\Lambda) = 60E_4(\Lambda)$ and $g_3 = g_3(\Lambda) = 140E_6(\Lambda)$. The numbers g_2, g_3 are called *invariants* of the lattice since for any $k \in \mathbb{C}^*$, $g_i(k\Lambda) = g_i(\Lambda)$ for $i = 2, 3$. The cubic polynomial $q(x) = 4x^3 - g_2x - g_3$ has discriminant $\Delta = g_2^3 - 27g_3^2$, which never vanishes for the invariants of the lattice Λ. A classical result states that a lattice can be determined by its invariants.

Theorem 2.1. *If a, b are complex numbers such that $a^3 - 27b^2 \neq 0$, then there exists a lattice Λ with $g_2(\Lambda) = a$ and $g_3(\Lambda) = b$.*

Consider $\Lambda = \langle \lambda_1, \lambda_2 \rangle$ and let $\lambda_3 := \lambda_1 + \lambda_2$. It follows from the condition $\Delta \neq 0$, that (2.2) has three distinct roots given by $e_i := \wp_\Lambda(\lambda_i/2)$. Using the standard notation for half-periods, let $\omega_i := \lambda_i/2$ for $i = 1, 2, 3$. Hence, the set of critical points of $\wp_\Lambda$ are given by the union of the residue classes of the half-periods.

2.1. **Dynamics of elliptic functions.** The Julia and Fatou sets of an elliptic function are invariant under the action of Λ (see [HK05]), that is $\mathcal{J}(f) = \mathcal{J}(f)+\Lambda$ and $\mathcal{F}(f) = \mathcal{F}(f)+\Lambda$.

As stated in the introduction, the singular set of an elliptic function consists only of a finite number of critical values, thus they are functions of finite type. Among all possible periodic Fatou components, we are only concerned in rotation domains. For $0 < r < s < \infty$, consider the standard annulus

$$A_{r,s} = \{z \in \mathbb{C} \mid r < |z| < s\}.$$

Definition 2.2. An n-periodic Fatou component $H \subset \mathbb{C}$ is a *Herman ring* for f if there exist an irrational angle $\theta \in [0, 1)$ and a conformal isomorphism $\varphi : H \to A_{1,r}$, so that φ conjugates $f^n|H$ to the action of the rigid rotation $z \mapsto e^{2\pi i\theta} z$ over $A_{1,r}$. The map φ is called *linearizing coordinates* and θ is the *rotation number* of f^n in H. The n-periodic component H defines a *cycle of Herman rings*, $\mathcal{H} = \{H_0, \ldots, H_{n-1}\}$, where $H_j := f^j(H)$ for $j = 0, \ldots, n-1$ and $f(H_j) = H_{j+1 \mod n}$.

Similarly, a *Siegel disk*, $\Delta \subset \mathbb{C}$, is an n-periodic Fatou component where $f^n|\Delta$ is conformally conjugate to an irrational rigid rotation over the unit disk under the linearizing coordinates $\varphi : \Delta \to \mathbb{D}$. Observe that for $z_0 := \varphi^{-1}(0)$, one has $f^n(z_0) = z_0$ and $(f^n)'(z_0) = e^{2\pi i\theta}$. We write $\Delta(z_0, \theta)$ to denote a Siegel disk with *center* at z_0 and rotation number θ.

For further reference we state the next result for a cycle of Siegel disks, which is based on Proposition 2.2 in [HK05]. Its proof extends verbatim for a cycle of Herman rings.

Proposition 2.3. *Let f be an elliptic function with respect to Λ. Assume f has an n-cycle of Siegel disks $\mathcal{S} = \{S_0, \ldots, S_{n-1}\}$ with $f(\mathcal{S}) = \mathcal{S}$ and $f(S_i) = S_{i+1 \mod n}$. Then,*

1. *Each S_i is contained in one fundamental domain of Λ.*
2. *If $z \in \mathcal{S}$ and $\lambda \in \Lambda^*$ then $z + \lambda \notin \mathcal{S}$.*

3. Elliptic functions of order two

Consider an elliptic function $g \in \mathcal{E}(\Lambda)$ of order 2. As shown in [Du73, Theorem 2.3], one can express g as the composition

$$g(z) = M \circ \wp_\Lambda \circ L(z),$$

with $M(z) = (az + b)/(cz + d)$ and $L(z) = z - z_0$, for constants $a, b, c, d, z_0 \in \mathbb{C}$, and $ad - bc \neq 0$. Without loss in generality, we consider throughout this work the conjugacy $f = L \circ g \circ L^{-1}$, so that $f(z) = S \circ \wp_\Lambda(z)$ and S is again a linear fractional transformation. Depending on the multiplicity of poles for f, an explicit expression can be obtained as follows.

- *Case 1: Double poles.* Since $f = S \circ \wp_\Lambda$ has a double pole at the origin, then, for some $A, k \in \mathbb{C}, A \neq 0$, we can express f as

$$f(z) = A(\wp_\Lambda(z) - \wp_\Lambda(k)), \tag{3.1}$$

where $\mathrm{Zeros}(f) = \{\pm k\} + \Lambda$. Moreover, $\mathrm{Crit}(f) = \mathrm{Crit}(\wp_\Lambda)$.

- *Case 2: Simple poles.* Assume f has simple poles at $v, w \in \mathbb{C}$. For simplicity, let $2h \equiv (v-w) \mod \Lambda$, so that $\mathrm{Poles}(f) = \{\pm h\} + \Lambda$. Then, for some $A, k \in \mathbb{C}, A \neq 0$, we can write

$$f(z) = A\left(\frac{\wp_\Lambda(z) - \wp_\Lambda(k)}{\wp_\Lambda(z) - \wp_\Lambda(h)}\right), \tag{3.2}$$

where $\mathrm{Zeros}(f) = \{\pm k\} + \Lambda$. In this case, $\mathrm{Crit}(f) = \{0, \omega_1, \omega_2, \omega_3\} + \Lambda$.

From now on, f denotes an order 2 elliptic function written either as in (3.1) or as in (3.2).

Remark 3.1. Regardless of the order of the poles, observe that $f = S \circ \wp_\Lambda$ is an even function. Thus, f exhibits the same critical point symmetry as $\wp_\Lambda$, namely $f(c - w) = f(c + w)$ for all $c \in \mathrm{Crit}(f)$ and any $w \in \mathbb{C}$.

Let Γ denote a simple closed curve over the complex plane with a given orientation, and denote by $D = D_\Gamma$ the bounded component of $\mathbb{C} \setminus \Gamma$. Fix a point $a \in \mathbb{C} \setminus \Gamma$; the notation $-\Gamma + a$ stands for the set of points $\{-w + a \mid w \in \Gamma\}$. The following result is straightforward.

Lemma 3.2. *Let $p \in D$ and $s \in \mathbb{C} \setminus \overline{D}$. Then, the set $-\Gamma + 2p$ is a simple closed curve that surrounds p and has nonempty intersection with Γ. Also, $-\Gamma + (s + p)$ is a simple closed curve that surrounds s.*

3.1. **Proof of Theorem 1.1.** Assume $\mathcal{H} = \{H_0, \ldots, H_{q-1}\}$ is a q-cycle of Herman rings for f, with $q \geq 1$. Denote by B_i the compact component of $\mathbb{C} \setminus H_i$. As a consequence of [FP12, Theorem A], at least one of the B_i's must contain a pole of f. We assume the q-cycle has been labeled in such a way that the bounded component B_0 contains a pole, and then set $H_{i+1 \mod q} = f(H_i)$.

Choose a base point $b \in H_0$ and let $\Gamma = \overline{\mathcal{O}_f^+(b)}$. Clearly, $\Gamma = \Gamma_0 \sqcup \ldots \sqcup \Gamma_{q-1}$, where each Γ_i is a smooth, simple closed curve in H_i with a given orientation, and f preserves those orientations. Let D_i be the open and bounded component of $\mathbb{C} \setminus \Gamma_i$.

Assume first that f has a double pole at the origin. Let $\lambda \in B_0$ and consider the curve $-\Gamma_0 + 2\lambda$, which also surrounds λ. Then, since f is even and Λ-periodic, we have that for any $w \in \Gamma_0$,

$$f(w) = f(\lambda + (w - \lambda)) = f(\lambda - (w - \lambda)) = f(2\lambda - w),$$

in other words, f sends the curves Γ_0 and $-\Gamma_0 + 2\lambda$ onto $\Gamma_1 \subset H_1$. From Lemma 3.2, the curves have nonempty intersection so they both lie in H_0. In consequence, $\Gamma_0 = -\Gamma_0 + 2\lambda$ as sets, (that is, Γ_0 is symmetric with respect to λ). To reach a contradiction, we only need to observe that w and $2\lambda - w$ are two distinct points in Γ_0 with the same image. Indeed, $w = 2\lambda - w$ if and only if $w = \lambda$, but this is impossible as Γ_0 lies in the Fatou set and the set of poles of f lie in the Julia set. We conclude that any order two elliptic function with a double pole cannot have a cycle of Herman rings.

The same arguments used for the double pole case can be applied to critical points: if there exists a curve Γ_i whose component D_i contains a critical point $c \in \mathrm{Crit}(f)$, then Γ_i must be symmetric with respect to c, that is $\Gamma_i = -\Gamma_i + 2c$. But then again, Γ_i contains two distinct points, w and $2c - w$, with the same image (for otherwise, $w = 2c - w$ if and only if $w = c \in H_i$, which is impossible). Thus, regardless of the multiplicity of poles, we conclude

Lemma 3.3. *Each open component D_i is disjoint from Crit(f).*

From now on, assume f has two simple poles on each fundamental domain.

Proposition 3.4. *For each i, the cardinality of $D_i \cap Poles(f)$ is at most equal to one. Thus, if the intersection is empty, f maps D_i onto D_{i+1} conformally; otherwise f maps D_i onto $\widehat{\mathbb{C}} \setminus \overline{D_{i+1}}$ as a univalent, meromorphic function that reverses orientation.*

Proof. By assumption, f has simple poles, and since each D_i does not contain critical points, then it is sufficient to show that D_i contains at most one simple pole.

Part 1. in Proposition 2.3 can be applied to any cycle of Herman rings, thus, each H_i lies in a fundamental domain of Λ. If there exists a component D_i that contains more than one pole, then it must contain the two non-equivalent poles of its fundamental domain. Without loss of generality, let those poles be $\zeta = h$ and $\zeta' = -h + \lambda$, for some $\lambda \in \Lambda$. The midpoint of the line segment that joins ζ with ζ' is the critical point $c = \lambda/2$, and by Lemma 3.3, c must lie in the unbounded component of Γ_i.

Consider the sets $-D_i+2c$ and $-\Gamma_i+2c$. Clearly, $-D_i+2c$ is a topological disk, symmetric to D_i with respect to c, so it must contain both ζ and ζ'. Hence $\Gamma_i \cap (-\Gamma_i + 2c) \neq \emptyset$. If these curves are equal, then c must lie in D_i, a contradiction. If the curves are not equal, the critical point symmetry of f implies that $f(\Gamma_i) = f(-\Gamma_i + 2c)$, which contradicts the univalence of $f^n|H_i$. □

Corollary 3.5. *Any elliptic function of order 2 and with simple poles cannot have a fixed Herman ring.*

Proof. Indeed, if f has a positively invariant Herman ring H and $\Gamma \subset H$ is any curve of its foliation, then the bounded component $D \subset \widehat{\mathbb{C}} \setminus \Gamma$ contains a simple pole. From the previous proposition, f must map Γ onto itself with reversed orientation, a contradiction. □

One can infer from the previous results the existence of an even number of curves in the cycle $\Gamma = \{\Gamma_0, \ldots, \Gamma_{q-1}\}$ each one surrounding a pole. Let $\Gamma_{i_1}, \ldots, \Gamma_{i_{2r}} \in \Gamma$ be those curves (which are pairwise disjoint by definition) and assume each Γ_{i_j} surrounds the pole ζ_{i_j}. Using Lemma 3.2 one can construct new curves $\Gamma'_{i_1}, \ldots, \Gamma'_{i_{2r}} \in f^{-1}(\Gamma)$ so that every Γ'_{i_j} surrounds the same given pole, say ζ_0. This can be achieved by defining

$$\Gamma'_{i_j} := -\Gamma_{i_j} + (\zeta_0 + \zeta_{i_j}) = \{-w + \zeta_0 + \zeta_{i_j} \mid w \in \Gamma_{i_j}\}. \tag{3.3}$$

Observe that $\Gamma'_{i_1}, \ldots, \Gamma'_{i_{2r}}$ are also pairwise disjoint. For otherwise, if $\Gamma'_{i_j} \cap \Gamma'_{i_k} \neq \emptyset$, then f maps this intersection onto a subset of $\Gamma_{i_j+1} \cap \Gamma_{i_k+1}$ which can only happen if and only if $i_j = i_k$. Finally, if from the beginning, all the Γ_{i_j} surround the same pole, then set $\Gamma'_{i_j} := \Gamma_{i_j}$ and denote by ζ_0 the common pole. Let D'_{i_j} denote the bounded component of $\mathbb{C} \setminus \Gamma'_{i_j}$ and relabel the curves so that $\Gamma'_{i_j} \subset D'_{i_{j+1}}$ for $j = 1, \ldots, 2r-1$. Clearly, $\overline{D'_{i_j}} \subset D'_{i_{j+1}}$ so we can define the annular open region

$$A_j = A(\Gamma'_{i_j}, \Gamma'_{i_{j+1}}) := D'_{i_{j+1}} \setminus \overline{D'_{i_j}},$$

for $j = 1, \ldots, 2r-1$. Each A_j contains at least a boundary component of either a Herman ring from the q-cycle, or either a boundary component of one of its preimages, and hence, it contains many prepoles. If $p \in A_j$ is a prepole of f, let o_p denote the non-negative integer so that $f^{o_p}(p) \in \mathrm{Poles}(f)$.

Lemma 3.6. *For each $j = 1, \ldots, 2r-1$, let $k_j := \min\{o_p \mid p \in A_j \text{ prepole}\}$. Assume $k_{j_0} := \min\{k_j \mid 1 \leq j \leq 2r-1\}$, that is, the minimum of the orders is achieved inside the annulus A_{j_0}. Then $1 \leq k_{j_0} \leq q-1$ and $k_{j_0} < k_j$ for all $j \neq j_0$.*

Proof. First, observe that $k_{j_0} > 0$. Furthermore, if $k_{j_0} \geq q$ then $f^q|_{A_{j_0}}$ acts conformally and thus $\{f^{nk_{j_0}}|A_{j_0}\}_{n\geq 1}$ is a normal family, which is impossible. Now assume the minimum is achieved in both A_{j_0} and A_i, with $j_0 < i$ (the case $i < j_0$ is similar). From Proposition 3.4, it follows $f^{k_{j_0}}$ acts conformally over both rings, so their images under $f^{k_{j_0}}$ are again annuli. There exist distinct poles ζ and ζ' so that $\zeta \in f^{k_{j_0}}(A_{j_0})$ and $\zeta' \in f^{k_{j_0}}(A_i)$. If $j_0+1 < i$ then $f^{k_{j_0}}(A_{j_0+1})$ is an annulus that lies in $\widehat{\mathbb{C}} \setminus (f^{k_{j_0}}(A_{j_0}) \cup f^{k_{j_0}}(A_i))$ and contains ∞. This implies that $k_{j_0+1} < k_{j_0}$ which contradicts the minimality of k_{j_0}. If $j_0 + 1 = i$, then the images of A_{j_0} and A_{j_0+1} under $f^{k_{j_0}}$ are two nested annuli that share one boundary component. Hence, the most exterior boundary curve of $f^{k_{j_0}}(A_{j_0}) \cup f^{k_{j_0}}(A_i)$ surrounds two poles, a contradiction with Proposition 3.4. □

Remark 3.7. The notation Γ'_{i_j} will imply a curve in $f^{-1}(\Gamma)$ defined as in (3.3) that surrounds a suitable choice of ζ_0, while Γ_{i_j} will denote a curve lying in Γ (thus q-periodic) that surrounds its pole ζ_{i_j}.

Lemma 3.8. *Assume each disk in the nested collection $D'_{i_1} \subset \ldots \subset D'_{i_{2r}}$ has a common simple pole in its interior and has no other poles or critical points. Then, for each $j = 1, \ldots, 2r-1$, the interior and exterior boundaries of A_j are mapped under $f^{k_{j_0}}$ to the exterior and interior boundaries of $f^{k_{j_0}}(A_j)$, respectively.*

Proof. From Proposition 3.4, f sends the collection $\{D'_{i_j}\}$ homeomorphically onto a new nested collection of disks in $\widehat{\mathbb{C}}$, where $\infty \in f(D'_{i_1}) \subset \ldots \subset f(D'_{i_{2r}})$. The boundary of each $f(D'_{i_j})$ is given by $f(\Gamma'_{i_j}) = \Gamma_{i_j+1} \in \Gamma$ (with addition mod q). Let D_{i_j+1} denote the bounded complementary component of Γ_{i_j+1}. Then for each $j = 1, \ldots, 2r-1$, the condition

$$\infty \in f(D'_{i_j}) \subset f(D'_{i_{j+1}}) \qquad \text{implies} \qquad D_{i_{j+1}+1} \subset D_{i_j+1}. \tag{3.4}$$

Since f sends $A_j = D'_{i_{j+1}} \setminus \overline{D'_{i_j}} = A(\Gamma'_{i_j}, \Gamma'_{i_{j+1}})$ conformally onto $f(A_j) = f(D'_{i_{j+1}}) \setminus \overline{f(D'_{i_j})}$, it follows from (3.4) that $f(A_j) = D_{i_j+1} \setminus \overline{D_{i_{j+1}+1}}$, that is $f(A_j) = A(\Gamma_{i_{j+1}+1}, \Gamma_{i_j+1})$. This proves that f sends the interior and exterior boundaries of A_j onto the exterior and interior boundaries of $f(A_j)$ respectively. Briefly, we will refer to the interchange of boundaries just described as "f flips the boundaries of A_j".

We want to show $f^{k_{j_0}}$ flips the boundaries of each A_j only once. If $k_{j_0} = 1$ this is clear, so assume $k_{j_0} > 1$. Lemma 3.6 implies that $f^{k_{j_0}}$ acts conformally on each $\overline{A_j}$, and since $\bigcup f(\overline{A_j}) = \overline{D_{i_1+1}} \setminus D_{i_{2r}+1}$, it suffices to prove that $D_{i_{2r}+1}$ does not contain poles nor prepoles of order less than $k_{j_0} - 1$. For the shake of contradiction, assume the existence of $p \in D_{i_{2r}+1}$ and a least integer $0 \leq m < k_{j_0} - 1$ so that $f^m(p) = \zeta$ is a pole. Since $p \in D_{i_{2r}+1} \subset \ldots \subset D_{i_1+1}$, we have $\zeta \in f^m(D_{i_{2r}+1}) = D_{i_{2r}+m+1} \subset \ldots \subset f^m(D_{i_1+1}) = D_{i_1+m+1}$. This implies $\{\Gamma_{i_j+m+1}\}$ is a collection of $2r$ curves in Γ that surround the common pole ζ. From the definition of $\{\Gamma'_{i_j}\}$ we must have $\{\Gamma'_{i_j}\} = \{\Gamma_{i_j+m+1}\}$ as sets (and by convention, $\Gamma'_{i_j} = \Gamma_{i_j} \in \Gamma$, so we drop the symbol $'$). Observe f^{m+1} permutes these curves in such a way that $\Gamma_{i_j+m+1} = f^{m+1}(\Gamma_{i_j}) = \Gamma_{i_{2r-j+1}}$ for each $j = 1, \ldots, 2r$. It follows that $f^{2(m+1)}(\Gamma_{i_j}) = f^{m+1}(\Gamma_{i_{2r-j+1}}) = \Gamma_{i_{2r-(2r-j+1)+1}} = \Gamma_{i_j}$. As a consequence, $\{f^{2n(m+1)}|A_j\}_{n\geq 1}$ forms a normal family, arriving at the sought contradiction. □

We now proceed with the final part of the proof of Theorem 1.1 for the case of simple poles. To ease the notation, the numbers k_{j_0} and j_0 defined in Lemma 3.6 are now simply denoted by k and j. Thus, $p_j \in A_j$ denotes the prepole with smallest order among all prepoles in $A_1 \cup \ldots \cup A_{2r-1}$. And since $1 \le k \le q-1$ then we can find unique integers $m_0, n_0 \ge 1$ with $n_0 < k$, so that

$$q = m_0 \cdot k + n_0. \tag{3.5}$$

Since $f^k|A_j$ acts conformally, then $f^k(A_j)$ is an annular domain that contains the pole $f^k(p_j)$, and from Lemma 3.8, the exterior boundary of $f^k(A_j)$, namely $f^k(\Gamma'_{i_j})$, surrounds that pole. Then we can set $\Gamma_{i_\ell} := f^k(\Gamma'_{i_j})$ for some $\ell \in \{1, \ldots, 2r-1\}$ (hence, $f^k(p_j) = \zeta_{i_\ell}$) and $\Gamma_s := f^k(\Gamma'_{i_{j+1}})$ for some $s \in \{0, \ldots, q-1\}$. Since $1 \le k \le q-1$, then either $\ell < j$ or $\ell > j$.

Case $\ell < j$: Define $\Gamma'_{i_j} := -\Gamma_{i_j} + \zeta_{i_\ell} + \zeta_{i_j}$ (so it surrounds ζ_{i_ℓ}) and consider the annular region $A := A(\Gamma_{i_\ell}, \Gamma'_{i_j})$. Lemma 3.6 and Lemma 3.8 guarantee that $f^k|A$ acts conformally and flips only once the boundaries of A, thus $f^k(A) = A(\Gamma_{i_\ell}, \Gamma_{i_t})$, where $\Gamma_{i_t} := f^k(\Gamma_{i_\ell})$ for some $\ell < t \le 2r$. If $t < j$ (resp. $t > j$), then Grötzsch inequality implies a contradiction: one has $\mathrm{mod}(A) = \mathrm{mod}(f^k(A))$ and at the same time $f^k(A)$ is essentially contained as a proper annulus of A (resp. A is essentially contained as a proper annulus of $f^k(A)$). On the other hand, if $t = j$ then $f^k(A) = A$ and hence, $\{f^{nk}|A\}_{n\ge 1}$ is a normal family, a contradiction.

Case $\ell > j$: Let Γ'_{i_j} and $\Gamma'_{i_{j+1}}$ denote curves in $f^{-1}(\Gamma)$ that surround the pole ζ_{i_ℓ} and write $A_j := A(\Gamma'_{i_j}, \Gamma'_{i_{j+1}})$. From Lemma 3.8, f^k flips the boundaries of A_j only once, hence $f^k(A_j) = A(\Gamma_s, \Gamma_{i_\ell})$. And by Lemma 3.6, $\zeta_{i_\ell} \in f^k(A_j)$, thus Γ_s cannot surround this pole, and in particular $\Gamma_s \ne \Gamma'_{i_j}, \Gamma'_{i_{j+1}}$. The proof now resides in the location of Γ_s with respect to Γ'_{i_j} and $\Gamma'_{i_{j+1}}$.

If $\Gamma_s \subset D'_{i_j}$, then A_j is essentially contained as a proper annulus of $f^k(A_j)$, which contradicts Grötzsch inequality.

If $\Gamma_s \subset A(\Gamma'_{i_{j+1}}, \Gamma_{i_\ell})$, then $\overline{A_j} \subsetneq f^k(A_j)$ non-essentially. Denote by D_s the bounded complementary component of Γ_s and let $A := A(\Gamma'_{i_{j+1}}, \Gamma_{i_\ell})$. Observe that A does not contain any prepole of order less or equal than k, thus f^k flips its boundaries and sends A isomorphically to the annular domain

$$f^k(A) = D_s \setminus \overline{f^k(D_{i_\ell})} = D_s \setminus \overline{D_{i_\ell+k}} = A(\Gamma_{i_\ell+k}, \Gamma_s),$$

where $\Gamma_{i_\ell+k} = f^k(\Gamma_{i_\ell}) \subset D_s$ and $i_\ell + k \in \{0, \ldots, q-1\}$ (addition mod q). Furthermore, since $\Gamma_s \subset A$, then $\Gamma_{s+k} = f^k(\Gamma_s)$ must lie inside $f^k(A)$, so in particular, $\Gamma_{s+k} \subset D_s \setminus \overline{D_{i_\ell+k}}$. See Figure 1.

Now consider the 3-connected set $E := D_{i_\ell} \setminus (\overline{D'_{i_j}} \cup \overline{D_s})$, which contains the prepole p_j of minimal order. Recalling that f^k maps $\Gamma_{i_\ell} \mapsto \Gamma_{i_\ell+k}$, $\Gamma_s \mapsto \Gamma_{s+k}$ and $\Gamma'_{i_j} \mapsto \Gamma_{i_\ell}$, then it maps E conformally onto the 3-connected set

$$f^k(E) = D_{i_\ell} \setminus (\overline{D_{i_\ell+k}} \cup \overline{D_{s+k}}).$$

To reach a contradiction, we now provide an extremal length argument (see [Ah73] or appendix in [KL09] for further reference). Let $G \subset E$ be the family of curves that connect Γ_{i_ℓ} to the boundaries Γ'_{i_j} and Γ_s. Similarly, consider the family $G' \subset D_s \setminus (\overline{D_{i_\ell+k}} \cup \overline{D_{s+k}})$ connecting the respective boundaries. Then, $S = f^k(G)$ defines a family of curves in $f^k(E)$ that *overflows* both G and G' (S overflows G if any curve in S contains some curve in G). From conformality of f^k one obtains equality of extremal lengths, namely $\mathcal{L}(S) = \mathcal{L}(G)$, while by the Series Law, $\mathcal{L}(S) \ge \mathcal{L}(G) + \mathcal{L}(G')$, a contradiction.

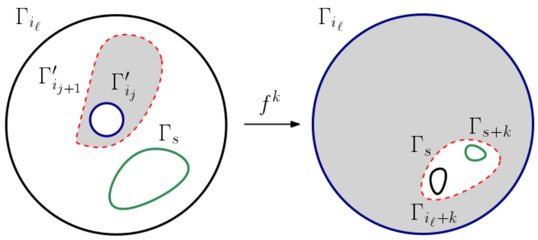

FIGURE 1. On the left, the configuration of curves $\Gamma'_{i_j}, \Gamma'_{i_{j+1}}$ and their f^k-images (namely, Γ_{i_ℓ} and Γ_s resp.) are shown. Since Γ_s is contained in the annular domain $A = A(\Gamma'_{i_j+1}, \Gamma_{i_\ell})$, its image under f^k maps onto a curve Γ_{s+k} contained in $A(\Gamma_{i_\ell+k}, \Gamma_s)$. Colors match their image.

The final case is when $\Gamma_s \subset A_j = A(\Gamma'_{i_j}, \Gamma'_{i_{j+1}})$ in such a way that the component $\overline{D_s}$ lies completely inside A_j and it is also the bounded complementary component of $f^k(A_j)$.

Since $\Gamma'_{i_{j+1}} \subset f^k(A_j)$ then we can find a simple closed curve $\eta \subset A_j$ sufficiently close to Γ'_{i_j} so that $f^k(\eta) = \Gamma'_{i_{j+1}}$. Denote by D_η the bounded complementary component of $\mathbb{C} \setminus \eta$ and define $C = A(\eta, \Gamma'_{i_j+1})$. Then, C is essentially contained in A_j and f^k sends C conformally onto $f^k(C) = A(\Gamma_s, \Gamma'_{i_{j+1}})$. Clearly $\Gamma_s \neq \eta$ for otherwise $f^k(C) = C$ and hence $\{f^{nk}|C\}_{n\geq 1}$ becomes a normal family. If Γ_s lies in D_η, then C is essentially contained as a subannulus in $f^k(C)$, contradicting Grötzsch inequality. So the remaining case to analyze is when $\Gamma_s \subset D'_{i_{j+1}} \setminus \overline{D_\eta}$.

Define $E_1 := D'_{i_{j+1}} \setminus (\overline{D_s} \cup \overline{D_\eta})$ and note that $f^k|E_1$ is conformal. Furthermore, $f^k(E_1)$ is a 3-connected domain bounded by $\Gamma'_{i_{j+1}}, \Gamma_s$ and Γ_{s+k}. We have several possible scenarios for the location of Γ_{s+k} with respect to the boundary components of E_1. First of all, Γ_{s+k} must lie in $D'_{i_{j+1}}$. Then, observe that Γ_{s+k} cannot be neither equal to Γ_s (as $k < q$) nor be in its interior (as E_1 is 3-connected). Then, we only need to analyze the location of Γ_{s+k} with respect to η:

(1) If Γ_{s+k} lies in D_η, then we can find two families of curves $G \subset E_1$ and $G' \subset D_\eta \setminus \overline{D_{s+k}}$ so that $S = f^k(G)$ in $f^k(E_1)$ overflows both G and G'. As before, conformality and the Series Law provide a contradiction.
(2) If $\Gamma_{s+k} = \eta$, then $f^k(E_1) = E_1$ which implies that $\{f^{nk}|E_1\}_{n\geq 1}$ is a normal family, again a contradiction.

We conclude that Γ_{s+k} lies in E_1 (see Figure 2). One can repeat the analysis above to conclude that for each $i = 1, \ldots, m_0 - 1$, the curve Γ_{s+ik} lies in E_i so the region

$$E_{i+1} := E_i \setminus \overline{D_{s+ik}},$$

is well defined. Furthermore, $f^k|E_i$ is conformal for each i and $E_{m_0} \subset E_{m_0-1} \subset \ldots \subset E_1 \subset C \subset A_j$. In particular, each disk D_{s+ik} also lies in A_j for all $i = 1, \ldots, m_0 - 1$.

If we apply f^{n_0} to the boundary curve $\Gamma_{s+(m_0-1)k}$ of E_{m_0}, it follows from equation (3.5) that

$$f^{n_0}(\Gamma_{s+(m_0-1)k}) = f^{n_0}(f^{m_0 k}(\Gamma_{i_{j+1}})) = \Gamma_{i_{j+1}},$$

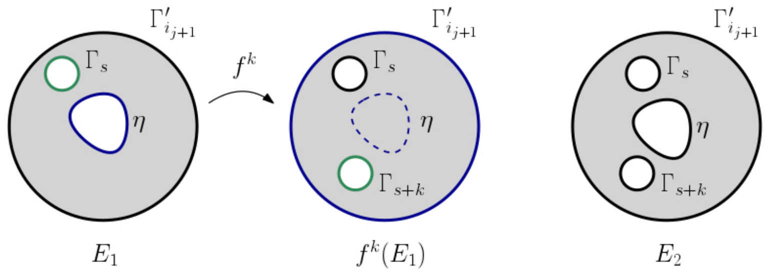

FIGURE 2. The only remaining possible location of Γ_{s+k} with respect to the boundaries of E_1 is shown. E_1 is the grey region bounded by $\Gamma'_{i_{j+1}}, \eta$ and Γ_s. Its conformal image, $f^k(E_1)$, is shown in grey and is bounded by $\Gamma'_{i_{j+1}} = f^k(\eta)$, $\Gamma_s = f^k(\Gamma'_{i_{j+1}})$ and $\Gamma_{s+k} = f^k(\Gamma_s)$ (colors match their image). The 4-connected region E_2 is obtained by removing the topological disk $\overline{D_{s+k}}$ from E_1.

since $\Gamma_s = f^k(\Gamma_{i_{j+1}}) = f^k(\Gamma'_{i_{j+1}})$. The curve $\Gamma_{i_{j+1}}$ surrounds the pole $\zeta_{i_{j+1}}$ by construction. Then, $\Gamma_{s+(m_0-1)k}$ must surround a prepole $p' \in D_{s+(m_0-1)k} \subset A_j$ with $o_{p'} < k$, a contradiction since k was the smallest order of prepoles in A_j. This concludes the case $\ell > j$ and the proof of Theorem 1.1.

4. Herman rings via quasiconformal surgery

In this section we describe the quasiconformal surgery between an elliptic function of order $o \geq 2$ and a rational function of degree $d \geq 2$, both with an invariant Siegel disk in their Fatou sets. The surgery produces an elliptic function of order $o + d - 1 \geq 3$ with an invariant Herman ring, thus answering in the positive the question of existence of Herman rings for elliptic functions.

4.1. **Preparing for surgery.** Let $f \in \mathcal{E}(\Lambda)$ be an elliptic function of order $o := o_f \geq 2$. Assume f has an invariant Siegel disk $\Delta = \Delta(0,\theta)$ centered at the origin with rotation number $\theta \in [0,1)$. Similarly, let W denote a rational map of degree $d \geq 2$ with an invariant Siegel disk $\tilde{\Delta} = \Delta(0, 1-\theta)$ centered at the origin and rotation by angle $1-\theta$. We also assume $\tilde{\Delta}$ does not contain the point at infinity. Denote the set of poles of f by $\mathrm{Poles}(f) = P_f + \Lambda$, where $P_f = \{\zeta_1, \ldots, \zeta_{o_f}\}$, listed with multiplicities. Let $Z_W = \{\eta_1, \ldots, \eta_d\}$ be the set of zeros of W, listed with multiplicities, in such a way that $\eta_1 = 0$ and thus $\eta_j \neq 0$ for $j = 2, \ldots, d$.

Consider the triplets $(f, \Delta, 0)$ and $(W, \tilde{\Delta}, 0)$. Each triplet has associated a linearizing coordinate

$$\varphi : \Delta \to \mathbb{D} \qquad \text{and} \qquad \tilde{\varphi} : \tilde{\Delta} \to \mathbb{D},$$

that conjugates $f|_{\Delta}$ and $W|_{\tilde{\Delta}}$ with rigid rotations of rotation numbers θ and $1-\theta$, both acting on $\mathbb{D}$. Fix $0 < r, \tilde{r} < 1$ and denote by $\gamma \subset \Delta$ and $\tilde{\gamma} \subset \tilde{\Delta}$ the invariant simple closed curves defined by $\varphi^{-1}(C_r)$ and $\tilde{\varphi}^{-1}(C_{\tilde{r}})$.

The circle inversion $L(z) = \tilde{r}r/z$ maps C_r bijectively to $C_{\tilde{r}}$ and, moreover, conjugates the rigid rotations restricted over these circles. Therefore, we can define a glueing map over the invariant curves γ and $\tilde{\gamma}$ by

$$h = \tilde{\varphi}^{-1}|_{\tilde{\gamma}} \circ L \circ \varphi|_{\gamma}.$$

From the above construction, h is a diffeomorphism and conjugates $f|_\gamma$ with $W|_{\tilde{\gamma}}$. One can construct a global glueing map $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that extends h analytically; Φ will be the result of four maps defined over different regions of the sphere. To begin, recall that if Γ is any simple closed curve in $\mathbb{C}$, then D_Γ denotes the bounded domain of $\mathbb{C} \setminus \Gamma$. Consider an annular neighborhood of $\gamma = \varphi^{-1}(C_r)$ defined by the preimage of an standard annulus, namely

$$A_\gamma := \varphi^{-1}(A_{r_1,r_2}),$$

where $0 < r_1 < r < r_2 < 1$. Define also $\gamma_{in} := \varphi^{-1}(C_{r_1})$, $\gamma_{out} := \varphi^{-1}(C_{r_2})$, $A_{in} := \varphi^{-1}(A_{r_1,r})$ and $A_{out} := \varphi^{-1}(A_{r,r_2})$. Analogous constructions under preimages by $\tilde{\varphi}$ and values $0 < \tilde{r}_1 < \tilde{r} < \tilde{r}_2 < 1$ determine the sets $A_{\tilde{\gamma}}, \tilde{A}_{in}, \tilde{A}_{out}, \tilde{\gamma}_{in}$ and $\tilde{\gamma}_{out}$.

Observe that $\overline{A}_\gamma$ is a closed and f-invariant annular neighborhood for γ, properly contained in Δ. The complementary components of $\overline{A}_\gamma$, namely $D_{\gamma_{in}}, \widehat{\mathbb{C}} \setminus \overline{D}_{\gamma_{out}}$ (and the respective complementary components for $\overline{A}_{\tilde{\gamma}}$) are simply connected proper domains of the sphere, so there exist conformal isomorphisms

$$R_{out} : \widehat{\mathbb{C}} \setminus \overline{D}_{\gamma_{out}} \to D_{\tilde{\gamma}_{in}} \qquad \text{and} \qquad R_{in} : D_{\gamma_{in}} \to \widehat{\mathbb{C}} \setminus \overline{D}_{\tilde{\gamma}_{out}}$$

that satisfy $R_{out}(\infty) = 0$ and $R_{in}(0) = \infty$. Moreover, since each invariant curve inside its Siegel disk is by definition analytic, there exist analytic extensions of the above maps into the boundary of their domains. Denote by $\widehat{R}_{out}$ the analytic extension of R_{out} over γ_{out} and by $\widehat{R}_{in}$ the extension of R_{in} over γ_{in}.

Two more maps must be defined over the subannuli of $\overline{A}_\gamma$. To do so, observe first that the restriction maps $h|_\gamma, \widehat{R}_{out}|_{\gamma_{out}}$ and $\widehat{R}_{in}|_{\gamma_{in}}$ define C^1-diffeomorphisms, and thus, they are all quasisymmetric functions. From Proposition 2.30, part (b) in [BF14], there exist two quasiconformal maps

$$h_{out} : \overline{A}_{out} \to \overline{\tilde{A}}_{in} \qquad \text{and} \qquad h_{in} : \overline{A}_{in} \to \overline{\tilde{A}}_{out}$$

with quasiconformal constants K_{out} and K_{in} respectively, defined in such a way that $h_{out}|_{\gamma_{out}} = \widehat{R}_{out}|_{\gamma_{out}}$, $h_{in}|_{\gamma_{in}} = \widehat{R}_{in}|_{\gamma_{in}}$ and $h_{out}|_\gamma = h_{in}|_\gamma = h|_\gamma$. Finally, define the global glueing map by

$$\Phi := \begin{cases} R_{out} & \text{on } \widehat{\mathbb{C}} \setminus \overline{D}_{\gamma_{out}}, \\ h_{out} & \text{on } \overline{A}_{out}, \\ h_{in} & \text{on } \overline{A}_{in}, \\ R_{in} & \text{on } D_{\gamma_{in}}. \end{cases}$$

Proposition 4.1. *The map $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfy the following properties:*

(1) $\Phi|_\gamma = h$.
(2) $\Phi(D_\gamma) = \widehat{\mathbb{C}} \setminus \overline{D}_{\tilde{\gamma}}$ *and* $\Phi(\widehat{\mathbb{C}} \setminus \overline{D}_\gamma) = D_{\tilde{\gamma}}$.
(3) $\Phi(0) = \infty$ *and* $\Phi(\infty) = 0$.
(4) Φ *is conformal everywhere except in a closed annular neighborhood of* γ.

Proof. The first property follows from the fact that $\Phi|_\gamma = h_{out}|_\gamma = h$. For the second, observe that $D_\gamma = A_{in} \sqcup \gamma_{in} \sqcup D_{\gamma_{in}}$, thus

$$\Phi(D_\gamma) = \tilde{A}_{out} \sqcup \tilde{\gamma}_{out} \sqcup \widehat{\mathbb{C}} \setminus \overline{D}_{\tilde{\gamma}_{out}} = \widehat{\mathbb{C}} \setminus \overline{D}_{\tilde{\gamma}}.$$

A similar decomposition applied to $\widehat{\mathbb{C}} \setminus \overline{D}_\gamma$ shows $\Phi(\widehat{\mathbb{C}} \setminus \overline{D}_\gamma) = D_{\tilde{\gamma}}$. The third property is straightforward. For the last property, recall that both h_{in} and h_{out} are, respectively, $K_{in}-$ and $K_{out}-$quasiconformal maps in the interior of their annular domains and agree with conformal maps on the boundary curves γ, γ_{in} and γ_{out}. On the complementary domains of $\overline{A}_\gamma$, Φ agrees with conformal maps R_{out} and R_{in}. Thus, Φ is conformal everywhere except in A_γ. $\square$

A K-*quasiregular function* can be defined as the composition of a K-quasiconformal map and a holomorphic function, regardless of the order of composition, (see for example Definition 1.33 and Proposition 1.37 in [BF14]). Our goal is to construct a K-quasiregular function $g : \mathbb{C} \to \widehat{\mathbb{C}}$ that is 2-periodic with respect to Λ and has an invariant annular domain. To do so, let

$$g(z) := \begin{cases} \Phi^{-1} \circ W \circ \Phi \circ T_\lambda^{-1}(z) & \text{if } z \in D_\gamma + \lambda, \lambda \in \Lambda, \\ f(z) & \text{otherwise.} \end{cases} \tag{4.1}$$

Theorem 4.2. *The function $g : \mathbb{C} \to \widehat{\mathbb{C}}$ is a quasiregular function outside a discrete set of poles, it is 2-periodic with respect to Λ and on each fundamental domain of Λ, it has exactly $o+d-1 \geq 3$ poles, counted with multiplicity. Moreover, g has an invariant annular domain where it is conjugated to a rigid rotation by angle θ.*

Proof. First, note that g is continuous at γ since $\Phi|_\gamma = h$ and h conjugates $f|_\gamma$ with $W|_{\tilde{\gamma}}$. From the second part of definition in (5), g is Λ-periodic in $\mathbb{C} \setminus \bigcup_{\lambda\in\Lambda}(D_\gamma + \lambda)$, and hence, continuous in $\gamma + \Lambda$. For an arbitrary point $z \in \bigcup_{\lambda\in\Lambda}(D_\gamma + \lambda)$, set $z = \zeta + \lambda$ for some $\lambda \in \Lambda$ and $\zeta \in D_\gamma$. The first part of (5) shows

$$g(z) = \Phi^{-1} \circ W \circ \Phi \circ T_\lambda^{-1}(\zeta + \lambda) = \Phi^{-1} \circ W \circ \Phi(\zeta) = g(\zeta).$$

Given any $\mu \in \Lambda$, let $\eta = \lambda + \mu \in \Lambda$, then, since $z + \mu = \zeta + \eta \in D_\gamma + \eta$,

$$g(z + \mu) = \Phi^{-1} \circ W \circ \Phi \circ T_\eta^{-1}(\zeta + \eta) = \Phi^{-1} \circ W \circ \Phi(\zeta) = g(\zeta),$$

thus $g(z + \mu) = g(z)$ by the previous step. We conclude that g is Λ-periodic in all $\mathbb{C}$.

It is clear that g is well-defined in all $\mathbb{C} \setminus \bigcup_{\lambda\in\Lambda}(D_\gamma + \lambda)$ except at the poles of f. Assume there exists a point $z \in D_\gamma$ for which $g(z) = \infty$. From (5), one obtains

$$W \circ \Phi(z) = \Phi(\infty) = 0, \qquad \text{if and only if} \qquad \Phi(z) \in Z_W \cap \widehat{\mathbb{C}} \setminus D_{\tilde{\gamma}},$$

which is equivalent to $z \in \Phi^{-1}(Z_W - \{0\})$. If z is a pole of g in $D_\gamma + \lambda$ for some $\lambda \in \Lambda$, then equivalently, $z \in \Phi^{-1}(Z_W - \{0\}) + \lambda$. We conclude

$$\text{Poles}(g) = \left(P_f \sqcup \Phi^{-1}(Z_W - \{0\})\right) + \Lambda, \tag{4.2}$$

and thus, g has exactly $o + d - 1$ poles on each fundamental domain of Λ. In particular, Poles(g) is a discrete set of the complex plane.

To show g is quasiregular, note that g is meromorphic in $\mathbb{C} \setminus \bigcup_{\lambda\in\Lambda}(D_\gamma + \lambda)$. Writing $\overline{D}_\gamma = \overline{A}_{\gamma_{in}} \sqcup D_{\gamma_{in}}$, the action of g in this domain can be expressed as

$$g|_{\overline{D}_\gamma} := \begin{cases} R_{in}^{-1} \circ W \circ R_{in} & \text{on } D_{\gamma_{in}}, \\ h_{in}^{-1} \circ W \circ h_{in} & \text{on } \overline{A}_{\gamma_{in}}. \end{cases}$$

On one hand, $g|_{D_{\gamma_{in}}}$ is the composition of a 1-quasiconformal map R_{in} and a holomorphic map $R_{in}^{-1} \circ W$, thus it is 1-quasiregular. On the other hand, $g|_{\overline{A}_{\gamma_{in}}}$ is $K_{in}-$quasiconformally conjugated to a holomorphic function W. Thus, g is CK_{in}^2-quasiregular in the complex plane (for some $C > 0$) outside its discrete set of poles.

Finally, the topological annulus $A = \Delta \cap \Phi^{-1}(\tilde{\Delta})$ is g-invariant. Indeed, both f restricted to $\Delta \setminus D_\gamma$ and W restricted to $D_\gamma \cap \Phi^{-1}(\tilde{\Delta})$ leave their domains invariant. It follows from the construction that $g|_A$ is conjugated to the rigid rotation $z \mapsto e^{2\pi i\theta} z$.

□

Corollary 4.3. *The function g has a unique essential singularity at $z = \infty$ and at no other point of $\widehat{\mathbb{C}}$.*

Proof. Since g coincides with the elliptic function f in $\mathbb{C} \setminus \bigcup_{\lambda \in \Lambda}(D_\gamma + \lambda)$, it inherits the essential singularity at infinity. In $D_\gamma \setminus \{0\}$, the function g is either conjugate to a rigid rotation or it is conformally conjugated to a rational function. In particular, since W is a rational map, then $g(0) = \Phi^{-1} \circ W \circ \Phi(0) = \Phi^{-1} \circ W(\infty) \in \widehat{\mathbb{C}}$, and the origin reduces to either a removable singularity or a pole for g. □

4.2. **Straightening g.** The g-invariant ring $A = \Delta \cap \Phi^{-1}(\tilde{\Delta})$ can be decomposed into two invariant subrings as $A = A_o \sqcup A_i$ with $A_o = \Delta \setminus D_\gamma$ and $A_i = D_\gamma \cap \Phi^{-1}(\tilde{\Delta})$. Let

$$X = \bigcup_{n \geq 0} g^{-n}(\mathrm{Poles}(g)) \cup \{\infty\}.$$

As it was shown in Theorem 4.2, the poles of g form a discrete set. By the σ-additivity of the Lebesgue measure, X has measure zero.

Theorem 4.4. *There exists a measurable Beltrami differential μ defined over the extended complex plane that has bounded dilatation, it is g-invariant and satisfies that for any $\lambda \in \Lambda$, $\mu(u + \lambda) = \mu(u)$.*

Proof. As customary, denote by μ_0 the constant Beltrami differential equal to zero. We define μ in several steps. First, using the partition $A = A_o \sqcup A_i$, let $\mu_A := \mu_0$ in A_o and set $\mu_A := \Phi^*(\mu_0)$ in A_i, in that way $\mu|_A = \mu_A$. We now define μ along the backward orbit of A under g: for any integer $n > 0$, let $\mu = (g^n)^*(\mu_A)$ on $g^{-n}(A)$. Finally, set $\mu = \mu_0$ everywhere else, so in particular, $\mu = \mu_0$ in X.

To see μ has bounded dilatation, first observe that g is conformal in A_o and K_{in}^2-quasiregular in A_i, thus, the dilatation of $\mu|_A$ is bounded. In the first pullback defined on $g^{-1}(A) \setminus A$, the dilatation of μ may increase by a factor of K_{in}, as one of the branches of g^{-1} may be K_{in}-quasiconformal. After that, all subsequent pullbacks are performed by holomorphic branches of g, so the dilatation of μ does not increase and hence, remains bounded. Thus, we obtain that μ is a g-invariant Beltrami differential of bounded dilatation and defined in the extended complex plane.

Finally, observe that for any $u \in \mathbb{C}$ and any $\lambda \in \Lambda$, one has $\mu(u) = T_\lambda^* \mu(u)$, since T_λ is conformal. Furthermore, the definition of pullback of a Beltrami differential yields

$$T_\lambda^* \mu(u) := \mu(T_\lambda(u)) \frac{\overline{\partial_z T_\lambda(u)}}{\partial_z T_\lambda(u)} = \mu(u + \lambda),$$

since $\partial_z T_\lambda \equiv 1$. Hence μ is Λ-periodic. □

The Measurable Riemman Mapping Theorem guarantees the existence of a unique K'-quasiconformal map $\psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $0 < K' = C' K_{in}^3 < \infty$, that solves the Beltrami equation $\psi^* \mu_0 = \mu$ and fixes 0, 1 and ∞. It follows from the proof of Theorem 4.2 that

$g^n|_{\mathbb{C}\setminus X}$ is at least CK_{in}^2-quasiregular for any $n \geq 1$. Then, Sullivan's Straightening Theorem implies that g is quasiconformally conjugate to a transcendental meromorphic function

$$\mathscr{G} = \psi \circ g \circ \psi^{-1} : \mathbb{C} \to \widehat{\mathbb{C}}.$$

We want to show that $\mathscr{G}$ is an elliptic function with respect to a double lattice.

Lemma 4.5. *Given $\lambda \in \Lambda$ and it associated translation $T_\lambda(z) = z + \lambda$, its conjugate map $\psi \circ T_\lambda \circ \psi^{-1}$ is equal to the translation $z \mapsto z + \psi(\lambda)$.*

Proof. The case $\lambda = 0$ is straightforward so assume $\lambda \neq 0$ and set $S_\lambda := \psi \circ T_\lambda \circ \psi^{-1}$. Since $S_\lambda^*(\mu_0) = \mu_0$ and fixes the point at infinity, then S_λ is a conformal automorphism of the plane. Moreover, $S_\lambda(0) = \psi(\lambda)$ and $S_\lambda(1) = \psi(1+\lambda)$, so S_λ must coincide with the affine transformation $z \mapsto (\psi(1+\lambda) - \psi(\lambda))z + \psi(\lambda)$. Furthermore, T_λ has a single fixed point of multiplicity two at infinity, and under the conjugacy, so is S_λ. This implies that for every $\lambda \in \Lambda$, the linear coefficient $\psi(1+\lambda) - \psi(\lambda)$ is equal to 1 and the conclusion follows. $\square$

Remark 4.6. The condition on the linear coefficient above entails the identity

$$\psi(1+\lambda) = 1 + \psi(\lambda), \text{ for any } \lambda \in \Lambda. \tag{4.3}$$

And from the identity $\psi \circ T_\lambda = T_{\psi(\lambda)} \circ \psi$, one has that for any $z \in \mathbb{C}$ and any $\lambda \in \Lambda$,

$$\psi(z+\lambda) = \psi(z) + \psi(\lambda). \tag{4.4}$$

The following result is a straightforward consequence of (4.4).

Lemma 4.7. *The set $\tilde{\Lambda} = \{\tilde{\lambda} := \psi(\lambda) \mid \lambda \in \Lambda\}$ is a double lattice with group representation $T(\tilde{\lambda}) = T_{\psi(\lambda)}$.*

Proof. The action $\psi|\Lambda : \Lambda \to \tilde{\Lambda}$ defines a group homomorphism by (4.4), so $\tilde{\Lambda}$ is a discrete additive group isomorphic to $\mathbb{Z} \times \mathbb{Z}$. The identity element is $0 = \psi(0)$ and for any $\tilde{\lambda} \in \tilde{\Lambda}$, its additive inverse is given by $-\tilde{\lambda} := \psi(-\lambda)$. $\square$

Theorem 2 *The function $\mathscr{G} : \mathbb{C} \to \widehat{\mathbb{C}}$ is a doubly periodic meromorphic function with respect to the lattice $\tilde{\Lambda}$, it has order $o + d - 1 \geq 3$ and possesses an invariant Herman ring.*

Proof. The conjugacy of $\mathscr{G}$ and g via the homeomorphism ψ preserves the number of poles on each fundamental domain. Also, the above construction guarantees that $\mathscr{A} = \psi(A)$ is an invariant Fatou component of the meromorphic function $\mathscr{G}$ where it is conjugated to the irrational rotation $z \mapsto e^{2\pi i\theta}z$. We are left to show that $\mathscr{G}$ is doubly periodic with respect to the lattice $\tilde{\Lambda}$. Indeed, from Lemma 4.5 and the Λ-periodicity of g shown in Theorem 4.2, we obtain

$$\begin{aligned} \mathscr{G} \circ T_{\psi(\lambda)} &= \psi \circ g \circ \psi^{-1} \circ (\psi \circ T_\lambda \circ \psi^{-1}), \\ &= \psi \circ g \circ T_\lambda \circ \psi^{-1}, \\ &= \psi \circ g \circ \psi^{-1}, \end{aligned}$$

hence $\mathscr{G}(z + \tilde{\lambda}) = \mathscr{G}(z)$ for all $z \in \mathbb{C}$ and all $\tilde{\lambda} \in \tilde{\Lambda}$. We conclude that $\mathscr{G} \in \mathcal{E}(\tilde{\Lambda})$ is an elliptic function of order $o + d - 1 \geq 3$ and with an invariant Herman ring. $\square$

5. A bound for the number of invariant Herman rings

For a given $m \geq 1$, let $H_1, \ldots, H_m$ be invariant Herman rings of the elliptic function $f \in \mathcal{E}(\Lambda)$ with order $o_f \geq 3$. From Theorem A in [FP12], there exists a pole on each bounded connected component of $\mathbb{C} \setminus \bigcup_{j=1}^m H_j$. We begin the proof of Theorem 1.3 by establishing first that residue classes of these poles can be chosen to be pairwise disjoint.

As before, let B_i denote the compact complementary component of H_i and let $\Gamma_i \subset H_i$ be an f-invariant curve.

Lemma 5.1. *There exists a collection of poles, $\zeta_1, \ldots, \zeta_m$, each one selected from a bounded component of $\mathbb{C} \setminus \bigcup_{j=1}^m H_j$ such that $\zeta_j \equiv \zeta_k \mod \Lambda$ if and only if $j = k$.*

Proof. It suffices to consider the case $m = 2$. Assume that B_1 contains the pole ζ while B_2 contains the pole $\zeta + \lambda$ for some $\lambda \in \Lambda$. From Λ-periodicity, either $H_2 - \lambda$ lies in the interior of B_1 or otherwise, $H_1 + \lambda$ lies in the interior of B_2. To fix ideas, consider the case $H_2 - \lambda \subset \operatorname{int}(B_1)$ (the other case follows by interchanging the labels). Let A denote the bounded annular domain defined by the curves $\Gamma_2 - \lambda$ and Γ_1. If $\lambda = 0$ then A must contain a pole η by Theorem A in [FP12]. Similarly, if $\lambda \neq 0$ but A contains a pole η, then it follows from Proposition 2.3 that η and ζ must belong to distinct residue classes, so in both cases we are done.

Let $\lambda \in \Lambda^*$ and assume A contains no other pole of f. Since $\overline{A}$ is compact and $f|\overline{A}$ is holomorphic (and thus continuous), the Maximum Modulus Principle implies that $z \mapsto |f(z)|$ attains its maximum at the boundary of A. Nevertheless, the image of ∂A under f lies in two distinct fundamental domains, and since $f|A$ is an open mapping, this contradicts the previous assertion. Thus, A must contain a pole whose residue class is disjoint from ζ. ◻

We are now able to provide an upper bound for the number of invariant Herman rings of an elliptic function.

Theorem 3. Let $\zeta_1, \ldots, \zeta_m$ be the poles obtained in Lemma 5.1. Denote by $H_{j_1}, \ldots, H_{j_k}$ the invariant Herman rings that are not contained in any bounded component B_i. In turn, for each $s = 1, \ldots, k$, the domain B_{j_s} contains at least n_{j_s} poles: namely, its own pole ζ_{j_s} and any other element $\zeta_i + \lambda \in [\zeta_i]$ for $\lambda \in \Lambda$ such that $H_i + \lambda$ lies inside B_{j_s}. Also, B_{j_s} may contain a pole whose residue class is disjoint from the set $\{\zeta_1, \ldots, \zeta_m\}$. Hence, $\#(B_{j_s} \cap \operatorname{Poles}(f)) \geq n_{j_s}$. From the definition of n_{j_s}, observe that $1 \leq n_{j_s} \leq m$ and $\sum_{s=1}^k n_{j_s} = m$.

Through quasiconformal surgery, one can turn each H_{j_s} into an invariant Siegel disk. We describe the main steps of the construction and refer the reader to [BF14, Chapter 7] for a detailed explanation.

Fix $s = 1$. For simplicity, write $H := H_{j_1}, B := B_{j_1}$ and for an f-invariant curve $\Gamma \subset H$, denote by D_Γ the bounded component of $\mathbb{C} \setminus \Gamma$. As stated in Definition 2.2, the conformal isomorphism $\varphi : H \to A_{1,r}$ conjugates f with the rigid rotation $R_\theta(z) = e^{2\pi i \theta}$, where $\theta \in [0, 1)$ is the rotation number of H. We can find a unique $r_0 \in (1, r)$ so that $\varphi(\Gamma) = C_{r_0}$. Since the restriction $\varphi^{-1}|C_{r_0} : C_{r_0} \to \Gamma$ is a C^1-diffeomorphism (hence quasisymmetric) there exists a quasiconformal extension $\Phi^{-1} : \overline{D(0, r_0)} \to \overline{D_\Gamma}$ where $\Phi^{-1} \equiv \varphi$ on Γ. Consider the function

$$g(z) := \begin{cases} \Phi^{-1} \circ R_\theta \circ \Phi \circ T_\lambda^{-1}(z) & \text{if } z \in D_\Gamma + \lambda, \lambda \in \Lambda, \\ f(z) & \text{otherwise.} \end{cases}$$

The proof of Theorem 4.2 extends almost verbatim to show that $g : \mathbb{C} \to \widehat{\mathbb{C}}$ is a quasiregular map outside a discrete set of poles given by

$$\text{Poles}(g) = \text{Poles}(f) \setminus \{[\zeta] \ : \ \zeta + \lambda \in B \cap \text{Poles}(f) \text{ for some } \lambda \in \Lambda\}.$$

The function g remains doubly-periodic with respect to Λ, it has at most $o_f - n_{j_1}$ poles on each fundamental domain, and it has an invariant disk where g is conjugated to R_θ. By Theorem 4.4, there exists a bounded g-invariant Beltrami differential μ, integrated by a quasiconformal mapping $\psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ (fixing $0, 1$ and ∞) so as in Theorem 1.2 one can conclude that

$$f_1 := \psi \circ g \circ \psi^{-1} : \mathbb{C} \to \widehat{\mathbb{C}}$$

is an elliptic function with respect to the lattice $\psi(\Lambda)$, has order at most $o_f - n_{j_1}$ and its Fatou set contains an invariant Siegel disk and $m - n_{j_1} \geq 0$ invariant Herman rings given by $\psi(H_{j_2}), \ldots, \psi(H_{j_k})$. From Liouville's Theorem,

$$2 \leq o_{f_1} \leq o_f - n_{j_1}.$$

If $n_{j_1} = m$ we are done. Otherwise, apply the surgery construction to f_1 and the ring $\psi(H_{j_2})$ to obtain a new elliptic function f_2 with two Siegel disks, $m - (n_{j_1} + n_{j_2})$ invariant Herman rings and order $2 \leq o_{f_2} \leq o_f - (n_{j_1} + n_{j_2})$. Repeating the process if necessary, one ends with an elliptic function f_k with exactly k invariant Siegel disks and no Herman rings. The order of f_k is $2 \leq o_{f_k} \leq o_f - m$ and the sought inequality follows.

□

A very useful consequence of Theorem 1.3 is the following.

Corollary 1 *If f has $m \geq 1$ invariant Herman rings, then the complement of $\bigcup_{i=1}^m (H_i \cup B_i) + \Lambda$ contains the residue classes of (at least) two simple poles or (at least) the residue class of a multiple pole.*

This result provides an even finer upper bound on the number of invariant Herman rings according to the poles and their multiplicities. To do that, consider the integer partitions of $o \geq 2$, that is, the collection of all positive integers that sum up to o. We can associate to each partition an elliptic function whose poles have the multiplicities described by the partition. And although each of these functions have the same order (and hence, have at most $o - 2$ invariant Herman rings), one can say more.

As an example, let $o = 4$. Its integer partitions are given by

$$\{(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)\}.$$

If $f_{(4)}$ denotes an order 4 elliptic function with a single pole of multiplicity 4, then by Corollary 1.4 it cannot have an invariant Herman ring. Similarly, if $f_{(3,1)}$ has a triple pole and a simple pole, then it has at most one invariant Herman ring which must surround a simple pole. The function $f_{(2,2)}$ has at most an invariant Herman ring that surrounds a double pole. For the rest of the partitions, their associated functions can have at most 2 invariant Herman rings, as established by Theorem 1.3. We conclude with the following.

Corollary 5.2. *Any elliptic function with a pole of full multiplicity cannot have invariant Herman rings.*

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