Submitted to
Topology Proceedings

A CLASS OF EVEN ELLIPTIC FUNCTIONS WITH NO HERMAN RINGS

MÓNICA MORENO ROCHA AND PABLO PÉREZ LUCAS

Abstract. In this note we study the dynamical and topological properties of Julia and Fatou sets of certain even elliptic functions. By computing their conformal class, we obtain sufficient conditions to show these functions do not exhibit Herman rings, extending known results for the Weiertrass $\wp$-function (Hawkins & Koss, 2004), and for $1/\wp$ over triangular lattices (Koss, 2009). As an application, we show Julia sets of $1/\wp$ over square lattices are either connected or totally disconnected.

1. Introduction

Consider an analytic map $f : U \to U$ defined over an annular domain U of the complex plane. U is called a *Herman ring* if the iterates of $f|U$ are analytically conjugate to an irrational rotation acting on a round, non-degenerated annulus. In general, it is a difficult problem to determine their existence since, in contrast to other types of Fatou domains, Herman rings are not associated to periodic orbits.

2010 *Mathematics Subject Classification.* Primary 37F10, 33E05, 37F20; Secondary 37F45.

Key words and phrases. Elliptic functions, Herman rings, connectivity, Julia sets.

In this note we give sufficient conditions on a class of even elliptic functions that prevents the presence of cycles of Herman rings. Elliptic functions can be described succinctly as double periodic transcendental meromorphic functions whose set of poles form an infinite and discrete set in $\mathbb{C}$. As meromorphic functions, they belong to the Speiser class $\mathcal{S}$ since any elliptic function has a finite number of critical values, with no other type of singular values. This implies the existence of only preperiodic Fatou components that eventually map into a cycle of superattracting, attracting, parabolic domains or cycles of Siegel disks or Herman rings. In Section 2 we provide a short introduction to elliptic functions and their dynamics over square lattices. We refer the reader to the classical expositions of [9] for background on elliptic functions, the survey in [2] and the book [8] for general information on dynamics of transcendental meromorphic functions.

Our first result is based on the work of Jane Hawkins and Lorelei Koss presented in [6] and [7], where they show (among many other results) that Weierstrass $\wp$-function has no cycle of Herman rings over any lattice. Theorem A is then derived after computing the *conformal equivalence class* of $\wp$ (discussed in Section 3).

Theorem A. Let Θ be any lattice. Then, for any choice of $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$ and $M \in \mathrm{Aut}(\widehat{\mathbb{C}})$, the even elliptic function defined with respect to $\alpha\Theta$,

$$g_{\alpha\Theta}(z) = M(\alpha^2 \wp_{\alpha\Theta}(z-\beta))$$

has no cycle of Herman rings.

Observe that $z \mapsto -1/\wp_\Theta(z)$ belongs to the conformal class of $\wp_\Theta$ for $\alpha = 1$, $\beta = 0$ and $M(z) = -1/z$ above. As a consequence, we can generalize previously known results by L. Koss for triangular lattices, [10], and P. Pérez Lucas for real rectangular lattices, [11].

Corollary 1.1. Over any given lattice Θ, the even elliptic function $z \mapsto 1/\wp_\Theta(z)$ does not exhibit cycles of Herman rings.

In Section 4 we derive another application of Theorem A, namely a Fundamental Dichotomy Theorem for $f_\Omega(z) = 1/\wp_\Omega(z)$, where Ω denotes any square lattice. This result provides an extension to the original dichotomy result of L. Koss for $1/\wp$ defined over triangular lattices, [10, Theorem 1.1].

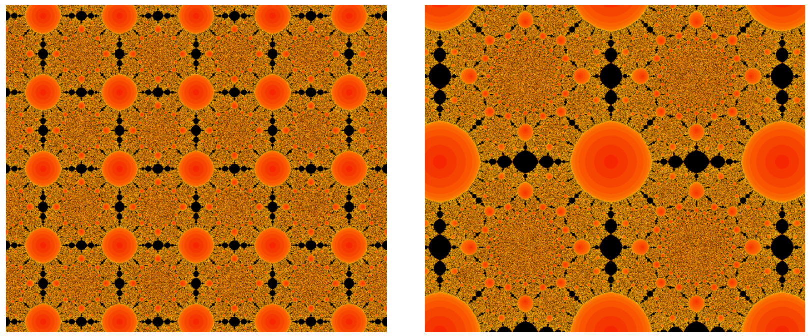

FIGURE 1. An example of a connected Julia set for $1/\wp$ on a real square lattice. Both images are centered at the origin, which is a superattracting fixed point. The basin of the origin appears in shades of orange. There is also an attracting cycle (in black) containing the orbits of the non-zero critical values.

Theorem B (Fundamental dichotomy over square lattices)**.** Let Ω be a square lattice. If all three critical values of f_Ω belong to the same Fatou component, then its Julia set is totally disconnected. Otherwise, the Julia set is connected.

The proof contained in Section 4 proceeds along the same lines of the proof of Theorem 1.1 in [10], although some major differences are implied by the symmetries and properties of $1/\wp$ on square lattices. A version of Theorem B for real square lattices appeared first in the master's thesis work of the second author (see [11]), here we present a generalization for any square lattice using a different and more direct approach.

Some examples of connected and totally disconnected Julia sets for $1/\wp$ over real square lattices are provided in Figures 1 and 2.

2. Background

Let f denote a non-constant, meromorphic function defined over the complex plane, $\mathbb{C}$, and taking values on the Riemann sphere, $\widehat{\mathbb{C}}$. The function f is called *double periodic* if its set of periods, Θ, is given by the additive subgroup $\{n\theta_1 + m\theta_2 \mid n, m \in \mathbb{Z}\}$, where

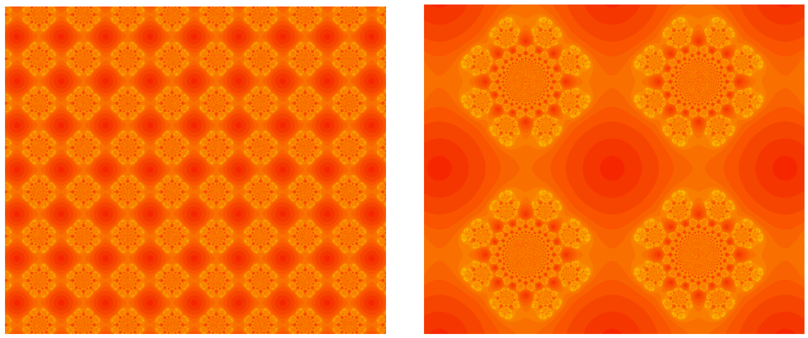

FIGURE 2. An example of a totally disconnected Julia set for $1/\wp$ on a real square lattice. Both images are centered around the origin. In this case, all critical values lie in the immediate basin of the origin, displayed in shades of orange.

$\theta_1, \theta_2 \in \mathbb{C}$ are $\mathbb{R}$-independent. We write $\Theta = \langle \theta_1, \theta_2 \rangle$ to denote the base $\{\theta_1, \theta_2\}$ generating Θ. In this sense, we say $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is an *elliptic function with respect to* Θ if any element $\zeta \in \Theta$ satisfies $f(z + \zeta) = f(z)$ for all $z \in \mathbb{C}$ (that is, ζ is a period of f).

Two lattices Θ and Ω are said to be *similar* if $\Theta = k\Omega$, for some $k \in \mathbb{C}^*$. Similarity is an equivalence relation between lattices and an equivalence class is usually known as a *shape*, [6]. One particular shape we are interested in consists of *square lattices*, that is, $\Omega = \langle \omega, i\omega \rangle$ for some $\omega \in \mathbb{C}^*$. Clearly, any square lattice is similar to a *real square lattice*, that is, when the generator is real and positive. From now on $\Lambda = \langle \lambda, i\lambda \rangle$ represents a real square lattice, Ω a square lattice and Θ any given lattice, possibly of a different shape as Ω.

A closed, connected subset $\mathcal{Q}$ of $\mathbb{C}$ is said to be a *fundamental region* for Θ if for any point $z \in \mathbb{C}$, there exists (at least) one point in $\mathcal{Q}$ that is congruent mod Θ to z while no two points in the interior of $\mathcal{Q}$ are congruent. We denote by $\mathcal{Q}_0$ the fundamental region for Θ with vertices in $0, \theta_1, \theta_1 + \theta_2$ and θ_2. Clearly, for any square lattice we can always find a square fundamental region $\mathcal{Q}$.

For any given lattice Θ, the Weierstrass $\wp$-function can be defined by the series

$$\wp_{\Theta}(z) = \frac{1}{z^2} + \sum_{\theta\in\Theta\setminus\{0\}} \left(\frac{1}{(z-\theta)^2} - \frac{1}{\theta^2} \right).$$

This is an even elliptic function with respect to Θ, it has degree 2 when restricted over a fundamental region or any translation $\mathcal{Q}+t$, for $t \in \mathbb{C}$. The set of poles, $\Pi(\wp)$, coincides with Θ and each of them has order 2. Its derivative, $\wp'$, is an odd elliptic function of order 3 with periods at Θ. Both functions satisfy a homogeneity property that will be important in the next section.

Proposition 2.1 (Homogeneity)**.** *For any given $k \in \mathbb{C}^*$, and any lattice Θ, the functions $\wp$ and $\wp'$ satisfy,*

$$\begin{aligned} \wp_{k\Theta}(ku) &= \frac{1}{k^2}\wp_{\Theta}(u), \\ \wp'_{k\Theta}(ku) &= \frac{1}{k^3}\wp'_{\Theta}(u). \end{aligned}$$

We denote by $\mathcal{E} = \mathcal{E}(\Theta)$ the set of all elliptic functions defined over a fixed lattice Θ. It is not difficult to show that $\mathcal{E}$ is in fact a field. Remarkably, $\mathcal{E}$ is equal to the rational field $\mathbb{C}(\wp, \wp')$, so any elliptic function $f \in \mathcal{E}$ can be written as

$$f(z) = R \circ \wp(z) + \wp'(z) \cdot (S \circ \wp(z)),$$

where R and S are rational maps with complex coefficients. Even elliptic functions coincide with the rational field $\mathbb{C}(\wp)$, see [9, Theorem 3.11.1].

The dynamical properties of $\wp$ over square lattices have been extensively studied by Hawkins and Koss in [6] and [7], and Clemons in [4] among other references. We collect a series of results that we will use extensively in the following sections and refer the reader to the above articles and chapter 3 in [9] for their proofs.

Theorem 2.2. *For a fixed $\omega \in \mathbb{C}^*$, let $\Omega = \langle \omega, i\omega \rangle$ denote a square lattice. The following properties hold for $\wp_\Omega$.*

(1) *$\wp_\Omega$ has simple critical points over the* half periods $\omega/2, i\omega/2$ *and $\omega/2 + i\omega/2$. Thus, the set of critical points of $\wp_\Omega$ are*

given by

$$Crit(\wp_\Omega) = \left\{\frac{\omega}{2}, \frac{i\omega}{2}, \frac{\omega + i\omega}{2}\right\} + \Omega.$$

(2) *Since $\wp_\Omega$ is double periodic, it has only three critical values,*

$$e_1 = \wp_\Omega\left(\frac{\omega}{2}\right), \ e_2 = \wp_\Omega\left(\frac{i\omega}{2}\right), \ e_3 = \wp_\Omega\left(\frac{\omega + i\omega}{2}\right).$$

The *Julia set* of an elliptic function $h \in \mathcal{E}$ is defined as

$$J(h) = \overline{\bigcup_{n\geq 0} h^{-n}(\infty)}.$$

The *Fatou set*, $F(h)$, is the complement of the Julia set in the plane and coincides with the largest open set where the iterates of h are both defined and form a normal family. We are interested in the following properties of the Julia and Fatou sets. Refer to [6], [7] and [4] for their proofs.

Theorem 2.3. *Let Θ be any lattice. Then, for any $h \in \mathcal{E}(\Theta)$,*

(1) $J(h) + \Theta = J(h)$ *and* $F(h) + \Theta = F(h)$.

(2) *If h is even, then* $(-1)J(h) = J(h)$ *and* $(-1)F(h) = F(h)$.

Moreover, if Ω is any square lattice, then

$$J(\wp_\Omega) = iJ(\wp_\Omega) \text{ and } F(\wp_\Omega) = iF(\wp_\Omega).$$

3. Conformal Class of an Elliptic Map

Fix a lattice $\Theta = \langle\theta_1, \theta_2\rangle$ and consider an elliptic function $h \in \mathcal{E}(\Theta)$, we would like to determine its conformal equivalence class. If there exist maps $L \in \mathrm{Aut}(\mathbb{C})$ and $M \in \mathrm{Aut}(\widehat{\mathbb{C}})$ so that for some lattice Σ and a function $g_\Sigma \in \mathcal{E}(\Sigma)$

$$g_\Sigma \circ L(z) = M \circ h_\Theta(z),$$

for all $z \in \mathbb{C}$, we say that g_Σ is *conformally equivalent* to h_Θ. In order to compute the conformal class of h_Θ, consider an affine map $L(z) = \alpha z + \beta$, with $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$, and M, a Möbius transformation. Then, the conjugacy equation becomes

$$g_\Sigma(z) = M \circ h_\Theta\left(\frac{1}{\alpha}(z - \beta)\right). \tag{3.1}$$

Since $\mathcal{E}(\Theta)$ is a rational field, then $M \circ h_\Theta \in \mathcal{E}(\Theta)$. We need to ensure that $h_\Theta \circ L^{-1}$ is a non-constant, double periodic meromorphic function defined over some lattice Σ that needs to be determined.

Consider first the simplest case when $h_\Theta = \wp_\Theta$. Let $w = L^{-1}(z)$. By the homogeneity property of the Weierstrass map,

$$h_\Theta \circ L^{-1}(z) = \wp_\Theta\Big(\frac{1}{\alpha}(z-\beta)\Big) = \alpha^2 \wp_{\alpha\Theta}(z-\beta).$$

That is $\Sigma = \alpha\Theta$ for some $\alpha \in \mathbb{C}^*$. Thus, not surprisingly, conformally equivalent maps to the Weierstrass $\wp$-function are defined over similar lattices and its conformal class reduces to maps of the form

$$g_{\alpha\Theta}(z) = M\big(\alpha^2 \wp_{\alpha\Theta}(z-\beta)\big),$$

for any choice of $\alpha \in \mathbb{C}^*, \beta \in \mathbb{C}$ and $M \in \mathrm{Aut}(\widehat{\mathbb{C}})$. Since the action of $g_{\alpha\Theta}(z)$ on $\mathcal{Q}_0 - \beta$ is the same as the action of $g_{\alpha\Lambda}(z+\beta)$ on $\mathcal{Q}_0$, we may select $\beta = 0$ for simplicity.

In a more general situation, consider $h_\Theta = R(\wp_\Theta) + \wp'_\Theta S(\wp_\Theta)$ with R and S rational maps with complex coefficients. A straightforward computation shows the following.

Proposition 3.1. *Given an elliptic function h_Θ, its conformal conjugacy class within $\mathcal{E}(\Theta)$ is determined by all those maps of the form*

$$g_{\alpha\Theta}(z) = M \circ R\big(\alpha^2\wp_{\alpha\Theta}(z)\big) + \alpha^3 \wp'_{\alpha\Theta}(z) S\big(\alpha^2 \wp_{\alpha\Theta}(z)\big),$$

for any $\alpha \in \mathbb{C}^*$ and any $M \in Aut(\widehat{\mathbb{C}})$.

Clearly, Theorem A is a consequence of the above proposition and Theorem 5.4 in [6], which states that for any lattice, $\wp$ has no Herman rings. For completeness we provide the details of our assertion next.

Proposition 3.2. *Every element in the conformal class of $\wp_\Theta$ has no Herman rings.*

Proof. We proceed by contradiction. For simplicity, we write $\wp = \wp_\Theta$ and $g = g_{\alpha\Theta}$ that represents an element on the conformal class of $\wp_\Theta$. Let $U \subset \mathbb{C}$ be a Herman ring for g and assume for now U is fixed by the action of g. Let $\varphi : U \to A$ be a conformal equivalence so that $\varphi \circ g = e^{i\zeta}\varphi$ in U, with $\zeta \in \mathbb{R} \setminus \mathbb{Q}$.

Since g is conformally equivalent to $\wp$, we have $M \circ \wp(z) = g \circ L(z)$ for all $z \in \mathbb{C}$. Let $V = L^{-1}(U)$ which is again an annular domain of the plane. Then, for all $z \in V$,

$$\wp(z) = M^{-1} \circ (\varphi^{-1}(e^{i\zeta}\varphi)) \circ L(z),$$

that is, $\wp$ acts conformally on V because it is a composition of conformal maps. Since connectivity number and moduli are conformal invariants, then for every $k \geq 1$, $\wp^k(V)$ is an annular domain of the Riemann sphere with modulus equal to $\mathrm{mod}(V)$.

Assume $J(\wp) \cap V \neq \emptyset$. Because any elliptic function is a ramified N-cover of the plane, it has no exceptional points, so by Montel's theorem, $\mathbb{C} \subset \wp^m(V)$ for some $m > 0$ sufficiently large. But this implies $\mathrm{mod}(\wp^m(V)) = \infty$, a contradiction. Hence V is a Fatou component of $\wp$. Since elliptic functions belong to the class $\mathcal{S}$, it cannot have wandering domains (see [2], §4), so for some $k \geq 0$, $V_k := \wp^k(V)$ is periodic under $\wp^m$ for some $m \geq 1$. By a result of A. Bolsch (see remark following Corollary 2 in [1]), V_k is either a Herman ring (which is impossible for $\wp$ by Theorem 5.4 in [6]) or a *tongue over itself*. This last condition implies that $\wp^m$ assumes in V_k every value on itself infinitely many times, contradicting the injectivity of the function.

The case when U belongs to a n-cycle of Herman rings follows similarly. Indeed, since $V = L^{-1}(U)$ does not have to be fixed by $\wp^n$ we can find $M_j \in \mathrm{Aut}(\widehat{\mathbb{C}})$ so that $M_j \circ \wp^j(V) = g^j(U)$ and in particular, for all $z \in V$

$$\wp^n(z) = M_n^{-1} \circ (\varphi^{-1}(e^{i\zeta}\varphi)) \circ L(z),$$

which is again conformal.

□

4. Fundamental dichotomy

As an application of Theorem A we show next that the Julia set of $f_\Omega(z) = 1/\wp_\Omega(z)$, with $\Omega = \langle \omega, i\omega \rangle$ any square lattice, can be either connected or totally disconnected. First, we recall several properties of f_Ω and its Julia and Fatou sets. Proofs for the real square case can be found in [11] or derived from the properties of $\wp_\Omega$ described in Theorem 2.2 and Theorem 2.3.

Proposition 4.1. *Let Ω be a square lattice. The following properties hold for $f_\Omega = 1/\wp_\Omega$.*

(1) *The critical points and poles of f_Ω are given by*
$$Crit(f_\Omega) = \left\{0, \frac{\omega}{2}, \frac{i\omega}{2}\right\} + \Omega,\ \Pi(f_\Omega) = \left\{\frac{\omega + i\omega}{2}\right\} + \Omega.$$
(2) *The critical values of f_Ω are three distinct complex values given by $0, v_\Omega := 1/e_1$ and $-v_\Omega$.*
(3) *The origin is a superattracting fixed point of f_Ω.*
(4) *$J(f_\Omega) = iJ(f_\Omega)$ and $F(f_\Omega) = iF(f_\Omega)$.*
(5) *$f_\Omega(\pm iz) = -f_\Omega(z)$ for all $z \in \mathbb{C}$.*

Any non-real square lattice $\Omega = \langle \omega, i\omega \rangle$ can be expressed as
$$\Omega = e^{i\theta}\Lambda = e^{i\theta}\langle \lambda, i\lambda \rangle,$$
where $\lambda = |\omega| > 0$ and $\theta = \arg(\omega) \in [0, 2\pi]$. So, by Proposition 2.1, we derive a natural relation between f_Ω and f_Λ: if $z \in \mathbb{C}/\Omega, u \in \mathbb{C}/\Lambda$ so that $z = e^{i\theta}u$, then

$$f_\Omega(z) = \frac{1}{\wp_{e^{i\theta}\Lambda}(e^{i\theta}u)} = \frac{e^{i2\theta}}{\wp_\Lambda(u)} = e^{i2\theta} f_\Lambda(u). \tag{4.1}$$

In particular, the critical values of f_Ω and f_Λ are related by
$$v_\Omega = f_\Omega\left(\frac{\omega}{2}\right) = f_\Omega\left(e^{i\theta}\frac{\lambda}{2}\right) = e^{i2\theta} f_\Lambda\left(\frac{\lambda}{2}\right) = e^{i2\theta} v_\Lambda,$$
and $-v_\Omega = -e^{i2\theta} v_\Lambda$.

Lemma 4.2. *Let R be the ray $\{t\omega \mid t \in \mathbb{R}\}$, and define $L = iR$. Then f_Ω sends R into the line segment $[0, v_\Omega]$ while L is sent into the line segment $[-v_\Omega, 0]$.*

Proof. Observe first that f_Λ sends the imaginary axis to the real axis. In particular, $\mathbb{R}$ is sent into the interval $[0, v_\Lambda] \subset \mathbb{R}$ while $i\mathbb{R}$ is sent to the interval $[-v_\Lambda, 0] \subset \mathbb{R}$. From equation (4.1) we observe that for any $t \in \mathbb{R}$
$$f_\Omega(t\omega) = f_\Omega(e^{i\theta}t\lambda) = e^{i2\theta} f_\Lambda(t\lambda).$$
From this and the relation between critical values, we see that f_Ω sends R into $e^{i2\theta}[0, v_\Lambda] = [0, v_\Omega]$. The action over L is similarly obtained. $\square$

Denote by $B(0)$ the immediate basin of the origin for the function f_Ω. From now on, the notation $[a, b]$ denotes a simple curve with endpoints a and b which is not necesarily a line segment.

Lemma 4.3 (4-fold symmetry)**.** *Let U be a Fatou component of f_Ω and denote by μ a primitive 4th root of unity. If there exist $z_0 \in U$ and an integer n so that $\mu^n z_0 \in U - \{z_0\}$, then $i^k z \in U$ for all $k \in \mathbb{Z}$.*

Proof. Let γ be a simple curve $[z_0, \mu^n z_0]$ completely contained in U and some $n \in \mathbb{Z}$. Assume first $\mu^n = i$ (the case $\mu^n = -i$ is similar). Since $i\gamma$ has iz_0 as a common endpoint with γ, thus it must lie completely inside U. Denote by Γ the union of γ and its rotation by i, this is a curve with endpoints at $\pm z_0$ and passing by iz_0. Since f_Ω is even, it follows that $-U$ is also a Fatou component. Moreover, the curve $-\Gamma$ lies in $-U$ and has the same endpoints as Γ. It follows that $U = -U$ and hence $i^k z_0 \in U$ for all $k \in \mathbb{Z}$.

If $\mu^n = -1$, then Γ is a closed curve passing by $\pm z_0$. In this case, the curve $i\Gamma \subset iU$ passes by $\pm iz_0$. Since both curves differ by an isometry, it follows that $\Gamma \cap i\Gamma \neq \emptyset$, so $U = iU$ and $i^k z_0 \in U$ for all $k \in \mathbb{Z}$. For any other point $z \in U$, let α denote the simple curve joining z_0 with z and apply the above arguments to $\gamma \cup \alpha$.

□

There are two immediate consequences of the 4-fold symmetry.

Corollary 4.4. The immediate basin of attraction of the origin has 4-fold symmetry.

Proof. For any $z \in B(0)$, let $\gamma = [0, z] \subset B(0)$ be a simple curve. By the symmetries in Proposition 4.1, $-\gamma \subset B(0)$, so z and $-z$ both lie in the immediate basin and by the above lemma, the result follows. □

Corollary 4.5. If the immediate basin of attraction of the origin contains another critical value, then it must contain all critical values.

Proof. This follows from the previous corollary and the second item in Proposition 4.1. □

Proposition 4.6. *If all critical values of f_Ω lie in the immediate basin of the origin, then $J(f_\Omega)$ is a Cantor set.*

The proof is a consequence of the following result due to Hawkins and Koss, [7, Theorem 3.15].

Theorem 4.7. If h is a hyperbolic elliptic function and W is a Fatou component of h which is a double toral band and contains all critical values, then $F(h) = W$ and $J(h)$ is a Cantor set.

A *double toral band* is a Fatou component that contains the boundary of a fundamental region, see [7, p.113] for further details.

Proof. (Prop. 4.6) By hypothesis, f_Ω is a hyperbolic map since all critical orbits are attracted to the origin. Let us show $B(0)$ is a double toral band. Consider a simple curve $[0, v_\Omega]$ inside $B(0)$. Since the origin is a fixed point and $B(0)$ is forward invariant, there exists a preimage curve $[0, \omega/2]$ that maps injectively onto $[0, v_\Omega]$. Clearly, this preimage must lie in $B(0)$. By the 4-fold symmetry, the curve $\gamma = [-\omega/2, 0] \cup [0, \omega/2]$ also lies in the immediate basin. The invariance of the Fatou set under translation by Ω implies that $\gamma+\omega$ remains in $B(0)$ since it joins $\omega/2, \omega$ and $3\omega/2$, so in particular $\omega \in B(0)$. Finally, from any given curve $[0, \omega]$ inside $B(0)$ we can generate a closed curve

$$[0, \omega] \cup [0, i\omega] \cup [\omega, \omega + i\omega] \cup [i\omega, \omega + i\omega]$$

that is completely contained in the immediate basin and defines the boundary of a fundamental region. Thus, $B(0)$ is a double toral band and by Theorem 4.7, the Julia set is a Cantor set.

□

The next result will be important in our construction. Its proof differs from the version given by L. Koss in [10] for triangular lattices. Compare also with the proof found in [11] where we use complex moduli tools for the case of real square lattices; here we provide a fairly straightforward argument for any square lattice.

Proposition 4.8. *If Ω denotes a square lattice, then there exists no Fatou component of f_Ω containing exactly two critical values.*

Proof. We have already seen that if $B(0)$ contains another critical value, it most contain all of them (Corollary 4.5). Assume the existence of a Fatou component U containing both $v = v_\Omega$ and $-v$, but not the origin. From this assumption we readily derive the following properties.

(1) *All positive iterates of U have 4-fold symmetry.* Indeed, it follows from Lemma 4.3 that $\pm iv$ also belong to U. Since f_Ω is even and by item 5 in Proposition 4.1, we see that $f_\Omega(U)$ contains $f_\Omega(v)$ and $-f_\Omega(v)$, implying its 4-fold symmetry. The same argument can be applied to subsequent iterates of U.
(2) $B(0)$ *is bounded.* Since U has 4-fold symmetry, there exists a simple closed curve Γ in U that connects the points $\pm z, \pm iz \in U$. Clearly, $B(0)$ lies in the bounded component of $\widehat{\mathbb{C}} \setminus \Gamma$.
(3) *Either U eventually maps into $B(0)$ or it belongs to an N-cycle of attracting, superattracting or parabolic components.* This is a consequence of the Classification Theorem for periodic Fatou components for maps in the class $\mathcal{S}$ (see [8, Theorem 4.5]) and the fact that the boundary of any rotation domain belongs to the accumulation set of postcritical orbits (see [5, Theorem 3.7.15]).

Let $N \geq 1$ be the smallest integer so that $f_\Omega^N(U) \subset B(0)$. It is not difficult to see that $f_\Omega^{-1}(B(0)) = B(0) + \Omega$, so there exists $\zeta \in \Omega$ for which $B(0)+\zeta \subseteq f_\Omega^{N-1}(U)$. By the 4-fold symmetry, it must also contain $B(0) - \zeta$, $B(0) + i\zeta$ and $B(0) - i\zeta$. But for $f_\Omega^{N-1}(U)$ to be connected, it is necessary that these copies are not pairwise disjoint, thus in particular, $B(0) + \zeta = B(0) - \zeta$, so $B(0) = B(0) + 2\zeta$. Since $B(0)$ is bounded, ζ has to be equal to 0. But then $B(0) \subseteq f_\Omega^{N-1}(U)$ and this contradicts the definition of N.

Finally, consider the case when U belongs to a periodic cycle of attracting, superattracting or parabolic components disjoint from $B(0)$, namely $\mathcal{U} = \{U_0, \ldots, U_{N-1}\}$ for some $N \geq 1$. Assume $U_0 = U$ and denote by γ a simple closed curve in U_0 containing $\pm v, \pm iv$. By (5) in Proposition 4.1, $f_\Omega(\gamma)$ is a connected set inside U_1 that contains the non-zero points $f_\Omega(v)$ and $-f_\Omega(v)$. The rays R and L defined in Lemma 4.2 define a partition of the plane into four symmetric quadrants q_1, q_2, q_3 and q_4. If $f_\Omega(v)$ lies in q_j then clearly $-f_\Omega(v)$ lies in $-q_j$, so $f_\Omega(\gamma)$ must intersect both R and L. This implies that $f_\Omega^2(\gamma)$ intersect $[0, v]$ and $[-v, 0]$, so in particular, the second iterate of γ cuts both R and L. As the same conclusion can be reached for $f_\Omega^n(\gamma)$ and all $n \geq 0$, the spherical diameter of these sets cannot converge to zero, contradicting the fact that f_Ω^{jN} converges uniformly to a constant on compact sets of U_j.

□

We now proceed to prove Theorem B. As we have observed, the Fatou set of f_Ω is never empty. By the previous proposition, every Fatou component contains one, three or no critical values. If $B(0)$ contains all critical values, Proposition 4.6 implies the Julia set is a Cantor set. Thus, it remains to show that, whenever a Fatou component contains at most one critical value, the Julia set is connected. To this end, we cite the following result.

Theorem 4.9 (Theorem 3.1 in [7])**.** *Let Θ be any given lattice. If each Fatou component of $\wp_\Theta$ contains either 0 or 1 critical value, then $J(\wp_\Theta)$ is connected.*

The proof of the above result is essentially the same for f_Ω, as both functions have exactly three critical values and act as double-branched coverings over fundamental regions. This concludes the proof of Theorem B.

Acknowledgements. We thank the organizers for the opportunity to present part of these results in the Dynamical Systems special session of the 47th Spring Topology and Dynamics Conference in Central Connecticut State University, New Britain, CT.

We also would like to thank Jesús Hernández for his assistance that allowed us to produce the computer images presented here. The first author was supported in part by Consejo Nacional de Ciencia y Tecnología (grant CB-2010-01, 153850).

References

[1] Andreas Bolsch, *Periodic Fatou components of meromorphic functions.* Bull. London Math. Soc. **31** (1999), 543–555.

[2] Walter Bergweiller, *Iteration of meromorphic functions.* Bull. American Math. Soc. (New Series) **29** (1993), 151–188.

[3] Ian N. Baker, Janina Kotus and Yinian Lü, *Iterates of meromorphic functions III: preperiodic domains.* Ergodic Th. and Dyn. Syst., **11** (1991), 603–618.

[4] Joshua J. Clemons. *Connectivity of Julia sets for Weierstrass elliptic functions on square lattices.* Proc. American Math. Soc., **140**, (2012), 1963–1972.

[5] Edson de Faria and Welington de Melo, *Mathematical tools in one-dimensional dynamics.* Cambridge Studies in Advance Mathematics, 115. Cambridge University Press, 2008.

[6] Jane Hawkins and Lorelei Koss, *Parametrized dynamics of the Weierstrass elliptic function*, Conf. Geom. and Dyn., **8** (2004), 1–35.

[7] Jane Hawkins and Lorelei Koss, *Connectivity properties of Julia sets of Weierstrass elliptic functions*, Topology Appl. **152** (2005), 107–137.

[8] Xin-Hou Hua and Chung-Chun Yang, *Dynamics of transcendental functions.* Asian Mathematics Series, 1. Gordon and Breach Science Publishers, 1998.

[9] Gareth A. Jones and David Singerman, *Complex functions: an algebraic and geometric viewpoint.* Cambridge University Press, 1997.

[10] Lorelei Koss, *A fundamental dichotomy for Julia sets of a family of elliptic functions.* Proc. American Math. Soc. **137** (2009), 3927–3938.

[11] Pablo Pérez Lucas, *Dinámica de la función elíptica* $h_\lambda = \frac{1}{\wp_\lambda}$ *parametrizada sobre retículas cuadradas reales.* Master thesis. Centro de Investigación en Matemáticas. February, 2013.

Centro de Investigación en Matemáticas, CIMAT, A.C.; Guanajuato, México, 36240
E-mail address: mmoreno@cimat.mx

Centro de Investigación en Matemáticas, CIMAT, A.C.; Guanajuato, México, 36240
E-mail address: perezppl@cimat.mx