Non-archimedean dynamics of rational functions

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Let $\mathbb{C}_K$ be a complete and algebraically closed non-archimedean field and let $R \in \mathbb{C}_K(z)$ be a rational function, the iterates of R define a dynamical system over the projective space $\mathbb{P}(\mathbb{C}_K)$ and also over the Berkovich projective line, $\mathbb{P}_B$, associated with $\mathbb{C}_K$. In this dissertation we study two problems related to the iteration of R over these spaces. The first problem consists in describing all possible Berkovich Fatou components associated to a quadratic rational function. The second problem deals with the construction of m-Herman rings from Siegel disks and provides an upper bound for the number of cycles of m-Herman rings.

- In Chapter 2 we provide a brief account of non-archimedean fields. Then we study the dynamics given by the family of iterations of a rational function $R \in \mathbb{C}_K(z)$ acting over the projective space $\mathbb{P}(\mathbb{C}_K)$. We also define the Fatou and Julia sets of R and state without proofs some of the most important results. The primary sources for this chapter are found in [3], [2] and [4].

- In Chapter 3 we define the Berkovich projective line $\mathbb{P}_B$ associated with a field $\mathbb{C}_K$ and study the dynamical system given by the iteration of a rational function $R \in \mathbb{C}_K(z)$ acting over $\mathbb{P}_B$. We define the Berkovich Fatou and Julia sets of R and we state without proofs some results used in the next chapters. The exposition is based on [1] and [3].

- In Chapter 4 we describe the Berkovich Fatou set for a quadratic rational function. We first prove that if a quadratic rational function has two repelling fixed points, then its Berkovich Fatou set consists of a unique attracting basin of Cantor type (Theorem A). Then, under some conditions over the field $\mathbb{C}_K$, we show that a quadratic rational function has at most one m-Herman ring (Theorem B). We also describe those situations when the Berkovich Fatou set contains attracting domains and conclude by establishing sufficient conditions that guarantee no wandering domains exist. These

results generalize and complement the description of the Fatou set for quadratic rational functions with coefficients over $\mathbb{C}_p$ found in [2].

- Chapter 5 contains our work on m-Herman rings for rational functions of degree at least 2. We first show how to construct a rational function $Q \in \mathbb{C}_K(z)$ with a n-cycle of m-Herman rings from a rational function $R \in \mathbb{C}_K(z)$ with a n-cycle of Siegel disks (Theorem C). We also describe an algorithm to construct rational functions with exactly $\deg(R) - 1$ cycles of m-Herman rings. We finish with the proof of Theorem D, which provides, under some conditions on the indifferent domain of R, an upper bound on the number of m-Herman rings, hence providing a partial answer to a question found in [4].

References

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