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# Inverse Function Theorem

In this paper we obtain an extension of the classical inverse function theorem from Banach spaces to some classes of locally convex spaces.

## I. INTRODUCTION

First we should remember the most typical forms of the inverse function theorem and some definitions.

Theorem 1: Let $c$ be a point of Banach space $X$, and $f$ be a function of class $C^1$ from a neighbourhood of the point $c$ into Banach space $Y$, such that $df(c)$ (the Frechet differential of $f$ at the point $c$) is an isomorphism of the spaces $X$ and $Y$. Then the function $f$ is a diffeomorphism from some neighbourhood of the point $c$ onto a neighbourhood of the point $f(c)$.

Let $f$ and $g$ be two functions defined on some neighbourhood of the point $c$ in the space $X$ with values in the space $Y$.

Definition 1. The functions $f$ and $g$ are called uniformly tangent at the point $c$ if for every $r>0$ there exists $s>0$ such that.

$$|(f-g)(x)-(f-g)(y)|\leqslant r|x-y| \text{ for } |x-c|\leqslant s \quad \text{and} \quad |y-c|\leqslant s\,.$$

Definition 2. The function $f$ is called uniformly differentiable at the point $c$ if there exists a continuous linear function $T_c$ from $X$ to $Y$, such that the functions $f$ and $T_c$ are uniformly tangent at the point $c$.

Remarks: If a function $f$ is uniformly differentiable at the point $c$, then it is differentiable in the Frechet sense and $T_c = df(c)$. Let $G$ be an open set in the space $X$.

A function $f$ is of class $C^1$ on $G$, if and only if it is uniformly differentiable at each point of set $G$. This follows easily from the mean value theorem. If we replace in Theorem 1 the class $C^1$ by uniform differentiability of the function $f$ at the point $c$, we get the weaker conclusion, namely that the function $f$ is a homeomorphism from some neighbourhood of the point $c$ onto a neighbourhood of the point $f(c)$ and the inverse function $f^{-1}$ is uniformly differentiable at the point $f(c)$ and $df^{-1}(f(c)) = (df(c))^{-1}$.

We reformulate this remark and we get:

Theorem 2. Let $f$ be a function defined on a neighbourhood of a point $c$ in a Banach space $X$ with values in a Banach space $Y$. We suppose that there exists an isomorphism $T$ from $Y$ to $X$, such that the functions $I_x - T \circ f$ and $0_x$ are uniformly tangent at the point $c$ (by $I_x$ and $0_x$ we denote identity and zero-function on $X$). Then $f$ is a homeomorphism from a neighbourhood of the point $c$ onto a neighbourhood of the point $f(c)$ and there exists an isomorphism $S$ from $X$ to $Y$, such that the functions $I_y - S \circ f^{-1}$ and $0_y$ are uniformly tangent at the point $f(c)$ and $S = T^{-1}$.

## II. SOME EXTENSIONS OF BASIC DEFINITIONS

Let $X$ be a locally convex Hausdorff space. By $Q$ we denote a family of continuous seminorms generating the topology of $X$. Let $f$ be a function from a subset of $X$ with values in $X$, let $A$ be a set contained in the domain of $f_1$ and $r$ be a positive number.

Definition 3. We say that the function $f$ is $r$-Lipschitzian on the set $A$ if for every $x \in A$, $y \in A$, $q \in Q$ we have $q(f(x)-f(y)) \leqslant rq(x-y)$. Let $f$ and $g$ be two functions defined in the neighbourhood of the point $c$ of space $X$ with values in $X$.

Definition 4: The functions $f$ and $g$ are called uniformly tangent at the point $c$ if for every $r > 0$ there exists a neighbourhood $A_r$ of the point zero in $X$, such that the function $f-g$ is $r$-Lipschitzian on the set $c + A_r$.

If $X$ is a Banach space, then the definitions 1 and 4 are equivalent. In what follows we denote by $X$ and $Y$ the locally convex Hausdorff spaces sequentially complete (i.e. such that each Cauchy sequence of $X$ (or $Y$) is convergent).

## III. INVERSE FUNCTION THEOREM

Let $f$ be a function defined on a neighbourhood of the point $c$ in $X$ with values in $Y$. We suppose that there exists an isomorphism $T$ from $Y$ to $X$ such that the functions $I_x - T \circ f$ and $0_x$ are uniformly tangent at the point $c$. Then $f$ is a homeomorphism from a neighbourhood of the point $c$ onto a neighbourhood of the point $f(c)$ and there exists an isomorphism $S$ from $X$ to $Y$, such that the functions $I_y - S \circ f^{-1}$ and $0_y$ are uniformly tangent at the point $f(c)$ and $S = T^{-1}$.

Banach lemma. Let $A$ be a closed subset of $X$, and $f$ be a function defined on $A$ with values in $X$, such that $f(A) \subset A$; Suppose that $f$ is $r$-Lipschitzian with $r < 1$. Then there exists exactly one point $x$ of the set $A$ such that $f(x) = x$.

Proof of lemma: Let $x_0$ be a point of $A$. For each $n$ we put $x_n = f^n(x_0)$. For $q \in Q$ ($Q$ is the family of seminorms from the definition of Lipschitz condition) and $p \geqslant 0$ we get $q(x_{n+1} - x_n) \leqslant rq(x_n - x_{n-1})$, $q(x_{n+1} - x_n) \leqslant r^n q(x_1 - x_0)$, $q(x_{n+p} - x_n) \leqslant r^n q(x_1 - x_0)/1 - r$, and so we have proved that the sequence $\{x_n\}$ is a Cauchy sequence. Therefore it is convergent to some point $x$ of the set $A$. Simultaneously the sequence $\{f(x_n)\}$ is convergent to $x$. The Lipschitz condition implies that the restiiction of $f$ to $A$ is continuous, and so the sequence $\{f(x_n)\}$ converges to $f(x)$, therefore $f(x) = x$. If $f(y) = y$ we get $q(x-y) = q(f(x)-f(y)) \leqslant rq(x-y)$ hence $q(x-y) = 0$ and $x = y$. The uniqueness follows.

Proof of Theorem. Our assumptions imply that there exists a neighbourhood $A_{1/2}$ of the point zero in $X$, such that the function $I_x - T \circ f$ is 1/2-Lipschitz on the set $c + A_{1/2}$. There exists $q \in Q$ and $r > 0$ such that the set $B = \{x \in X : q(x) \leqslant r\}$ is contained in $A_{1/2}$. Let $g$ be the restriction of $I_x - T \circ f$ to the set $c + B$, then $g(x) - g(c) \in 1/2B$ for $x - c \in B$. We put $C = f(c) + 1/2T^{-1}(B)$. On the set $(c+B) \times C$ we define function $F$ by the formula: $F(x, y) = g(x) + T(y)$. For each $y \in C$ we have the following inclusion $F(c+B, y) \subset c+B$, because if $x \in B$, then $F(c+x, y) = g(c) + g(x) - g(c) + T(y)$ is contained in the set $(c - T \circ f(c)) + 1/2B + (T \circ f(c) + 1/2B) = c + B$. Let us remark that for $y \in C$, $x_1 \in c+B$, $x_2 \in c+B$, $q \in Q$ we have $q(F(x_1, y) - F(x_2, y)) \leqslant 1/2q(x_1 - x_2)$. By our lemma applied to the function $F(\cdot, y)$ we get that for each $y \in C$ there exists an unique point $x_y \in c+B$ such that $F(x_y, y) = x_y$. It follows that the function $f$ is one-to-one from the set $c+B$ onto $C$. So we have proved the existence of inverse function $f^{-1}$ (we have $f^{-1}(y) = x_y$). The continuity of $g$ and $T^{-1}$ implies the continuity of $f$. For each $q \in Q$, $y_1 \in C$, $y_2 \in C$, we have inequalities:

$$q(f^{-1}(y_1) - f^{-1}(y_2)) = q(F(f^{-1}(y_1)), y_1) - F(f^{-1}(y_2), y_2))$$
$$\leqslant q(g(f^{-1}(y_1)) - g(f^{-1}(y_2))) + q(T(y_1 - y_2)) \leqslant (1/2)q(f^{-1}(y_1) - f^{-1}(y_2)) + q(T(y_1 - y_2))$$

and so $q(f^{-1}(y_1) - f^{-1}(y_2)) \leqslant 2|T|q(y_1 - y_2)$. This proves the continuity of $f^{-1}$. We have proved that $f$ is a homeomorphism. Now we shall prove that the functions $I_y - T^{-1} \circ f^{-1}$ and $0_y$ are uniformly tangent at the point $f(c)$. Let us fix $r: 0 < r < 1$. Then there exists a neighbourhood $A_r$ of the point zero in $X$, such that we have: $q(x_1 - x_2 - (T \circ f(x_1) - T \circ f(x_2))) \leqslant rq(x_1 - x_2)$ for $q \in Q$, $x_1 \in c + A_r$, $x_2 \in c + A_r$, hence $q(T \circ f(x_1) - T \circ f(x_2) - (x_1 - x_2)) \leqslant (r/1-r)q(T \circ f(x_1) - T \circ f(x_2))$. Let us put $y_1 = f(x_1)$, $y_2 = f(x_2)$. We get for $y_1 \in f(c + A_r)$ and $y_2 \in f(c + A_r)$; $q \in Q$ $q(y_1 - y_2 - (T^{-1} \circ f^{-1}(y_1) - T^{-1} \circ f^{-1}(y_2))) \leqslant (r/1-r)q(y_1 - y_2)$. The last inequality proves our thesis because if $r$ tends to zero, $r/1-r$ must also tend to zero.

Remark. The implicit function theorem may be proved in the same way, taking the Banach lemma as a basic.