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John W. Hagood, Department of Mathematics and Statistics, Northern Arizona University, PO Box 5717, Flagstaff, AZ 86011-5717, USA.
email: `john.hagood@nau.edu`

THE LEBESGUE DIFFERENTIATION THEOREM VIA NONOVERLAPPING INTERVAL COVERS

Abstract

A short proof is given for the Lebesgue Differentiation Theorem using a variation of the Heine-Borel covering property, without reliance on sophisticated approaches such as Vitali covers and the rising sun lemma.

In this paper we use a variation of the Heine-Borel covering property to prove the theorem due to Lebesgue that every monotone function $f : [a, b] \to \mathbb{R}$ is differentiable almost everywhere. The approach is more accessible than typical treatments that use Vitali covers, the rising sun lemma or other methods [1, 2, 3, 4, 5]. Throughout λ represents Lebesgue measure on the real line.

A family of nondegenerate compact intervals $\mathcal{C}$ is a **right adapted interval cover** of a set $E \subseteq \mathbb{R}$ if for each $x \in E$ there is an interval $[L(x), R(x)] \in \mathcal{C}$ such that $L(x) < x < R(x)$ and $[s, R(x)] \in \mathcal{C}$ for all $s \in [L(x), x]$. The term **left adapted interval cover** is defined similarly, and we refer to either of these as an **adapted interval cover**. We say that a family of compact intervals is **nonoverlapping** if the interiors of the intervals are pairwise disjoint.

Covering Lemma. *If $\mathcal{C}$ is an adapted interval cover of a compact set $K \subseteq \mathbb{R}$, then there is a finite collection of nonoverlapping intervals in $\mathcal{C}$ that covers K.*

Proof. Without loss of generality, suppose that $\mathcal{C}$ is right adapted. Let $a = \min K$ and $b = \max K$ and let A be the set of all $t \in [a, b]$ such that $\mathcal{C}$ has a finite nonoverlapping subcover of $[a, t] \cap K$. Then $a \in A$, so A is nonempty. Let $\beta = \sup A$. We first show that $\beta \in K$. Otherwise, β lies in a component (c, d) of $[a, b] \backslash K$ and there is a finite collection $\mathcal{D}$ of nonoverlapping intervals in $\mathcal{C}$ that covers $[a, c] \cap K$. Then $\mathcal{D}$ can be modified by deleting extraneous

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intervals to the right of c and adding $[d, R(d)]$. This contradicts $\beta < d$, so $\beta \in K$.

Now let $t \in (L(\beta), \beta] \cap A$ and choose any finite nonoverlapping collection $\mathcal{D}$ of intervals in $\mathcal{C}$ that covers $[a, t] \cap K$. If $[r, s]$ is the right-most interval of $\mathcal{D}$ that contains t, then either $s \geq b$ in which case $b \in A$ as desired or $s \leq \beta$ and $\mathcal{D}$ can be modified to include $[s, R(\beta)]$. Then $\min\{R(\beta), b\} \in A$, which is impossible unless $b = \beta \in A$. $\square$

The key to the proof of the main theorem is a growth lemma for monotone functions in terms of Dini derivates. As usual, the upper right-hand Dini derivate is given by

$$D^+ f(x) = \inf_{\alpha > 0} \sup_{0 < h < \alpha} \frac{f(x+h) - f(x)}{h}$$

and the other derivates D_+, D^- and D_- are defined similarly. It is well known that the derivates of a monotone function are measurable.

Growth Lemma. *Suppose that f is strictly increasing on $[a, b]$. Let C be the set of points in (a, b) at which f is continuous and let E be a Borel subset of C.*

(a) For any Dini derivate D, if $Df(x) > q$ on E, then $\lambda(f(E)) \geq q\lambda(E)$.

(b) For any Dini derivate D, if $Df(x) < p$ on E, then $\lambda(f(E)) \leq p\lambda(E)$.

PROOF. Part (a): The proofs for D^+ and D^- are similar and the other two cases are then consequences, so we proceed with D^+. Suppose that $D^+ f > q$ on a Borel set $E \subseteq C$. Since f is strictly increasing, $f(E)$ is Borel measurable. Let $\varepsilon > 0$, and choose a compact set $K \subseteq E$ and an open set $U \supseteq f(E)$ such that $\lambda(E \backslash K) < \varepsilon$ and $\lambda(U \backslash f(E)) < \varepsilon$.

Construct a right adapted interval cover $\mathcal{C}$ of K as follows. For each $x \in K$, f is continuous at x so there is an open interval $I \subseteq (a, b)$ about x such that $f(I) \subseteq U$. Choose a number $R(x) \in I$ satisfying $x < R(x)$ and

$$f(R(x)) - f(x) > q(R(x) - x).$$

Using continuity at x, choose $L(x) \in I$ such that $L(x) < x$ and

$$f(R(x)) - f(s) > q(R(x) - s)$$

whenever $L(x) \leq s \leq x$. Let

$$\mathcal{C} = \{[s, R(x)] : x \in K, L(x) \leq s \leq x\}.$$

Then $\mathcal{C}$ is a right adapted interval cover of K, so there is a finite set of nonoverlapping intervals $\{[c_i, d_i]\}_{i=1}^n$ that covers K and associated points $x_i \in K$ such that $L(x_i) \le c_i \le x_i < d_i = R(x_i)$. The intervals $[f(c_i), f(d_i)]$ are also nonoverlapping and lie in U. Then

$$\lambda(f(E)) > \lambda(U) - \varepsilon \ge \lambda(\cup_{i=1}^n [f(c_i), f(d_i)]) - \varepsilon = \sum_{i=1}^n (f(d_i) - f(c_i)) - \varepsilon$$
$$> \sum_{i=1}^n q(d_i - c_i) - \varepsilon \ge q\lambda(K) - \varepsilon > q\lambda(E) - \varepsilon(1+q).$$

Thus, $\lambda(f(E)) \ge q\lambda(E)$.

Part(b): Let $\varepsilon > 0$ and choose a compact set $K \subseteq f(E)$ and an open set $U \supseteq E$ such that $\lambda(f(E)\backslash K) < \varepsilon$ and $\lambda(U\backslash E) < \varepsilon$. Now K must have the form

$$K = [f(\alpha_0), f(\beta_0)]\backslash \bigcup_i (f(\alpha_i), f(\beta_i))$$

for some finite or countable set of points $\alpha_i, \beta_i \in E$, so

$$f^{-1}(K) = [\alpha_0, \beta_0]\backslash \bigcup_{i\ge 1} (\alpha_i, \beta_i)$$

which is a closed subset of E. This permits us to apply the above technique to $f^{-1}(K)$ and U to show that $\lambda(f(E)) \le p\lambda(E)$. □

The proof of the main theorem now follows from two consequences of the growth theorem. In the setting of the lemma, if

$$A = \{x \in C : Df(x) = \infty\}$$

for any Dini derivate D, then for any positive real number q,

$$f(b) - f(a) \ge \lambda(f(A)) \ge q\lambda(A).$$

Thus $\lambda(A) = 0$, so that the Dini derivates of f are finite a.e. Second, all sets of the form

$$B = \{x \in C : D_+ f(x) < p < q < D^- f(x)\}$$

satisfy $q\lambda(B) \le \lambda(f(B)) \le p\lambda(B)$ so that $\lambda(B) = 0$. That is, $D^- f \le D_+ f$ a.e. and similarly $D^+ f \le D_- f$ a.e. The case of a general monotone function follows from the strictly increasing case in standard fashion. Thus we have the

Lebesgue Differentiation Theorem. *If $f : [a, b] \to \mathbb{R}$ is monotone, then f is differentiable almost everywhere.*

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