Annals of Mathematics

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Source: *The Annals of Mathematics*, Second Series, Vol. 20, No. 4 (Jul., 1919), pp. 292-296
Published by: Annals of Mathematics
Stable URL: http://www.jstor.org/stable/1967124
Accessed: 02/03/2009 12:09

NOTE ON THE DERIVATIVES WITH RESPECT TO A PARAMETER OF THE SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS.

By T. H. Gronwall.

1. For a single equation, Dr. Ritt has solved the problem indicated in the title by a very simple and direct method which presupposes only that the solution of the equation has been proved to exist, but makes no use of the properties of the particular type of approximations on which this existence proof is based. Being founded on the integrability of a linear differential equation of the first order by quadratures, Dr. Ritt's proof cannot be extended immediately to a system of equations. It is the purpose of the present note to show how this difficulty may be overcome; similar proofs of the differentiability of the solution with respect to the initial values of both dependent and independent variables are also given.

2. In the following, the subscripts λ, μ, ν will take the values $1, 2, \cdots, n$. Let a be a parameter, and consider the system of differential equations

$$\frac{dy_\nu}{dx} = f_\nu(x; y_1, \cdots, y_n; a) \tag{1}$$

with the initial conditions

$$y_\nu = y_\nu{}^0 \qquad \text{for} \qquad x = x_0; \tag{2}$$

we assume that all the functions

$$f_\nu, f_{\mu\nu} = \frac{\partial f_\nu}{\partial y_\mu} \qquad \text{and} \qquad f_{0\nu} = \frac{\partial f_\nu}{\partial a}$$

are continuous for

$$x_0 \leqq x \leqq x_0 + h, \qquad y_\nu{}^0 - k_\nu \leqq y_\nu \leqq y_\nu{}^0 + k_\nu, \qquad a_1 \leqq a \leqq a_2. \tag{3}$$

We also assume that it has already been proved that the equations (1) possess a unique system of solutions with the given initial conditions, defined on the interval $(x_0, x_0 + h)$ and each y_ν assuming, for $x_0 \leqq x \leqq x_0 + h$ and every a in (a_1, a_2), only values between $y_\nu{}^0 - k_\nu$ and $y_\nu{}^0 + k_\nu$. Then *the partial derivatives $\partial y_\nu/\partial a$ all exist and satisfy the differential equations*

$$\frac{d}{dx}\left(\frac{\partial y_\nu}{\partial a}\right) = \sum_\mu f_{\mu\nu}(x; y_1, \cdots, y_n; a)\frac{\partial y_\mu}{\partial a} + f_{0\nu}(x; y_1, \cdots, y_n; a) \tag{4}$$

with the initial conditions

$$\frac{\partial y_\nu}{\partial a} = 0 \quad \text{for} \quad x = x_0. \tag{5}$$

We begin by proving the following lemma: *when, for* $x_0 \leqq x \leqq x_0 + h$, *the continuous function* $z = z(x)$ *satisfies the inequalities*

$$0 \leqq z \leqq \int_{x_0}^{x} (Mz + A)dx, \tag{6}$$

where the constants M and A are positive or zero, then

$$0 \leqq z \leqq Ahe^{Mh}, \qquad (x_0 \leqq x \leqq x_0 + h). \tag{7}$$

In analogy to the process of integrating a linear differential equation of the first order, we make $z = e^{M(x-x_0)} \cdot \zeta$; let the maximum of ζ on the closed interval $(x_0, x_0 + h)$ occur at $x = x_1$. For this value of x, (6) gives

$$0 \leqq e^{M(x_1-x_0)} \cdot \zeta_{\max} \leqq \int_{x_0}^{x_1} (Me^{M(x_1-x_0)}\zeta + A)dx,$$

whence by the first theorem of the mean

$$0 \leqq e^{M(x_1-x_0)}\zeta_{\max} \leqq \zeta_{\max} \int_{x_0}^{x_1} Me^{M(x-x_0)}dx + \int_{x_0}^{x_1} Adx$$

$$= \zeta_{\max}\,(e^{M(x_1-x_0)} - 1) + A(x_1 - x_0),$$

or finally

$$0 \leqq \zeta_{\max} \leqq A(x_1 - x_0) \leqq Ah,$$

from which (7) follows at once.

By (1) and (2) we have

$$y_\nu = y_\nu^0 + \int_{x_0}^{x} f_1(x;\ y_1,\ \cdots,\ y_n;\ a)dx;$$

denoting by y_ν' the solutions of (1) where the parameter a has been replaced by a' $(a_1 \leqq a' \leqq a_2)$ but the initial conditions remain unchanged, it follows that

$$y_\nu' - y_\nu = \int_{x_0}^{x} [f_\nu(x;\ y_1',\ \cdots,\ y_n';\ a) - f_\nu(x;\ y_1,\ \cdots,\ y_n;\ a)]dx$$

and consequently

$$\frac{y_\nu' - y_\nu}{a' - a} = \int_{x_0}^{x} \left[\sum_\mu \bar{f}_{\mu\nu} \cdot \frac{y_\mu' - y_\nu}{a' - a} + \bar{f}_{0\nu} \right] dx, \tag{8}$$

where $\bar{f}_{\mu\nu} = f_{\mu\nu}\ (x;\ \eta_{1\mu\nu},\ \cdots,\ \eta_{n\mu\nu};\ \alpha_{\mu\nu})$, $\bar{f}_{0\nu} = f_{0\nu}\ (x;\ \eta_{10\nu},\ \cdots,\ \eta_{n0\nu};\ \alpha_{0\nu})$ and the arguments $\eta_{\lambda\mu\nu}$, $\alpha_{\mu\nu}$ are intermediate between y_λ and y_λ', a and a' respectively.

Now denote by M a constant not less than any of the expressions $n|f_{\mu\nu}|$, and by A a constant not less than any of the expressions $|f_{0\nu}|$ for all values of x, y_1, $\cdots$, y_n, a in the region (3). Moreover, denote by $z = z(x, a, a')$, where x is variable, the greatest of the n expressions

$$\left|\frac{y_\nu' - y_\nu}{a' - a}\right|;$$

for fixed values of a and a' ($\neq a$), this z is evidently a continuous function of x in $(x_0, x_0 + h)$. With these definitions of M, A and z, (8) immediately leads to (6), the lemma becomes applicable, and (7) gives

$$\left|\frac{y_\nu' - y_\nu}{a' - a}\right| \leqq Ahe^{Mh}. \tag{9}$$

In particular, it follows that the y_ν are uniformly continuous functions of a. By the existence theorem, the system of differential equation

$$\frac{dY_\nu}{dx} = \sum_\mu f_{\mu\nu}(x;\ y_1, \cdots, y_n;\ a)Y_\mu + f_{0\nu}(x;\ y_1, \cdots, y_n;\ a) \tag{4'}$$

with the initial conditions

$$Y_\nu = 0 \quad \text{for} \quad x = x_0 \tag{5'}$$

has a unique solution for x in $(x_0, x_0 + h)$ and a in (a_1, a_2). Our theorem is therefore proved if we show that

$$\frac{\partial y_\nu}{\partial a} = Y_\nu. \tag{10}$$

From (8), (4') and (5') it follows at once that

$$\begin{aligned} \frac{y_\nu' - y_\nu}{a' - a} - Y_\nu = \int_{x_0}^{x} \Bigg[&\sum_\mu \bar{f}_{\mu\nu}\left(\frac{y_\mu' - y_\mu}{a' - a} - Y_\mu\right) \\ &+ \sum_\mu (\bar{f}_{\mu\nu} - f_{\mu\nu})Y_\mu + \bar{f}_{0\nu} - f_{0\nu}\Bigg] dx \end{aligned} \tag{11}$$

and on account of the continuity of y_ν in respect to a and the continuity of $f_{\mu\nu}, f_{0\nu}$ in their arguments, we may make each of the n expressions

$$\left|\sum_\mu (\bar{f}_{\mu\nu} - f_{\mu\nu})Y_\mu + \bar{f}_{0\nu} - f_{0\nu}\right|$$

less than any assigned ϵ by taking $|a' - a|$ sufficiently small. Denoting now by $z = z(x, a, a')$ the greatest of the n expressions

$$\left|\frac{y_\nu' - y_\nu}{a' - a} - Y_\nu\right|,$$

this z is a continuous function of x for fixed values of a and a', (11) leads to

(6) with $A = \epsilon$, and (7) gives

$$\left| \frac{y_\nu' - y_\nu}{a' - a} - Y_\nu \right| < \epsilon h e^{Mh}$$

uniformly for a, a' in (a_1, a_2), and this last inequality is equivalent to (10), which proves the theorem.

3. We now proceed to prove that *the partial derivatives $\partial y/\partial y_\lambda^0$ all exist and satisfy the differential equations*

$$\frac{d}{dx}\left(\frac{\partial y_\nu}{\partial y_\lambda^0}\right) = \sum_\mu f_{\mu\nu}(x;\ y_1,\ \cdots,\ y_n;\ a)\frac{\partial y_\mu}{\partial y_\lambda^0} \tag{12}$$

with the initial conditions

$$\frac{\partial y_\nu}{\partial y_\lambda^0} = \epsilon_{\lambda\nu} = \begin{cases} 1, \lambda = \nu \\ 0, \lambda \neq \nu \end{cases} \quad \text{for} \quad x = x_0. \tag{13}$$

This follows immediately from the preceding theorem by the substituting

$$y_\nu = z_\nu + y_\nu^0,$$

which gives the differential equations

$$\frac{dz_\nu}{dx} = f_\nu(x;\ z_1 + y_1^0,\ \cdots,\ z_n + y_n^0;\ a)$$

with the initial conditions $z_\nu = 0$ for $x = x_0$. Regarding y_λ^0 as the parameter in the preceding theorem, we find

$$\frac{d}{dx}\left(\frac{\partial z_\nu}{\partial y_\lambda^0}\right) = \sum_\mu f_{\mu\nu}\left(\frac{\partial z_\mu}{\partial y_\lambda^0} + \epsilon_{\lambda\mu}\right)$$

with the initial conditions $\partial z_\nu/\partial y_\lambda^0 = 0$ for $x = x_0$, whence (12) and (13).

4. Finally, supposing the continuity and existence conditions which hold in (3) to be fulfilled also in the extended region

$$x_0 - h_1 \leqq x \leqq x_0 + h, \qquad y_\nu^0 - k \leqq y_\nu \leqq y_\nu + k, \qquad a_1 \leqq a \leqq a_2,$$

the partial derivatives $\partial y_\nu/\partial x_0$ all exist and satisfy the differential equations

$$\frac{d}{dx}\left(\frac{\partial y_\nu}{\partial x_0}\right) = \sum_\mu f_{\mu\nu}(x;\ y_1,\ \cdots,\ y_n;\ a)\frac{\partial y_\mu}{\partial x_0} \tag{14}$$

with the initial conditions

$$\frac{\partial y_\nu}{\partial x_0} = -f_\nu(x_0;\ y_1^0,\ \cdots,\ y_n^0;\ a) \quad \text{for} \quad x = x_0. \tag{15}$$

Let y_ν' be the solutions of (1) when x_0 is replaced by x_0', all y_ν^0 remaining the same; then we obtain in the same manner as (8) was found

$$\frac{y_\nu' - y_\nu}{x_0' - x_0} = \int_{x_0}^{x} \frac{f_\nu(x;\, y_1',\, \cdots,\, y_n';\, a) - f_\nu(x;\, y_1,\, \cdots,\, y_n;\, a)}{x_0' - x_0}\, dx$$
$$- \frac{1}{x_0' - x_0} \int_{x_0}^{x_0'} f_\nu(x;\, y_1',\, \cdots,\, y_n';\, a)\, dx$$
$$= \int_{x_0}^{x} \sum_\mu \bar{f}_{\mu\nu} \frac{y_\mu' - y_\mu}{x_0' - x_0}\, dx - \bar{f}_\nu.$$

Using the auxiliary system

$$\frac{dY_\nu}{dx} = \sum_\mu f_{\mu\nu} Y_\mu$$

with the initial conditions

$$Y_\nu = -f_\nu(x_0;\, y_1^0,\, \cdots,\, y_n^0;\, a) \qquad \text{for} \qquad x = x_0,$$

the proof then proceeds exactly as before.